Maruo, K.
Osaka J. Math.
**29** (1992), 455–470

# ON THE EXISTENCE OF SOLUTIONS TO WAVE INTEGRODIFFERENTIAL EQUATIONS WITH SUBDIFFERENTIAL OPERATORS

Dedicated to Professor Hiroki Tanabe on his sixtieth birthday

KENJI MARUO

(Received December 25, 1989)

## 0. Introduction

In this paper we consider the following integrodifferential equation

$$(0.1)\qquad \begin{cases} \dfrac{d^2u}{dt^2}(t)+\partial\psi u(t)+\partial\varphi u(t)+\displaystyle\int_0^t a(t-s)\,\partial\varphi u(s)ds \ni f(t, u(t)) \\ u(0)=a\,, \qquad \dfrac{du}{dt}(0)=b \end{cases}$$

in a real Hilbert space $H$. Here $\psi$ and $\varphi$ are lower semicontinuous proper convex functions from $H$ to $[0, \infty]$, and $\partial\psi$ and $\partial\varphi$ are the subdifferentials of $\psi$ and $\varphi$ respectively. The functions $a(\cdot)$ and $f(\cdot, \cdot)$ are continuous from $[0, T]$ to $(-\infty, \infty)$ and from $[0, T]\times H$ to $H$.

Our purpose here is to prove the existence of a global solution on $[0, T]$ of the initial value problem (0.1). In the case of $a(t)\equiv 0$ K. Maruo [3] proved the existence of a solution to the above equation under some restrictions. Moreover, we showed that this class of equations contains vibrating string equations with unilateral constraints which were deeply investigated by M. Schatzman [4], A. Bamberger and M. Schatzman [1] and C. Citrini and L. Amerio in [5]. We will extend the result of [3] to the equation containing a delay term which corresponds to vibrating string with not only a unilateral constraint but also a memory (see the example of section 4). In a general situation it seems to be difficult to solve the above initial value problem (0.1). Hence we will seek a solution which satisfies (0.1) in some generalized sense as in [3].

The outline of the present paper is as follows. In section 1 we list the notations and state the assumptions and theorem. In section 2 we obtain an energy estimate to Yosida approximate solutions of the initial value problem (0.1). In section 3 we prove our theorem. In section 4 we show an example.

The auther would like to express his hearty gratitude to the referee for

kind and helpful advices.

## 1. Assumptions, theorem and notations.

We list some notations which will be used throughout the paper. Let $X_1$, $X_2$ and $V$ be real Banach spaces and $V^*$ the dual space of $V$. We use the same notation ( , ) as the inner product of $H$ to denote the pairings between $X_1$, $V$ and their corresponding duals. We denote the norm of a Banach space $S$ by $|\cdot|_S$ and use the usual notations $L_p(0, T; S)$, $C([0, T]; S)$ etc. to denote variable spaces of functions with values in $S$. By $\partial\varphi_\lambda$ and $\varphi_\lambda(\cdot)$ we denote Yosida approximations of $\partial\varphi$ and $\varphi(\cdot)$ respectively, i.e $\partial\varphi_\lambda x=\lambda^{-1}(x-J_\lambda^\varphi x)$ and $\varphi_\lambda(x)=(2\lambda)^{-1}|x-J_\lambda^\varphi x|_H^2+\varphi(J_\lambda^\varphi x)$ where $J_\lambda^\varphi=(1+\lambda\partial\varphi)^{-1}$. The notations $d^\pm u/dt$ denote the left and right derivatives of $u$ in $H$.

Next we state the assumptions and theorem.

The Banach spaces $V$, $X_1$, $H$ and $X_2$ hold the following properties.
(A-1) The following inclusion relations hold:

$$V\subset X_1\subset H\subset X_2 \quad \text{and} \quad X_2\subset \{\text{the dual space } X_1\}$$

where each inclusion mapping is continuous. Moreover, $X_1$ is separable, the imbedding mapping $V\to X_1$ is compact, and $V$ is reflexive and dense in $H$.

We introudce the assumptions of $\psi(\cdot)$ (see [2]).

A-2) $\psi(\cdot)$ is a lower semicontinuous, convex function from Domain $D(\psi)=V$ to $[0, \infty]$ and the subdifferential $\partial\psi$ of $\psi(\cdot)$ is single valued and bounded from $V$ to $V^*$. Moreover they satisfy the following conditions.

(1) The function $\psi$ is coercive in the sense that $\lim_{|x|_V\to\infty} \psi(x)/|x|_V=\infty$.

(2) Suppose we are given a sequence of functions $\{u_n\}\subset W_\infty^1(0, T; H)\cap L_\infty(0, T; V)$ such that
$u_n\to u$ in $C([0, T]; H)$,
$u_n\to u$ in the weak star topology of $L_\infty(0, T; V)$.
Then a subsequence $\{u_{n_k}\}$ can be extracted so that $\partial\psi u_{n_k}\to\partial\psi u$ in the weak star topology of $L_\infty(0, T; V^*)$.

REMARK. In view of the coerciveness condition (1) $\psi$ is lower semicontinuous also in the topology of $H$.

Next we state the assumptions of $\varphi$.

