ON THE EQUILIBRIUM MEASURE OF RECURRENT MARKOV PROCESSES

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(Received February 21, 1977)

0. Introduction

In the case of planar Brownian motion, if we denote \( h(x, y) = -\frac{1}{\pi} \log |x - y| \), the following results are well known (see [13], [16]). (i) If \( F \) is a non-polar compact set, then there exists a probability measure \( \xi_F \) on \( F \) such that \( \int h(x, y) \xi_F(dy) \) equals a constant \( R(F) \) on \( F \) except on a polar set. The measure \( \xi_F \) and the constant \( R(F) \) are respectively called the equilibrium measure and Robin's constant of \( F \). (ii) A compact set \( F \) is non-polar if and only if there exists a non-zero finite measure \( \xi \) on \( F \) such that \( \int h(x, y) \xi(dy) \) is locally bounded.

In this paper we shall be concerned with the similar problem for recurrent Hunt processes with strong Feller resolvent. In our case, in place of \( h(x, y) \), we shall use a density \( g(x, y) \) of a potential kernel \( G(x, dy) \) of \( X \) relative to the invariant measure \( \mu(dy) \). Unfortunately, our density \( g(x, y) \) is not equal to \( h(x, y) \) in the case of planar Brownian motion but equal to \( h(x, y) + f(x) + g(y) \) with some locally bounded functions \( f \) and \( g \) (see §4).

Now we shall outline the contents of this paper. Let \( X \) be a recurrent Hunt process with strong Feller resolvent and \( \mu \) an invariant measure of \( X \). If we are given a certain finite non-negative continuous additive functional \( A \) of \( X \) then we can construct a potential kernel \( G \) of \( X \) by means of time change and killing based upon \( A \) ([4], [12]). In this paper we shall suppose, for simplicity, that \( A_t = \int_0^t I_C(X_s)ds \) for an arbitrary fixed non-null compact set \( C \) but the similar argument can be applicable for a large class of functionals \( A \).

In section 1, some preliminary results are established. Among others, a potential kernel \( K_A \) and an invariant measure \( \nu_A \) of the time changed process by \( A \) are described. In section 2, for any other finite non-negative continuous additive functional \( B \), a potential kernel \( K_B \) and an invariant measure \( \nu_B \) of the time changed process by \( B \) are constructed by making use of \( K_A \) and \( \nu_A \). In section 3, let us introduce the duality hypothesis that there exists a dual process \( \hat{X} \) (of \( X \) relative to \( \mu \)) satisfying those regularity conditions like \( X \).
We shall then construct a kernel function \( g(x, y) \) such that \( g(\cdot, y) \) [resp. \( g(x, \cdot) \)] is finely [resp. confinedly] continuous, finite except on a polar set and \( K_B(x, dy) = g(x, y) \nu_B(dy) \) [resp. \( K_B(dx, y) = g(x, y) \nu_B(dx) \)] for all continuous additive functionals \( B \), where \( K_B \) is the potential kernel of the time changed process of \( \hat{X} \) by the dual functional \( \hat{B} \) of \( B \). In particular, when \( B_t = t \), we have \( G(x, dy) = g(x, y) \mu(dy) \) and \( \hat{G}(dx, y) = g(x, y) \mu(dx) \), where \( G \) is the potential kernel of \( \hat{X} \) associated with \( \hat{A}_t = \int_0^t I_c(\hat{X}_s) \, ds \) in the sense of section 2. In this sense, our function \( g(x, y) \) may be called the potential kernel function associated with \((G, \hat{G})\).

In section 4, we introduce the notion of potential kernel function \( h(x, y) \) in a more general sense and then establish a relation between \( h(x, y) \) and \( g(x, y) \).

In section 5, we shall show the equilibrium principle. This means that, if \( F = \text{supp}(J^G) \), then there is a probability measure \( \xi_F \) on \( \text{supp}(J^G) \) such that
\[
\int g(x, y) \xi_F(dy) = R(F) \quad \text{on } F.
\]
In our case, the equilibrium measure \( \xi_F \) and Robin's constant \( R(F) \) have intuitive probabilistic meanings. If \( X \) and \( \hat{X} \) are equivalent, the results of section 5 have simpler forms and the analogous potential principles to classical potential theory hold. This case is treated in section 6. There a characterization of the equilibrium measure by means of energy is also given.

1. Notations and preliminary results

Let \( E \) be a locally compact Hausdorff space with countable base, \( \mathcal{E} \) the Borel \( \sigma \)-field on \( E \) and \( \mathcal{E}^\ast \) the \( \sigma \)-field obtained by the universal completion of \( \mathcal{E} \). If \( \mathcal{A} \) is a \( \sigma \)-field of subsets of \( E \) then the classes of all bounded \( \mathcal{A} \)-measurable functions, all bounded non-negative \( \mathcal{A} \)-measurable functions and all bounded \( \mathcal{A} \)-measurable functions with compact support are denoted by \( \mathcal{B}_\mathcal{A} \), \( \mathcal{B}_\mathcal{A}^+ \) and \( \mathcal{B}_\mathcal{A}_c \) respectively.

Throughout in this paper, let \( X=(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x) \) be a recurrent Hunt process on \( E \) with strong Feller resolvent, that is, a Hunt process satisfying

(i) (Recurrence); For all \( f \in \mathcal{B}_\mathcal{A}_c \), \( G^\Phi f(x)=E^x[\int_0^\infty f(X_t) \, dt] \equiv 0 \) or \( \equiv \infty \) on \( E \).

(ii) (Strong Feller property of resolvent); For all \( p>0 \) and \( f \in \mathcal{B}_\mathcal{A} \), \( G^\Phi f(x)=E^x[\int_0^\infty e^{-pt} f(X_t) \, dt] \) is bounded continuous.

In this case, it is well known that there exists a unique (except a constant multiple) invariant Radon measure \( \mu \) of \( X \), which is positive on every open sets (see [1], [2]). Let \( \Phi \) be the family of all non-negative continuous additive functionals (abbreviated CAF) \( A=(A_t)_{t \geq 0} \) of \( X \) such that \( A_t < \infty \) a.s. for all \( t < \infty \) and let \( \Phi^+ \) be the subfamily of functionals \( A \in \Phi \) which are not equivalent to the zero functional. If \( A \in \Phi^+ \) then \( P^x(A_t=\infty)=1 \) for all \( x \) ([1]). For \( A \in \Phi^+ \) and \( p \geq 0 \) we define a kernel \( K^\Phi_{lt} \) by
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(1.1) \( K_\mu f(x) = E^\mu \left[ \int_0^\infty e^{-\beta t} f(X_t) dA_t \right] \).

Note that \( (K_\mu)_\beta \) is the resolvent of the time changed process of \( X \) by \( A \). Moreover, for \( A, B \in \Phi^+ \) and \( p, q \geq 0 \) we define two auxiliary kernels \( U^p_{\lambda;B} \) and \( V^q_{\lambda;B} \) as follows:

(1.2) \( U^p_{\lambda;B} f(x) = E^\mu \left[ \int_0^\infty e^{-pA_t} e^{-qB_t} f(X_t) dA_t \right] \).

(1.3) \( V^q_{\lambda;B} f(x) = E^\mu \left[ \int_0^\infty e^{-pA_t} e^{-qB_t} f(X_t) dB_t \right] \).

Obviously, \( U^p_{\lambda;B} = V^q_{\lambda;B} \). The family \( (U^p_{\lambda;B})_{p>0} \) is the resolvent of the time changed process by \( A \) of the \( e^{-qB_t} \)-subprocess of \( X \). If \( B_t = t \), we shall write \( K^p_{\mu A} \) for \( U^p_{\lambda;A} \) and \( G^q_{\mu B} \) for \( V^q_{\lambda;B} \), i.e.,

(1.4) \( K^p_{\mu A} f(x) = E^\mu \left[ \int_0^\infty e^{-pA_t} f(X_t) dA_t \right] \).

(1.5) \( G^q_{\mu B} f(x) = E^\mu \left[ \int_0^\infty e^{-qB_t} f(X_t) dB_t \right] \).

Note that \( U^p_{\lambda;B} = K^p_{\mu A} = G^q_{\mu B} \) and \( V^q_{\lambda;B} = G^q_{\mu B} \) (the resolvent of \( \mu \)).

In the sequel, if there is no danger of confusion, the suffices \( A, B \) will be often omitted.

Lemma 1.1 (Nagasawa-Sato [10; theorem 2.1 and 2.2]). Write \( U^p_{\mu q} \) and \( V^q_{\mu p} \) for \( U^p_{\lambda;B} \) and \( V^q_{\lambda;B} \). For all \( p, p' > 0 \), \( q, q' \geq 0 \) and \( f \in \mathcal{E}^* \),

(1.6) \( U^p_{\mu q} f - U^p_{\mu q'} f + (p-p') U^p_{\mu q'} f + (q-q') V^q_{\mu p} U^p_{\mu q'} f = 0 \),

(1.7) \( V^q_{\mu p} f - V^q_{\mu p'} f + (p-p') V^q_{\mu p'} f + (q-q') U^p_{\mu q} V^q_{\mu p'} f = 0 \).

If, in particular, \( U^{p_0}_{\mu q} f \) [resp. \( V^{q_0}_{\mu p} f \)] is bounded for some \( q_0 \geq 0 \) then \( U^{p_0}_{\mu q} f \) [resp. \( V^{q_0}_{\mu p} f \)] is bounded for all \( q \geq 0 \) and \( (1.6) \) [resp. \( (1.7) \)] holds for all \( p, p', q, q' \geq 0 \) satisfying \( p+q > 0 \) and \( p'+q' > 0 \).

Lemma 1.2 ([12; lemma 2.2]). There exists an increasing sequence \( \{E_n\}_{n \geq 1} [\text{resp. } \{F_n\}_{n \geq 1}] \) of subsets in \( \mathcal{E}^* \) such that \( \bigcup \{E_n\} = E \) [resp. \( \bigcup \{F_n\} = E \)] and \( U^{p_0}_{\mu q} (\cdot, E_n) [\text{resp. } V^{q_0}_{\mu p} (\cdot, F_n)] \) is bounded for all \( n \geq 1 \).

Lemma 1.3 (Blumenthal-Getoor [3; III, section 5]). If \( A \in \Phi^+ \) then \( G^p_{\mu A} (\cdot, F) \) is bounded for all compact set \( F \) and \( p > 0 \).

A set \( C \) is said to be null if it is a set of potential zero relative to \( (G^p)_p \).

