Yajima, T.
Osaka J. Math.
6 (1969), 435-446

ON A CHARACTERIZATION OF KNOT GROUPS OF SOME SPHERES IN R^4

Dedicated to Professor A. Komatu for his 60th birthday

TAKESHI YAJIMA

(Received February 18, 1969)

1. Introduction

A characterization of knot groups of S^n in S^{n+2} was given by M. A. Kervaire [6] in the case of $n\geqq 3$, and several approaches[1] were established in that of $n=1$. In the case of $n=2$, S. Kinoshita [7] gave a sufficient condition for the existence of a 2-sphere in S^4 whose knot group has a given Alexander polynomial, and Kervaire also gave a sufficient condition for existence of a 2-sphere in a homotopy 4-sphere (Theorem 2, [6]).

Though some necessary conditions were given by Kervaire, it still seems to be difficult to characterize the knot group of S^2 in S^4. In this note we shall concern a special type of knotted 2-spheres in R^4, and prove the following theorem:

Theorem. *In order that a given group G is isomorphic to the knot group of some ribbon 2-knot in R^4, it is necessary and sufficient that G has a Wirtinger presentation*[2] *such that*

(1) $G/[G, G]\cong Z$,

(2) *the deficiency of G equals to* 1.

Ribbon 2-knots are a special kind of locally flat 2-spheres defined in [12] as *simply knotted spheres*, and Z is the additive group of integers.

Since we restrict presentations of G within Wirtinger presentations, the condition (1) assures that the weight of G is 1. On the other hand, E.S. Rapaport [9] proved that the condition (2) implies $H_2(G)=0$, therefore the necessary conditions of Kervaire follow from our conditions. Moreover, in the Theorem 2 of Kervaire, if we restrict presentations of G within the Wirtinger fashion, then homotopy 4-sphere can be substituted by 4-sphere.

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1) [8], Chapter 9.

2) A group presentation $G=(x_1, \cdots, x_n: r_1, \cdots, r_m)$ is called in this note a Wirtinger presentation, if each relator is described in a form $x_i=w_{i,j}x_jw_{i,j}^{-1}$, where $w_{i,j}$ is a word of the free group $F[\boldsymbol{x}]$, $\boldsymbol{x}=(x_1, \cdots, x_n)$. c.f. [1], p. 86.

In the last section we shall discuss, as an appendix, some property of knot groups of spheres in general.

2. Ribbon knots

A *ribbon* D is a singular 2-disk immersed in R^3, whose singularities $\sigma_1, \cdots, \sigma_n$ are all of ribbon type[3]. To define more precisely, let Δ be a 2-disk and f be a immersion of Δ into R^3. If f satisfies the conditions:

(i) $f\,|\,\partial\Delta$ is a homeomorphism,

(ii) each component of singularities of $D=f(\Delta)$ is a simple arc σ_i $(i=1, \cdots, n)$,

(iii) $f^{-1}(\sigma_i)$ consists of two simple arcs $s_{i,1}$ and $s_{i,2}$ such that $\partial s_{i,1}\subset\partial\Delta$ and $s_{i,2}\subset \mathrm{Int}\,\Delta$,

then D is a ribbon and $k=\partial D$ is called a *ribbon knot.*

Let $\sum=\sigma_1\cup\cdots\cup\sigma_n$. Then $D-(k\cup\sum)$ consists of $n+1$ domains D_0, $D_1, \cdots, D_n$. Let $\nu(D_i)$ be the number of components of $\partial\bar{D}_i\cap\sum$. We call $\bar{D}$ a terminal band if $\nu(\bar{D}_i)=1$, an ordinary band if $\nu(D_i)=2$, and a branched band if $\nu(D_i)\geqq 3$.

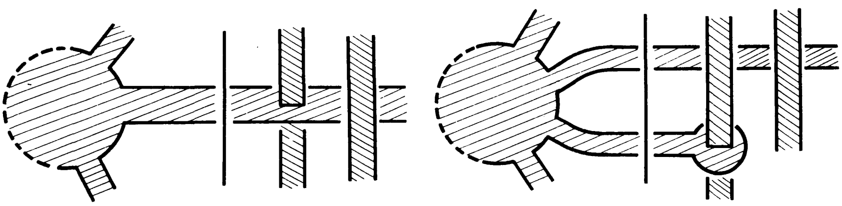
(2.1) *A ribbon knot k can be deformed into the following situation*:

(1) *D has only one branched band,*

(2) σ_i $(i=1, \cdots, n)$ *is contained in the interior of some terminal band,*

(3) *every band does not twist in the projection of k.*

Proof. By sliding bands along k, we can easily prove (1). If there exist a singularity σ_i such that it is contained in the interior of an ordinary band or that of a branched band as shown in Fig. 1, (a), then deform k into (b). By repeating such a deformation of k we can prove (2).