A-3) There exists $z\in V$ scuh that, for any $x\in H$,

$$(\partial\varphi_\lambda x, x-z)\geq C_1|\partial\varphi_\lambda x|_{X_2}-C_2\{\varphi_\lambda(x)+\psi(x)+1\}$$

where $C_1$ and $C_2$ are positive constants independent of $x$ and $\lambda$.

The function $f(t, x)$ from $[0, T]\times H$ to $H$ satisfies the following conditions.
A-4)
(1) For each $x\in H$ $f(\cdot, x)$ is continuous in $H$ in $[0, T]$.
(2) The following inequalities hold:
$|f(t, x)-f(t, y)|_H\leq C|x-y|_H$,
$|f(t, x)|_H\leq C\{1+|x|_H\}$
for any $x, y\in H$ and any $t\in[0, T]$ where $C$ is a constant independent of $x, y$ and $t$.

Let $k(t)$ be the solution of the following integral equation

$$k(t) = a^+(t)-\int_0^t a^-(s)k(t-s)ds, \qquad 0\leq t\leq T \tag{1.1}$$

where $a^+(t)=\mathrm{Max}\{a(t), 0\}$ and $a^-(t)=\mathrm{Min}\{a(t), 0\}$. As is easliy seen the solution $k(t)$ exists, is unique and nonnegative.

A-5) The function $a(t)$ is real valued and belongs to $C^1([0, T])$.

Furthermore, we assume the following condition either A-6) or A-7).

A-6) The function $a(\cdot)$ belongs to $C^2([0, T])$ and the following inequalities hold:

$$\underset{0\leq t\leq T}{\mathrm{Max}}\int_0^t \{k(t-s)\int_0^s a^+(\xi)d\xi-a^-(s)\}ds<1 \quad \text{and} \quad \int_0^T k(s)ds<1\,.$$

In addition to A-3) we assume that

A-7) For any positive $\varepsilon$ there exists a constant $C_\varepsilon$ such that

$$|(\partial\psi x, x-y)|\leq\varepsilon\psi(y)+C_\varepsilon(\psi(x)+1) \quad \text{for any } x, y\in H\,.$$

REMARK. If $\int_0^T |a(s)|ds<1$ and $a(\cdot)\in C^2([0, T])$ then the assumption A-6) is satisfied. Indeed, integrating both sides of (1.1) over $[0, T]$ and noting that $a^+(t)=|a(t)|+a^-(t)$ we have

$$\int_0^T k(s)ds\leq\int_0^T |a(s)|ds+\int_0^T a^-(s)ds-\int_0^T a^-(s)ds\int_0^T k(s)ds\,,$$

which implies

$$\int_0^T k(s)ds<1\,.$$

Therefore

$$\int_0^t\{k(t-s)\int_0^s a^+(\xi)d\xi-a^-(s)\}ds<\int_0^T |a(\xi)|d\xi<1\,.$$

With regard to the type of the initial value problem (0.1) we consider solutions in the following sense.

DEFINITION. We say that a function $u\in C([0, T]; X_1)\cap W^1_\infty(0, T; H)$ is the solution of the initial value problem (0.1) if the following conditions are satisfied:

1) $\varphi(u(t))+|u(t)|_V$ is bounded in $[0, T]$.
2) There exists a linear functional $F$ on $C([0, T]; X_1)$ such that

$$F(v-u)\leq\int_0^T\varphi(v(s))ds-\int_0^T\varphi(u(s))ds$$

for any $v\in C([0, T]; X_1)$ and

$$F(v(\cdot))+\int_\cdot^T a(s-\cdot)v(s)ds)$$
$$=\int_0^T(\frac{du}{ds}(s), \frac{dv}{ds}(s))ds+\int_0^T(f(s, u(s))-\partial\psi u(s), v(s))ds$$
$$+(b, v(0))-(\frac{d^-}{dt}u(T), v(T))$$

for any $v\in C([0, T]; X_1)\cap L_1(0, T; V)\cap W^1_1(0, T; H)$.

3) The initial conditions are satisfied in the following sense

$$u(0)=a, \quad b-\frac{d^+}{dt}u(0)\in\partial I_K a$$

where $K$ is the closure of the effective domain of $\varphi$, $I_K$ is the indicator function of $K$ and $\partial I_K$ is the subdifferantial of $I_K$.

Now we state our theorem.

**Theorem.** *Let the initial values $a$ and $b$ be given so that*

$$a\in V\cap D(\varphi) \quad \textit{and} \quad b\in H\,.$$

*Then under the assumptions* A-1), A-2), A-3), A-4) *and* A-6) *or* A-1), A-2), A-3), A-4), (A-5) *and* A-7) *we have at least one solution to the initial value problem* (0.1).

## 2. Approximate solutions.

To begin with we prove some lemmas concerning the properties of the subdifferential $\partial\psi$. Throughout this paper we assume the conditions stated in our Theorem.

