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ON THE EQUILIBRIUM MEASURE OF RECURRENT MARKOV PROCESSES

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0. Introduction

In the case of planar Brownian motion, if we denote $h(x,y)=-\frac{1}{\pi}log|x-y|$, the following results are well known (see [13], [16]). (i) If F is a non-polar compact set, then there exists a probability measure ξ_F on F such that $\int h(x,y)\xi_F(dy)$ equals a constant $R(F)$ on F except on a polar set. The measure ξ_F and the constant $R(F)$ are respectively called the quilibruim measure and Robin's constant of F. (ii) A compact set F is non-polar if and only if there exists a non-zero finite measure ξ on F such that $\int h(x,y)\xi(dy)$ is locally bounded.

In this paper we shall be concerned with the similar problem for recurrent Hunt processes with strong Feller resolvent. In our case, in place of $h(x,y)$, we shall use a density $g(x,y)$ of a potential kernel $G(x,dy)$ of X relative to the invariant measure $\mu(dy)$. Unfortunately, our density $g(x,y)$ is not equal to $h(x,y)$ in the case of planar Brownian motion but equal to $h(x,y)+f(x)+g(y)$ with some locally bounded functions f and g (see §4).

Now we shall outline the contents of this paper. Let X be a recurrent Hunt process with strong Feller resolvent and μ an invariant measure of X. If we are given a certain finite non-negative continuous additive functional A of X then we can construct a potential kernel G of X by means of time change and killing based upon A ([4], [12]). In this paper we shall suppose, for simplicity, that $A_t=\int_0^t I_C(X_s)ds$ for an arbitrary fixed non-null compact set C but the similar argument can be applicable for a large class of functionals A.

In section 1, some preliminary results are established. Among others, a potential kernel K_A and an invariant measure ν_A of the time changed process by A are described. In section 2, for any other finite non-negative continuous additive functional B, a potential kernel K_B and an invariant measure ν_B of the time changed process by B are constructed by making use of K_A and ν_A. In section 3, let us introduce the duality hypothesis that there exists a dual process $\hat{X}$ (of X relative to μ) satisfying those regularity conditions like X.

We shall then construct a kernel function $g(x, y)$ such that $g(\cdot, y)$ [resp. $g(x, \cdot)$] is finely [resp. cofinely] continuous, finite except on a polar set and $K_B(x, dy)=g(x, y)\nu_B(dy)$ [resp. $\hat{K}_B(dx, y)=g(x, y)\nu_B(dx)$] for all continuous additive functionals B, where $\hat{K}_B$ is the potential kernel of the time changed process of $\hat{X}$ by the dual functional $\hat{B}$ of B. In particular, when $B_t=t$, we have $G(x, dy)=g(x, y)\mu(dy)$ and $\hat{G}(dx, y)=g(x, y)\mu(dx)$, where $\hat{G}$ is the potential kernel of $\hat{X}$ associated with $\hat{A}_t=\int_0^t I_C(\hat{X}_s)\,ds$ in the sense of section 2. In this sense, our function $g(x, y)$ may be called the potential kernel function associated with $(G, \hat{G})$. In section 4, we introduce the notion of potential kernel function $h(x, y)$ in a more general sense and then establish a relation between $h(x, y)$ and $g(x, y)$. In section 5, we shall show the equilibrium principle. This means that, if $F=supp\,(B)$, then there is a probability measure ξ_F on $supp\,(\hat{B})$ such that $\int g(x, y)\xi_F(dy)=R(F)$ on F. In our case, the equilibrium measure ξ_F and Robin's constant $R(F)$ have intuitive probablistic meanings. If X and $\hat{X}$ are equivalent, the results of section 5 have simpler forms and the analogous potential principles to classical potential theory hold. This case is treated in section 6. There a characterization of the equilibrium measure by means of energy is also given.

1. Notations and preliminary results

Let E be a locally compact Hausdorff space with countable base, $\mathcal{E}$ the Borel σ-field on E and $\mathcal{E}^*$ the σ-field obtained by the universal completion of $\mathcal{E}$. If $\mathcal{A}$ is a σ-field of subsets of E then the classes of all bounded $\mathcal{A}$-measurable functions, all bounded non-negative $\mathcal{A}$-measurable functions and all bounded $\mathcal{A}$-measurable functions with compact support are denoted by $b\mathcal{A}$, $b\mathcal{A}_+$ and $b\mathcal{A}_c$, respectively.

Throughout in this paper, let $X=(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a recurrent Hunt process on E with strong Feller resolvent, that is, a Hunt process satisfying

(i) (Recurrence); *For all* $f\in b\mathcal{E}_+$, $G^0f(x)=E^x[\int_0^\infty f(X_t)\,dt]\equiv 0$ *or* $\equiv\infty$ *on* E.

(ii) (Strong Feller property of resolvent); *For all* $p>0$ *and* $f\in b\mathcal{E}$, $G^pf(x)=E^x[\int_0^\infty e^{-pt}f(X_t)\,dt]$ *is bounded continuous.*

In this case, it is well known that there exists a unique (except a constant multiple) invariant Radon measure μ of X, which is positive on every open sets (see [1], [2]). Let Φ be the family of all non-negative continuous additive functionals (abbreviated CAF) $A=(A_t)_{t\geqq 0}$ of X such that $A_t<\infty$ *a.s.* for all $t<\infty$ and let Φ^+ be the subfamily of functionals $A\in\Phi$ which are not equivalent to the zero functional. If $A\in\Phi^+$ then $P^x(A_\infty=\infty)=1$ for all x ([1]). For $A\in\Phi^+$ and $p\geqq 0$ we define a kernel K_A^p by

$$(1.1) \qquad K_A^p f(x) = E^x[\int_0^\infty e^{-pA_t} f(X_t) dA_t] .$$

Note that $(K_A^p)_{p>0}$ is the resolvent of the time changed process of X by A.

Moreover, for $A, B \in \Phi^+$ and $p, q \geqq 0$ we define two auxiliary kernels $U_{A,B}^{p,q}$ and $V_{A,B}^{p,q}$ as follows:

$$(1.2) \qquad U_{A,B}^{p,q} f(x) = E^x[\int_0^\infty e^{-pA_t - qB_t} f(X_t) dA_t] ,$$

$$(1.3) \qquad V_{A,B}^{p,q} f(x) = E^x[\int_0^\infty e^{-pA_t - qB_t} f(X_t) dB_t] .$$

Obviously, $U_{A,B}^{p,q} = V_{B,A}^{q,p}$. The family $(U_{A,B}^{p,q})_{p>0}$ is the resolvent of the time changed process by A_t of the e^{-qB_t}-subprocess of X. If $B_t = t$, we shall write $K_A^{p,q}$ for $U_{A,\{t\}}^{p,q}$ and $G_A^{p,q}$ for $V_{A,\{t\}}^{p,q}$, i.e.,

$$(1.4) \qquad K_A^{p,q} f(x) = E^x[\int_0^\infty e^{-pA_t - qt} f(X_t) dA_t] ,$$

$$(1.5) \qquad G_A^{p,q} f(x) = E^x[\int_0^\infty e^{-pA_t - qt} f(X_t) dt] .$$

Note that $U_{A,B}^{p,0} = K_A^{p,0} = K_A^p$, $V_{A,B}^{0,q} = K_B^q$ and $G_A^{0,q} = G^q$ (the resolvent of X).

In the sequel, if there is no danger of confusion, the suffices A, B will be often omitted.

Lemma 1.1 (Nagasawa-Sato [10; theorem 2.1 and 2.2]). *Write $U^{p,q}$ and $V^{p,q}$ for $U_{A,B}^{p,q}$ and $V_{A,B}^{p,q}$. For all $p>0$, $p'>0$ $q \geqq 0$, $q' \geqq 0$ and $f \in b\mathcal{E}^*$,*

$$(1.6) \qquad U^{p,q}f - U^{p',q'}f + (p-p')\, U^{p,q}U^{p',q'}f + (q-q')\, V^{p,q}U^{p',q'}f = 0 ,$$

$$(1.7) \qquad V^{q,p}f - V^{q',p'}f + (p-p')\, V^{q,p}V^{q',p'}f + (q-q')\, U^{q,p}V^{q',p'}f = 0 .$$

If, in particular, $U^{0,q_0}|f|$ [resp. $V^{q_0,0}|f|$] is bounded for some $q_0 \geqq 0$ then $U^{0,q}|f|$ [resp. $V^{q,0}|f|$] is bounded for all $q>0$ and (1.6) *[resp.* (1.7)*] holds for all $p, p', q, q' \geqq 0$ satifysing $p+q>0$ and $p'+q'>0$.*

Lemma 1.2 ([12; lemma 2.2]). *There exists an increasing sequence $\{E_n\}_{n\geqq 1}$ [resp. $\{F_n\}_{n\geqq 1}$] of subsets in $\mathcal{E}^*$ such that $\bigcup_{n\geqq 1} E_n = E$ [resp. $\bigcup_{n\geqq 1} F_n = E$] and $U^{0,1}(\cdot, E_n)$ [resp. $V^{1,0}(\cdot, F_n)$] is bounded for all $n \geqq 1$.*

Lemma 1.3 (Blumenthal-Getoor [3; III, section 5]). *If $A \in \Phi^+$ then $G_A^{p,0}(\cdot, F)$ is bounded for all compact set F and $p>0$.*

A set C is said to be null if it is a set of potential zero relative to $(G^p)_{p>0}$. Let C be an arbitrary (but fixed) non-null compact subset of E and let us assume that μ is normalized on C as $\mu(C)=1$.

In the remainder of this paper, unless otherwise stated, the CAF $A=(A_t)_{t\geq 0}\in\Phi^+$ always represents the CAF defined by

$$(1.8)\qquad A_t=\int_0^t I_C(X_s)\,ds\,.$$

Then for every $B\in\Phi^+$,

$$U^{0,1}_{A,B}1(x)=E^x\Big[\int_0^\infty e^{-B_t}I_C(X_t)\,dt\Big]=G^{1,0}_B(x,C)$$

is bounded by lemma 1.3. Moreover we have

Lemma 1.4. *For any p, $q>0$ and $f\in b\mathcal{E}^*$, the functions $K^p_A f$ and $G^{p,q}_A f$ are bounded continuous. In case $f\in b\mathcal{E}^*_c$, $G^{p,0}_A f$ is bounded continuous for all $p>0$.*

Proof. Drop the suffix A in the related kernels. For any $p>0$ and $f\in b\mathcal{E}^*$ we have, from (1.7),

$$G^p f-G^{p,p}f-pK^{0,p}G^{p,p}f=0\,.$$

Since $K^{0,p}g=G^p(I_C g)$ for any $g\in b\mathcal{E}^*$ the function

$$\begin{aligned}G^{p,p}f&=G^p f-pG^p(I_C G^{p,p}f)\\&=G^p(f-pI_C G^{p,p}f)\end{aligned}$$

is bounded continuous by the strong Feller property of G^p. Therefore,

$$G^{p,q}f=G^{p,p}(f+(q-p)\,G^{p,p}G^{p,q}f)$$

is bounded continuous. If $f\in b\mathcal{E}^*_c$ then $G^{p,0}f$ is bounded by lemma 1.3, so that the above equality for $q=0$ shows that $G^{p,0}f$ is bounded continuous.