(a) (b)

Fig. 1

If a band has twists, even number of them can be cancelled by the operation in Fig. 2, and the last one, if exists, can be cancelled by a rotation of its terminal band. Thus we have completed the proof.

Now we shall explain another construction of ribbon knots for a convenience in the next section. Let $A, C_1, \cdots, C_\lambda$ be unlinking trivial circles in R^3.

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3) [5] p. 172.

Fig. 2

Take disjoint small arcs $\alpha_1, \cdots, \alpha_\lambda$ on A, and a small arc γ_i on C_i $(i=1, \cdots, \lambda)$. For every i, connect α_i with γ_i by a non-twisting narrow band B_i which may run through C_j $(j=1, \cdots, \lambda)$, or may get tangled with itself or with other bands. Then we have a ribbon knot

$$k=(A \cup C_1 \cup \cdots \cup C_\lambda) \cup (\partial B_1 \cup \cdots \cup \partial B_\lambda) - (\alpha_1 \cup \cdots \cup \alpha_\lambda \cup \gamma_1 \cup \cdots \cup \gamma_\lambda).$$

Conversely, in virtue of (2.1), we have the following proposition:

(2.2) *Every ribbon knot can be constructed as the above fashion.*

Let R^3_t be the hyperplane parpendicular to the x_4-axis of R^4 at $x_4=t$, and let k be a ribbon knot in R^3_0. We can attach[4] a 2–disk D_+ in the halfspace $H^4_+=\{R^3_t \mid 0 \leqq t\}$ to the knot k such that $\partial D_+=k$ and it does not contain any minimal point. The attaching of the disk D_+ to the knot k is completed by the saddle point transformations[5] on the band B_i $(i=1, \cdots, \lambda)$. Let D_- be the disk in $H^4_-=\{R^3_t \mid t \leqq 0\}$ which is the mirror image of D_+ with respect to R^3_0. The sphere $S=D_+ \cup D_-$ was called in [12] a *simply knotted sphere.*

The definition of ribbons is extended to *n-ribbons* as follows: Let Δ^n be a n-dimensional ball and f be a immersion of Δ^n into R^{n+1}. We call $D^n=f(\Delta^n)$ a n-ribbon, if f satisfies the following conditions:

(i) $f \mid \partial \Delta^n$ is a homeomorphism,

(ii) each component of singularities of D^n is a $(n-1)$-ball σ_i,

(iii) $f^{-1}(\sigma_i)$ consists of two $(n-1)$-balls $s^{n-1}_{i,1}$ and $s^{n-1}_{i,2}$ such that $\partial s^{n-1}_{i,1} \subset \partial \Delta^n$ and $s^{n-1}_{i,2} \subset \operatorname{Int} \Delta^n$.

It is easily seen, for instance by the projection method in [11], that for every simply knotted sphere S^2 there exists a 3–ribbon D^3 such that $\partial D^3=S^2$. Therefore we may use the name ribbon 2–knots[6] proposed by F. Hosokawa instead of simply knotted spheres.

3. Proof of the theorem

We shall begin with the proof of that the conditions (1), (2) are necessary. Suppose that there are n overpasses on a band B. Let $c_1, \cdots, c_n$ be generators of the knot group each of which corresponds to one of the overpasses and let

4) [4], p. 133 or [12], (3.1).

5) [4], p. 133.

6) c.f. [13].

$a_0, \cdots, a_n$ and $b_0, \cdots, b_n$ be generators corresponding to the successive positive and the negative sides of the band B respectively (Fig. 3, $\varepsilon_i = \pm 1$).