**Lemma 1.** *Let $g$ be a continuous mapping from $C([0, T]; H)$ to $L_2(0, T; H)$ such that the following inequality holds:*

$$\int_0^t|g(v)(s)-g(w)(s)|_H^2\,ds$$
$$\leq C\int_0^t|v(s)-w(s)|_H^2\,ds$$

*for any* $v, w \in C([0, T]; H)$ *and* $t \in [0, T]$.
*Then there exists a solution* $u \in L_\infty(0, T; V) \cap W^1_\infty(0, T; H) \cap W^2_\infty(0, T; V^*)$ *of the following equation*

$$(2.1)\qquad \begin{cases} \dfrac{d^2u}{dt^2} + \partial\psi u = g(u) & \text{on } [0, T]\times V^*, \\ u(0) = a\,, \quad \dfrac{du}{dt}(0) = b\,. \end{cases}$$

*Moreover the solution satisfies the following energy inequality*

$$(2.2)\qquad \begin{cases} 2^{-1}\left|\dfrac{d^\pm}{dt}u(t)\right|_H^2 + \psi(u(t)) \le 2^{-1}|b|_H^2 + \psi(a) \\ + \displaystyle\int_0^t \left(g(u)(s)\,, \ \frac{du}{ds}(s)\right)ds \qquad \textit{for any} \quad t\in(0, T)\,. \end{cases}$$

Proof. We consider the following approximate equation to the inital value problem (2.1), for any $\mu>0$,

$$(2.3)\qquad \begin{cases} \dfrac{d^2}{dt^2}u_\mu + \partial\psi_\mu u_\mu = g(u_\mu) & \text{on } [0, T]\times V^*, \\ u_\mu(0) = a\,, \quad \dfrac{d}{dt}u_\mu(0) = b\,. \end{cases}$$

Here $\psi_\mu$ is the Yosida approximation of $\psi$ considered as a convex function on $H$ which is lower semicontinuous also in the topology of $H$ (Remark after A-2)). Taking the inner products of both sides of (2.3) with $(d/dt)u_\mu(t)$ and integrating the resultant equality over $[0, t]$, we have

$$2^{-1}\left|\frac{d}{dt}u_\mu(t)\right|_H^2 + \psi_\mu(u_\mu(t)) = 2^{-1}|b|_H^2 + \psi_\mu(a)$$
$$+ \int_0^t \left(g(u_\mu)(s), \frac{d}{ds}u_\mu(s)\right)ds \qquad \text{for any} \quad t\in(0, T)\,.$$

Using Gronwall's lemma and the assumptions of the lemma we see that the functions $|(d/dt)u_\mu(t)|_H$ and $\psi_\mu(u_\mu(t))$ are uniformly bounded on $[0, T]$. Then using (1) in A-2) we see that $|J^\psi_\mu u_\mu(t)|_V$ are uniformly bounded on $[0, T]$. From A-1) we know that $\{J^\psi_\mu u_\mu(t)\}_\mu$ is relatively compact in $H$ for each fixed $t$. Combining the uniform boundedness of $|(d/dt)u_\mu(t)|_H$ and the above result and using Ascoli-Arezela's theorem we obtain that there exists a subsequence of $\{J^\psi_\mu u_\mu(t)\}$ such that

$$\lim_{j\to\infty} J^\psi_{\mu_j} u_{\mu_j}(t) = u(t) \quad \text{in} \quad C([0, T]; H)\,.$$

Moreover, since both functions $|(d/dt)u_\mu(t)|_H$ and $|J^\psi_\mu u_\mu(t)|_V$ are uniformly bounded it follows that $u(t)\in W^1_\infty(0, T; H)\cap L_\infty(0, T; V)$. Noting the equation (2.3), (2) in A-2) and the above resultants we know that $u$ belongs to $W^2_\infty(0, T; V^*)$. Thus we complete the proof.

We consider the Yosida approximate equations of (0.1) with $\varphi_\lambda$ in place of $\varphi$:

$$(2.4)\qquad \begin{cases} \dfrac{d^2}{dt^2}u_\lambda+\partial\psi u_\lambda+\partial\varphi_\lambda u_\lambda+\displaystyle\int_0^\cdot a(\cdot-s)\partial\varphi_\lambda u_\lambda(s)ds=f(\cdot, u_\lambda)\,,\\ u_\lambda(0)=a\,,\quad \dfrac{d}{dt}u_\lambda(0)=b\,. \end{cases}$$

We set

$$g(u)(t)=-\partial\varphi_\lambda u(t)-\int_0^t a(t-s)\partial\varphi_\lambda u(s)ds+f(t, u(t))\,.$$

From the assumption A-4) and the Lipschitz continuity of $\partial\varphi_\lambda$ it follows that the mapping $g$ satisfies the hypothesis of Lemma 1. Hence we have the following lemma.

**Lemma 2.** *For each $\lambda>0$ there exists a solution of the equation* (2.4) *in $V^*$. Moreover the following energy inequality holds:*

$$(2.5)\qquad \begin{cases} 2^{-1}\left|\dfrac{d^\pm}{dt}u_\lambda(t)\right|_H^2+\psi(u_\lambda(t))+\varphi_\lambda(u_\lambda(t))\\ \le 2^{-1}|b|_H^2+\psi(a)+\varphi_\lambda(a)+\displaystyle\int_0^t\Big(f(s, u_\lambda(s)), \dfrac{d}{ds}u_\lambda(s)\Big)ds\\ -\displaystyle\int_0^t\int_0^s a(s-\xi)\Big(\partial\varphi_\lambda(u_\lambda(\xi)), \dfrac{d}{ds}u_\lambda(s)\Big)d\xi\, ds \end{cases}$$

*for any $t\in(0, T)$.*

Next we show that the functions $|(d^\pm/dt)u_\lambda(t)|_H$, $\varphi_\lambda(u_\lambda(t))$ and $\psi(u_\lambda(t))$ are uniformly bounded in $t$ and $\lambda$.