Since $(K^p_A)_{p>0}$ is a strong Feller resolvent by lemma 1.4, the mapping $x\to K^1_A(x,\cdot)$ of the compact set C into the space of measures over C is strongly continuous by a theorem of Mokobodzki (see Meyer [9]). Since, in addition, $K^1_A(x,\cdot)$ are equivalent for all $x\in E$, we have

$$\sup_{x,y\in C}\frac{1}{2}\|K^1_A(x,\cdot)-K^1_A(y,\cdot)\|\equiv a<1\,.$$

Thus there exists a unique invariant probability measure ν_A of K^1 such that

$$(1.9)\qquad \sup_{x\in E}\|(K^1_A)^{n+1}(x,\cdot)-\nu_A(\cdot)\|\leqq 2a^n$$

for all $n\geqq 0$ ([7; lemma 1.3]). Therefore the kernel

$$(1.10)\qquad K_A(x,F)=\sum_{n=1}^\infty[(K^1_A)^n(x,F)-\nu_A(F)]$$

is well defined and satisfies

$$(I-K_A^1)K_Af=K_A^1f-\langle \nu_A, f\rangle$$

for all $f\in b\mathcal{E}^*$.

Lemma 1.5. *The kernel* K_A *defined by* (1.10) *satisfies*

$$\lim_{p\to 0}\sup_{x\in E}\left\|K_A^p(x,\cdot)-\frac{\nu_A(\cdot)}{p}-K_A(x,\cdot)\right\|=0, \tag{1.11}$$

in particular,

$$\lim_{p\to 0}\sup_{x\in E}\|pK_A^p(x,\cdot)-\nu_A(\cdot)\|=0. \tag{1.12}$$

Proof. From the resolvent equation for (K_A^p) we have

$$\begin{aligned}K_A^p(x,\cdot)&=K_A^1\sum_{n=0}^{\infty}(1-p)^n(K_A^1)^n(x,\cdot)\\&=\sum_{n=1}^{\infty}(1-p)^{n-1}\{(K_A^1)^n(x,\cdot)-\nu_A(\cdot)\}+\frac{\nu_A(\cdot)}{p}\end{aligned}$$

for all $x\in E$ and $0<p<1$. Thus it follows that

$$\begin{aligned}&\left\|K_A^p(x,\cdot)-\frac{\nu_A(\cdot)}{p}-K_A(x,\cdot)\right\|\\&\leqq\sum_{n=1}^{\infty}\{1-(1-p)^{n-1}\}\|(K_A^1)^n(x,\cdot)-\nu_A(\cdot)\|\\&\leqq\sum_{n=1}^{\infty}\{1-(1-p)^{n-1}\}\,2a^{n-1}=2\left\{\frac{1}{1-a}-\frac{1}{1-a(1-p)}\right\}.\end{aligned}$$

Therefore the lemma follows.

2. An invariant measure and a potential kernel of (K_B^p)

Similarly to [4] and [12], for any $B\in\Phi^+$, an invariant measure ν_B and a potential kernel K_B of $(K_B^p)_{p>0}$ can be constructed by making use of ν_A and K_A defined in section 1. In [12], we have treated only the case of $B_t=t$ but the same arguments are valid for all $B\in\Phi^+$. We shall outline it in the form of our present use.

For any $B\in\Phi^+$ define the measure ν_B by

$$\nu_B=\nu_A V_{A,B}^{1,0} \tag{2.1}$$

Then ν_B charges no semipolar set and satisfies the following properties.

Lemma 2.1. *The measure* ν_B *is a* σ*-finite invariant measure of* $(K_B^p)_{p>0}$. *In particular,* $\nu_{\{t\}}=\mu$.

Proof. (cf. [12; theorem 2.7]) Since $V_{A,B}^{1,0}(\cdot, F_n)$ is bounded for all n by lemma 1.2, ν_B is σ-finite. Integrating the equality

$$V_{A,B}^{1,0}-K_B^p+K_A^1K_B^p-pV_{A,B}^{1,0}K_B^p=0$$

by ν_A we have

$$\nu_B=p\nu_BK_B^p\,,$$

that is, ν_B is an invariant measure of (K_B^p).

By the uniqueness of the invariant measure of X, $\nu_{\{t\}}$ is a constant multiple of μ, say,

$$\nu_{\{t\}}=\nu_AG_A^{1,0}=b\mu$$

for some constant b. Since $\mu(C)=1$ we have

$$b=\nu_AG_A^{1,0}(C)=\nu_AK^1(C)=\nu_A(C)=1\,,$$

Hence $\nu_{\{t\}}=\mu$.

Lemma 2.2 (cf. [4; proposition 2]). *For any B, $B'\in\Phi^+$,*

(2.2) $\qquad \nu_{B'}=p\nu_BV_{B,B'}^{p,0}\,.$

In particular, ν_B is the measure associated with B in the sense of Revuz ([14]). *Moreover it holds that $\nu_A=\mu|_C$, where $\mu|_C$ is the restriction of μ to C.*

Proof. Similarly to lemma 1.1, we can prove easily that

(2.3) $\qquad V_{A,B'}^{q,0}-V_{B,B'}^{p,0}+qK_A^qV_{B,B'}^{p,0}-pV_{A,B}^{q,0}V_{B,B'}^{p,0}=0$

for sufficiently many $f\in b\mathcal{E}^*$. Letting $q=1$ and integrating by ν_A, (2.2) follows. Set $B_t=t$ at (2.2) then $\nu_{B'}=p\nu_{\{t\}}V_{\{t\},B'}^{p,0}=p\mu K_{B'}^{0,p}$ by lemma 2.1. Hence $\nu_{B'}$ is the measure associated with B'. In particular, when $B'=A$, it follows that $\langle\nu_A,f\rangle=p\langle\mu, K_A^{0,p}f\rangle=p\langle\mu, G^pI_Cf\rangle=p\langle\mu G^p, I_Cf\rangle=\langle\mu, I_Cf\rangle=\langle\mu|_C, f\rangle$.

Define a kernel K_B by

(2.4) $\qquad K_B(x,\,\cdot)=K_AV_{A,B}^{1,0}(x,\,\cdot)+V_{A,B}^{1,0}(x,\,\cdot)-\nu_B(\cdot)\,.$

In case $B_t=t$ we shall denote K_B by G, which is the kernel we have constructed in [12]. Obviously, $K_B(x,\,\cdot)$ is a σ-finite signed measure on E and, for any $n\geqq 1$, the total variation of $K_B(x,\,\cdot)$ on F_n are uniformly bounded for all $x\in E$ by (1.9) ana lemma 1.2. Similarly, for any compact set F, the total variation of $G(x,\,\cdot)$ on F is uniformly bounded for all $x\in E$ by lemma 1.3. If we denote the total variation of a measure on F by $\|\cdot\|_F$, then the following theorem holds.

Theorem 2.3. *For all $n\geqq 1$,*

(2.5) $\qquad \limsup_{p\to 0}\sup_{x\in E}\left\|V_{A,B}^{p,0}(x,\,\cdot)-\frac{\nu_B(\cdot)}{p}-K_B(x,\,\cdot)\right\|_{F_n}=0\,,$

and in partciular,

(2.6) $$\lim_{p\to 0}\sup_{x\in E}\|pV^{p,0}_{A,B}(x,\ \cdot)-\nu_B(\cdot)\|_{F_n}=0\,.$$

If $B_t=t$ then we can take arbitrary compact set in place of F_n.

Proof. Write $V^{p,0}$ for $V^{p,0}_{A,B}$. For any Borel subset D of F_n

$$V^{p,0}(x,\ D)-V^{1,0}(x,\ D)+pK^p_A V^{1,0}(x,\ D)-K^p_A V^{1,0}(x,\ D)=0$$

from (1.7). This can be written, by noting (2.1),

$$\{V^{p,0}(x,\ D)-\frac{\nu_B(D)}{p}\}-V^{1,0}(x,\ D)+pK^p_A V^{1,0}(x,\ D)$$
$$-(K^p_A-\frac{1}{p}\nu_A)V^{1,0}(x,\ D)=0\,.$$

Thus we have

$$\|V^{p,0}(x,\ \cdot)-\frac{\nu_B(\cdot)}{p}-K_B(x,\ \cdot)\|_{F_n}$$
$$\leqq\|\{pK^p_A(x,\ \cdot)-\nu_A(\cdot)\}V^{1,0}\|_{F_n}$$
$$+\|\{K^p_A(x,\ \cdot)-\frac{\nu_A(\cdot)}{p}-K_A(x,\ \cdot)\}V^{1,0}\|_{F_n}\,.$$

This proves the theorem from lemma 1.5.

Corollary 1. *If $f\in b\mathcal{E}^*$ vanishes outside of some F_n, then*

(2.7) $$(I-pK^p_B)K_Bf=K^p_Bf-U^{0,p}_{A,B}1\langle\nu_B,f\rangle$$

for all $p>0$. If $V^{1,0}_{A,B}1$ is bounded, then

(2.8) $$K_B(I-pK^p_B)f=K^p_Bf-\langle\nu_A,\ K^p_Bf\rangle$$

for all $p>0$ and $f\in b\mathcal{E}^*$.

Proof. Suppose that $V^{1,0}1$ is bounded, then obviously (2.5) holds for E in place of F_n, so we have

$$K_B(I-pK^p_B)f(x)=\lim_{q\to 0}(V^{q,0}-\frac{1}{q}\nu_B)(I-pK^p_B)f(x)$$
$$=\lim_{q\to 0}V^{q,0}(I-pK^p_B)f(x)=\lim_{q\to 0}(K^p_Bf-qK^p_AK^q_Bf)(x)$$
$$=K^p_Bf(x)-\langle\nu_A,\ K^p_Bf\rangle\,,$$

from (1.13). The proof of (2.7) is similar.

Let us denote

(2.9) $$N_B=\{f;f\in b\mathcal{E}^*,=0 \textit{ outside of some } F_n \textit{ and } \langle\nu_B,f\rangle=0\}\,,$$

(2.10) $\quad N=\{f; f\in b\mathcal{E}^*, \langle\mu, f\rangle=0\}$.