Let \( C \) be an arbitrary (but fixed) non-null compact subset of \( E \) and let us assume that \( \mu \) is normalized on \( C \) as \( \mu(C) = 1 \).
In the remainder of this paper, unless otherwise stated, the CAF $A = (A_t)_{t \geq 0} \in \Phi^+$ always represents the CAF defined by

$$A_t = \int_0^t I_C(X_s) \, ds.$$  

Then for every $B \in \Phi^+$,

$$U^{0,1}_{A,B} 1(x) = E [\int_0^\infty e^{-Bt} I_C(X_t) \, dt] = G^{1,0}_B(x, C)$$

is bounded by lemma 1.3. Moreover we have

**Lemma 1.4.** For any $p, q > 0$ and $f \in b\mathcal{C}^*$, the functions $K^p_A f$ and $G^p A f$ are bounded continuous. In case $f \in b\mathcal{C}^*_1$, $G^{p,0}_A f$ is bounded continuous for all $p > 0$.

**Proof.** Drop the suffix $A$ in the related kernels. For any $p > 0$ and $f \in b\mathcal{C}^*$ we have, from (1.7),

$$G^p f - G^{p,0}_A f = p K^p A G^p f = 0.$$  

Since $K^{p,0} g = G^p(I_c g)$ for any $g \in b\mathcal{C}^*$ the function

$$G^{p,0}_A f = G^p f - p G^p(I_c G^{p,0}_A f) = G^p(f - p I_c G^{p,0}_A f)$$

is bounded continuous by the strong Feller property of $G^p$. Therefore,

$$G^{p,0}_A f = G^p(f + (q - p) G^{p,0}_A G^{p,0}_A f)$$

is bounded continuous. If $f \in b\mathcal{C}^*_1$ then $G^{p,0}_A f$ is bounded by lemma 1.3, so that the above equality for $q = 0$ shows that $G^{p,0}_A f$ is bounded continuous.

Since $(K^p_A)_{p > 0}$ is a strong Feller resolvent by lemma 1.4, the mapping $x \mapsto K^p_A(x, \cdot)$ of the compact set $C$ into the space of measures over $C$ is strongly continuous by a theorem of Mokobodzki (see Meyer [9]). Since, in addition, $K^p_A(x, \cdot)$ are equivalent for all $x \in E$, we have

$$\sup_{x, y \in E} \frac{1}{2} || K^p_A(x, \cdot) - K^p_A(y, \cdot) || \equiv a < 1.$$  

Thus there exists a unique invariant probability measure $\nu_A$ of $K^1$ such that

$$\sup_{x \in E} ||(K^p_A)^{n+1}(x, \cdot) - \nu_A(\cdot)|| \leq 2a^n$$

for all $n \geq 0$ ([7; lemma 1.3]). Therefore the kernel

$$K_A(x, F) = \sum_{n=1}^\infty [(K^p_A)^n(x, F) - \nu_A(F)]$$

(1.10)
is well defined and satisfies

\[(I - K_A^f)K_A^f = K_A^f - \langle \nu_A, f \rangle\]

for all \(f \in bC^*\).

**Lemma 1.5.** The kernel \(K_A\) defined by (1.10) satisfies

\[(1.11) \quad \limsup_{p \to 0, x \in E} \| K_A^p(x, \cdot) - \nu_A(\cdot) - K_A(x, \cdot) \| = 0,
\]

in particular,

\[(1.12) \quad \limsup_{p \to 0, x \in E} \| pK_A^p(x, \cdot) - \nu_A(\cdot) \| = 0.
\]

**Proof.** From the resolvent equation for \((K^*_A)\) we have

\[K_A^*(x, \cdot) = K_A^* \sum_{n=0}^{\infty} (1-p)^n (K_A^*)^n(x, \cdot)\]

\[= \sum_{n=1}^{\infty} (1-p)^{n-1} \{ (K_A^*)^n(x, \cdot) - \nu_A(\cdot) \} + \frac{\nu_A(\cdot)}{p}\]

for all \(x \in E\) and \(0 < p < 1\). Thus it follows that

\[\| K_A^p(x, \cdot) - \nu_A(\cdot) - K_A(x, \cdot) \| \]

\[\leq \sum_{n=1}^{\infty} \{ 1 - (1-p)^{n-1} \} \| (K_A^*)^n(x, \cdot) - \nu_A(\cdot) \|\]

\[\leq \sum_{n=1}^{\infty} \{ 1 - (1-p)^{n-1} \} 2a^{n-1} = 2 \left( \frac{1}{1-a} - \frac{1}{1-a(1-p)} \right).\]

Therefore the lemma follows.

**2. An invariant measure and a potential kernel of \((K_B^p)\)**

Similarly to [4] and [12], for any \(B \in \Phi^+\), an invariant measure \(\nu_B\) and a potential kernel \(K_B\) of \((K_B^p)_{p>0}\) can be constructed by making use of \(\nu_A\) and \(K_A\) defined in section 1. In [12], we have treated only the case of \(B_i = 1\) but the same arguments are valid for all \(B \in \Phi^+\). We shall outline it in the form of our present use.

For any \(B \in \Phi^+\) define the measure \(\nu_B\) by

\[(2.1) \quad \nu_B = \nu_AV_{A,B}^1\]

Then \(\nu_B\) charges no semipolar set and satisfies the following properties.

**Lemma 2.1.** The measure \(\nu_B\) is a \(\sigma\)-finite invariant measure of \((K_B^p)_{p>0}\). In particular, \(\nu_{[i]} = \mu\).

**Proof.** (cf. [12; theorem 2.7]) Since \(V_{A,B}^1(\cdot, F_0)\) is bounded for all \(n\) by
lemma 1.2, \( \nu_B \) is \( \sigma \)-finite. Integrating the equality
\[
V^1_{\lambda,0} - K^\lambda_B + K^\lambda_B - pV^1_{\lambda,0} = 0
\]
by \( \nu_A \) we have
\[
\nu_B = p\nu_B K^\lambda_B,
\]
that is, \( \nu_B \) is a constant multiple of \( \sigma \), say,
\[
\nu_{\{t\}} = \nu_A G^1_{\lambda} = b \mu
\]
for some constant \( b \). Since \( \mu(C) = 1 \) we have
\[
b = \nu_A G^1_{\lambda}(C) = \nu_A K^\lambda(C) = \nu_A(C) = 1,
\]
Hence \( \nu_{\{t\}} = \mu \).

**Lemma 2.2** (cf. \([4; \text{proposition 2}]\)). For any \( B, B' \in \Phi^+ \),
\[
(2.2) \quad \nu_B = p\nu_B V^0_B B.
\]
In particular, \( \nu_B \) is the measure associated with \( B \) in the sense of Revuz ([14]). Moreover it holds that \( \nu_A = \mu \mid C \), where \( \mu \mid C \) is the restriction of \( \mu \) to \( C \).

**Proof.** Similarly to lemma 1.1, we can prove easily that
\[
(2.3) \quad V^0_{\lambda} - V^0_{B',B} + qK^\lambda_B V^0_{B',B} - pV^0_{\lambda} V^0_{B',B} = 0
\]
for sufficiently many \( f \in b \). Letting \( q = 1 \) and integrating by \( \nu_A \), (2.2) follows. Set \( B_t = t \) at (2.2) then \( \nu_{B'} = p\nu_{\{t\}} V^0_{B',B} = p\mu K^\lambda_{B'} \) by lemma 1.1. Hence \( \nu_{B'} \) is the measure associated with \( B' \). In particular, when \( B' = A \), it follows that \( \langle \nu_A, f \rangle = p\langle \mu, G^I_{C} f \rangle = p\langle \mu, G^* f \rangle = p\langle \mu, I_c f \rangle = \langle \mu, I_c f \rangle = \langle \mu \mid C, f \rangle \).

Define a kernel \( K_B \) by
\[
(2.4) \quad K_B(x, \cdot) = K_A V^1_{\lambda,0}(x, \cdot) + V^1_{\lambda,0}(x, \cdot) - \nu_B(\cdot).
\]
In case \( B_t = t \) we shall denote \( K_B \) by \( G \), which is the kernel we have constructed in [12]. Obviously, \( K_B(x, \cdot) \) is a \( \sigma \)-finite signed measure on \( E \) and, for any \( n \geq 1 \), the total variation of \( K_B(x, \cdot) \) on \( F_n \) are uniformly bounded for all \( x \in E \) by (1.9) and lemma 1.2. Similarly, for any compact set \( F \), the total variation of \( G(x, \cdot) \) on \( F \) is uniformly bounded for all \( x \in E \) by lemma 1.3. If we denote the total variation of a measure on \( F \) by \( || \cdot ||_F \), then the following theorem holds.

**Theorem 2.3.** For all \( n \geq 1 \),
\[
(2.5) \quad \limsup_{p \to 0} \sup_{x \in F} || V^0_{\lambda,0}(x, \cdot) - \nu_B(\cdot) - K_B(x, \cdot) ||_F = 0,
\]
and in particular,

\[(2.6) \quad \limsup_{p \to 0} \|pV_{\beta,0}^p(x, \cdot) - \nu_B(\cdot)\|_{F_n} = 0.\]

If \(B_t = t\) then we can take arbitrary compact set in place of \(F_n\).

Proof. Write \(V_{\beta,0}^p\) for \(V_{\beta,0}^p\). For any Borel subset \(D\) of \(F_n\),

\[V_{\beta,0}^p(x, D) - V_{\beta,0}^p(x, D) + pK_{\beta}V_{\beta,0}^p(x, D) - K_{\beta}V_{\beta,0}^p(x, D) = 0\]

from (1.7). This can be written, by noting (2.1),

\[
\{V_{\beta,0}^p(x, D) - \frac{\nu_B(D)}{p} - V_{\beta,0}^p(x, D) + pK_{\beta}V_{\beta,0}^p(x, D)

- (K_{\beta} - \frac{1}{p} \nu_B)V_{\beta,0}^p(x, D) = 0.\]

Thus we have

\[
\|V_{\beta,0}^p(x, \cdot) - \frac{\nu_B(\cdot)}{p} - K_{\beta}(x, \cdot)\|_{F_n}
\leq \|\{pK_{\beta}(x, \cdot) - \nu_B(\cdot)\}V_{\beta,0}^p\|_{F_n}

+ \|\{K_{\beta}(x, \cdot) - \frac{\nu_B(\cdot)}{p} - K_{\beta}(x, \cdot)\}V_{\beta,0}^p\|_{F_n}.
\]

This proves the theorem from lemma 1.5.

**Corollary 1.** If \(f \in bC^\ast\) vanishes outside of some \(F_n\), then

\[(2.7) \quad (I - pK_{\beta})K_{\beta}f = K_{\beta}f - U_{\beta,0}f1_{\langle \nu_B, f \rangle}\]

for all \(p > 0\). If \(V_{\beta,0}^p\) is bounded, then

\[(2.8) \quad K_{\beta}(I - pK_{\beta})f = K_{\beta}f - \langle \nu_A, K_{\beta}f \rangle\]

for all \(p > 0\) and \(f \in bC^\ast\).