For a while we assume the assumtion A-6).

We set

$$w^\pm(t)=\int_0^t a^\pm(t-s)(\partial\varphi_\lambda u_\lambda(s), u_\lambda(t)-u(s))ds\,,$$

$$\tilde{w}^\pm(t)=\int_0^t -(\dot{a})^\mp(t-s)(\partial\varphi_\lambda u_\lambda(s), u_\lambda(t)-u_\lambda(s))ds$$

where $(\dot{a})(t)=da(t)/dt$.

**Lemma 3.** *There exists a constant $M$ such that*

(2.6) $$w^+(t)\geq -M\cdot\int_0^t\{|\frac{d}{ds}u_\lambda(s)|_H^2+1+\psi(u_\lambda(s))+\varphi_\lambda(u_\lambda(s))\}ds$$
$$-\int_0^t k(s)ds\cdot\psi(u_\lambda(t))$$
$$-\int_0^t k(t-s)\int_0^s a^+(\xi)d\xi\,ds\cdot\varphi_\lambda(u_\lambda(t))$$

*where $k(t)$ is the function in the assumption* A-6). *Furthermore*

(2.7) $$w^-(t)\geq\int_0^t a^-(t-s)\{\varphi_\lambda(u_\lambda(t))-\varphi_\lambda(u_\lambda(s))\}ds\,.$$

Proof. Inductively we define functions $h_n(t)$ as follows:

(2.8) $$h_1(t)=a^+(t)\,,\quad h_{n+1}(t)=\int_0^t -a^-(s)h_n(t-s)ds$$

where $n=1, 2, 3, \cdots$. Then we have

(2.9) $$0\leq h_n(t)\leq M^n\cdot t^{n-1}/(n-1)\,!$$

where $M=\underset{0\leq s\leq T}{\text{Max}}\,|a(s)|$.

From (2.8) we know

$$\sum_{n=1}^\infty h_n(t)=a^+(t)+\int_0^t -a^-(s)\sum_{n=1}^\infty h_n(t-s)ds\,.$$

In view of the uniqueness of the solution of the integral equation (1.1) we have $k(t)=\sum_{n=1}^\infty h_n(t)$. Moreover from (2.8) we see that the functions $k(t)$ and $(d/dt)k(t-s)$ are uniformly bounded in $0\leq t\leq T$.

For any natural number $n$ we set

(2.10) $$w_n(t)=\int_0^t h_n(t-s)(\partial\varphi_\lambda u_\lambda(s),\ u_\lambda(t)-u_\lambda(s))ds,$$
$$f_1^n(t)=\int_0^t h_n(t-s)(-\frac{d^2}{ds^2}u_\lambda(s),\ u_\lambda(t)-u_\lambda(s))ds,$$
$$f_2^n(t)=\int_0^t h_n(t-s)(-\partial\psi u_\lambda(s),\ u_\lambda(t)-u_\lambda(s))ds,$$
$$f_3^n(t)=\int_0^t h_n(t-s)(f(s,\ u_\lambda(s)),\ u_\lambda(t)-u_\lambda(s))ds,$$
$$f_4^n(t)=\int_0^t h_n(t-s)(-\int_0^s a^+(s-\mu)(\partial\varphi_\lambda u_\lambda(\mu),\ u_\lambda(t)-u_\lambda(\mu))d\mu ds$$

and

$$f_5^n(t)=\int_0^t h_n(t-s)(-\int_0^s a^-(s-\mu)(\partial\varphi_\lambda u_\lambda(\mu),\ u_\lambda(\mu)-u_\lambda(s))d\mu ds.$$

From (2.4) we see

$$\partial\varphi_\lambda u_\lambda = -\frac{d^2}{dt^2}u_\lambda - \partial\psi u_\lambda + f(\cdot, u_\lambda) - \int_0^\cdot a^+(\cdot-\mu)\partial\varphi_\lambda u_\lambda(\mu)d\mu$$
$$- \int_0^\cdot a^-(\cdot-\mu)\partial\varphi_\lambda u_\lambda(\mu)d\mu .$$

Substituting this in (2.10), noting (2.8) and using Fubini's theorem we get

$$w_n(t) = \sum_{i=1}^5 f_i^n(t) + \int_0^t h_n(t-s)w_1(s)ds + w_{n+1}(t) .$$

In view of (2.9) we see

$$|w_n(t)| \leq M^n T^n/n! \cdot \operatorname*{Max}_{0\leq s\leq t\leq T} |(\partial\varphi_\lambda u_\lambda(s), u_\lambda(t)-u_\lambda(s))| .$$

Thus it follows that

$$w_1(t) = \sum_{i=1}^5 \sum_{n=1}^\infty f_i^n(t) + \int_0^t k(t-s)w_1(s)ds .$$

Set $L_i(t) = \sum_{n=1}^\infty f_i^n(t)$, $i=1, 2, 3, 4, 5$, and $L(t) = \sum_{i=1}^5 L_i(t)$. Solving the above integral equation we get the following equality

$$w_1(t) = L(t) + \int_0^t \mathscr{K}(t-s)L(s)ds , \tag{2.11}$$

where $\mathscr{K}(t)$ is a positive continuous function in $0\leq s\leq t\leq T$. With the aid of an integration by parts we get

$$\begin{aligned} L_1(t) &= k(t)(b, u_\lambda(t)-a) \\ &\quad + \int_0^t (d/ds)k(t-s)((d/ds)u_\lambda(s), u_\lambda(t)-u_\lambda(s))ds \\ &\quad - \int_0^s k(t-s)|(d/ds)u_\lambda(s)|_H^2 ds . \end{aligned}$$