Definition. If a kernel H on E satisfies the condition that (i) for any $f\in N_B$ [resp. $f\in N$], $Hf\in b\mathcal{E}^*$ and that (ii) for any $f\in N_B$ [resp. $f\in N$], $(I-pK_B^p)\,Hf=K_B^pf$ [resp. $(I-pG^p)Hf=G^pf$] for all $p>0$, then we shall say that *H is a potential kernel of $(K_B^p)_{p>0}$ [resp. X].*

Corollary 2. *The kernels K_B and G are the potential kernels of $(K_B^p)_{p>0}$ and X, respectively.*

Corollary 3. *For every compact set F, the function $G(\cdot, F)$ is finely continuous.*

Proof. Set $B_t=t$ at (2.7)

$$G(x, F)=pG^pG(x, F)+G^p(x, F)-(K_A^{0,p}1(x))\,\mu(F)\,.$$

Since $K_A^{0,p}1$ is p-excessive, the result is obvious.

3. Hypothesis of duality and the kernel function $g(x, y)$

In this section we shall assume that there exists a Hunt process $\hat{X}$ with strong Feller resolvent $\hat{G}^p$ such that X and $\hat{X}$ are in duality relative to μ. It follows that $\hat{X}$ is also recurrent and μ is the invariant measure of $\hat{X}$.

Let $\hat{\Phi}^+$ be the family of all non-zero non-negative finite continuous additive functionals of $\hat{X}$. For any $\hat{A}$, $\hat{B}\in\hat{\Phi}^+$, we define $\hat{U}_{A,B}^{p,q}$ etc. by

$$f\hat{U}_{A,B}^{p,q}(x)=\hat{E}^x[\int_0^\infty e^{-p\hat{A}_t-q\hat{B}_t}f(\hat{X}_t)\,d\hat{A}_t]$$

etc. (in general, a kernel with respect to the dual process $\hat{X}$ is written such as $\hat{K}(D, x)$, so that $\hat{K}$ operates to function from the right side and to measure from the left).

By Revuz [14; theorem VII. 1], for any $B\in\Phi^+$, there exists a polar set P_B and a CAF $\hat{B}\in\hat{\Phi}^+$ of $\hat{X}$ restricted to $E-P_B$ such that $\nu_B=\hat{K}_B^{0,1}\,\mu$. Also, by [14; theorem VII. 2], there exists a jointly measurable kernel function $g_B^{p,q}(x, y)$ satisfying

(i) *$g_B^{p,q}(\cdot, y)$ [resp. $g_B^{p,q}(x, \cdot)$] is finely [resp. cofinely] continuous and q-excessive [resp. q-coexcessive] relative to the resolvent $(G_B^{p,q})_{q>0}$ [resp. $(\hat{G}_B^{p,q})_{q>0}$] for all $p>0$ and $y\in E-P_B$ [resp. $x\in E$],*

(ii) *For all $p, q>0$ and $x\in E$, $K_B^{p,q}(x, dy)=g_B^{p,q}(x, y)\nu_B(dy)$, $G_B^{p,q}(x, dy)=g_B^{p,q}(x, y)\mu(dy)$ and for all $p, q>0$ and $y\in E-P_B$, $\hat{K}_B^{p,q}(dx, y)=g_B^{p,q}(x, y)\,\nu_B(dx)$, $\hat{G}_B^{p,q}(dx, y)=g_B^{p,q}(x, y)\,\mu(dx)$.*

As before, the set C with $\mu(C)=1$ is fixed and A is given by (1.8). If $B=A$, P_B may be supposed to be empty and the dual CAF of A is given exactly by

$$(3.1) \qquad \hat{A}_t = \int_0^t I_C(\hat{X}_s)ds\,.$$

In the following, unless otherwise stated, $\hat{A}$ always represents this CAF and we shall drop the suffix A in $g_A^{p,q}$. Further, we shall denote $g^q(x, y)$ for $g_A^{0,q}(x, y)$, which is Kunita-Watanabe's potential kernel function. Note that $g_B^{0,q}(x, y)=g^q(x, y)$ for all $B\in\Phi^+$. Form the resolvent equation (1.6), for any $q>0$,

$$(3.2) \qquad \begin{aligned} g^{1,q}(x, y) &= g^q(x, y)-K_A^{1,q}g^q(x, y) \\ &= g^q(x, y)-g^q\hat{K}_A^{1,q}(x, y) \end{aligned}$$

on $\{(x, y);\ g^q(x, y)<\infty\}$. Hence for any $y\in E$, $K_A^{1,q}g^q(x, y)=g^q\hat{K}_A^{1,q}(x, y)$ *a.a.* $x(\mu)$. Since both sides of the equality are q-excessive, it holds for all $x, y\in E$ (cf. Getoor [5; theorem 2.5]).

Lemma 3.1. *For all $x\in E$ and $B\in\Phi^+$,*

$$(3.3) \qquad V_{A,B}^{1,0}(x, dy) = g^{1,0}(x, y)\nu_B(dy)\,.$$

Proof. Set $A_t'=A_t+qt$. Replacing A', $\{t\}$, B for A, B, B' in (2.3) we have

$$V_{A',B}^{1,0}f = V_{\{t\},B}^{q,0}f-K_{A'}^1V_{\{t\},B}^{q,0}f+qV_{A',\{t\}}^{1,0}V_{\{t\},B}^{q,0}f\,,$$

for sufficiently many functions f. Noting that $V_{\{t\},B}^{q,0}=K_B^{0,q}$, $K_{A'}^1=K_A^{1,q}+qG_A^{1,q}$ and $V_{A',\{t\}}^{1,0}=G_A^{1,q}$, it follows that

$$\begin{aligned} E^x[\int_0^\infty e^{-A_t-qt}f(X_t)\,dB_t] &= V_{A',B}^{1,0}f(x) \\ &= K_B^{0,q}f(x)-K_A^{1,q}K_B^{0,q}f(x) \\ &= \int\{g^q(x, y)-K_A^{1,q}g^q(x, y)\}\,f(y)\nu_B(dy) \\ &= \int\{g^q(x, y)-K_A^{1,q}g^q(x, y)\}\,f(y)\nu_B(dy) \\ &= \int\{g^{1,q}(x, y)f(y)\}\,\nu_B(dy) \end{aligned}$$

The last equality follows from (3.2) since ν_B has no mass on the polar set $\{y;\ g^q(x, y)=\infty\}$. Letting $q\to 0$ we have the result.

Dually, if $\hat{B}$ is the dual CAF of B then

$$(3.4) \qquad \hat{V}_{A,B}^{1,0}(dx, y) = g^{1,0}(x, y)\nu_B(dx) \qquad \textit{for all } y\in P_B\,.$$

Hence we have

Corollary. *For all $f, g\in b(\mathcal{E}^*)^+$,* (3.5) $$\int f(x)V_{A,B}^{1,0}g(x)\nu_A(dx)=\int f\hat{U}_{A,B}^{1,0}(y)g(y)\nu_B(dy)$$

(3.6) $$\int f\hat{V}_{A,B}^{1,0}(y)g(y)\nu_A(dy)=\int f(x)U_{A,B}^{1,0}g(x)\nu_B(dx)\,.$$

Since

$$K_A(x, dy)=K_B^1(x, dy)-\nu_A(dy)+K_AK_A^1(x, dy)\,,$$

it is easy to show that, for each x, $K_A(x, \cdot)$ is absolutely continuous relative to ν_A and its density is given by

$$g^{1,0}(x, y)-1+K_Ag^{1,0}(x, y)$$

up to a set of ν_A-measure 0.

However, in order to solve the problem proposed in the introduction, we have to choose a more elaborated density $g(x, y)$. To do this, we need one more preliminary observation.

For all $x, y\in E$ and $n\geqq 1$, set

(3.7) $$f_n(x, y)=(\hat{K}_A^1)^{n-1}g^{1,0}(x, y)-1=g^{1,0}(K_A^1)^{n-1}(x, y)-1\,,$$

then

(3.8) $$\begin{aligned} f_n(x, y)\nu_A(dy)&=(K_A^1)^n(x, dy)-\nu_A(dy)\\ f_n(x, y)\nu_A(dx)&=(\hat{K}_A^1)^n(dx, y)-\nu_A(dx)\,. \end{aligned}$$

Since

$$\int\sum_{n=1}^{\infty}|f_n(x, y)|\nu_A(dy)=\sum_{n=1}^{\infty}\|(K_A^1)^n(x, \cdot)-\nu_A(\cdot)\|<\infty$$

for all $x\in E$ from (1.9) and (3.8), the series $\sum_{n=1}^{\infty}f_n(x, y)$ converges absolutely for $a.a.x(\nu_A)$. Similarly for all $y\in E$, $\sum_{n=1}^{\infty}f_n(x, y)$ converges absolutely for $a.a.x(\nu_A)$. Also

$$\begin{aligned}\int_D\sum_{n=1}^{\infty}f_n(x, y)\nu_A(dy)&=\sum_{n=1}^{\infty}\int_D f_n(x, y)\nu_A(dy)\\ &=\sum_{n=1}^{\infty}\{(K_A^1)^n(x, D)-\nu_A(D)\}=K_A(x, D)\end{aligned}$$

for all $D\in\mathcal{E}$, that is, $\sum_{n=1}^{\infty}f_n(x, \cdot)$ is a density of $K_A(x, \cdot)$ relative to ν_A. Dually, $\sum_{n=1}^{\infty}f_n(\cdot, y)$ is a density of

(3.9) $$\hat{K}_A(\cdot, y)=\sum_{n=1}^{\infty}\{(\hat{K}_A^1)^n(\cdot, y)-\nu_A(\cdot)\}$$

relative to ν_A. Here the proof of the strong convergence of (3.9) is similar to (1.9).

Lemma 3.2. *There exists a Borel subset Γ of $E\times E$ satisfying the following conditions.*

(i) *Set $\Gamma_y=\{x;\,(x,y)\in\Gamma\}$ and $\hat{\Gamma}_x=\{y;\,(x,y)\in\Gamma\}$, then Γ_y^c and $\hat{\Gamma}_x^c$ are polar for all $x, y\in E$.*

(ii) *For all $(x,y)\in\Gamma$, $\sum_{n=1}^{\infty}f_n(x,y)$ converges absolutely, $|K_A|g^{1,0}(x,y)<\infty$ and $g^{1,0}|\hat{K}_A|(x,y)<\infty$, where $|K_A|(x,\cdot)$ is the total variation measure of $K_A(x,\cdot)$.*

(iii) *For all $(x,y)\in\Gamma$,*

$$\begin{aligned}(3.10)\qquad \sum_{n=1}^{\infty}f_n(x,y) &= g^{1,0}(x,y)-1+K_A g^{1,0}(x,y)\\ &= g^{1,0}(x,y)-1+g^{1,0}\hat{K}_A(x,y)\,.\end{aligned}$$

We define the kernel function $g(x,y)$ by

$$\begin{aligned}(3.11)\qquad g(x,y) &= (3.10) && \text{if } (x,y)\in\Gamma\\ &=\infty && \text{if } (x,y)\notin\Gamma\,.\end{aligned}$$

By the lemma, it is easy to see that the function $g(\cdot,y)$ [resp. $g(x,\cdot)$] is finely [resp. cofinely] continuous on the fine [resp. cofine] open set Γ_y [resp. $\hat{\Gamma}_x$] for all y [resp. all x]$\in E$.