Proof. Suppose that \(V_{\beta,0}^p\) is bounded, then obviously (2.5) holds for \(E\) in place of \(F_n\), so we have

\[
K_{\beta}(I - pK_{\beta})f(x) = \lim_{x \to 0} (V_{\beta,0}^p - \frac{1}{q})(I - pK_{\beta})f(x)
\]

\[
= \lim_{x \to 0} V_{\beta,0}^p(I - pK_{\beta})f(x) = \lim_{x \to 0} (K_{\beta}f - qK_{\beta}K_{\beta}f)(x)
\]

\[
= K_{\beta}f(x) - \langle \nu_A, K_{\beta}f \rangle,
\]

from (1.13). The proof of (2.7) is similar.

Let us denote

\[(2.9) \quad N_{\beta} = \{f; f \in bC^\ast, = 0 \text{ outside of some } F_n \text{ and } \langle \nu_B, f \rangle = 0\},\]
If a kernel $H$ on $E$ satisfies the condition that (i) for any $f \in N_B$ [resp. $f \in N$], $Hf \in bC^*$ and that (ii) for any $f \in N_B$ [resp. $f \in N$], $(I-pK_B^p)Hf = K_B^pf$ [resp. $(I-pG^p)Hf = G^pf$] for all $p > 0$, then we shall say that $H$ is a potential kernel of $(K_B^p)_{p>0}$ [resp. $X$].

**Corollary 2.** The kernels $K_B$ and $G$ are the potential kernels of $(K_B^p)_{p>0}$ and $X$, respectively.

**Corollary 3.** For every compact set $F$, the function $G(\cdot, F)$ is finely continuous.

**Proof.** Set $B_t = t$ at (2.7)

$$G(x, F) = pG^pG(x, F) + G^p(x, F) - (K_B^p1(x))\mu(F).$$

Since $K_B^p1$ is $p$-excessive, the result is obvious.

### 3. Hypothesis of Duality and the Kernel Function $g(x, y)$

In this section we shall assume that there exists a Hunt process $\hat{X}$ with strong Feller resolvent $\hat{G}^p$ such that $X$ and $\hat{X}$ are in duality relative to $\mu$. It follows that $\hat{X}$ is also recurrent and $\mu$ is the invariant measure of $\hat{X}$.

Let $\Phi^+$ be the family of all non-zero non-negative finite continuous additive functionals of $\hat{X}$. For any $\hat{A}, \hat{B} \in \Phi^+$, we define $\hat{G}^{\hat{A}, \hat{B}}$ etc. by

$$f\hat{\mathcal{U}}^{\hat{A}, \hat{B}}(x) = E^x\left[\int_0^t e^{-\hat{A}_s}e^{\hat{B}_s}f(\hat{X}_s)\,d\hat{\mathcal{A}}_s\right]$$

etc. (in general, a kernel with respect to the dual process $\hat{X}$ is written such as $\hat{K}(D, x)$, so that $\hat{K}$ operates to function from the right side and to measure from the left).

By Revuz [14; theorem VII. 1], for any $B \in \Phi^+$, there exists a polar set $P_B$ and a CAF $\hat{B} \in \Phi^+$ of $\hat{X}$ restricted to $E - P_B$ such that $\nu_B = \hat{K}_B^0.1\mu$. Also, by [14; theorem VII. 2], there exists a jointly measurable kernel function $g_{\hat{B}}^p(x, y)$ satisfying

(i) $g_{\hat{B}}^p(\cdot, y)$ [resp. $g_{\hat{B}}^p(x, \cdot)$] is finely [resp. cofinely] continuous and $q$-excessive [resp. $q$-coexcessive] relative to the resolvent $(\hat{G}^p_{\hat{B}})^{q>0}$ [resp. $(G^p_{\hat{B}})^{q>0}$] for all $p > 0$ and $y \in E - P_B$ [resp. $x \in E$],

(ii) For all $p, q > 0$ and $x \in E, K_{\hat{B}}^q(x, dy) = g_{\hat{B}}^p(x, y)\nu_B(dy), G_{\hat{B}}^q(x, dy) = g_{\hat{B}}^p(x, y)\mu(dy)$ and for all $p, q > 0$ and $y \in E - P_B, \hat{K}_{\hat{B}}^q(dx, y) = g_{\hat{B}}^p(x, y)\nu_B(dx), \hat{G}_{\hat{B}}^q(dx, dy) = g_{\hat{B}}^p(x, y)\mu(dx)$.

As before, the set $C$ with $\mu(C) = 1$ is fixed and $A$ is given by (1.8). If $B = A, P_B$ may be supposed to be empty and the dual CAF of $A$ is given exactly by
In the following, unless otherwise stated, \( \hat{A} \) always represents this CAF and we shall drop the suffix \( A \) in \( g^q_{A^*} \). Further, we shall denote \( g^q(x, y) \) for \( g^q_{A^*}(x, y) \), which is Kunita-Watanabe’s potential kernel function. Note that \( g^q_{A^*}(x, y) = g^q(x, y) \) for all \( B \in \Phi^+ \). Form the resolvent equation (1.6), for any \( q > 0 \),

\[
\begin{align*}
(3.2) \quad g^q(x, y) &= g^q_{A^*}(x, y) - K^g_{A^*} g^q(x, y) \\
&= g^q(x, y) - g^q K^g_{A^*}(x, y)
\end{align*}
\]
on \( \{(x, y); g^q(x, y) < \infty\} \). Hence for any \( y \in E, K^q_{A^*} g^q(x, y) = g^q K^q_{A^*} g^q(x, y) \) a.a. \( x(\mu) \). Since both sides of the equality are \( q \)-excessive, it holds for all \( x, y \in E \) (cf. Getoor [5; theorem 2.5]).

**Lemma 3.1.** For all \( x \in E \) and \( B \in \Phi^+ \),

\[
(3.3) \quad V^q_{A^*, B}(x, dy) = g^{1-q}(x, y) \nu_B(dy).
\]

**Proof.** Set \( A' = A + qt \). Replacing \( A', \{t\}, B \) for \( A, B, B' \) in (2.3) we have

\[
V^q_{A^*, B} f = V^q_{A^*} f - K^q_{A^*} V^q_{A^*} f + q V^q_{A^*} V^q_{A^*} f,
\]
for sufficiently many functions \( f \). Noting that \( V^q_{A^*} f = K^q_{B^*} f, K^q_{A^*} = K^q_{A^*} + qG^q_{A^*} \) and \( V^{q}_{A^*} f = G^q_{A^*} \), it follows that

\[
E^q \left[ \int_0^\infty e^{-A^* t - tf(X_t)} dB_t \right] = V^q_{A^*, B} f(x)
\]

\[
= K^q_{B^*} f(x) - K^q_{A^*} f(x) \\
= \int \{g^q(x, y) - K^q_{A^*} g^q(x, y)\} f(y) \nu_B(dy)
= \int \{g^q(x, y) - K^q_{A^*} g^q(x, y)\} f(y) \nu_B(dy)
= \int \{g^{1-q}(x, y) f(y)\} \nu_B(dy)
\]
The last equality follows from (3.2) since \( \nu_B \) has no mass on the polar set \( \{y; g^q(x, y) = \infty\} \). Letting \( q \to 0 \) we have the result.

Dually, if \( \hat{B} \) is the dual CAF of \( B \) then

\[
(3.4) \quad \hat{V}^q_{A^*, B}(dx, y) = g^{1-q}(x, y) \nu_B(dx) \quad \text{for all } y \in P_B.
\]

Hence we have

**Corollary.** For all \( f, g \in b(\mathcal{C}^*)^+ \),
\begin{align}
\left\langle 3.5 \right\rangle & \quad \int f(x) V_{\lambda, \nu}^{1, 0} g(x) \nu_A(dx) = \int f U_{\lambda, \nu}^{1, 0}(y) g(y) \nu_B(dy) \\
\left\langle 3.6 \right\rangle & \quad \int f \dot{V}_{\lambda, \nu}^{1, 0}(y) g(y) \nu_A(dy) = \int f(x) U_{\lambda, \nu}^{1, 0} g(x) \nu_B(dx).
\end{align}

Since
\[ K_A(x, dy) = K_B(x, dy) - \nu_A(dy) + K_A K_B(x, dy), \]
it is easy to show that, for each \( x \), \( K_A(x, \cdot) \) is absolutely continuous relative to \( \nu_A \) and its density is given by
\[ g^{1,0}(x, y) = 1 + K_A g^{1,0}(x, y) \]
up to a set of \( \nu_A \)-measure 0.

However, in order to solve the problem proposed in the introduction, we have to choose a more elaborated density \( g(x, y) \). To do this, we need one more preliminary observation.

For all \( x, y \in E \) and \( n \geq 1 \), set
\[ f_n(x, y) = (K_{\lambda}^{1})^{n-1} g^{1,0}(x, y) = 1 = g^{1,0}(K_{\lambda}^{1})^{n-1}(x, y) = 1, \]
then
\[ f_n(x, y) \nu_A(dy) = (K_{\lambda}^{1})^{n}(x, dy) - \nu_A(dy) \]
\[ f_n(x, y) \nu_A(dx) = (K_{\lambda}^{1})^{n}(dx, y) - \nu_A(dx). \]

Since
\[ \int \sum_{n=1}^{\infty} |f_n(x, y)| \nu_A(dy) = \sum_{n=1}^{\infty} \|(K_{\lambda}^{1})^{n}(x, \cdot) - \nu_A(\cdot)\| < \infty \]
for all \( x \in E \) from (1.9) and (3.8), the series \( \sum_{n=1}^{\infty} f_n(x, y) \) converges absolutely for \( a.a.x(\nu_A) \). Similarly for all \( y \in E \), \( \sum_{n=1}^{\infty} f_n(x, y) \) converges absolutely for \( a.a.x(\nu_A) \).

Also
\[ \int_D \sum_{n=1}^{\infty} f_n(x, y) \nu_A(dy) = \sum_{n=1}^{\infty} \int_D f_n(x, y) \nu_A(dy) \]
\[ = \sum_{n=1}^{\infty} \{(K_{\lambda}^{1})^{n}(x, D) - \nu_A(D)\} = K_A(x, D) \]
for all \( D \in \mathcal{E} \), that is, \( \sum_{n=1}^{\infty} f_n(x, \cdot) \) is a density of \( K_A(x, \cdot) \) relative to \( \nu_A \). Dually, \( \sum_{n=1}^{\infty} f_n(\cdot, y) \) is a density of
\[ \hat{K}_A(\cdot, y) = \sum_{n=1}^{\infty} \{(\hat{K}_{\lambda}^{1})^{n}(\cdot, y) - \nu_A(\cdot)\} \]
relative to \( \nu_A \). Here the proof of the strong convergence of (3.9) is similar to (1.9).
Lemma 3.2. There exists a Borel subset $\Gamma$ of $E \times E$ satisfying the following conditions.