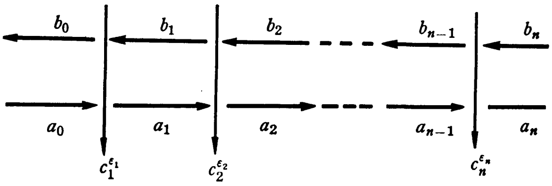

Fig. 3

Then we have for these crossings $2n$ relations

$$\begin{aligned} r_i &= a_{i-1}^{-1} c_i^{\varepsilon_i} a_i c_i^{-\varepsilon_i} = 1\,, \\ s_i &= c_i^{\varepsilon_i} b_i^{-1} c_i^{-\varepsilon_i} b_{i-1} = 1\,. \end{aligned} \qquad (i=1, \cdots, n)$$

As a whole these relations are equivalent to

$$r_i = 1, \quad s_i' = b_{i-1} a_{i-1}^{-1} c_i^{\varepsilon_i} a_i b_i^{-1} c_i^{-\varepsilon_i} = 1\,, \qquad (i=1, \cdots, n).$$

Therefore if we adjoin a new relation $a_0 = b_0$ to them, then we have $a_i = b_i$ $(i=1, \cdots, n)$.

(3.1) *In the situation of Fig.* 3, $2n+1$ *relations*

$$a_0 = b_0\,, \quad r_i = 1\,, \quad s_i = 1\,, \qquad (i=1, \cdots, n)$$

are equivalent to

$$a_0 = b_0\,, \quad r_i = 1\,, \quad a_i = b_i\,, \qquad (i=1, \cdots, n)\,.$$

Now let k be an oriented ribbon knot in a situation of (2.2), and let G_k be the knot group of k. Suppose that a band B_i runs under trivial circles $C_1^{(i)}, \cdots C_{\mu_i}^{(i)}$ in this order. Then the band B_i is divided into $\mu_i + 1$ parts. We shall call each part of B_i a *section* of B_i. Suppose that the j-th section, namely the part between $C_j^{(i)}$ and $C_{j+1}^{(i)}$, runs under bands $B_{j,1}^{(i)}, \cdots, B_{j,\nu(i,j)}^{(i)}$. Let $x_{i,j,k}$ and $y_{i,j,k}$ $(j=0, \cdots, \mu_i;\ k=0, \cdots, 2\nu(i,j))$ be generators of G_k which correspond to the positive sides and the negative sides of the band B_i respectively, and let $z_{i,0}, z_{i,1}, \cdots, z_{i,2\tau_i}$ be generators corresponding to the successive

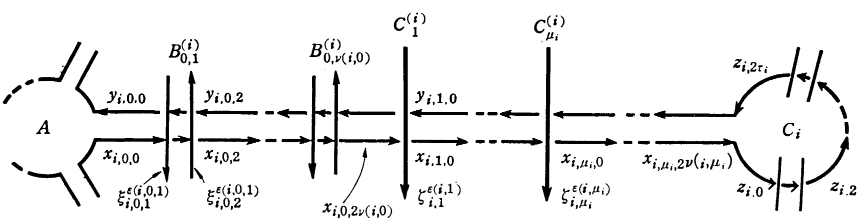

Fig. 4

arcs of C_i as illustrated in Fig. 4, where $\xi_{i,j,2r-1}$ and $\xi_{i,j,2r}$ are generators of the boundaries of $B^{(i)}_{j,r}$ ($r=1, \cdots, \nu(i, j)$), and where $\zeta_{i,j}$ corresponds to the arc of $C^{(i)}_j$ ($j=1, \cdots, \mu_i$).

On the above situation we have $\sum_{i=1}^{\lambda}\sum_{j=0}^{\mu_i} 2(2\nu(i, j)+1)$ generators concerning to the bands and $\sum_{i=1}^{\lambda}(2\tau_i+1)$ generators concerning to the trivial circles. On the other hand the defining relations are classified as follows:

R_1: $\sum_{i=1}^{\lambda}\sum_{j=0}^{\mu_i} 4\nu(i, j)$ relations each of which corresponds to a crossing point of a band B_i and some band B_j,

R_2: $\sum_{i=1}^{\lambda} 2\tau_i$ relations each of which corresponds to a crossing point of a circle C_i and a band B_j ($j=1, \cdots, \lambda$), where C_i is the underpass.

R_3: $\sum_{i=1}^{\lambda} 2\mu_i$ relations each of which corresponds to a crossing point of a band B_i and a circle C_j, where B_i is the under crossing band,

R_4: λ relations each of which corresponds to the identification of a boundary of a band B_i and that of the neighbouring band.