Proof. Noting that,

$$\begin{aligned}|f_{n+1}(x,y)| &= |K_A^1\{(K_A^1)^{n-1}g^{1,0}-1\}(x,y)|\\ &= \Big|\int K_A^1(x,dz)\Big\{\iint (K_A^1)^{n-2}(z,du)g^{1,0}(u,v)g^{1,0}(v,y)\nu_A(dv)\\ &\qquad -\int g^{1,0}(v,y)\nu_A(dv)\Big\}\Big|\\ &= \Big|\int K_A^1(x,dz)\int\{(K_A^1)^{n-2}g^{1,0}(z,v)-1\}g^{1,0}(v,y)\nu_A(dv)\Big|\\ &= |K_A^1 f_{n-1}\hat{K}_A^1(x,y)|\leqq K_A^1|f_{n-1}|\hat{K}_A^1(x,y)\end{aligned}$$

for $n\geqq 2$ and

$$\begin{aligned}&\int|(K_A^1)^n(x,dz)-\nu_A(dz)|g^{1,0}(z,y)\\ &\quad=\int|(K_A^1)^{n-1}g^{1,0}(x,z)-1|g^{1,0}(z,y)\nu_A(dz)\\ &\quad=\int|K_A^1\{(K_A^1)^{n-2}g^{1,0}-1\}(x,z)|g^{1,0}(z,y)\nu_A(dz)\\ &\quad\leqq K_A^1|f_{n-1}|\hat{K}_A^1(x,y)\end{aligned}$$

for $n\geqq 2$, let us define the set Γ by

$$\Gamma = \{(x, y);\ |f_1|(x, y)+|f_2|(x, y)+\sum_{n=1}^{\infty} K_A^1|f_n|\hat{K}_A^1(x, y)<\infty\}\ .$$

Then the proofs of (ii) and (iii) are obvious. For the proof of (i) set

$$\xi_x(dy) = \delta_x(dy)+K_A^1(x, dy)+\sum_{n=1}^{\infty}(K_A^1|f_n|)(x, y)\nu_A(dy)\ .$$

Then

$$\xi_x(E) = 2+\sum_{n=1}^{\infty}\int K_A^1(x, dy)|(K_A^1)^n(y, \cdot)-\nu_A(\cdot)|<\infty\ .$$

Moreover, it is easy to see that,

$$\Gamma = \{(x, y);\ \int\xi_x(dz)g^{1,0}(z, y)<\infty\}\ .$$

Hence $\hat{\Gamma}_x^c$ is polar if and only if $\int\xi_x(dz)g^{1,0}(z, y)<\infty$ except on a polar set. Since $g^{1,0}$ is the potential kernel function of the e^{-A_t}-subprocess (which is a transient Hunt process on E (13; III.3.16])) of X, the potential $\int\xi_x(dz)g^{1,0}(z, y)$ of the bounded measure ξ_x is finite except on a polar set if it is finite for $a.a.y(\mu)$ ([3; VI.2.3]). Since, for all $f\in b\mathcal{E}_c^+$,

$$\begin{aligned}&\iint\xi_x(dz)g^{1,0}(z, y)f(y)\mu(dy)\\&\quad=\int\xi_x(dz)G_A^{1,0}f(z)\leqq||G_A^{1,0}f||\xi_x(E)<\infty\ ,\end{aligned}$$

by lemma 1.3, $\int\xi_x(dz)g^{1,0}(z, y)<\infty a.a.y(\mu)$. Therefore $\hat{\Gamma}_x^c$ is polar. Similarly Γ_y^c is ploar.

Suppose we are given a CAF $B\in\Phi^+$ and let $\hat{B}$ be its dual CAF. Just as (2.4), define a kernel $\hat{K}_B$ by

$$\hat{K}_B(dx, y) = \hat{V}_{A,B}^{1,0}\hat{K}_A(dx, y)+\hat{V}_{A,B}^{1,0}(dx, y)-\nu_B(dx)\ , \tag{3.12}$$

for $y\notin P_B$, where $\hat{K}_A$ is the kernel defined by (3.9). In the case $B_t=t$ denote $\hat{G}_B$ by $\hat{G}$. For these kernels, the dual results of section 2 are valid.

Theorem 3.3. *For all $x\in E$, $y\in E$ and $z\in E-P_B$,*

$$\begin{aligned}&K_A(x, dy)=g(x, y)\nu_A(dy),\ \hat{K}_A(dx, y)=g(x, y)\nu_A(dx)\\&K_B(x, dy)=g(x, y)\nu_B(dy)\quad\textit{and}\quad\hat{K}_B(dx, z)=g(x, z)\nu_B(dx)\ .\end{aligned} \tag{3.13}$$

Proof. The first two equalities have been already proved. For the proof of the third equality, take a function $f\in b\mathcal{E}^+$ such that $V_{A,B}^{1,0}f$ is bounded. Then, since ν_B charges no polar set, it follows from lemma 3.1 and 3.2 that

$$\iint |g(x, y)| f(y) \nu_B(dy) \leqq V_{A,B}^{1,0} f(x) + \langle \nu_B, f \rangle + |K_A| V_{A,B}^{1,0} f(x) < \infty .$$

Hence

$$\int g(x, y) f(y) \nu_B(dy)$$
$$= V_{A,B}^{1,0} f(x) - \langle \nu_B, f \rangle + K_A V_{A,B}^{1,0} f(x) = K_B f(x) .$$

The last equality follows similarly.

Corollary. *For all $x \in E$ and $y \in E$,*

$$(3.14) \qquad G(x, dy) = g(x, y) \mu(dy) \quad \text{and} \quad \hat{G}(dx, y) = g(x, y) \mu(dx) .$$

For a measure ξ on E, let us denote

$$(3.15) \qquad G^{1,0}\xi(x) = \int g^{1,0}(x, y) \xi(dy) ,$$

$$(3.16) \qquad G\xi(x) = \int g(x, y) \xi(dy) ,$$

if they are well defined.

Let X_A and $\hat{X}_A$ be the subprocesses of X and $\hat{X}$ by the multiplicative functionals $M_t = e^{-A_t}$ and $\hat{M}_t = e^{-\hat{A}_t}$, respectively. Then a set is polar if and only if it is polar relative to X_A or $\hat{X}_A$. Moreover, as we have seen at lemma 1.4, the resolvents $(G_A^{1,p})_{p>0}$ and $(\hat{G}_A^{1,p})_{p>0}$ of the processes X_A and $\hat{X}_A$ are strong Feller, so that, it is well known that a compact set F is non-polar if and only if $G^{1,0}\xi$ is locally bounded for some non-zero finite measure ξ on F. Also, it is well known that if $G^{1,0}\xi$ is locally bounded then ξ charges no polar set (see [3; p. 285]). Hence we have the following theorem.

Theorem 3.4. *If F is a compact subset of E, then F is non-polar if and only if there exists a non-zero finite measure ξ on F such that $\int |g(x, y)| \xi(dy)$ is locally bounded.*

Proof. It is enough to prove that $G^{1,0}\xi$ is locally bounded if and only if $\int |g(x, y)| \xi(dy)$ is locally bounded.

If $G^{1,0}\xi$ is locally bounded for some non-zero finite measure ξ then ξ charges no polar set and hence, in particular, $\xi(\hat{\Gamma}_x^c) = 0$ for all $x \in E$. So, it follows that,

$$\int |g(x, y)| \xi(dy) \leqq G^{1,0}\xi(x) + \xi(E) + |K_A| G^{1,0}\xi(x) .$$

In the right side of the inequality, since $G^{1,0}\xi$ is bounded on the compact set C, the last two terms are bounded. Therefore, $\int |g(x, y)| \xi(dy)$ is locally bounded.

Conversely, if $\int|g(x,y)|\xi(dy)$ is locally bounded then $\xi(\hat{\Gamma}_x^c)=0$ from the definition of $g(x,y)$. Therefore, for any $x\in E$,

$$g(x,y)=g^{1,0}(x,y)-1+\int g^{1,0}(x,z)g(z,y)\nu_A(dz)$$

$a.a.y(\xi)$. Thus

$$\begin{aligned}G^{1,0}\xi(x) &\leqq \int|g(x,y)|\xi(dy)+\xi(E)\\ &\quad+\int g^{1,0}(x,z)\{\int|g(z,y)|\xi(dy)\}\nu_A(dz)\,.\end{aligned}$$

Since $G^{1,0}\nu_A(x)=K_A^1 1(x)=1$,

$$\begin{aligned}&\int g^{1,0}(x,z)\{\int|g(z,y)|\xi(dy)\}\,\nu_A(dz)\\ &\leqq \sup_{z\in C}\int|g(z,y)|\xi(dy)\,.\end{aligned}$$

Therefore the theorem is proved.

4. Potential kernel functions

By the corollary of theorem 3.3, we shall say that $g(x,y)$ is *the potential kernel function associated with* $(G,\hat{G})$. Moreover the kernel function $g(x,y)$ satisfies several regularity conditions (corollaries 2 and 3 of theorem 2.3, lemma 3.2).

We now extend the notion of potential kernel functions.

Definition. An $\mathcal{E}^*\times\mathcal{E}^*$-measurable kernel function $h(x,y)$ is said to be a *potential kernel function* if the following conditions are satisfied.

(i) Set $H(x,dy)=h(x,y)\mu(dy)$ and $\hat{H}(dx,y)=h(x,y)\mu(dx)$. Then H and $\hat{H}$ are the potential kernels of X and $\hat{X}$ such that Hf and $f\hat{H}$ are well defined and locally bounded for all $f\in b\mathcal{E}_c^*$. Moreover, the functions $H(\cdot,F)$ and $\hat{H}(F,\cdot)$ are finely and cofinely continuous for any compact set F, respectively.

(ii) The sections $(\Gamma_h)_y^c$ and $(\hat{\Gamma}_h)_x^c$ (see §3) of the set $\Gamma_h^c=\{(x,y);\ |h(x,y)|=\infty\}$ are polar sets and the functions $h(\cdot,y)$ and $h(x,\cdot)$ are finely and cofinely continuous on the fine and cofine open sets $(\Gamma_h)_y$ and $(\hat{\Gamma}_h)_x$ for all $x,y\in E$, respectively.