(i) Set $\Gamma = \{ (x, y); (x, y) \in \Gamma \}$ and $\hat{\Gamma} = \{ y; (x, y) \in \Gamma \}$, then $\Gamma$ and $\hat{\Gamma}$ are polar for all $x, y \in E$.

(ii) For all $(x, y) \in \Gamma$, $\sum_{n=1}^{\infty} f_n(x, y)$ converges absolutely, $|K_A|g^{1,0}(x, y) < \infty$ and $g^{1,0} |K_A|(x, y) < \infty$, where $|K_A|(x, \cdot)$ is the total variation measure of $K_A(x, \cdot)$.

(iii) For all $(x, y) \in \Gamma$,

$$
\sum_{n=1}^{\infty} f_n(x, y) = g^{1,0}(x, y) - 1 + K_A g^{1,0}(x, y)
$$

We define the kernel function $g(x, y)$ by

$$
g(x, y) = (3.10) \quad \text{if } (x, y) \in \Gamma
$$

$$
g(x, y) = \infty \quad \text{if } (x, y) \notin \Gamma.
$$

By the lemma, it is easy to see that the function $g(\cdot, y)$ [resp. $g(x, \cdot)$] is finely [resp. cofinely] continuous on the fine [resp. cofine] open set $\Gamma$ [resp. $\hat{\Gamma}$] for all $y$ [resp. all $x$] $\in E$.

Proof. Noting that,

$$
|f_{n+1}(x, y)| = |K_A \{(K_A)^{n-1}g^{1,0} - 1\}(x, y) |
$$

$$
= \left| \int K_A(x, dz) \left\{ \int (K_A)^{n-2}(z, du) g^{1,0}(u, v) g^{1,0}(v, y) \nu_A(dv) 
\right. 
\right.
$$

$$
- \left. \left. \int g^{1,0}(v, y) \nu_A(dv) \right\} \right| 
$$

$$
= \left| \int K_A(x, dz) \left\{ (K_A)^{n-2}g^{1,0}(z, \cdot) - 1 \right\} g^{1,0}(v, y) \nu_A(dv) \right| 
$$

$$
= |K_A f_{n-1} \hat{K}_A(x, y)| \leq K_A |f_{n-1}| \hat{K}_A(x, y)
$$

for $n \geq 2$ and

$$
\int |(K_A)^n(x, dz) - \nu_A(dz)| g^{1,0}(x, y)
$$

$$
= \int |(K_A)^n - 1| g^{1,0}(x, y) \nu_A(dz)
$$

$$
= \int |K_A \{(K_A)^{n-2}g^{1,0} - 1\}(x, z)| g^{1,0}(z, y) \nu_A(dz)
$$

$$
\leq K_A |f_{n-1}| \hat{K}_A(x, y)
$$

for $n \geq 2$, let us define the set $\Gamma$ by
Then the proofs of (ii) and (iii) are obvious. For the proof of (i) set
\[ \xi_a(dy) = \delta_a(dy) + K_a(x, dy) + \sum_{n=1}^{\infty} (K_a f_n)(x, y) \nu_a(dy). \]
Then
\[ \xi_a(E) = 2 + \sum_{n=1}^{\infty} \int K_a(x, dy) (K_a)^n(y, \cdot) - \nu_a(\cdot) < \infty. \]
Moreover, it is easy to see that,
\[ \Gamma = \{(x, y); \xi_a(dx) g^{1,0}(x, y) < \infty \}. \]
Hence \( \Gamma_x \) is polar if and only if \( \int \xi_a(dx) g^{1,0}(x, y) < \infty \) except on a polar set.

Suppose we are given a CAF \( B \in \Phi^+ \) and let \( \hat{B} \) be its dual CAF. Just as (2.4), define a kernel \( \hat{K}_B \) by
\[ \hat{K}_B(dx, y) = \hat{V}_{A}^{1,0} \hat{K}_A(dx, y) + \hat{V}_{A}^{1,0}(dx, y) - \nu_B(dx), \]
for \( y \in P_B \), where \( \hat{K}_A \) is the kernel defined by (3.9). In the case \( B_i = t \) denote \( \hat{G}_B \) by \( \hat{G} \). For these kernels, the dual results of section 2 are valid.

**Theorem 3.3.** For all \( x \in E, y \in E \) and \( z \in E - P_B \),
\[ \begin{align*}
K_A(x, dy) &= g(x, y) \nu_A(dy), \\
\hat{K}_A(dx, y) &= g(x, y) \nu_A(dx) \\
K_B(x, dy) &= g(x, y) \nu_B(dy) \quad \text{and} \quad \hat{K}_B(dx, z) = g(x, z) \nu_B(dx).
\end{align*} \]
\[ \iint |g(x, y)| f(y) \nu_B(dy) \leq V^{1,0}_A f(x) + \langle v_B, f \rangle + |K_A| V^{1,0}_A f(x) < \infty. \]

Hence
\[ \int g(x, y) f(y) \nu_B(dy) = V^{1,0}_A f(x) - \langle v_B, f \rangle + K_A V^{1,0}_A f(x) = K_B f(x). \]

The last equality follows similarly.

**Corollary.** For all \( x \in E \) and \( y \in E \),
\[ (3.14) \quad G(x, dy) = g(x, y) \mu(dy) \quad \text{and} \quad \hat{G}(dx, y) = g(x, y) \mu(dx). \]

For a measure \( \xi \) on \( E \), let us denote
\[ (3.15) \quad G^{1,0}\xi(x) = \int g^{1,0}(x, y) \xi(dy), \]
\[ (3.16) \quad G\xi(x) = \int g(x, y) \xi(dy), \]
if they are well defined.

Let \( X_A \) and \( \dot{X}_A \) be the subprocesses of \( X \) and \( \dot{X} \) by the multiplicative functionals \( M_t = e^{-A_t} \) and \( \dot{M}_t = e^{-\dot{A}_t} \), respectively. Then a set is polar if and only if it is polar relative to \( X_A \) or \( \dot{X}_A \). Moreover, as we have seen at lemma 1.4, the resolvents \( (G^{1,0})_{\rho>0} \) and \( (\dot{G}^{1,0})_{\rho>0} \) of the processes \( X_A \) and \( \dot{X}_A \) are strong Feller, so that, it is well known that a compact set \( F \) is non-polar if and only if \( G^{1,0}\xi \) is locally bounded for some non-zero finite measure \( \xi \) on \( F \). Also, it is well known that if \( G^{1,0}\xi \) is locally bounded then \( \xi \) charges no polar set (see [3; p. 285]). Hence we have the following theorem.

**Theorem 3.4.** If \( F \) is a compact subset of \( E \), then \( F \) is non-polar if and only if there exists a non-zero finite measure \( \xi \) on \( F \) such that \( \int |g(x, y)| \xi(dy) \) is locally bounded.

Proof. It is enough to prove that \( G^{1,0}\xi \) is locally bounded if and only if \( \int |g(x, y)| \xi(dy) \) is locally bounded.

If \( G^{1,0}\xi \) is locally bounded for some non-zero finite measure \( \xi \) then \( \xi \) charges no polar set and hence, in particular, \( \xi(\hat{F}_e) = 0 \) for all \( x \in E \). So, it follows that,
\[ \int |g(x, y)| \xi(dy) \leq G^{1,0}\xi(x) + \xi(E) + |K_A| G^{1,0}\xi(x). \]

In the right side of the inequality, since \( G^{1,0}\xi \) is bounded on the compact set \( C \), the last two terms are bounded. Therefore, \( \int |g(x, y)| \xi(dy) \) is locally bounded.
Conversely, if \( |g(x, y)| \xi(dy) \) is locally bounded then \( \xi(\tilde{\Gamma}^\xi_x) = 0 \) from the definition of \( g(x, y) \). Therefore, for any \( x \in E \),

\[
g(x, y) = g^{1,0}(x, y) - 1 + \int g^{1,0}(x, z)g(x, y)\nu_A(dz)
\]

a.a.\( y(\xi) \). Thus

\[
G^{1,0}(x) \equiv \int |g(x, y)| \xi(dy) + \xi(E) + \int g^{1,0}(x, z)\{\int |g(z, y)| \xi(dy)\} \nu_A(dz).
\]

Since \( G^{1,0}\nu_A(x) = K_11(x) = 1 \),

\[
\int g^{1,0}(x, z)\{\int |g(z, y)| \xi(dy)\} \nu_A(dz) \leq \sup_{x \in E} \int |g(x, y)| \xi(dy).
\]

Therefore the theorem is proved.

4. Potential kernel functions

By the corollary of theorem 3.3, we shall say that \( g(x, y) \) is the potential kernel function associated with \( (G, \tilde{\Gamma}) \). Moreover the kernel function \( g(x, y) \) satisfies several regularity conditions (corollaries 2 and 3 of theorem 2.3, lemma 3.2).

We now extend the notion of potential kernel functions.

**Definition.** An \( \mathcal{E}^* \times \mathcal{E}^* \)-measurable kernel function \( h(x, y) \) is said to be a potential kernel function if the following conditions are satisfied.

(i) Set \( H(x, dy) = h(x, y) \mu(dy) \) and \( \hat{H}(dx, y) = h(x, y) \mu(dx) \). Then \( H \) and \( \hat{H} \) are the potential kernels of \( X \) and \( \hat{X} \) such that \( Hf \) and \( f\hat{H} \) are well defined and locally bounded for all \( f \in b\mathcal{E}^* \). Moreover, the functions \( H(\cdot, F) \) and \( \hat{H}(F, \cdot) \) are finely and cofinely continuous for any compact set \( F \), respectively.

(ii) The sections \( (\Gamma_h)^c \) and \( (\hat{\Gamma}_h)^c \) (see §3) of the set \( \Gamma_h = \{(x, y); |h(x, y)| = \infty\} \) are polar sets and the functions \( h(\cdot, y) \) and \( h(x, \cdot) \) are finely and cofinely continuous on the fine and cofine open sets \( (\Gamma_h)^c \) and \( (\hat{\Gamma}_h)^c \), for all \( x, y \in E \), respectively.

We shall show how any potential kernel function \( h(x, y) \) is related to \( g(x, y) \). Recall that \( \Gamma^c = \{(x, y); |g(x, y)| = \infty\} \).