Noting that

$$\begin{aligned} |u_\lambda(t)-a|_H &= |\int_0^t (d/ds)u_\lambda(s)ds|_H \\ &\leq 2^{-1}\int_0^t (1+|(d/ds)u_\lambda(s)|_H^2)ds \end{aligned}$$

we obtain

$$|L_1(t)| \leq C\int_0^t (|\frac{d}{ds}u_\lambda(s)|_H^2 + |u_\lambda(t)-u_\lambda(s)|_H^2 + 1)ds . \tag{2.12}$$

Using the assumption A-4) and Schwarz's inequality we see

$$|L_3(t)| \leq C\int_0^t (|u_\lambda(t)-u_\lambda(s)|_H^2 + |u_\lambda(s)|_H^2 + 1)ds . \tag{2.13}$$

The definition of the subdifferential yeilds

$$L_2(t) \geq \int_0^t k(t-s)\psi(u_\lambda(s))ds - \int_0^t k(t-s)ds \cdot \psi(u_\lambda(t)), \tag{2.14}$$

$$\begin{aligned} L_4(t) \leq & \int_0^t \int_0^s k(t-s)a^+(s-\mu)\varphi_\lambda(u_\lambda(\mu))d\mu ds \\ & - \int_0^t \int_0^s k(t-s)a^+(s-\mu)d\mu ds \cdot \varphi_\lambda(u_\lambda(t)), \end{aligned} \tag{2.15}$$

$$\begin{aligned} L_5(t) \geq & -\int_0^t \int_0^s k(t-s)a^-(s-\mu)\varphi(u_\lambda(\mu))d\mu ds \\ & + \int_0^t \int_0^s k(t-s)a^-(s-\mu)\varphi(u_\lambda(s))d\mu ds. \end{aligned} \tag{2.16}$$

Combining (2.11), (2.12), (2.13), (2.14), (2.15), (2.16) and the assumption A-6) we obtain (2.6). The equlity (2.7) is a direct consequence of the difinition of the subdifferential.

Noting Lemma 3 and using the argument of the proof of Lemma 3 we can establish the following lemma, where $\tilde{k}(t)$ is the solution of

$$\tilde{k}(t) = -(\dot{a})^-(t) + \int_0^t -a^-(s)\tilde{k}(t-s)ds.$$

**Lemma 4.** *There exists a constant $M$ such that*

$$\begin{aligned} \tilde{w}^+(t) \geq & -M \cdot \int_0^t \{|\frac{d}{ds}u_\lambda(s)|_H^2 + 1 + \psi(u_\lambda(s)) + \varphi_\lambda(u_\lambda(s))\} ds \\ & - \int_0^t \tilde{k}(t-s)ds \cdot \psi(u_\lambda(t)) \\ & - \int_0^t \tilde{k}(t-s) \int_0^s a^+(\xi)d\xi ds \cdot \varphi_\lambda^t(t). \end{aligned}$$

*Moreover*

$$\tilde{w}^-(t) \geq \int_0^t (\dot{a})^+(t-s)\{\varphi_\lambda(u_\lambda(s) - \varphi_\lambda(u_\lambda(t))\} ds.$$

**Proposition 5.** *Under the assumptions* A-1), A-2), A-3), A-4) *and* A-6) *the funcions $|(d/dt)u_\lambda(t)|_H$, $\varphi_\lambda(u_\lambda(t))$ and $\psi(u_\lambda(t))$ are uniformly bounded in $\lambda$ and $t$.*

Proof. Using Fubini's theorem and the integration by parts we see

$$\begin{aligned} & \int_0^t \int_0^s a(s-\xi)((\partial\varphi_\lambda u_\lambda(\xi)), \frac{d}{ds}u_\lambda(s))d\xi ds \\ & = \int_0^t \int_\xi^t a(s-\xi)(\partial\varphi_\lambda(u_\lambda(\xi)), \frac{d}{ds}(u_\lambda(s) - u_\lambda(\xi)))dsd\xi \\ & = w^+(t) + w^-(t) + \int_0^t (\tilde{w}^+(s) + \tilde{w}^-(s))ds. \end{aligned}$$

Combining the inequality (2.5), Lemma 3, Lemma 4 and the above equality and using the assumption A-6) and Gronwall's inequality we complete the proof.