R_5: 2λ relations each of which corresponds to the identification of a band B_i and C_i.

As a whole, the number of defining relations equals the number of generators.

To get a presentation of the knot group G_S of the ribbon 2-knot from G_k, it is sufficient to adjoin the new relations

(3.2) $$x_{i,0,0} = y_{i,0,0} \qquad (i=1, \cdots, \lambda-1)$$

to the defining relations of G_k. Since every band B_i does not run through the circle A, (3.2) and R_4 induce the relation $x_{\lambda,0,0}=y_{\lambda,0,0}$.

In virtue of (3.1) we have:

(3.3) The Wirtinger presentation of G_k and (3.2) implies $x_{i,j,k}=y_{i,j,k}$ for every i, j, k.

If we apply (3.3) to the presentation of G_S, it can be reduced more simply as follows: Suppose that the generators concerning a crossing of a band B_i and a band B_j are illustrated as in Fig. 5.

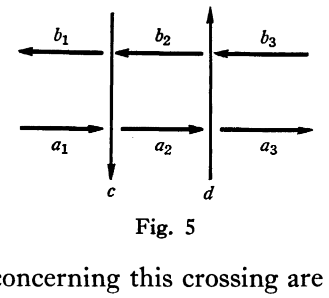

Fig. 5

Then the relations of G_S concerning this crossing are

$$a_1^{-1}ca_2c^{-1}=1,\quad a_2^{-1}d^{-1}a_3d=1\,,$$
$$a_1=b_1\,,\quad a_2=b_2\,,\quad a_3=b_3\,,\quad c=d\,.$$

If we eliminate a_2 and b_2 from these relations by Tietze transformation, then we get relations

$$a_1=b_1=a_3=b_3\,,\quad c=d\,.$$

As a consequence of the above fact, if we use a new generator $x_{i,j}$ $(=x_{i,j,0}=x_{i,j,2}=\cdots=y_{i,j,0}=y_{i,j,2}=\cdots)$ corresponding to each j-section of the band B_i, then all relations of R_1 are cancelled and the number of relations in R_3 becomes $\sum_{i=1}^{\lambda}\mu_i$.

A similar consideration as the above enables us to eliminate all relations of R_2, and generators $z_{i,0}$, $z_{i,2}$, $\cdots$ become a single generator z_i. Moreover, in virtue of R_5, one of generators z_i and x_{i,μ_i} is cancelled for each i. Similarly we can put $x_{1,0}=x_{2,0}=\cdots=x_{\lambda,0}$ by R_4.

Consequently, the number of generators is $\sum_{i=1}^{\lambda}(\mu_i+1)-(\lambda-1)=\sum_{i=1}^{\lambda}\mu_i+1$ and the number of defining relations is $\sum_{i=1}^{\lambda}\mu_i$.

(3.4) *G_S has a presentation such that each generator corresponds to some section of a band, where* 0*–sections belong together a single section, and each relation is the same one as G_k which corresponds to the crossing point of the positive side of some band B_i and a overpass C_j.* Therefore the number of generators exceeds the number of defining relations by one.

Since we restrict the presentation of G_S within wirtinger presentations, the condition (1) assures that the deficiency of G_S is not greater than 1.

(3.5) *If a group G is isomorphic to the knot group of some ribbon* 2*–knot, then G has a Wirtinger presentation of deficiency* 1.

(3.6) *The first elementary ideal*[7] *of the knot group of a ribbon* 2*–knot is principal.*
This follows immediately from (3.5) by [9].

REMARK. R.H. Fox and J.W. Milnor [3] proved that the Alexander polynomial of any slice knot, and of corse any ribbon knot, has a form $\Delta(t)=f(t)f(t^{-1})$. H. Terasaka also proved the same for ribbon knots in a part of [10]. I have proved previously in [12] that the Alexander polynomial $\Delta_S(t)$ of the ribbon 2–knot S constructed from a ribbon knot k equals $f(t)$, where $\Delta_k(t)=f(t)f(t^{-1})$. But my proof does not hold true for all ribbon 2–knots, because Terasaka's proof was for a special type of ribbon knots. Recently K. Yonebayashi [14] gave a perfect proof of Fox-Milnor's theorem and also my theorem according

7) [4], p. 127.

to Terasaka's idea. However the existence of a sphere which differs from ribbon 2-knots (Theorem 3, [12]) is a immediate consequence of (3.6).