**Theorem 4.1.** If \( h(x, y) \) is a potential kernel function of \( X \), then
(4.1) \[ g(x, y) = h(x, y) - H(x, C) - \hat{H}(C, y) + H(C, C), \]
for all \((x, y) \in \Gamma \cap \Gamma_x\), where \(H(C, C) = \int_C H(x, C) \mu(dx)\).

Proof. If \(f \leq N\) then, by (i), \(Gf - Hf\) is bounded and satisfies \((I - pG^*)(Gf - Hf) = 0\), so that, \(Gf - Hf\) equals a constant on \(E\). Particularly, set \(f = I_{F^c} - \mu(F) I_{\mathcal{C}} \in N\) for a relatively compact set \(F \in \mathcal{E}^*\) then, since \(G(\cdot, C) = 0\),

\[
G(x, F) - H(x, F) + H(x, C) \mu(F) = a
\]
for some constant \(a\). Integrating both sides of (4.2) by \(\nu_x = \mu|_C\) and noting that \(\nu_x G = 0\), we have

\[-H(C, F) + H(C, C) \mu(F) = a.
\]

Thus,

\[G(x, F) = H(x, F) - H(x, C) \mu(F) - H(C, F) + H(C, C) \mu(F).
\]

Therefore, for all \(x \in E\), (4.1) holds for \(a.a. y(\mu)\). Since \(\rho^p\) is equivalent to \(\hat{G}^{(\cdot, y)}\) for all \(p > 0\) and \(y \in E([1])\), \(\rho\) charges all cofine open sets. Hence, for all \(x \in E\), (4.1) holds for cofinely dense \(y \in E\). Since both sides of (4.1) are cofinely continuous relative to \(y\) on the cofine open set \(\hat{\Gamma}_x \cap (\hat{\Gamma}_x)^c\), (4.1) holds for all \(y \in \hat{\Gamma}_x \cap (\hat{\Gamma}_x)^c\).

If \(B \in \Phi^+\) then, since the associated measure \(\nu_B\) of \(B\) has no mass on any semipolar set, we have

**Corollary 1.** If \(h(x, y)\) is a potential kernel function of \(X\), then the kernels \(H_B(x, dy) = h(x, y) \nu_B(dy)\) and \(\hat{H}_B(dx, y) = h(x, y) \nu_B(dx)\) are potential kernels of \((K_B^P)\) and \((K_B^P)\) respectively.

**Corollary 2.** Let \(h(x, y)\) be a potential kernel function such that \(\Gamma_h \subset \Gamma\), then a compact subset \(F\) of \(E\) is non-polar if and only if \(\int |h(x, y)| \xi(dy)\) is locally bounded for some non-zero finite measure \(\xi\) on \(F\). In particular, if \(X\) and \(X\) are equivalent, then \(F\) is non-polar iff \(\int |h(x, y)| \xi(dy)\) is bounded on \(F\) for some \(\xi\) as above.

Proof. It is enough to show that \(\int |g(x, y)| \xi(dy)\) is locally bounded if and only if \(\int |h(x, y)| \xi(dy)\) is locally bounded.

If \(\int |g(x, y)| \xi(dy)\) is locally bounded, then \(\xi\) charges no polar set by theorem 3.4 and, in particular, \(\xi(\hat{\Gamma}_x \cap (\hat{\Gamma}_x)^c) = 0\). Hence it follows from (4.1) that \(\int |h(x, y)| \xi(dy)\) is locally bounded.

Conversely, if \(\int |h(x, y)| \xi(dy)\) is locally bounded, then \(\xi\) has no mass on \((\hat{\Gamma}_x)^c\) for all \(x \in E\), so that (4.1) holds \(a.a. y(\xi)\) for all \(x \in E\). Hence \(\int |g(x, y)| \xi(dy)\) is locally bounded. If \(X\) and \(X\) are equivalent, then all semipolar sets are polar.
Hence \( \sup_{x \in \mathbb{R}} G^{1.0}(x) = \sup_{x \in \mathcal{F}} G^{1.0}(x) \) ([3]), so that the last part of the corollary is obvious from the proof of theorem 3.4.

**Remark.** If \( h(x, y) \) is a potential kernel function of \( X \), then the kernel function \( h'(x, y) \) defined by

\[
(4.3) \quad h'(x, y) = \begin{cases} 
   h(x, y) & \text{if } (x, y) \in \Gamma \cap \Gamma_a \\
   \infty & \text{if } (x, y) \notin \Gamma \cap \Gamma_a
\end{cases}
\]

is a potential kernel function of \( X \). For this kernel function, the hypothesis \( \Gamma_a \subset \Gamma \) of the corollary 2 holds obviously.

**Remark.** So far we have fixed a compact set \( \mathcal{C} \) and assumed that \( \mu(\mathcal{C}) = 1 \). If we delete such normalization condition, the only minor change is necessary; \( \nu_\mathcal{A} \) equals \( [\mu(\mathcal{C})]^{-1} \mu |_{\mathcal{C}} \) for \( \mu |_{\mathcal{C}} \) and \( \nu \mathcal{B} \) equals \( [\mu(\mathcal{C})]^{-1} \mu \) for \( \mu \). It then follows that \( G(x, dy) = g(x, y)[\mu(C)]^{-1} \mu(dy) \).

For two compact sets \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \), let \( G_1, G_2 \) and \( g_1, g_2 \) be their associated potential kernels and kernel functions. Let \( \mu \) be an arbitrary invariant measure (not necessarily normalized either on \( \mathcal{C}_1 \) or \( \mathcal{C}_2 \)). By an argument similar to the proof of theorem 4.1, we have \( G_1(x, F) = G_2(C_2, F) = G_2(x, F) - \frac{\mu(F)}{\mu(C_2)} G_2(x, C_1) \). By the preceding remark, we obtain the following relation:

\[
(4.4) \quad \frac{g_1(x, y)}{\mu(C_1)} \frac{1}{\mu(C_2)} G_1(C_2, y) = \frac{g_2(x, y)}{\mu(C_2)} \frac{1}{\mu(C_1)} G_2(x, C_1)
\]
on \( \Gamma \Gamma_1 \cap \Gamma \Gamma_2 \).

5. *Equilibrium measure*

Let \( \mathcal{F} \) be the family of all non-empty relatively compact sets \( \mathcal{F} \) which is the fine support of some CAF \( B \in \Phi^+ \). In this section we shall fix a set \( F \in \mathcal{F} \) and the corresponding CAF \( B \). Let \( \{\hat{\mathcal{F}}_n\}_{n \geq 1} \) be an increasing sequence satisfying that \( \hat{\mathcal{F}}_n = E \) and \( \hat{\mathcal{V}}_n \mathcal{B}U^{n} \hat{\mathcal{F}}_n, \cdot) \) are bounded for all \( n \). The existence of such a sequence is the same as in lemma 1.2. Define the continuous additive functionals \( B^* \in \Phi \) by \( B^* = \int_0^1 \mathcal{I}_{\hat{\mathcal{F}}_n} \mathcal{B}(X_n) dB^* \). Then the fine support of each \( B^* \) is relatively compact. The kernels defined by \( A \) and \( B^* \) are denoted by \( U^* \mathcal{A} \) and \( V^* \mathcal{A} \). By the definition of \( B^*, \mathcal{V}_n^1 \mathcal{A} \) and \( |f| \) and \( |\hat{V}_n^1 \mathcal{A}| \) are bounded for all \( f \in b \mathcal{C}^* \).

Let \( \nu_{\mathcal{B}} \) be the measure associated with \( B \) as before and set \( \nu_{\mathcal{B}}(\cdot) = \nu_{\mathcal{B}}(\cdot \cap F \cap \hat{\mathcal{F}}_n) \). The fine support of \( \nu_{\mathcal{B}} \) is equal to \( F \) (see [14; remark II.2]) and \( \nu_{\mathcal{B}} \) is the measure associated with \( B^* \). Write \( K_n \) for \( K_{\mathcal{B}_n} \). It follows that \( K_n f \) is well defined and bounded for all \( f \in b \mathcal{C}^* \).

**Lemma 5.1.** If \( B^* \neq 0 \) then, for all \( p > 0 \),

\[
\int_{\mathbb{R}} |f(x)|^p dK_n(x) = \int_{\mathcal{F}} |g(x, y)|^p d\nu_{\mathcal{B}}(x, y) \quad (\text{for a.e. } x, y) \quad (\text{for a.e. } x, y)
\]
(5.1) \[ pK_a(U_n^{0,1}) + U_n^{0,1} = R_n(p) \]

is a finite constant on \( E \).

Proof. If \( B^* = 0 \) then \( B^* \subseteq \Phi^+ \), so that, by theorem 2.3, formulas (1.6) and (2.2), and lemma 1.5,

\[
pK_a(U_n^{0,1})(x) = p \lim_{t \to 0} \{ V_n^{0,1}(x, \cdot) - \frac{1}{q} \} U_n^{0,1} \]

\[
= \lim_{t \to 0} \{ pV_n^{0,1} U_n^{0,1}(x) - \frac{1}{q} \} \]

\[
= \lim_{t \to 0} \{ qK_n U_n^{0,1}(x) - U_n^{0,1}(x) \} \]

\[
= \nu A U_n^{0,1} - U_n^{0,1}(x) .
\]

Therefore,

(5.2) \[ pK_a(U_n^{0,1})(x) + U_n^{0,1} = \nu A U_n^{0,1} = R_n(p) \]

is a constant.

Let \( T_F \) be the hitting time of the set \( F \), \( \tau = \inf \{ t; B_t > 0 \} \) and \( \tau^* = \inf \{ t; B_t^* > 0 \} \), where \( \inf \phi = \infty \). Then, \( T_F = \tau \) a.s. (see [3; proposition V.3.5]) and \( \tau^* \downarrow \tau \) a.s. as \( n \uparrow \infty \). Since \( R_n(p) = E^\tau \mathcal{A} \left[ e^{-p \sum_{i=0}^{\tau} I_c(X_i)} dt \right] \) decreases when \( n \) or \( p \) increases, the limit

(5.3) \[ R(F) = \lim_{n \to \infty} \lim_{p \to \infty} R_n(p) = \lim_{p \to \infty} \lim_{n \to \infty} R_n(p) \]

exists and it is finite since \( B \equiv 0 \) (see the proof of lemma 5.2 below).

**Definition.** We shall call the constant \( R(F) \) as Robin's constant of \( F \) (relative to the potential kernel function \( g(x, y) \)).