We shall show how any potential kernel function $h(x,y)$ is related to $g(x,y)$. Recall that $\Gamma^c=\{(x,y);\ |g(x,y)|=\infty\}$.

Theorem 4.1. *If $h(x,y)$ is a potential kernel function of X, then*

$$(4.1)\qquad g(x, y) = h(x, y) - H(x, C) - \hat{H}(C, y) + H(C, C),$$

for all $(x, y) \in \Gamma \cap \Gamma_h$, *where* $H(C, C) = \int_C H(x, C)\,\mu(dx)$.

Proof. If $f \in N$ then, by (i), $Gf - Hf$ is bounded and satisfies $(I - pG^p)(Gf - Hf) = 0$, so that, $Gf - Hf$ equals a constant on E. Particularly, set $f = I_F - \mu(F) I_C \in N$ for a relatively compact set $F \in \mathcal{E}^*$ then, since $G(\cdot, C) = 0$,

$$(4.2)\qquad G(x, F) - H(x, F) + H(x, C)\,\mu(F) = a$$

for some constant a. Integrating both sides of (4.2) by $\nu_A = \mu|_C$ and noting that $\nu_A G = 0$, we have

$$-H(C, F) + H(C, C)\,\mu(F) = a\,.$$

Thus,

$$G(x, F) = H(x, F) - H(x, C)\,\mu(F) - H(C, F) + H(C, C)\,\mu(F)\,.$$

Therefore, for all $x \in E$, (4.1) holds for $a.a.y(\mu)$. Since μ is equivalent to $\hat{G}^p(\cdot, y)$ for all $p > 0$ and $y \in E([1])$, μ charges all cofine open sets. Hence, for all $x \in E$, (4.1) holds for cofinely dense $y \in E$. Since both sides of (4.1) are cofinely continuous relative to y on the cofine open set $\hat{\Gamma}_x \cap (\hat{\Gamma}_h)_x$, (4.1) holds for all $y \in \hat{\Gamma}_x \cap (\hat{\Gamma}_h)_x$.

If $B \in \Phi^+$ then, since the associated measure ν_B of B has no mass on any semipolar set, we have

Corollary 1. *If $h(x, y)$ is a potential kernel function of X, then the kernels $H_B(x, dy) = h(x, y)\,\nu_B(dy)$ and $\hat{H}_B(dx, y) = h(x, y)\,\nu_B(dx)$ are potential kernels of (K_B^p) and $(\hat{K}_B^p)$, respectively.*

Corollary 2. *Let $h(x, y)$ be a potential kernel function such that $\Gamma_h \subset \Gamma$, then a compact subset F of E is non-polar if and only if $\int |h(x, y)|\,\xi(dy)$ is locally bounded for some non-zero finite measure ξ on F. In particular, if X and $\hat{X}$ are equivalent, then F is non-polar iff $\int |h(x, y)|\,\xi(dy)$ is bounded on F for some ξ as above.*

Proof. It is enough to show that $\int |g(x, y)|\,\xi(dy)$ is locally bounded if and only if $\int |h(x, y)|\,\xi(dy)$ is locally bounded.

If $\int |g(x, y)|\,\xi(dy)$ is locally bounded, then ξ charges no polar set by theorem 3.4 and, in particular, $\xi(\{\hat{\Gamma}_x \cap (\hat{\Gamma}_h)_x\}^c) = 0$. Hence it follows from (4.1) that $\int |h(x, y)|\,\xi(dy)$ is locally bounded.

Conversely, if $\int |h(x, y)|\,\xi(dy)$ is locally bounded, then ξ has no mass on $(\hat{\Gamma}_h)_x^c$ for all $x \in E$, so that (4.1) holds $a.a.y(\xi)$ for all $x \in E$. Hence $\int |g(x, y)|\,\xi(dy)$ is locally bounded. If X and $\hat{X}$ are equivalent, then all semipolar sets are polar.

Hence $\sup_{x\in E} G^{1,0}\xi(x)=\sup_{x\in F} G^{1,0}\xi(x)$ ([3]), so that the last part of the corollary is obvious from the proof of theorem 3.4.

Remark. If $h(x, y)$ is a potential kernel function of X, then the kernel function $h'(x, y)$ defined by

$$(4.3)\qquad \begin{aligned} h'(x, y) &= h(x, y) \qquad \textit{if}\ \ (x, y)\in\Gamma\cap\Gamma_h \\ &= \infty \qquad\qquad\ \textit{if}\ \ (x, y)\notin\Gamma\cap\Gamma_h \end{aligned}$$

is a potential kernel function of X. For this kernel function, the hypothesis $\Gamma_{h'}\subset\Gamma$ of the corollary 2 holds obviously.

Remark. So far we have fixed a compact set C and assumed that $\mu(C)=1$. If we delete such normalization condition, the only minor change is necessary; ν_A equals $[\mu(C)]^{-1}\mu|_C$ for $\mu|_C$ and $\nu_{\{t\}}$ equals $[\mu(C)]^{-1}\mu$ for μ. It then follows that$G(x, dy)=g(x, y)[\mu(C)]^{-1}\mu(dy)$.

For two compact sets C_1 and C_2, let G_1, G_2 and g_1, g_2 be their associated potential kernels and kernel functions. Let μ be an arbitrary invariant measure (not necessarily normalized either on C_1 or C_2). By an argument similar to the proof of theorem 4.1, we have $G_1(x, F)-\dfrac{G_1(C_2, F)}{\mu(C_2)}=G_2(x, F)-\dfrac{\mu(F)}{\mu(C_1)}G_2(x, C_1)$. By the preceding remark, we obtain the following relation:

$$(4.4)\qquad \frac{g_1(x, y)}{\mu(C_1)}-\frac{1}{\mu(C_2)}G_1(C_2, y) = \frac{g_2(x, y)}{\mu(C_2)}-\frac{1}{\mu(C_1)}G_2(x, C_1)$$

on $\Gamma g_1\cap\Gamma g_2$.

5. Equilibrium measure

Let $\boldsymbol{F}$ be the family of all non-empty relatively compact sets F which is the fine support of some CAF $B\in\Phi^+$. In this section we shall fix a set $F\in\boldsymbol{F}$ and the corresponding CAF B. Let $\{\hat{F}_n\}_{n\geq 1}$ be an increasing sequence satisfying that $\cup\,\hat{F}_n=E$ and $\hat{V}^{1,0}_{A,B}(\hat{F}_n, \cdot)$ are bounded for all n. The existence of such a sequence is the same as in lemma 1.2. Define the continuous additive functionals $B^n\in\Phi$ by $B^n_t=\int_0^t I_{F_n\cap\hat{F}_n}(X_s)dB_s$. Then the fine support of each B^n is relatively compact. The kernels defined by A and B^n are denoted by $U^{p,q}_n$ and $V^{p,q}_n$. By the definition of B^n,$V^{1,0}_n|f|$ and $|f|\,\hat{V}^{1,0}_n$ are bounded for all $f\in b\mathcal{E}^*$. Let ν_B be the measure assoicated with B as before and set $\nu_n(\cdot)=\nu_B(\cdot\cap F_n\cap\hat{F}_n)$. The fine support of ν_B is equal to F (see [14; remark II.2]) and ν_n is the measure associated with B^n. Write K_n for K_{B^n}. It follows that K_nf is well defined and bounded for all $f\in b\mathcal{E}^*$.

Lemma 5.1. *If $B^n\neq 0$ then, for all $p>0$,*

$$(5.1) \qquad pK_n(U_n^{0,p}1)+U_n^{0,p}1\equiv R_n(p)$$

is a finite constant on E.

Proof. If $B^n\neq 0$ then $B^n\in\Phi^+$, so that, by theorem 2.3, formulas (1.6) and (2.2), and lemma 1.5,

$$\begin{aligned} pK_n(U_n^{0,p}1)(x) &= p \lim_{q\to 0} \{V_n^{q,0}(x, \cdot)-\frac{1}{q}\nu_n\}\, U_n^{0,p}1 \\ &= \lim_{q\to 0} \{pV_n^{q,0}U_n^{0,p}1(x)-\frac{1}{q}\} \\ &= \lim_{q\to 0} \{qK_A^q U_n^{0,p}1(x)-U_n^{0,p}1(x)\} \\ &= \nu_A U_n^{0,p}1-U_n^{0,p}1(x)\,. \end{aligned}$$

Therefore,

$$(5.2) \qquad pK_n(U_p^{0,p}1)(x)+U_n^{0,p}1(x) = \nu_A U_n^{0,p}1 = R_n(p)$$

is a constant.

Let T_F be the hitting time of the set F, $\tau=inf\,\{t;\, B_t>0\}$ and $\tau^n=inf\,\{t;\, B_t^n>0\}$, where $inf\,\phi=\infty$. Then, $T_F=\tau$ *a.s.* (see [3; proposition V.3.5]) and $\tau^n\downarrow\tau$ *a,.s.* as $n\uparrow\infty$. Since $R_n(p)=E^{\nu_A}[\int_0^\infty e^{-pB_t^n}I_C(X_t)\,dt]$ decreases when n or p increases, the limit

$$(5.3) \qquad R(F) = \lim_{n\to\infty}\lim_{p\to\infty} R_n(p) = \lim_{p\to\infty}\lim_{n\to\infty} R_n(p)$$

exists and it is finite since $B\neq 0$ (see the proof of lemma 5.2 below).

DEFINITION. We shall call the constant $R(F)$ as Robin's constant of F (relative to the potential kernel function $g(x, y)$).

Lemma 5.2. $R(F)=E^{\nu_A}[\int_0^{T_F} I_C(X_t)\,dt]$.

Proof. Since $B\neq 0$, $U_n^{0,1}1$ is bounded for all large n. Hence, for all $p\geqq 1$ and large n,

$$R_n(p) \leqq R_n(1) = \nu_A U_n^{0,1}1<\infty\ ,$$

Therefore, by the Lebesque theorem,

$$\begin{aligned} \lim_{p\to\infty} R_n(p) &= \lim_{p\to\infty} E^{\nu_A}[\int_0^\infty e^{-pB_t^n}I_C(X_t)\,dt] \\ &= E^{\nu_A}[\int_0^\infty (\lim_{p\to\infty} e^{-B_t^n})I_C(X_t)\,dt] \\ &= E^{\nu_A}[\int_0^{\tau_n} I_C(X_t)\,dt]\,. \end{aligned}$$

Letting $n\to\infty$ we have the result.

REMARK. From the lemma 5.2, Robin's constant $R(F)$ of F does not depend on the choice of B.

Lemma 5.3. *If $F\in F$ then there exists a probability measure ξ_F on $\bar{F}$ such that*

$$\lim_{n\to\infty}\lim_{p\to\infty} V_n^{1,0}(pU_n^{0,p}1)(x) = G^{1,0}\xi_F(x) \tag{5.4}$$

for a.a.$x(\mu)$. Moreover, $V_n^{1,0}(pU_n^{0,p}1)(x)$ are uniformly bounded for all $x\in E$, $p\geqq 1$ and large n.