Now we shall prove that the condition of the theorem is sufficient. Let G be an arbitrary group which has a Wirtinger presentation

$$
(3.7)\quad \begin{aligned}
&G=(x_1,\cdots,x_n : r_1,\cdots,r_m),\\
&r_1\colon x_{i_1}=x_{j_1}^{\varepsilon_1}x_{k_1}x_{j_1}^{-\varepsilon_1},\\
&r_2\colon x_{i_2}=x_{j_2}^{\varepsilon_2}x_{k_2}x_{j_2}^{-\varepsilon_2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&r_m\colon x_{i_m}=x_{j_m}^{\varepsilon_m}x_{k_m}x_{j_m}^{-\varepsilon_m},
\end{aligned}
\qquad (\varepsilon_i=\pm 1,\ i=1,\cdots,m)
$$

and satisfies the condition (1) of the theorem.

To show the system of defining relations schematically, it is convenient to use a diagram constructed as follows: Take n vertices $X_1,\cdots,X_n$ corresponding to generators $x_1,\cdots,x_n$ in this order. For each relation r_l, $l=1,\cdots,m$, connect X_{k_l} and X_{i_l} by an arrow labelled such that

$$X_{k_l}\xrightarrow{x_{j_l}^{\varepsilon_l}}X_{i_l}\quad\text{or}\quad X_{k_l}\xleftarrow{x_{j_l}^{-\varepsilon_l}}X_{i_l}.$$

Then we have the *diagram of presentation* of (3.7).

Since G satisfies the condition (1), the diagram of the presentation of (3.7) is connected. If G satisfies the condition (2), then the diagram does not contain any closed circuit. Therefore the diagram forms a tree. Then it is easy to construct a ribbon 2-knot, whose knot group has the presentation (3.7), by the projection method mentioned in [12] (Fig. 6). But we shall explain how to construct a ribbon knot in the situation of (2.2) from the presentation (3.7).

Suppose that $(x_1,\cdots,x_\lambda)$ be the aggregate of distinct generators each of which is contained in $(x_{j_1},\cdots,x_{j_m})$, and that x_n is not contained in $(x_{j_1},\cdots,x_{j_m})$. Since for each element $X_i (i=1,\cdots,n-1)$, there exists uniquely determined path

$$X_n\xrightarrow{x_{i,1}^{\varepsilon_{i,1}}}X_{i'}\xrightarrow{x_{i,2}^{\varepsilon_{i,2}}}\cdots\xrightarrow{x_{i,l_i}^{\varepsilon_{i,l_i}}}X_i$$

of the diagram, we have induced relations of (3.7)

$$
(3.8)\qquad s_i\colon x_i=w_ix_nw_i^{-1},\quad w_i=x_{i,l_i}^{\varepsilon_{i,l_i}}\cdots x_{i,2}^{\varepsilon_{i,2}}x_{i,1}^{\varepsilon_{i,1}},\quad (i=1,\cdots,m,\ m=n-1)
$$

where w_i is a word on $(x_1,\cdots,x_\lambda)$. It is easy to see that (3.8) is equivalent to (3.7), and that $s_{\lambda+1},\cdots,s_m$ can be eliminated. Therefore we have a equivalent presentation

$$(3.9)\qquad G=(x_1,\cdots,x_\lambda,x_n\colon s_1,\cdots,s_\lambda).$$

Now we shall define a ribbon knot k such that the knot group G_S is isomorphic to G, where the sphere S is constructed from k. Suppose that $A, C_1, \cdots, C_\lambda$; $\alpha_1, \cdots, \alpha_\lambda$; $\gamma_1, \cdots, \gamma_\lambda$ be the same as mentioned in the section 2, and that they are all oriented anti-clockwise. After having A correspond to x_n and C_i to x_i $(i=1, \cdots, \lambda)$, let $C_{i,1}, \cdots, C_{i,\mu_i}$ be the sequence of C_j $(j=1, \cdots, \lambda)$, where $C_{i,k}$ $(k=1, \cdots, \mu_i)$ corresponds to $x_{i,k}$ in (3.8). Connect α_i with γ_i by a non-twisting band B_i such that B_i runs successively under $C_{i,1}, \cdots, C_{i,\mu_i}$ as shown