**Lemma 5.2.** \( R(F) = E^\tau \mathcal{A} \left[ \int_0^{T_F} I_c(X_i) dt \right] \).

Proof. Since \( B \equiv 0 \), \( U_n^{0,1} \) is bounded for all large \( n \). Hence, for all \( p \geq 1 \) and large \( n \),

\[
R_n(p) \leq R_n(1) = \nu A U_n^{0,1} < \infty ,
\]

Therefore, by the Lebesque theorem,

\[
\lim_{p \to \infty} \lim_{n \to \infty} R_n(p) = E^\tau \mathcal{A} \left[ \int_0^{T_F} e^{-p \sum_{i=0}^\tau I_c(X_i)} dt \right]
\]

\[
= E^\tau \mathcal{A} \left[ \lim_{p \to \infty} e^{-p \sum_{i=0}^\tau I_c(X_i)} dt \right]
\]

\[
= E^\tau \mathcal{A} \left[ \int_0^{T_F} I_c(X_i) dt \right] .
\]
Letting $n \to \infty$ we have the result.

**Remark.** From the lemma 5.2, Robin's constant $R(F)$ of $F$ does not depend on the choice of $B$.

**Lemma 5.3.** If $F \subseteq F$ then there exists a probability measure $\xi_F$ on $F$ such that

\begin{equation}
\lim_{n \to \infty} \lim_{p \to \infty} V_n^{1,0}(pU_n^{0,1})(x) = G^{1,0}\xi_F(x)
\end{equation}

for a.a.$x(\mu)$. Moreover, $V_n^{1,0}(pU_n^{0,1})(x)$ are uniformly bounded for all $x \in E$, $p \geq 1$ and large $n$.

**Proof.** From (1.6), for all $p > 0$ and $n \geq 1$ such that $B^n = 0$,

$$V_n^{1,0}(pU_n^{0,1})(x) = 1 - U_n^{0,1}(x) + K_n U_n^{0,1}(x).$$

As in the proof of lemma 5.2, $U_n^{0,1}(x)$ are uniformly bounded for all $x \in E$, $p \geq 1$ and $n \geq 1$ such that $B^n = 0$, and

$$\lim_{n \to \infty} \lim_{p \to \infty} U_n^{0,1}(x) = U^{0,1}(x) \equiv E^T\left[\int_0^T I_c(X_t) dt\right].$$

Hence,

$$\lim_{n \to \infty} \lim_{p \to \infty} V_n^{1,0}(pU_n^{0,1})(x) = 1 - U^{0,1}(x) + K_1 U^{0,1}(x),$$

boundedly. Define a measure $\xi_{p,n}$ on the compact set $F$ by

$$\xi_{p,n}(dy) = pU_n^{0,1}(y)\nu_n(dy)$$

for $p > 0$ and $n \geq 1$ such that $B^n = 0$, then

$$\xi_{p,n}(E) = \langle \nu_n, pU_n^{0,1} \rangle = \langle \nu_A, 1 \rangle = 1.$$

Thus there exists a sequence $p_k \to \infty$ such that \( \{\xi_{p_k,n}\}_{k \geq 1} \) converges weakly to a probability measure $\xi_n$ on $F$ as $k \to \infty$, for all $n$. Therefore, we can choose a subsequence $\{\xi_{p_k,n}\}$ of $\{\xi_n\}$ which converges weakly to a probability measure $\xi_F$ on $F$ as $m \to \infty$. Taking an arbitrary function $f \in bC_1$, we have

$$\int f(x)\{1 - U^{0,1}(1) + K_1 U^{0,1}(x)\} \mu(dx)$$

$$= \lim_{n \to \infty} \lim_{p \to \infty} \int f(x)V_n^{1,0}(pU_n^{0,1})(x) \mu(dx)$$

$$= \lim_{n \to \infty} \lim_{p \to \infty} \int f(x)G_{p,n}(x) \mu(dx)$$

$$= \lim_{n \to \infty} \lim_{p \to \infty} \int fG_{p,n}(y)\xi_{p,n}(dy)$$

$$= \int fG_{p,n}(y)\xi_F(dy) = \int f(x)G^{1,0}\xi_F(x) \mu(dx).$$
where we used the boundedness and continuity of $fG^{1,0}$, which follows from the dual facts of lemmas 1.3 and 1.4. Therefore,

\[(5.5) \quad 1 - U^{0,1}(x) + K^{1}U^{0,1}(x) = G^{1,0} \xi_{F}(x), \quad \text{for } a.a. x(\mu).\]

Let $\hat{B}$ be the dual CAF of $B$ as in section 3 and let $\hat{F}$ be the cofine support of $B$. As before, $\hat{F}$ is the cofine support of $\nu_{B}$. Set $\hat{\tau} = \inf \{ t; \hat{B}_{t} > 0 \}$, then $\hat{\tau} = \hat{T}_{F}$ a.s. $\hat{\mathbb{P}}^{x}$ for all $x \in E - P_{B}$, where $\hat{T}_{F}$ is the hitting time of $\hat{F}$ relative to $\hat{X}$.

**Lemma 5.4.** For all $f \in bC_{*}$,

\[(5.6) \quad \int f(y) \xi_{F}(dy) = \hat{E}^{x}_{\mathbb{A}}[f(\hat{X}_{\hat{\tau}})].\]

In particular, $\xi_{F}$ is a probability measure on $\hat{F}$ which attains no mass on any polar set.

**Proof.** It is enough to show the equality (5.6) for $f \in C_{c}$. If $f \in C_{c}$, then by the corollary of lemma 3.1,

\[
\int f(y) \xi_{F}(dy) = \lim_{m \to \infty} \left( \int f(y) p_{h} U^{0,1}_{m+1}(y) \nu_{m}(dy) \right) = \lim_{m \to \infty} \lim_{k \to \infty} \left( \int f(y) \hat{p} \hat{p}_{k}^{0,1}(y) \nu_{A}(dy) \right) = \lim_{m \to \infty} \lim_{k \to \infty} \left( \hat{E}^{x}_{\mathbb{A}} \left[ \int_{0}^{\infty} e^{-u} f(\hat{X}_{\hat{\tau}^{m}(y/\hat{p}_{k})}) du \right] \right),
\]

where $(\hat{B}^{*}) = \int_{I_{\hat{F}}} e^{1/n} d\hat{B}_{u}$ is the dual CAF of $B^{*}$ and $\hat{\tau}_{n}(s) = \inf \{ u; (\hat{B}^{*})_{u} > s \}$. Since, for all $n \geq 1, \hat{\tau}_{n}(s) \to \hat{\tau} = \hat{\tau}_{0}(0)$ a.s. as $s \to 0$,

\[
\int f(y) \xi_{F}(dy) = \lim_{m \to \infty} \hat{E}^{x}_{\mathbb{A}} \left[ \int_{0}^{\infty} e^{-u} f(\hat{X}_{\hat{\tau}^{m}}) du \right] = \lim_{m \to \infty} \hat{E}^{x}_{\mathbb{A}}[f(\hat{X}_{\hat{\tau}^{m}})].
\]

Also, since $\hat{\tau}_{m} \to \hat{\tau}$ a.s. when $m \to \infty$, the lemma follows.

**Theorem 5.5** (Equilibrium principle). Let $F \in C_{*}$ be a relatively compact subset of $E$ and suppose that there exists a CAF $B \in \Phi^{+}$ with fine support $F$. Then there exists a unique probability measure $\xi_{F}$ on $F$ such that

\[(5.7) \quad G^{1,0} \xi_{F}(x) = \text{a constant on } F.
\]

Here, $\hat{F}$ is the cofine support of the dual CAF $\hat{B}$ of $B$ and the constant is equal to Robin's constant $R(F)$ of $F$. The measure $\xi_{F}$ is given by (5.6) and called the equilibrium measure of $F$. 
Proof. Let us show that the measure \( \xi_F \) in lemma 5.3 satisfies

\[ G\xi_F(x) + U^{0,-1}(x) = R(F) \quad \text{everywhere on } E. \]  

This proves (5.7) since \( U^{0,-1}(x) = E^t [\int_0^t I_c(X_s) \, ds] = 0 \) on \( F \). From (2.2), (2.4) and lemma 5.1,

\[
R_n(p) = K_n(p U_n^{0,1})(x) + U_n^{0,1}(x)
\]

\[
= K_n V^{?,1}_n(p U_n^{0,1})(x) + V^{1,0}_n(p U_n^{0,1})(x)
\]

\[
- p U_n^{0,1} + U_n^{0,1}(x)
\]

\[
= K_n V^{1,0}_n(p U_n^{0,1})(x) + V^{1,0}_n(p U_n^{0,1})(x)
\]

\[
- 1 + U_n^{0,1}(x),
\]

since \( p U_n^{0,1} = \nu_A(C) = 1 \). Let \( p \to \infty \) and \( n \to \infty \), then, as we have seen in lemmas 5.2 and 5.3, \( R_n(p) \to R(F) \) and \( V^{1,0}_n(p U_n^{0,1})(x) \to G^{1,0}\xi_F(x) \) a.a. \( x(\mu) \), boundedly. Since \( K_A(x, \cdot) \) is a bounded signed measure and which is absolutely continuous relative to \( \mu \), we have, for a.a. \( x(\mu) \),

\[ R(F) = K_A G^{1,0}\xi_F(x) + G^{1,0}\xi_F(x) - 1 + U^{0,-1}(x) \]

\[ = \int \left\{ K_A g^{1,0}(x, y) + g^{1,0}(x, y) - 1 \right\} \xi_F(dy) + U^{0,-1}(x) \]

\[ = G\xi_F(x) + U^{0,-1}(x), \]

from lemmas 3.2 and 5.4. Denote

\[ \xi(dy) = \xi_F(dy) + \sum_{i=1}^{\infty} \left\{ (K_A)^n - \nu_A \right\} g^{1,0}(dy, x) \xi_F(dx), \]

then \( \xi \) is a bounded signed measure on \( F \) and

\[ G\xi_F(x) = G^{1,0}\xi_F(x) - 1. \]

Since \( G^{1,0}\xi_F(x) \) is bounded, \( G^{1,0}\xi_F(x) \) is the difference of two bounded excessive functions relative to \( (G_A^{\lambda})_{\lambda>0} \). Therefore,

\[ \lim_{p \to \infty} p G_A^{1,0}\xi_F(x) = G^{1,0}\xi_F(x) \quad \text{for all } x \in E. \]

Moreover,

\[ p G_A^{1,0}1(x) = E^t [\int_0^\infty \exp (- \int_0^t I_c(X_s) \, ds - t) \, dt] \to 1, \]

as \( p \to \infty \) for all \( x \in E \) and \( G_A^{1,0}(x, \cdot) \) is absolutely continuous relative to \( \mu \), for all \( x \in E \) and \( p>0 \). Thus, operating \( p G_A^{1,0} \) to both sides of (5.9) and letting \( p \to \infty \), we have

\[ R(F) = G\xi_F(x) + \lim_{p \to \infty} p G_A^{1,0}U^{0,-1}(x) \quad \text{for all } x \in E. \]
Therefore, it is enough to show that

\[(5.10) \quad \lim_{p \to 0} \rho G_A^{1,p} U^{0,\infty}_1(x) = U^{0,\infty}_1(x) \quad \text{for all } x \in E.\]