Proof. From (1.6), for all $p>0$ and $n\geqq 1$ such that $B^n\neq 0$,

$$V_n^{1,0}(pU_n^{0,p}1)(x) = 1-U_n^{0,p}1(x)+K_A^1U_n^{0,p}1(x)\,.$$

As in the proof of lemma 5.2, $U_n^{0,p}1(x)$ are uniformly bounded for all $x\in E$, $p\geqq 1$ and $n\geqq 1$ such that $B^n\neq 0$, and

$$\lim_{n\to\infty}\lim_{p\to\infty} U_n^{0,p}1(x) = U^{0,\infty}1(x) \equiv E^x[\int_0^T I_C(X_t)\,dt]\,.$$

Hence,

$$\lim_{n\to\infty}\lim_{p\to\infty} V_n^{1,0}(pU_n^{0,p}1)(x) = 1-U^{0,\infty}1(x)+K_A^1U^{0,\infty}1(x)\,,$$

boundedly. Define a measure $\xi_{p,n}$ on the compact set $\bar{F}$ by
$\xi_{p,n}(dy)=pU_n^{0,p}1(y)\nu_n(dy)$ for $p>0$ and $n\geqq 1$ such that $B^n\neq 0$, then

$$\xi_{p,n}(E) = \langle\nu_n,\, pU_n^{0,p}1\rangle = \langle\nu_A,\, 1\rangle = 1\,.$$

Thus there exists a sequence $p_k\to\infty$ such that $\{\xi_{p_k,n}\}_{k\geqq 1}$ converges weakly to a probability measure ξ_n on $\bar{F}$ as $k\to\infty$, for all n. Therefore, we can choose a subsequence $\{\xi_{n_m}\}$ of $\{\xi_n\}$ which converges weakly to a probability measure ξ_F on $\bar{F}$ as $m\to\infty$. Taking an arbitrary function $f\in b\mathcal{E}_c^*$, we have

$$\begin{aligned}
&\int f(x)\{1-U^{0,\infty}1(x)+K_A^1U^{0,\infty}1(x)\}\,\mu(dx)\\
&=\lim_{m\to\infty}\lim_{k\to\infty}\int f(x)V_{n_m}^{1,0}(p_kU_{n_m}^{0,p_k}1)(x)\,\mu(dx)\\
&=\lim_{m\to\infty}\lim_{k\to\infty}\int f(x)\,G^{1,0}\xi_{p_k,n_m}(x)\,\mu(dx)\\
&=\lim_{m\to\infty}\lim_{k\to\infty}\int f\hat{G}_A^{1,0}(y)\,\xi_{p_k,n_m}(dy)\\
&=\int f\hat{G}_A^{1,0}(y)\,\xi_F(dy)=\int f(x)\,G^{1,0}\xi_F(x)\,\mu(dx)\,,
\end{aligned}$$

where we used the boundedness and continuity of $f\hat{G}_A^{1,0}$, which follows from the dual facts of lemmas 1.3 and 1.4. Therefore,

$$(5.5) \qquad 1-U^{0,\infty}1(x)+K_A^1U^{0,\infty}1(x) = G^{1,0}\xi_F(x)\,, \qquad \text{for } a.a.x(\mu)\,.$$

Let $\hat{B}$ be the dual CAF of B as in section 3 and let $\hat{F}$ be the cofine support of B. As before, $\hat{F}$ is the cofine support of ν_B. Set $\hat{\tau}=\mathit{inf}\,\{t;\,\hat{B}_t>0\}$, then $\hat{\tau}=\hat{T}_F$ *a.s.* $\hat{P}^x$ for all $x\in E-P_B$, where $\hat{T}_F$ is the hitting time of $\hat{F}$ relative to $\hat{X}$.

Lemma 5.4. *For all $f\in b\mathcal{E}^*$,*

$$(5.6) \qquad \int f(y)\xi_F(dy) = \hat{E}^{\nu_A}[f(\hat{X}_{\hat{\tau}})]\,.$$

In particular, ξ_F is a probability measure on $\hat{F}$ which attains no mass on any polar set.

Proof. It is enough to show the equality (5.6) for $f\in C_c$. If $f\in C_c$, then by the corollary of lemma 3.1,

$$\begin{aligned}\int f(y)\xi_F(dy) &= \lim_{m\to\infty}\lim_{k\to\infty}\int f(y)p_kU_{n_m}^{0,p_k}1(y)\nu_{n_m}(dy)\\ &= \lim_{m\to\infty}\lim_{k\to\infty}\int p_kf\hat{V}_{n_m}^{0,p_k}(y)\nu_A(dy)\\ &= \lim_{m\to\infty}\lim_{k\to\infty}\hat{E}^{\nu_A}[p_k\int_0^\infty e^{-p_k(\hat{B}^{n_m})_t}f(\hat{X}_t)\,d(\hat{B}^{n_m})_t]\\ &= \lim_{m\to\infty}\lim_{k\to\infty}\hat{E}^{\nu_A}[\int_0^\infty e^{-u}f(\hat{X}_{\hat{\tau}_{n_m}(u/p_k)})\,du]\,,\end{aligned}$$

where $(\hat{B}^n)_t=\int_0^t I_{F_n\cap F_n}(\hat{X}_u)\,d\hat{B}_u$ is the dual CAF of B^n and $\hat{\tau}_n(s)=\mathit{inf}\,\{u;\,(\hat{B}^n)_u>s\}$. Since, for all $n\geqq 1$, $\hat{\tau}_n(s)\to\hat{\tau}_n\equiv\hat{\tau}_n(0)$ *a.s. as* $s\to 0$,

$$\begin{aligned}\int f(y)\xi_F(dy) &= \lim_{m\to\infty}\hat{E}^{\nu_A}[\int_0^\infty e^{-u}f(\hat{X}_{\hat{\tau}_{n_m}})\,du]\\ &= \lim_{m\to\infty}\hat{E}^{\nu_A}[f(\hat{X}_{\hat{\tau}_{n_m}})]\,.\end{aligned}$$

Also, since $\hat{\tau}_{n_m}\to\hat{\tau}$ *a.s.* when $m\to\infty$, the lemma follows.

Theorem 5.5 (Equilibrium principle). *Let $F\in\mathcal{E}^*$ be a relatively compact subset of E and suppose that there exists a CAF $B\in\Phi^+$ with fine support F. Then there exists a unique probability measure ξ_F on $\hat{F}$ such that*

$$(5.7) \qquad G\xi_F(x) = a \text{ constant on } F.$$

Here, $\hat{F}$ is the cofine support of the dual CAF $\hat{B}$ of B and the constant is equal to Robin's constant $R(F)$ of F. The measure ξ_F is given by (5.6) *and called the equilibrium measure of F.*

Proof. Let us show that the measure ξ_F in lemma 5.3 satisfies

(5.8) $\quad G\xi_F(x)+U^{0,\infty}1(x)=R(F) \qquad$ everywhere on E.

This proves (5.7) since $U^{0,\infty}1(x)=E^x[\int_0^\tau I_C(X_t)\,dt]=0$ on F. From (2.2), (2.4) and lemma 5.1,

$$\begin{aligned} R_n(p) &= K_n(pU_n^{0,p}1)(x)+U_n^{0,p}1(x) \\ &= K_A V_n^{1,0}(pU_n^{1,p}1)(x)+V_n^{1,0}(pU_n^{0,p}1)(x) \\ &\quad -p\nu_n U_n^{0,p}1+U_n^{0,p}1(x) \\ &= K_A V_n^{1,0}(pU_n^{0,p}1)(x)+V_n^{1,0}(pU_n^{0,p}1)(x) \\ &\quad -1+U_n^{0,p}1(x)\,, \end{aligned}$$

since $p\nu_n U_n^{0,p}1=\nu_A(C)=1$. Let $p\to\infty$ and $n\to\infty$, then, as we have seen in lemmas 5.2 and 5.3, $R_n(p)\to R(F)$ and $V_n^{1,0}(pU_n^{0,p}1)(x)\to G^{1,0}\xi_F(x)$ *a.a.* $x(\mu)$, boundedly,. Since $K_A(x,\ \cdot)$ is a bounded signed measure and which is absolutely continuous relative to μ, we have, for $a.a.x(\mu)$,

$$\begin{aligned} (5.9) \qquad R(F) &= K_A G^{1,0}\xi_F(x)+G^{1,0}\xi_F(x)-1+U^{0,\infty}1(x) \\ &= \int\{K_A g^{1,0}(x,y)+g^{1,0}(x,y)-1\}\,\xi_F(dy)+U^{0,\infty}1(x) \\ &= G\xi_F(x)+U^{0,\infty}1(x)\,, \end{aligned}$$

from lemmas 3.2 and 5.4. Denote

$$\xi(dy)=\xi_F(dy)+\int\sum_{n=1}^{\infty}\{(\hat{K}_A^1)^n-\nu_A\}\,g^{1,0}(dy,z)\,\xi_F(dz)\,,$$

then ξ is a bounded signed measure on $\bar{F}$ and

$$G\xi_F(x) \doteq G^{1,0}\xi(x)-1\,.$$

Since $G^{1,0}\xi_F(x)$ is bounded, $G^{1,0}\xi(x)$ is the difference of two bounded excessive functions relative to $(G_A^{1,p})_{p>0}$. Therefore,

$$\lim_{p\to\infty} pG_A^{1,p}G^{1,0}\xi(x)=G^{1,0}\xi(x) \qquad \text{for all } x\in E\,.$$

Moreover,

$$pG_A^{1,p}1(x)=E^x[\int_0^\infty exp\,(-\int_0^{t/p} I_C(X_s)\,ds-t)\,dt]\to 1\,,$$

as $p\to\infty$ for all $x\in E$ and $G_A^{1,p}(x,\ \cdot)$ is absolutely continuous relative to μ, for all $x\in E$ and $p>0$. Thus, operating $pG_A^{1,p}$ to both sides of (5.9) and letting $p\to\infty$, we have

$$R(F)=G\xi_F(x)+\lim_{p\to\infty} pG_A^{1,p}U^{0,\infty}1(x) \qquad \text{for all } x\in E\,.$$

Therefore, it is enough to show that

(5.10) $\lim_{p} pG_A^{1,p}U^{0,\infty}1(x) = U^{0,\infty}1(x)$ for all $x\in E$.