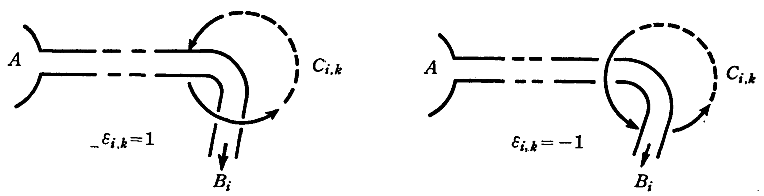

Fig. 6

in Fig. 6. From the ribbon, thus constructed, we have a ribbon knot k in the situation of (2.2). However k is not uniquely determined from (3.9).

It can be easily seen from (3.4), that the ribbon 2-knot constructed from k has the knot group isomorphic to G.

(3.10) *If a group G satisfies the condition* (1), (2) *of the theorem, then there exists a ribbon 2-knot S such that the knot group G_S is isomorphic to G.*

As an application of the theorem, we shall give an alternating proof of Kinoshita's theorem [7]. The idea of the proof is due to Terasaka [10].

(3.11) *For each polynomial $f(t)$ with $f(1)=\pm 1$, there exists a ribbon 2-knot S whose Alexander polynomial $\Delta_S(t)$ is equal to $f(t)$.*

Proof. If a group presentation $G=(x_1, x_2: x_2=wx_1w^{-1})$, w is a word on (x_1, x_2), can be constructed such that the Alexander polynomial of G equals $f(t)$, then (3.11) follows from the theorem.

Suppose that $f(1)=1$, and split $f(t)$ into the form

$$f(t)=(1-t)g(t)+1\,, \quad g(t)=a_0+a_1t+\cdots+a_mt^m.$$

For every integer n, define a word u_n such that

$$u_n=\begin{cases} \underbrace{(x_1x_2^{-1})\cdots(x_1x_2^{-1})}_{n} & \text{if } n>0, \\ 1 & \text{if } n=0, \\ \underbrace{(x_2x_1^{-1})\cdots(x_2x_1^{-1})}_{-n} & \text{if } n<0, \end{cases}$$

and put

$$w=\prod_{k=0}^{m} x_2^k u_{a_k} x_2^{-k}.$$

Then we have, by the free differential calculus[8], $\left(\dfrac{\partial w}{\partial x_1}\right)^{\psi\varphi}=g(t)$ and $w^{\psi\varphi}=1$. If we put $r=wx_1w^{-1}x_2^{-1}$, then it follows that

$$\begin{aligned}\left(\frac{\partial r}{\partial x_1}\right)^{\psi\varphi} &= \left(\frac{\partial w}{\partial x_1}\right)^{\psi\varphi}+w^{\psi\varphi}+\left(wx_1\frac{\partial w^{-1}}{\partial x_1}\right)^{\psi\varphi} \\ &= \left(\frac{\partial w}{\partial x_1}\right)^{\psi\varphi}+w^{\psi\varphi}-(wx_1w^{-1})^{\psi\varphi}\left(\frac{\partial w}{\partial x_1}\right)^{\psi\varphi} \\ &= (1-t)\left(\frac{\partial w}{\partial x_1}\right)^{\psi\varphi}+1=f(t)\,.\end{aligned}$$

Thus (3.11) is proved.

4. Peripheral subgroups

Let k be a knot in R^3, and U_k be a regular neighbourhood of k. Let G_k be a knot group of k, where the base point is chosen on the torus $T_k=\partial\bar{U}_k$. Then the inclusion mapping

$$f\colon (\bar{U}_k-k)\to(R^3-k)$$

induces a homomorphism

$$f^*\colon \pi_1(\bar{U}_k-k)\to G_k\,.$$

The subgroup $H_k^*=f^*(\pi_1(\bar{U}_k-k))$ of G_k is known[9] as the *peripheral subgroup* of G_k.

If k is not a trivial knot then f^* is a monomorphism[10]. Since T_k is a deformation retract of $\bar{U}_k-k$, $H_k^*\cong\pi_1(T_k)$ and it is a free abelian group of rank 2. We can choose a meridian m and a longitude l as generators of H_k^* such that m is null-homotopic in $\bar{U}$ and l is null-homologous in R^3-U_k.