**Proposition 6.** *Under the assumptions of Proposition* 5 *there exists a constant $M$ independent of $\lambda$ and $t$ such that*

$$\int_0^T |\partial\varphi_\lambda u_\lambda(s)|_{X_2} ds \leq M .$$

Proof. We set

$$y(t) = \int_0^t (\partial\varphi_\lambda(u_\lambda(s)), u_\lambda(s)-z)ds$$

where $z$ is the element in the assumption A-3).
In view of (2.4) we get

$$\begin{aligned} y(t) &= \int_0^t (-\frac{d^2}{ds^2}u_\lambda(s)-\partial\psi u_\lambda(s)+f(s, u_\lambda(s)), u_\lambda(s)-z)ds \\ &- \int_0^t\int_0^s a^+(s-\xi)\,(\partial\varphi_\lambda u_\lambda(\xi), u_\lambda(s)-u_\lambda(\xi))\,d\xi ds \\ &- \int_0^t\int_0^s a^-(s-\xi)\,(\partial\varphi_\lambda u_\lambda(\xi), u_\lambda(s)-u_\lambda(\xi))\,d\xi ds \\ &- \int_0^t\int_0^s a^+(s-\xi)\,(\partial\varphi_\lambda u_\lambda(\xi), u_\lambda(\xi)-z)\,d\xi ds \\ &- \int_0^t\int_0^s a^-(s-\xi)\,(\partial\varphi_\lambda u_\lambda(\xi), u_\lambda(\xi)-z)\,d\xi ds \\ &= I_1+I_2+I_3+I_4+I_5 . \end{aligned}$$

Using the integration by parts, the definition of the usbdifferential and the assumption A-4) we get

$$I_1 \leq \int_0^t \{|\frac{d}{ds} u_\lambda(s)|_H^2+1+\psi(u_\lambda(s))+\varphi_\lambda(u_\lambda(s))\}\,ds .$$

From the difinition of the subdifferential it follows

$$I_3+I_4 \leq \int_0^t\int_0^s |a(t-s)|\,\{\varphi_\lambda(u_\lambda(s))+\varphi_\lambda(u_\lambda(\xi))\}\,d\xi ds .$$

From Proposition 5 it follows $I_1+I_3+I_4 \leq$ Constant. Combining $I_2=-\int_0^t w^+(s)ds$, Lemma 3 and Proposition 5 we see $I_2 \leq$ Constant.
Using the integration by parts we see

$$I_5 = -\int_0^t a^-(t-\xi)y(\xi)d\xi ds .$$

Then we get

$$y(t) \leq \text{Const} - \int_0^t a^-(t-s)y(s)ds\,.$$

Combining the assumption A-3) and Proposition 5 we know that $(y(t)+N)$ is a positive function on $[0, T]$ where $N$ is some large positive number. Then using Gronwall's lemma and the above inequality we get

$$(y(t)+N) \leq \text{Const}\,.$$

Using a similar method to the proof of lemma 3 of [3] and combining the above inequality and Proposition 5 we complete the proof.

Next we assume the assumption A-5) and A-7).

We define $w_n(t, \xi)$ by

$$\text{(2.18)} \qquad \begin{aligned} w_1(t, \zeta) &= \int_\zeta^t a(s-\zeta)\frac{d}{ds}u_\lambda(s)ds \quad \text{and} \\ w_{n+1}(t, \zeta) &= \int_\zeta^t -a(s-\zeta)w_n(t, s)ds \quad \text{inductively}\,. \end{aligned}$$

**Lemma 7.** *We have the following inequalities*

$$|w_n(t, \zeta)|_H \leq A^{n-1} \cdot M_\lambda(t) \cdot (t-\zeta)^{n-1}/(n-1)\,!$$

*where* $A = \operatorname*{Max}_{0\leq t\leq T} |a(t)|$, $L = \operatorname*{Max}_{0<t<T} |\frac{d}{dt}a(t)|$ *and*

$$M_\lambda(t) = (2A+TL) \operatorname*{Max}_{0\leq s\leq t} |u_\lambda(s)|_H\,.$$

Proof. With the aid of the integation by parts we see

$$|w_1(t, \xi)| \leq M_\lambda(t)\,.$$

The remaing part can be established by induction.

REMARK. From Lemma 2 we know $w_n(t, \cdot) \in L_\infty(0, T; V)$ for each $t, n = 1, 2, 3, \cdots$.

We set, for $n = 1, 2, 3, \cdots$,

$$f_{n,1}(t) = -\int_0^t (w_n(t, \zeta), \frac{d^2}{d\zeta^2}u_\lambda(\zeta))d\zeta\,,$$

$$f_{n,2}(t) = -\int_0^t (w_n(t, \zeta), \partial\psi u_\lambda(\zeta))d\zeta \quad \text{and}$$

$$f_{n,3}(t) = \int_0^t (w_n(t, \zeta), f(\zeta, u_\lambda(\zeta))d\zeta\,.$$

**Lemma 8.** *We get the following equality*

$$\int_0^t\int_0^s a(s-\zeta)\,(\partial\varphi_\lambda u_\lambda(\zeta),\ \frac{d}{ds}u_\lambda(s))d\zeta ds=\sum_{u=1}^{\infty}F_n(t)$$

*where* $F_n(t)=\{f_{n,1}(t)+f_{n,2}(t)+f_{n,3}(t)\}$.

Proof. Using the equation (2.5) and Fubini's theorem we get

$$\int_0^t(w_n(t,\xi),\ \partial\varphi_\lambda u_\lambda(\xi))d\xi=F_n(t)+$$
$$+\int_0^t(w_{n+1}(t,\xi),\ \partial\varphi_\lambda u_\lambda(\xi))d\xi\ .$$

Noting that

$$\int_0^t\int_0^s a(s-\zeta)\,(\partial\varphi_\lambda u_\lambda(\zeta),\ \frac{d}{ds}u_\lambda(s))d\zeta ds$$
$$=\int_0^t(w_1(t,\xi),\ \partial\varphi_\lambda u_\lambda(\xi))d\xi$$

and Lemma 7 we can prove this lemma.