Let $p>1$ then,

$$\begin{aligned} & pG_A^{1,p}U^{0,\infty}1(x) \\ &= E^x[\int_0^\infty p\, exp\, \{-\int_0^t I_C(X_s)\,ds-pt\} E^{X_t}[\int_0^\tau I_C(X_u)\,du]\,dt] \\ &\leqq E^x[\int_0^\infty pe^{-pt}\{\int_t^{t+\tau\circ\theta_t} I_C(X_u)\,du\}\,dt] \\ &\leqq E^x[\int_0^\infty e^{-t}\{\int_{t/p}^{(t/p)+\tau\circ\theta_{t/p}} I_C(X_u)\,du\}\,dt] \\ &\leqq E^x[\int_0^\infty e^{-t}\{\int_t^{t+\tau\circ\theta_t} I_C(X_u)\,du\}\,dt]+1 \\ &\leqq \|U^{0,\infty}1\|+1\,. \end{aligned}$$

Thus, noting that $\lim_{t\to 0}(t+\tau\circ\theta_t)=\tau$ (see [3; p. 214]), by the Lebesgue theorem,

$$\begin{aligned} & \lim_{p\to\infty} pG_A^{1,p}\, U^{0,\infty}1(x) \\ &= \lim_{p\to\infty} E^x[\int_0^\infty exp\, \{-\int_0^{t/p} I_C(X_s)\,ds-t\}\,dt\int_{t/p}^{(t/p)+\tau\circ\theta_{t/p}} I_C(X_u)\,du] \\ &= \int_0^\infty e^{-t}E^x[\int_0^\tau I_C(X_u)\,du]\,dt = U^{0,\infty}1(x)\,. \end{aligned}$$

Now, it remains only the proof of the uniqueness. Let ξ be a bounded signed measure on $\hat{F}$ satisfying $\xi(E)=0$ and $G\xi(x)=a$, for some constant a, on F. For the proof of uniqueness we claim that $\xi=0$. Integrating both sides of $G\xi(x)=a$ $(x\in F)$ by $f(x)\nu_n(dx)$, we have

(5.11) $\int f\hat{K}_n(y)\xi(dy) = a\langle\nu_n, f\rangle$

for all $f\in b\mathcal{E}^*$ and $n\geqq 1$, where $\hat{K}_n(dx, y)=g(x, y)\nu_n(dx)$ as before. Set $f=g(I-p\hat{K}^p_{B^n})$ for a bounded continuous function g. It follows, from the dual result of (2.8), that

$$f\hat{K}_n(y) = g(I-p\hat{K}^p_{B^n})\hat{K}_n(y) = g\hat{K}^p_{B^n}(y)-\langle g\hat{K}^p_{B^n}, \nu_A\rangle\,,$$

for all $y\notin P_{B^n}$, $n\geqq 1$ and $p>0$. Substituting this function into (5.11), we have

$$\int g\hat{K}^p_{B^n}(y)\xi(dy) = 0 \qquad \text{for all } n\geqq 1 \text{ and } p>0\,,$$

because $\xi(E)=0$, $\langle g(I-p\hat{K}^p_{B^n}), \nu_n\rangle=0$ and ξ vanishes outside of $\hat{F}\subset E-P_B\subset E-P_{B^n}$. Therefore, similarly to the proof of lemma 5.4, we have

$$\int \hat{E}^y[g(\hat{X}_{\hat{\tau}})]\,\xi(dy) = 0\,.$$

This implies that $\int g(y)\,\xi(dy)=0$, since $\hat{P}^y[\hat{\tau}=0]=1$ for all $y\in\hat{P}$.

6. Symmetric case

In this section we shall assume, in addition, that $g^p(x, y)=g^p(y, x)$ for all $p>0$ and $x, y\in E$, that is, X and $\hat{X}$ are equivalent. In this case, as is well known (see [3; proposition VI. 4. 10]), any semipolar set is polar. Hence, for every compact set F, the set $F-F^r$ is polar, where F^r is the set of all regular points of F (see [3; II. 3.3]). Therefore F is a projective set (see [3; V. 4.5]). Hence, by considering the projection of CAF $\{t\}$, there exists a CAF B such that

$$E^x[e^{-T_F}] = E^x[\int_0^\infty e^{-t}dB_t] \tag{6.1}$$

and $supp\,(B)=F^r$ ([3; V. 4.6 and 4.7]), where $supp\,(B)$ is the fine support of B. Obviously, F is a polar set if and only if the corresponding CAF B is zero. Let $T=inf\,\{t\ ;\ B_t=\infty\}$. We have

$$1 \geqq E^x[e^{-T_F}] \geqq E^x[\int_0^T e^{-t}\,dB_t]\geqq E^x[e^{-T}B_T]\,.$$

This implies that $T=\infty$ *a.s.* P^x for all $x\in E$, that is, $B\in\Phi$. Let $\hat{B}$ be the dual CAF of B then, under our present hypothesis, the corresponding polar set P_B may be supposed empty (see the proof of [14 ; VII. 1]) and the cofine support $\hat{F}$ of $\hat{B}$ is equal to ${}^rF=F^r$, since the fine and cofine topologies conicide, where ${}^rF$ is the set of all coregular points of F. Therefore, by theorem 5.5 we have

Theorem 6.1. *If F is a non-polar compact subset of E, then there exists a unique probability measure ξ_F on F^r such that*

$$G\xi_F(x) = R(F) \quad on\ F^r\,. \tag{6.2}$$

Here, the measure ξ_F and the constant $R(F)$ are given by

$$\xi_F(dy) = \hat{P}^{\nu_A}[\hat{X}_{\hat{T}_F}\in dy] \quad and \tag{6.3}$$

$$R(F) = E^{\nu_A}[\int_0^{T_F} I_C(X_t)\,dt]\,, \tag{6.4}$$

respectively. The measure ξ_F is called the equilibrium measure of F (relative to the potential kernel function $g(x, y)$).

Corollary. *Under the hypothesis of theorem* 6.1 *there exists a unique probability measure ξ_F on F such that $G\xi_F$ is bounded on F and satisfies* (6.2).

Proof. It is enough to prove the uniqueness. Suppose that a measure ξ on F satisfies the conditions of the corollary. Then since $G\xi$ is bounded on F, ξ charges no polar set (see the proofs of theorem 3.4 and corollary 2 of theorem 4.1). Hence ξ is a measure on F^r, so that the corollary follows from theorem 6.1.

REMARK. By the proof of the corollary, the result of the corollary may be replaced by the following result. "There exists a unique probability measure on F which attains no mass on any polar set and satisfies (6.2)".

By using the relation (4.1) of $g(x, y)$ and an arbitrary potential kernel function $h(x, y)$, we would like to investigate the equilibrium principle relative to $h(x, y)$. At present, however, we can get only a partial result on this problem; we have to impose very strong conditions on $h(x, y)$ and we do not know even if the logarithmic potential kernel function of planar Brownian motion satisfies these conditions. Our conditions are the following.

(H1) For every compact set D,

$$\lim_{p\to 0} \sup_{x,y\in D} |g^p(x, y)-\phi(p)-h(x, y)| = 0 \tag{6.5}$$

for some function ϕ and a potential kernel function h.

(H2) For all $p>0$ and bounded continuous function f, $K_B^p f$ is continuous, where B is a CAF with fine support F^r, as before.

To find the equilibrium measure ξ relative to h, we shall attempt a formal calculation. Suppose that a probability measure ξ on F satisfies $H\xi(x)= \int h(x, y)\xi(dy)=a$ *on* F^r for some constant a. Then, from (4.1), for all $x\in F^r$

$$\begin{aligned} G\xi(x) &= H\xi(x)-H(x, C)-\int \hat{H}(C, y)\xi(dy)+H(C, C) \\ &= -H(x, C)+a-\int \hat{H}(C, y)\xi(dy)+H(C, C)\,. \end{aligned}$$

Operating $I-pK_B^p$ and integrating by $fd\nu_B$, it follows that

$$\langle f, (I-pK_B^p)G\xi\rangle_{\nu_B} = -\langle f, (I-pK_B^p)H(\cdot, C)\rangle_{\nu_B}\,. \tag{6.6}$$

The left side of (6.6) becomes

$$\begin{aligned} \langle f, (I-pK_B^p)G\xi\rangle_{\nu_B} &= \langle f(I-p\hat{K}_B^p)\hat{K}_B, \xi\rangle \\ &= \langle f\hat{K}_B^p, \xi\rangle-\langle f\hat{K}_B^p, \nu_A\rangle \end{aligned}$$

from the dual formula of (2.8).

On the other hand, the right side of (6.6) becomes

$$-\langle f, (I-pK_B^p)H(\cdot, C)\rangle_{\nu_B} = -\lim_{q\to 0}\langle f, (I-pK_B^p)(G^q(\cdot, C)-\phi(q))\rangle_{\nu_B}$$
$$= -\lim_{q\to 0}\langle f, (I-pK_B^p)G^q(\cdot, C)\rangle_{\nu_B} = \lim_{q\to 0}\langle f, -G_B^{p,0}(\cdot, C)+qG_B^{p,0}G^q(\cdot, C)\rangle_{\nu_B}$$
$$= -\langle f\hat{K}_B^p, \nu_A\rangle + \lim_{q\to 0}\langle f, qG_B^{p,0}G^q(\cdot, C)\rangle_{\nu_B}.$$

Hence (6.6) becomes

$$\langle f\hat{K}_B^p, \xi\rangle = \lim_{q\to 0}\langle f, qG_B^{p,0}G^q(\cdot, C)\rangle_{\nu_B}.$$

Multiplying p and letting $p\to\infty$ we have

$$\langle f, \xi\rangle = \lim_{p\to\infty}\lim_{q\to 0} pq\langle f, G_B^{p,0}G^q(\cdot, C)\rangle_{\nu_B}. \tag{6.7}$$

Theorem 6.2. *Let F be a non-polar compact subset of E. Under the hypothesis* (H1) *and* (H2), *there exists a unique probability measure ξ on F^r such that*

$$H\xi = \text{a constant on } F^r. \tag{6.8}$$

Proof. Since

$$\langle 1, pqG_B^{p,0}G^q(\cdot, C)\rangle_{\nu_B} = pq\langle 1\hat{K}_B^p\hat{G}^q, \nu_A\rangle = 1,$$

the measure $pqG_B^{p,0}G^q(x, C)\nu_B(dx)$ is a probability measure on F for any $p, q>0$. Hence, for all $p>0$, we can choose a sequence $q_n\to 0$ and a probability measure ξ_p on F such that $pq_nG_B^{p,0}G^{q_n}(x, C)\nu_B(dx)\to\xi_p(dx)$, weakly. Similarly, there exists a sequence $p_m\to\infty$ and a probability measure ξ on F such that $\xi_{p_m}\to\xi$, weakly. From the hypothesis (H2), for all bounded continuous function f,

$$\langle p_k f\hat{K}_B^{p_k}, \xi\rangle = \lim_m \lim_n \langle p_k f\hat{K}_B^{p_k}, p_m q_n G_B^{p_m,0}G^{q_n}(\cdot, C)\rangle_{\nu_B}$$
$$= \lim_m \lim_n \langle p_k p_m q_n f\hat{K}_B^{p_k}\hat{K}_B^{p_m}\hat{G}^{q_n}, \nu_A\rangle$$
$$= \lim_m \lim_n \frac{1}{p_m-p_k}\langle p_k p_m q_n f(\hat{K}_B^{p_k}-\hat{K}_B^{p_m})\hat{G}^{q_n}, \nu_A\rangle$$
$$= \lim_m \lim_n \frac{1}{p_m-p_k} p_k p_m q_n\langle f\hat{K}_B^{p_k}\hat{G}^{q_n}, \nu_A\rangle$$
$$= \lim_n p_k q_n\langle f\hat{K}_B^{p_k}\hat{G}^{q_n}, \nu_A\rangle = \langle f, \xi_k\rangle.$$

Hence, letting $k\to\infty$, it follows that

$$\lim_k \langle p_k f\hat{K}_B^{p_k}, \xi\rangle = \langle f, \xi\rangle,$$

that is, $\xi=\hat{E}^\xi[f(\hat{X}_{\hat{T}})]=E^\xi[f(X_{T_F})]$. So that ξ is a measure on F^r.