Suppose that a regular projection of a knot k is given. Then we have a Wirtinger presentation

$$(4.1)\qquad \begin{aligned} &G_k=(x_1,\cdots,x_n\colon r_1,\cdots,r_n)\,,\\ &r_1\colon x_1=x_{i_1}^{\varepsilon_1}x_2x_{i_1}^{-\varepsilon_1}\,,\\ &r_2\colon x_2=x_{i_2}^{\varepsilon_2}x_3x_{i_2}^{-\varepsilon_2}\,,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &r_n\colon x_n=x_{i_n}^{\varepsilon_n}x_1x_{i_n}^{-\varepsilon_n}\,,\end{aligned}$$

where each one of relations is an induced relation of the others. Concerning this presentation, there exist one-to-one correspondences between the generators

8) [4], p. 124.
9) [2].
10) [8], p. 67.

and the overpasses of k, and also between the relations and the crossing points of k. We may call such a presentation of G a *faithful presentation* with respect to the knot projection.

Suppose that (4.1) is a faithful presentation of a projection of k, and that x_1 is chosen as the representative of m. Then we have that

$$l = x_{i_1}^{\varepsilon_1} \cdots x_{i_n}^{\varepsilon_n} \cdot x_1^{-(\varepsilon_1 + \cdots + \varepsilon_n)}$$

is the above mentioned longitude. It is known that l is contained in $G_k^{(2)}$ and $l \neq 1$ if k is not trivial.

Similarly, we can define the peripheral subgroup of a knot group $G_S = \pi_1(R^4 - S)$, where S is a locally flat 2-sphere in R^4. In this case, since $\bar{U}_S = D^2 \times S^2$, we have $\pi_1(\bar{U}_S - S) = Z$. Therefore any null-homologous loop in $\bar{U}_S - S$ must shrink to a point.

Let G be a group such that $G/[G, G] \cong Z$ and that it has a Wirtinger presentation (3.7) with $m > n-1$. Then the diagram of the presentation must contain several closed circuits. Let

$$(4.2) \qquad L: X_{i_0} \xrightarrow{x_{j,1}^{\varepsilon_1}} X_{i_1} \xrightarrow{x_{j,2}^{\varepsilon_2}} \cdots \xrightarrow{x_{j,k}^{\varepsilon_k}} X_{i_0}, \quad \varepsilon_h = \pm 1, \ h = 1, \cdots, k$$

be one of these circuits. We call the element

$$l = x_{j,1}^{\varepsilon_1} x_{j,2}^{\varepsilon_2} \cdots x_{j,k}^{\varepsilon_k} x_{i_0}^{-(\varepsilon_1 + \cdots + \varepsilon_k)}$$

of G the *l-element* corresponding to L.

If (3.7) is a faithful presentation of G_S, then a small letter $x_{j,h}$ ($h = 1, \cdots, k$) in (4.2) corresponds to the oversurface[11] of the surface $X_{i,h-1} \cup X_{i,h}$. Therefore every l-element is equivalent to a null-homologous loop in $\bar{U}_s - S$.

(4.3) If (3.7) is a faithful presentation of G_S, then every l-element derived from the diagram must be equal to 1.

In virtue of [11], Theorem (4.4), we have a faithful presentation of G_S for every projection of S. On the other hand, in order that a group G is isomorphic to the knot group of some sphere in R^4, it seems necessary that every l-element derived from the diagram is equal to 1. But it is still an open question. For example, if $G = (x, y, u, v\colon x = yvy^{-1},\ v = x^{-1}yx,\ y = x^{-1}ux,\ u = yxy^{-1})$, which is a presentation of the knot group of the Fox's sphere[12], then we can easily verify that $l = y^2x^{-2} = 1$. Notice that if the Wirtinger presentation contains some induced relations then the above conjecture does not hold.

Between the Kervaire's condition and l-elements of G_S we have the following connection:

(4.4) *Let G be a group such that $G/[G, G] \cong Z$ and that it has a Wirtinger presentation. If all l-elements of the diagram are equal to* 1, *then* $H_2(G) = 0$.

11) c.f. [11], §4.
12) [4], p. 136, *Example* 12.