Let \(p > 1\) then,

\[
\rho G_A^{1,p} U^{0,\infty}_1(x) \\
= E^*\left[ \int_0^\infty \rho \exp \left\{ -\int_0^t I_c(X_s) ds - pt \right\} E^x \left[ \int_0^\tau I_c(X_u) du \right] dt \right] \\
\leq E^*\left[ \int_0^\infty e^{-pt} \left\{ \int_t^{t+\tau} I_c(X_u) du \right\} dt \right] \\
\leq E^*\left[ \int_0^\infty e^{-t} \left\{ \int_{t/p}^{(t/p)+\tau} I_c(X_u) du \right\} dt \right] \\
\leq E^*\left[ \int_0^\infty e^{-t} \left\{ \int_{t/p}^{t+\tau} I_c(X_u) du \right\} dt \right] + 1 \\
\leq ||U^{0,\infty}_1|| + 1.
\]

Thus, noting that \(\lim_{t \to \tau} (t+\tau \theta) = \tau\) (see [3; p. 214]), by the Lebesgue theorem,

\[
\lim_{p \to 0} \rho G_A^{1,p} U^{0,\infty}_1(x) \\
= \lim_{p \to 0} \int_0^\infty \rho \exp \left\{ -\int_0^t I_c(X_s) ds - pt \right\} E^x \left[ \int_0^\tau I_c(X_u) du \right] dt \\
= \int_0^\infty e^{-t} E^x \left[ \int_0^\tau I_c(X_u) du \right] dt = U^{0,\infty}_1(x).
\]

Now, it remains only the proof of the uniqueness. Let \(\xi\) be a bounded signed measure on \(\mathcal{F}\) satisfying \(\xi(E) = 0\) and \(G\xi(x) = a\), for some constant \(a\), on \(F\). For the proof of uniqueness we claim that \(\xi = 0\). Integrating both sides of \(G\xi(x) = a\) \((x \in F)\) by \(f(x)\nu_n(dx)\), we have

\[(5.11) \quad \int f\hat{K}_n(y) \xi(dy) = a \langle \nu_n, f \rangle\]

for all \(f \in bC^*\) and \(n \geq 1\), where \(\hat{K}_n(dx, y) = g(x, y)\nu_n(dx)\) as before. Set \(f = g(I-p\hat{K}_n)\) for a bounded continuous function \(g\). It follows, from the dual result of (2.8), that

\[
f\hat{K}_n(y) = g(I-p\hat{K}_n)\hat{K}_n(y) = g\hat{K}_n^*(y) - \langle g\hat{K}_n^*, \nu_A \rangle,
\]

for all \(y \in P_n^*, n \geq 1\) and \(p > 0\). Substituting this function into (5.11), we have

\[
\int g\hat{K}_n^*(y) \xi(dy) = 0 \quad \text{for all } n \geq 1 \text{ and } p > 0,
\]

because \(\xi(E) = 0\), \(\langle g(I-p\hat{K}_n^*), \nu_n \rangle = 0\) and \(\xi\) vanishes outside of \(\mathcal{F} \subset E - P_n \subset E - P_n^*\). Therefore, similarly to the proof of lemma 5.4, we have
This implies that \( \int \hat{g}(\hat{X}_t) \xi(dy) = 0 \), since \( \hat{P}^{\tau_t} = 1 \) for all \( y \in \hat{P} \).

6. Symmetric case

In this section we shall assume, in addition, that \( g^p(x, y) = g^p(y, x) \) for all \( p > 0 \) and \( x, y \in E \), that is, \( X \) and \( \hat{X} \) are equivalent. In this case, as is well known (see [3; proposition VI. 4. 10]), any semipolar set is polar. Hence, for every compact set \( F \), the set \( F - F' \) is polar, where \( F' \) is the set of all regular points of \( F \) (see [3; II. 3.3]). Therefore \( F \) is a projective set (see [3; V. 4.5]). Hence, by considering the projection of CAF \( \{ t \} \), there exists a CAF \( B \) such that

\[
E'[e^{-T_F}] = E'[\int_0^\infty e^{-t} dB_t]
\]

and \( \text{supp (B)} = F' \) ([3; V. 4.6 and 4.7]), where \( \text{supp (B)} \) is the fine support of \( B \). Obviously, \( F \) is a polar set if and only if the corresponding CAF \( B \) is zero. Let \( T = \inf \{ t ; B_t = \infty \} \). We have

\[
1 \geq E'[e^{-T_F}] \geq E'[\int_0^T e^{-t} dB_t] \geq E'[e^{-T_B}] .
\]

This implies that \( T = \infty \) a.s. \( P^x \) for all \( x \in E \), that is, \( B \in \Phi \). Let \( \hat{B} \) be the dual CAF of \( B \) then, under our present hypothesis, the corresponding polar set \( P_{\hat{B}} \) may be supposed empty (see the proof of [14 VII. 1]) and the cofine support \( \hat{F} \) of \( \hat{B} \) is equal to \( 'F = F' \), since the fine and cofine topologies coincide, where \( 'F \) is the set of all coregular points of \( F \). Therefore, by theorem 5.5 we have

**Theorem 6.1.** If \( F \) is a non-polar compact subset of \( E \), then there exists a unique probability measure \( \xi_F \) on \( F' \) such that

\[
G^{\xi_F}(x) = R(F) \quad \text{on } F' .
\]

Here, the measure \( \xi_F \) and the constant \( R(F) \) are given by

\[
\xi_F(dy) = \hat{P}^{\tau_t} [X_{\hat{t}} \in dy] \quad \text{and}
\]

\[
R(F) = E^{\tau_t} [\int_0^{\tau_F} I_c(X_t) dt] ,
\]

respectively. The measure \( \xi_F \) is called the equilibrium measure of \( F \) (relative to the potential kernel function \( g(x, y) \)).

**Corollary.** Under the hypothesis of theorem 6.1 there exists a unique probability measure \( \xi_F \) on \( F \) such that \( G^{\xi_F} \) is bounded on \( F \) and satisfies (6.2).
Proof. It is enough to prove the uniqueness. Suppose that a measure \( \xi \) on \( F \) satisfies the conditions of the corollary. Then since \( G\xi \) is bounded on \( F \), \( \xi \) charges no polar set (see the proofs of theorem 3.4 and corollary 2 of theorem 4.1). Hence \( \xi \) is a measure on \( F' \), so that the corollary follows from theorem 6.1.

REMARK. By the proof of the corollary, the result of the corollary may be replaced by the following result. "There exists a unique probability measure on \( F \) which attains no mass on any polar set and satisfies (6.2)".

By using the relation (4.1) of \( g(x, y) \) and an arbitrary potential kernel function \( h(x, y) \), we would like to investigate the equilibrium principle relative to \( h(x, y) \). At present, however, we can get only a partial result on this problem; we have to impose very strong conditions on \( h(x, y) \) and we do not know even if the logarithmic potential kernel function of planar Brownian motion satisfies these conditions. Our conditions are the following.

(H1) For every compact set \( D \),

\[
\lim_{p \to 0} \sup_{x, y \in D} |g^p(x, y) - \phi(y) - h(x, y)| = 0
\]

for some function \( \phi \) and a potential kernel function \( h \).

(H2) For all \( p > 0 \) and bounded continuous function \( f \), \( K_f^p f \) is continuous, where \( B \) is a CAF with fine support \( F' \), as before.

To find the equilibrium measure \( \xi \) relative to \( h \), we shall attempt a formal calculation. Suppose that a probability measure \( \xi \) on \( F \) satisfies \( H\xi(x) = h(x, y)\xi(dy) = a \) on \( F' \) for some constant \( a \). Then, from (4.1), for all \( x \in F' \)

\[
G\xi(x) = H\xi(x) - H(x, C) - \int H(C, y)\xi(dy) + H(C, C)
\]

Operating \( I - pK^p \) and integrating by \( fd\nu_B \), it follows that

\[
\langle f, (I - pK^p)G\xi \rangle_{\nu_B} = -\langle f, (I - pK^p)H(\cdot, C) \rangle_{\nu_B}.
\]

The left side of (6.6) becomes

\[
\langle f, (I - pK^p)G\xi \rangle_{\nu_B} = \langle f(I - pK^p)K, \xi \rangle = \langle fK, \xi \rangle - \langle fK, \nu_A \rangle
\]

from the dual formula of (2.8).

On the other hand, the right side of (6.6) becomes


\[ \langle f, (I-pK^b)H(\cdot, C) \rangle_{\nu_B} = -\lim_{t \to 0} \langle f, (I-pK^b)(G^t(\cdot, C) - \phi(q)) \rangle_{\nu_B} \]

\[ = -\lim_{t \to 0} \langle f, (I-pK^b)G^t(\cdot, C) \rangle_{\nu_B} = \lim_{t \to 0} \langle f, -G^b G^t(\cdot, C) + qG^b G^t(\cdot, C) \rangle_{\nu_B} \]

\[ = -\langle f\dot{K}_b^b, \nu_A \rangle + \lim_{t \to 0} \langle f, qG^b G^t(\cdot, C) \rangle_{\nu_B} . \]

Hence (6.6) becomes

\[ \langle f\dot{K}_b^b, \xi \rangle = \lim_{t \to 0} \langle f, qG^b G^t(\cdot, C) \rangle_{\nu_B} . \]

Multiplying \( p \) and letting \( p \to \infty \) we have

\[ \langle f, \xi \rangle = \lim_{p \to \infty} \langle pqG^b G^t(\cdot, C) \rangle_{\nu_B} . \]

**Theorem 6.2.** Let \( F \) be a non-polar compact subset of \( E \). Under the hypothesis (H1) and (H2), there exists a unique probability measure \( \xi \) on \( F' \) such that

\[ H\xi = \text{a constant on } F' . \]

Proof. Since

\[ \langle 1, pqG^b G^t(\cdot, C) \rangle_{\nu_B} = pq\langle 1\dot{K}_b^b G^t, \nu_A \rangle = 1 , \]

the measure \( pqG^b G^t(x, C)\nu_B(dx) \) is a probability measure on \( F \) for any \( p, q > 0 \). Hence, for all \( p > 0 \), we can choose a sequence \( q_n \to 0 \) and a probability measure \( \xi_n \) on \( F \) such that \( pq_nG^b G^t(x, C)\nu_B(dx) \to \xi_n(dx) \), weakly. Similarly, there exists a sequence \( p_n \to \infty \) and a probability measure \( \xi \) on \( F \) such that \( \xi_n \to \xi \), weakly. From the hypothesis (H2), for all bounded continuous function \( f \),

\[ \langle p_n f\dot{K}_b^b, \xi_n \rangle = \lim_{m \to 0} \langle p_n f\dot{K}_b^b, p_m q_n G^b G^t(\cdot, C) \rangle_{\nu_B} \]

\[ = \lim_{m \to 0} \langle p_n p_m q_n f\dot{K}_b^b G^t, \nu_A \rangle \]

\[ = \lim_{m \to 0} \frac{1}{p_m - p_n} \langle p_n p_m q_n f\dot{K}_b^b G^t, \nu_A \rangle \]

\[ = \lim_{m \to 0} \frac{1}{p_m - p_n} \langle p_n q_n f\dot{K}_b^b G^s, \nu_A \rangle = \langle f, \xi_n \rangle . \]

Hence, letting \( k \to \infty \), it follows that

\[ \lim_k \langle p_n f\dot{K}_b^b, \xi_n \rangle = \langle f, \xi \rangle , \]

that is, \( \xi = \hat{E}^f[ f(X_{\infty}) ] - E^f[ f(X_{T_{\infty}}) ] \). So that \( \xi \) is a measure on \( F' \).