To prove (6.8), let f be a bounded continous function with compact support. By restricting the CAT B as in section 5, we may suppose that $V_{A,B}^{1,0}1$ is bounded. Then, since $f\hat{G}$ is bounded and continuous from (2.7), we have

$$\begin{aligned}\langle f, G\xi\rangle_\mu &= \langle f\hat{G}, \xi\rangle = \lim_m \lim_n \langle f\hat{G}, p_m q_n G_B^{p_m,0} G^{q_n}(\cdot, C)\rangle_{\nu_B}\\
&= \lim_m \lim_n p_m \langle f\hat{G}, G_B^{p_m,0}(\cdot, C) - G^{q_n}(\cdot, C) + p_m K_B^{p_m} G^{q_n}(\cdot, C)\rangle_{\nu_B}\\
&= \lim_m \lim_n p_m \langle f, K_B G_B^{p_m,0}(\cdot, C) - K_B(I - p_m K_B^{p_m}) G^{q_n}(\cdot, C)\rangle_\mu\\
&= \lim_m \lim_n p_m \langle f, K_B G_B^{p_m,0}(\cdot, C) - K_B^{p_m} G^{q_n}(\cdot, C) + \nu_A K_B^{p_m} G^{q_n}(C)\rangle_\mu\end{aligned}$$

from (2.8). By the definition of K_B and H,

$$\begin{aligned}\lim_m p_m K_B G_B^{p_m,0}(x, C) &= \lim_m \lim_{q\to 0} p_m \left\{V_{A,B}^{q,0} - \frac{\nu_B}{q}\right\} U_{A,B}^{0,p_m} 1(x)\\
&= \lim_m \lim_{q\to 0} \left\{p_m V_{A,B}^{q,0} U_{A,B}^{0,p_m} 1(x) - \frac{1}{q}\right\}\\
&= \lim_m \lim_{q\to 0} \left\{K_A^q 1 - U_{A,B}^{0,p_m} 1 + q K_A^q U_{A,B}^{0,p_m} 1 - \frac{1}{q}\right\}(x)\\
&= \lim_m \{-U_{A,B}^{0,p_m} 1 + \nu_A U_{A,B}^{0,p_m} 1\}(x)\\
&= -E^x\left[\int_0^{T_F} I_C(X_s)\, ds\right] + E^{\nu_A}\left[\int_0^{T_F} I_C(X_s)\, ds\right],\end{aligned}$$

$$\begin{aligned}&\lim_m \lim_n p_m \{K_B^{p_m} G^{q_n}(x, C) - \nu_A K_B^{p_m} G^{q_n}(C)\}\\
&= \lim_m \lim_n p_m [K_B^{p_m}\{G^{q_n}(x, C) - \phi(q_n)\} - \nu_A K_B^{p_m}\{G^{q_n}(\cdot, C) - \phi(q_n)\}]\\
&= \lim_m p_m \{K_B^{p_m} H(x, C) - \nu_A K_B^{p_m} H(C)\}\\
&= E^x[H(X_{T_F}, C)] - E^{\nu_A}[H(X_{T_F}, C)]\,.\end{aligned}$$

Hence

$$\begin{aligned}\langle f, G\xi\rangle_\mu = \langle f, &-E^{\cdot}\left[\int_0^{T_F} I_C(X_s)\, ds\right] - E^{\cdot}[E(X_{T_F}, C]\rangle_\mu\\
&+ \langle f, 1\rangle_\mu E^{\nu_A}\left[\int_0^{T_F} I_C(X_s)\, ds + H(X_{T_B}, C)\right].\end{aligned}$$

Therefore

$$G\xi(x) = -E^x\left[\int_0^{T_F} I_C(X_s)\, ds + H(X_{T_F}, C)\right] + E^{\nu_A}\left[\int_0^{T_F} I_C(X_s)\, ds + H(X_{T_F}, C)\right]$$

for $a.a.x$ (μ). In particular,

$$G\xi(x) = -H(x, C) + E^{\nu_A}\left[\int_0^{T_F} I_C(X_s)\, ds + H(X_{T_F}, C)\right]$$

for $a.a.x \in F^r$ (μ), and hence for all $x \in F^r$. Hence, by (4.1), (6.8) holds. If ξ_1 and ξ_2 are measures on F^r satisfying (6.8), then $G(\xi_1 - \xi_2)$ equals to a constant on F^r. Hence $\xi_1 = \xi_2$ by the proof of theorem 5.5.

In the classical case, the equilibrium measure is characterized as the measure which minimize the energy. In our case, the analogous result holds. Denote $\mathfrak{M}$ the family of all bounded signed measures ξ on E with compact support such that $\int |g(x, y)| |\xi|(dy)$ is bounded, $\mathcal{M}^+=\{\xi\geqq 0;\ \xi\in\mathcal{M}\}$ and $\mathcal{M}^0=\{\xi\in\mathcal{M};\ \xi(E)=0\}$. For $\xi, \zeta\in\mathcal{M}$, define the mutual energy of ξ and ζ by

$$(6.9)\qquad (\xi, \zeta)=\iint g(x, y)\,\xi(dx)\zeta(dy)\,.$$

Denote (ξ, ξ) by $I(\xi)$ and call it the energy of ξ.

Lemma 6.3. *If $\xi\in\mathcal{M}^0$, then $I(\xi)$ is non-negative. Moreover, $I(\xi)=0$ if and only if $\xi=0$.*

Proof. Suppose that $\xi\in\mathcal{M}^0$. Since $G^{1,0}|\xi|(x)$ is bounded,

$$\begin{aligned}&\int\sum_{n=1}^{\infty}|(K_A^1)^n-\nu_A|(x, dz)G^{1,0}|\xi|(z)\\ &\qquad\leqq \|G^{1,0}|\xi|\|\sum_{n=1}^{\infty}\|(K_A^1)(x, \cdot)-\nu_A\|\end{aligned}$$

converges uniformly in x. Hence for any $\varepsilon>0$ there exists a number N such that

$$\Big|\sum_{n=N}^{\infty}\int\{(K_A^1)^n-\nu_A\}\ (x, dz)G^{1,0}\xi(z)\Big|<\varepsilon$$

for all $x\in E$. From our definition of $g(x, y)$, for $(x, y)\in\Gamma$,

$$\begin{aligned}g(x, y)=g^{1,0}(x, y)-1+&\sum_{n=1}^{N-1}\{(K_A^1)^n-\nu_A\}g^{1,0}(x, y)\\ &+\varepsilon(x, y, N)\,,\end{aligned}$$

where $\varepsilon(x, y, N)=\sum_{n=N}^{\infty}\{(K_A^1)^n-\nu_A\}g^{1,0}(x, y)$. Since $\xi(\hat{\Gamma}_x)=0$, for all $x\in E$,

$$\begin{aligned}I(\xi)=&\iint g^{1,0}(x, y)\xi(dx)\xi(dy)+\sum_{n=1}^{N-1}\iint(K_A^1)^n g^{1,0}(x, y)\xi(dx)(dy)\\ &+\iint\varepsilon(x, y, N)\xi(dx)\xi(dy)\,.\end{aligned}$$

From the resolvent equation (1.7), we have

$$g^{2,0}(x, y)-g^{1,0}(x, y)+K_A^2 g^{1,0}(x, y)=0\,.$$

This combined with $g^{1,0}(x, y)\geqq g^{2,0}(x, y)$, we have

$$K_A^2 g^{2,0}(x, y)\leqq K_A^2 g^{1,0}(x, y)\leqq g^{1,0}(x, y)\,,$$

so that

$$\int g^{2,0}(x, z)g^{2,0}(z, y)\nu_A(dz)\leqq g^{1,0}(x, y)\,.$$

Hence we have, from the symmetry of $g^2(x, y)$

$$\iint g^{1,0}(x, y)\xi(dx)\xi(dy)\geqq\int\Big\{\int g^{2,0}(x, y)\xi(dy)\Big\}^2\nu_A(dx)\geqq 0\,.$$

Similarly it follows that

$$\sum_{n=1}^{N-1}\iint (K_A^1)^n g^{1,0}(x,y)\xi(dx)\xi(dy)\geqq 0\,.$$

Therefore $I(\xi)\geqq -\varepsilon$ and hence $I(\xi)\geqq 0$.

Suppose that $I(\xi)=0$. By a routine argument, we have $|(\xi,\zeta)|^2\leqq I(\xi)I(\zeta)$ for all $\zeta\in\mathcal{M}^0$. Hence $(\xi,\zeta)=0$ for all $\zeta\in\mathcal{M}^0$. This implies that $G\xi$ equlas to a constant on E. Integrating by ν_A, we can see that the constant is equals to 0. Hence $\xi=0$.

Theorem 6.4. *The equilibrium measure ξ_F of a compact set F is the unique measure which attains the*

(6.10) $\quad min\ \{I(\xi);\ \xi\in\mathcal{M}^+,\ \xi(E)=1,$ *support of* $\xi\subseteqq F\}$,

and Robin's *constant $R(F)$ equals the minimum value of* (6.10).

Proof. The proof is similar to the classical case [16]. If a measure ξ satisfies the conditions of (6.10), then, since $G\xi_F=R(F)$ on F except a polar subset of F and ξ charges no polar set,

$$\begin{aligned} I(\xi) &= I(\xi-\xi_F)-I(\xi_F)+2(\xi,\xi_F)\\ &= I(\xi-\xi_F)+R(F)\,. \end{aligned}$$

Since $\xi-\xi_F\in\mathcal{M}^0$, this implies the result by lemma 6.2.

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