**Lemma 9.** *There exists a constant C independent of* $\lambda$ *and t such that*

$$|f_{n,1}(t)|+|f_{n,3}(t)|\leq$$
$$C\{(At)^{n-2}/(n-2)\,!\}\,(\int_0^t|\frac{d}{ds}u_\lambda(s)|^2ds+1)$$

*for* $n=1,2,3,\cdots$, *where we set* $(At)^{-1}/(-1)!=1$.

Proof. In view of Lemma 7 and the assumption A-4) we find

$$(2.19)\qquad |f_{n,3}(t)|\leq(At)^{n-1}M_\lambda(t)\int_0^t C(1+|u_\lambda(s)|_H)ds/(n-1)\,!\ .$$

On the other hand there exist a constant $K$ independet of $\lambda$ and $t$ such that

$$(2.20)\qquad M_\lambda(t),\ C(1+|u_\lambda(t)|_H)\leq K(\int_0^t|\frac{d}{ds}u_\lambda(s)|_H+1)ds\ .$$

The desired result on $f_{n,3}(t)$ follows from (2.19) and (2.20). Using the integration by parts and noting $w_1(t,t)=0$ we see

$$|f_{1,1}(t)|\leq|w_1(t,0)|_H|b|_H+\int_0^t|a(0)|\,|\frac{d}{ds}u_\lambda(s)|_H^2ds$$
$$+\int_0^t\int_\xi^t|\frac{da}{ds}(s-\xi)|\,|\frac{d}{ds}u_\lambda(s)|_H|\frac{d}{d\xi}u_\lambda(\xi)|_H dsd\xi\ .$$

Noting Lemma 7, (2.20) and choosing a constant $M$ so large that $M\geq((LT+A)T+(K+AT)|b|_H)$ we get the required inequality for $f_{1,1}(t)$. Noting the following equalities

$$\frac{d}{ds}w_n(t, s) = a(0)w_{n-1}(t, s)+\int_s^t \frac{d}{ds}a(\xi-s)w_{n-1}(t, \xi)d\xi .$$

and lemma 7 we have

$$|\frac{d}{ds}w_n(t, s)|_H \leq (A(t-s))^{n-2}M_\lambda(t)/(n-2)!\{A+LT\}$$

where $n\geq 2$.
On the other hand it follows

$$|f_{n,1}(t)| \leq |w_n(t, 0)|_H |b|_H+\int_0^t |\frac{d}{ds}w_n(t, s)|_H |\frac{d}{ds}u_\lambda(s)|_H ds .$$

Then using Lemma 7, the above two inequalities and (2.20) we know

$$|f_{n,1}(t)| \leq (At)^{n-2}M_\lambda(t)/(n-2)!\{AT|b|_H+(A+LT)\int_0^t |\frac{d}{ds}u_\lambda(s)|_H ds\}$$
$$\leq M(At)^{n-2}/(n-2)!\{1+\int_0^t |\frac{d}{ds}u_\lambda(s)|_H^2 ds\}$$

where $M$ is a positive large number independent of $\lambda$, $t$ and $n$. Our required inequalities for $f_{n,1}(t)$ are obtained.

**Lemma 10.** *For any $\varepsilon>0$, there exists a constant $K_\varepsilon$ independent of $n$ and $t$ such that*

$$|f_{n,2}(t)| \leq \{\varepsilon(At)^{n-1}/(n-1)!\}\psi(u_\lambda(t))+$$
$$\{K_\varepsilon(At)^{n-1}/(n-1)!\}\{\int_0^t \psi(u_\lambda(s))ds+1\}.$$

Proof. From (2.18) we see that the functions $w_n(t, \xi)$ are equal to

$$(-1)^{n-1}\int_\xi^t\int_{\xi_1}^t \cdots \int_{\xi_{n-1}}^t a(\xi_1-\xi)a(\xi_2-\xi_1)\cdots a(\xi_{n-1}-\xi_{n-2})w_1(t, \xi_{n-1})d\mathcal{S}_n$$

where $d\mathcal{S}_n=d\xi_{n-1}d\xi_{n-2}\cdots d\xi_1$ and $n=1, 2, \cdots$.
From (2.18) we have the following equality

$$w_1(t, \xi_{n-1}) = a(t-\xi_{n-1})u_\lambda(t)-a(0)u_\lambda(\xi_{n-1})-\int_{\xi_{n-1}}^t \dot{a}(s-\xi_{n-1})u_\lambda(s)ds$$
$$= a(t-\xi_{n-1})(u_\lambda(t)-u_\lambda(\xi_{n-1}))-\int_{\xi_{n-1}}^t \dot{a}(s-\xi_{n-1})(u_\lambda(s)-u_\lambda(\xi_{n-1}))ds$$

where $\dot{a}=(d/dt)a(t)$.
Using the above two lemmas and the assumption A-7) and noting

$$u_\lambda(\cdot)-u_\lambda(\xi_{n-1}) = (u_\lambda(\cdot)-u_\lambda(\xi))+(u_\lambda(\xi)-u_\lambda(\xi_{n-1}))$$

we obtain the following inequalities

$$|(w_n(t,\xi),\partial\psi u_\lambda(\xi))|$$
$$\leq A^{n-1}\int_\xi^t\cdots\int_{\xi_{n-2}}^t(\varepsilon\psi(u_\lambda(t))+(C_\varepsilon+1)\psi(u_\lambda(\xi))+C_1\psi(u_\lambda(\xi_{n-1}))+(C_\varepsilon+C_1))d\mathfrak{s}_n$$
$$+A^{n-2}L\int_\xi^t\cdots\int_{\xi_{n-1}}^t(\psi(u_\lambda(s))+(C_1+1)\psi(u_\lambda(\xi))+C_1\psi(u_\lambda(\xi_{n-1}))+2C_1)dsd\mathfrak{s}_n\,.$$