Proof. Suppose that (3.7) is the Wirtinger presentation of G. By the condition of G we have that $m \geqq n-1$. If $m=n-1$, then the proposition is obvious by [9]. Suppose that $m>n-1$, and take a maximal tree $\mathcal{T}$ in the diagram. Assume that $\boldsymbol{r}_1=(r_1, \cdots, r_{n-1})$ is the aggrgate of relations each of which corresponds to a oriented segment of $\mathcal{T}$ and that $\boldsymbol{r}_2=(r_n, \cdots, r_m)$ is that of remaining relations.

Fix a vertex of $\mathcal{T}$, say X_n, as a base. Then for every vertex X_i $(i=1, \cdots, n-1)$ there exists a uniquely determined path

$$P_i\colon X_n \xrightarrow{x_{i,1}^{\varepsilon_{i,1}}} \cdots \xrightarrow{x_{i,k_i}^{\varepsilon_{i,k_i}}} X_i\,, \qquad \varepsilon_{i,h}=\pm 1\ (h=1, \cdots, k_i),$$

in $\mathcal{T}$. Corresponding to these pathes, we get induced relations of $\boldsymbol{r}_1$

$$s_i\colon x_i = v_i x_n v_i^{-1}\,, \quad v_i = x_{i,k_i}^{\varepsilon_{i,k_i}} \cdots x_{i,1}^{\varepsilon_{i,1}} \qquad (i=1, \cdots, n-1).$$

It is easy to check that $\boldsymbol{r}_1$ is equivalent to $\boldsymbol{s}_1=(s_1, \cdots, s_{n-1})$.

Since $\mathcal{T}$ is maximal, we can choose closed circuits $L_n, \cdots, L_m$ in the diagram such that $L_j (j=n, \cdots, m)$ consists of the segment corresponding to r_j and a path in $\mathcal{T}$. Let l_j be one of l-element for L_j. Then $(\boldsymbol{r}_1, r_j)$ induces a relation

$$s_j\colon x_{i_j} = l_j x_{i_j} l_j^{-1}\,, \qquad (j=n, \cdots, m)$$

for some x_{i_j}. It is also easily checked that $(\boldsymbol{r}_1, r_j)$ is equivalent to $(\boldsymbol{r}_1, s_j)$ for every $j=n, \cdots, m$.

Consequently we have a Wirtinger presentation

$$(4.5) \qquad \begin{aligned} &G = (x_1, \cdots, x_n\colon s_1, \cdots, s_m)\,, \\ &s_1 = v_1 x_n v_1^{-1} x_1^{-1}\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &s_{n-1} = v_{n-1} x_n v_{n-1}^{-1} x_{n-1}^{-1}\,, \\ &s_n = l_n x_{i_n} l_n^{-1} x_{i_n}^{-1}\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &s_m = l_m x_{i_m} l_m^{-1} x_{i_m}^{-1}\,, \end{aligned}$$

where v_i $(i=1, \cdots, n-1)$ is some word of the free group $F=F[\boldsymbol{x}]$, $\boldsymbol{x}=(x_1, \cdots, x_n)$ and l_j $(j=n, \cdots, m)$ is a l-element of (3.7).

Let R be the kernel of the mapping $\varphi\colon F \to G$. It is sufficient to prove $[F, F] \cap R \subset [F, R]$ for that $H_2(G)=0$. Suppose that a word c is contained in $[F, F] \cap R$. Then we have

$$c = \prod_j w_j s_{i_j}^{\varepsilon_j} w_j^{-1}\,, \qquad w_j \in F,\ \varepsilon_j = \pm 1.$$

Since $l_j \in R$, s_j is contained in $[F, R]$ for $j=n, \cdots, m$. Therefore there exist integers $p_1, \cdots, p_{n-1}$ such that

$$c \equiv \prod_j s_{i_j}^{\varepsilon_j} \qquad \mod [F, R]$$
$$\equiv s_1^{p_1} \cdots s_{n-1}^{p_{n-1}} \qquad \mod [F, R] .$$

The assumption $c \in [F, F]$ implies that $s_1^{p_1} \cdots s_{n-1}^{p_{n-1}} \in [F, F]$. Therefore the exponent sum of each generator must equal 0, that is $p_1 = \cdots = p_{n-1} = 0$. Hence $c \equiv 1 \mod [F, R]$. Thus we have completed the proof.

OSAKA CITY UNIVERSITY

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