To prove (6.8), let \( f \) be a bounded continuous function with compact support. By restricting the CAT \( B \) as in section 5, we may suppose that \( V^b_{1,0} \) is bounded. Then, since \( fG \) is bounded and continuous from (2.7), we have
\[
\langle f, G_\xi \rangle_\mu = \langle f, G_\xi \rangle = \lim_{m \to \infty} \langle f, G_\xi, \mu \rangle = \lim_{m \to \infty} \langle f, G_\xi, \nu_\mu \rangle
\]

\[
= \lim_{m \to \infty} \lim_{n \to \infty} \langle f, G_k G_{k_n} G_{k_n}^* (\cdot, C) \rangle _\nu_n
\]

\[
= \lim_{m \to \infty} \lim_{n \to \infty} \langle f, K_k G_{k_n} G_{k_n}^* (\cdot, C) - K_k (I - p_k K_k) G_{k_n}^* (\cdot, C) \rangle _\nu_n
\]

\[
= \lim_{m \to \infty} \lim_{n \to \infty} \langle f, K_k G_{k_n} G_{k_n}^* (\cdot, C) - K_k G_{k_n}^* (\cdot, C) + \nu_k K_k G_{k_n} G_{k_n}^* (\cdot, C) \rangle _\nu_n
\]

from (2.8). By the definition of \( K_k \) and \( H \),

\[
\lim_{m \to \infty} \lim_{n \to \infty} \langle f, K_k G_{k_n} G_{k_n}^* (\cdot, C) \rangle _\nu_n
\]

\[
= \lim_{m \to \infty} \lim_{n \to \infty} \left\{ p_m \left\{ V_{A_k, B_k}^0 U_{A_k, B_k}^0 1(x) \right\} - \frac{\nu_B}{q} \right\}
\]

\[
= \lim_{m \to \infty} \lim_{n \to \infty} \left\{ K_k 1 - U_{A_k, B_k}^0 1 + q K_k U_{A_k, B_k}^0 1 - \frac{1}{q} \right\}(x)
\]

\[
= - E'\left[ \int_0^{T_f} I_c(X_s) ds \right] + E'\left[ \int_0^{T_f} I_c(X_s) ds \right]
\]

\[
\lim_{m \to \infty} \lim_{n \to \infty} \langle f, K_k G_{k_n} G_{k_n}^* (\cdot, C) - \nu_k K_k G_{k_n} G_{k_n}^* (\cdot, C) \rangle _\nu_n
\]

\[
= \lim_{m \to \infty} \lim_{n \to \infty} \left\{ K_k \{ G_{k_n} (\cdot, C) - \phi(q_n) \} - \nu_k K_k \{ G_{k_n} (\cdot, C) - \phi(q_n) \} \right\}
\]

\[
= \lim_{m \to \infty} \lim_{n \to \infty} \left\{ K_k \{ H(x, C) - \nu_k H(C) \} \right\}
\]

\[
= E'[H(X_{T_f}, C)] - E'[H(X_{T_f}, C)]
\]

Hence

\[
\langle f, G_\xi \rangle_\mu = \langle f, - E'\left[ \int_0^{T_f} I_c(X_s) ds \right] - E'[H(X_{T_f}, C)] \rangle _\mu
\]

\[
+ \langle f, 1 \rangle_\mu E'\left[ \int_0^{T_f} I_c(X_s) ds + H(X_{T_f}, C) \right].
\]

Therefore

\[
G_\xi(x) = - E'\left[ \int_0^{T_f} I_c(X_s) ds + H(X_{T_f}, C) \right] + E'\left[ \int_0^{T_f} I_c(X_s) ds + H(X_{T_f}, C) \right]
\]

for a.a. \( x \) (\( \mu \)). In particular,

\[
G_\xi(x) = - H(x, C) + E'\left[ \int_0^{T_f} I_c(X_s) ds + H(X_{T_f}, C) \right]
\]

for a.a. \( x \in F' (\mu) \), and hence for all \( x \in F' \). Hence, by (4.1), (6.8) holds. If \( \xi_1 \) and \( \xi_2 \) are measures on \( F' \) satisfying (6.8), then \( G(\xi_1 - \xi_2) \) equals to a constant on \( F' \). Hence \( \xi_1 = \xi_2 \) by the proof of theorem 5.5.

In the classical case, the equilibrium measure is characterized as the measure which minimize the energy. In our case, the analogous result holds. Denote \( \mathfrak{M} \) the family of all bounded signed measures \( \xi \) on \( E \) with compact support.
such that $\int |g(x, y)|\xi(dy)$ is bounded, $\mathcal{M}^+=\{\xi\geq 0; \xi\in\mathcal{M}\}$ and $\mathcal{M}^0=\{\xi\in\mathcal{M}; \xi(E)=0\}$. For $\xi, \zeta\in\mathcal{M}$, define the mutual energy of $\xi$ and $\zeta$ by

$$
(\xi, \zeta) = \int \int g(x, y)\xi(dx)\zeta(dy).
$$

Denote $(\xi, \zeta)$ by $I(\xi)$ and call it the energy of $\xi$.

**Lemma 6.3.** If $\xi\in\mathcal{M}^0$, then $I(\xi)$ is non-negative. Moreover, $I(\xi)=0$ if and only if $\xi=0$.

Proof. Suppose that $\xi\in\mathcal{M}^0$. Since $G^{1,0}|\xi|(x)$ is bounded,

$$
\int \sum_{n=1}^{\infty} |(K_\lambda)^n-\nu\lambda|(x, dz)G^{1,0}|\xi|(z)
\leq||G^{1,0}|\xi||\sum_{n=1}^{\infty} ||(K_\lambda)^n(x, \cdot)-\nu\lambda||
$$

converges uniformly in $x$. Hence for any $\varepsilon>0$ there exists a number $N$ such that

$$
|\sum_{n=N}^{\infty} \int \{(K_\lambda)^n-\nu\lambda\}(x, dz)G^{1,0}\xi(z)| < \varepsilon
$$

for all $x\in E$. From our definition of $g(x, y)$, for $(x, y)\in \Gamma$,

$$
g(x, y) = g^{1,0}(x, y) - 1 + \sum_{n=1}^{\infty} \{(K_\lambda)^n-\nu\lambda\}g^{1,0}(x, y) + \varepsilon(x, y, N),
$$

where $\varepsilon(x, y, N) = \sum_{n=N}^{\infty} \{(K_\lambda)^n-\nu\lambda\}g^{1,0}(x, y)$. Since $\xi(\hat{\Gamma})=0$, for all $x\in E$,

$$
I(\xi) = \int \int g^{1,0}(x, y)\xi(dx)\xi(dy) + \sum_{n=1}^{\infty} \int \int (K_\lambda)^n g^{1,0}(x, y)\xi(dx)(dy)
+ \int \int \varepsilon(x, y, N)\xi(dx)\xi(dy).
$$

From the resolvent equation (1.7), we have

$$
g^{2,0}(x, y) - g^{1,0}(x, y) + K_\lambda g^{1,0}(x, y) = 0.
$$

This combined with $g^{1,0}(x, y) \geq g^{2,0}(x, y)$, we have

$$
K_\lambda g^{2,0}(x, y) \leq K_\lambda g^{1,0}(x, y) \leq g^{1,0}(x, y),
$$

so that

$$
\int g^{2,0}(x, z)g^{2,0}(z, y)\nu\lambda(dz) \leq g^{1,0}(x, y).
$$

Hence we have, from the symmetry of $g^2(x, y)$

$$
\int \int g^{1,0}(x, y)\xi(dx)\xi(dy) \geq \int \{\int g^{2,0}(x, y)\xi(dy)\}^2\nu\lambda(dx) \geq 0.
$$
Similarly it follows that
\[ \sum_{n=0}^{\infty} \int (K_{t_n}^* g^{1,0}(x, y)) \xi(dx) \xi(dy) \geq 0. \]

Therefore \( I(\xi) \geq -\varepsilon \) and hence \( I(\xi) \geq 0. \)

Suppose that \( I(\xi) = 0. \) By a routine argument, we have \( \lvert (\xi, \zeta) \rvert \leq I(\xi) I(\zeta) \) for all \( \zeta \in \mathcal{M}_0. \) Hence \( (\xi, \zeta) = 0 \) for all \( \zeta \in \mathcal{M}_0. \) This implies that \( G_\xi \)
equals a constant on \( E. \) Integrating by \( v_A, \) we can see that the constant is equals to 0. Hence \( \xi = 0. \)

**Theorem 6.4.** The equilibrium measure \( \xi_F \) of a compact set \( F \) is the unique measure which attains the
\[
(6.10) \quad \min \{ I(\xi); \xi \in \mathcal{M}_0^+, \xi(E) = 1, \text{support of } \xi \subseteq F \},
\]
and Robin's constant \( R(F) \) equals the minimum value of \( (6.10). \)

Proof. The proof is similar to the classical case [16]. If a measure \( \xi \)
satisfies the conditions of \( (6.10), \) then, since \( G_{\xi F} = R(F) \) on \( F \) except a polar subset of \( F \) and \( \xi \) charges no polar set,
\[
I(\xi) = I(\xi - \xi_F) - I(\xi_F) + 2(\xi, \xi_F) = I(\xi - \xi_F) + R(F).
\]

Since \( \xi - \xi_F \in \mathcal{M}_0, \) this implies the result by lemma 6.2.

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**References**