Then it follows

$$|f_{n,2}(t)|\leq\varepsilon(At)^{n-1}/(n-1)!\cdot\psi(u_\lambda(t))$$
$$+(C_\varepsilon+C_1+1)(At)^{n-2}A/(n-1)!\cdot\int_0^t\psi(u_\lambda(s))ds+(C_\varepsilon+C_1)(tA)^{n-1}/(n-1)!$$
$$+LA^{n-2}t^{n-1}/(n-1)!\cdot 2(C_1+2)\int_0^t\psi(u_\lambda(s))ds\,.$$

Therefore the proof of the lemma is complete.

Combining Lemmas 8, 9 and 10 and the inequality (2.5), choosing $\varepsilon$ sufficently small and using Gronwall's lemma we get the following proposition.

**Proposition 11.** *Under the assumptions* A-1), A-2), A-3), A-4), A-7) *the functions* $|\frac{d}{dt}u_\lambda(t)|_H$, $\varphi_\lambda(u_\lambda(t))$ *and* $\psi(u_\lambda(t))$ *are uniformly bounded in* $\lambda$ *and* $t$.

Noting the above proposition and using a similar argment to the proof of Proposition 6 we have the following lemma.

**Proposition 12.** *Under the assumptions of Proposition* 11 *there exists a constant* $M$ *independent of* $\lambda$ *and* $t$ *such that*

$$\int_0^T|\partial\varphi_\lambda u_\lambda(s)|_{X_2}ds\leq M\,.$$

## 3. Proof of Theorem.

We set

$$F_\lambda(v)=\int_0^T(\partial\varphi_\lambda u_\lambda s),v(s))ds\qquad\text{for any}\quad v\in C([0,T];X_1)\,.$$

From the definition of $F_\lambda$ and Fubini's theorem we get the following lemma.

**Lemma 13.** *We have the following equality*

$$\int_0^T\int_0^\zeta a(\zeta-s)(\partial\varphi_\lambda u_\lambda(s),v(\zeta))dsd\zeta=F_\lambda(\int_\cdot^T a(\zeta-\cdot)v(\zeta)d\zeta)\,.$$

Combining Propositions 5, 6 and lemma 13 or Proposition 11, 12 and Lemma 13 and using the argument of the proof of Theorem in [3] we obtain our theorem.

## 4. Example. (see the example in [3])

Put $H=L_2(0, 1)$, $X_1=C([0, 1])$, $X_2=L_1(0, 1)$ and $V=\mathring{H}_1(0, 1)$. Then from Sobolev's imbedding theorem the assumption A-1) follows.

We consider the following symmetric sesqulinear form $a(u, v)$ defined on $V\times V$.

$$1)\quad a(u, u)\geq\delta|u|_V^2$$
$$2)\quad |a(u, v)|\leq K|u|_V|v|_V$$

for any $u, v\in V$ where $\delta$ and $K$ are some positive constants. We put $\psi(u)=a(u, u)$. Then it is easy to prove that $\psi(\cdot)$ satisfies the assumption A-2). Moreover we know that it satisfies the assumption A-7).

Set

$$K=\{f\in L_2(0, 1); f(x)\geq r(x)\quad a.e\ x\in[0, 1]\}$$

where $r\in C([0, 1])$ and $r(0), r(1)<1$.
Let $\varphi=I_K$ be the indicator function of $K$. Then we show that the Yosida approximation of $\partial\varphi$ satisfies the assumption A-3). We choose a function $\theta\in C^1([0, 1])$ such that $\theta(0)=\theta(1)=0$ and $\theta(x)-r(x)>\sigma>0$ for any $x\in[0, T]$. In the assumption A-3) we define $z$, $c_1$ and $c_2$ as $\theta$, $\sigma$ and 0 respectively.
Since

$$\partial\varphi_\lambda f(x)=\begin{cases}0 & \text{if } f(x)\geq r(x)\\ \lambda^{-1}(f(x)-r(x)) & \text{if } f(x)<r(x)\,,\end{cases}$$

and $f(x)<r(x)$ implies

$$\theta(x)-f(x)>\theta(x)-r(x)>\sigma\,,$$

we have

$$(\partial\varphi_\lambda f, f-\theta)\geq\sigma|\partial\varphi_\lambda f|_{X_2}\,.$$

Therefore we have the assumption A-3).
Now we can consider the term $\partial\varphi u$ as a unilateral constraint and the integral term in the equation (0.1) as a memory term. Then we can regard the initial value problem (0.1) as the vibrating equations with a unilateral constraint and a memory term.

---

### References

[1] A. Bamberger and M. Schatzman: *New results on the vibrating string with a continuous obstacle*, SIAM J. Math. Anal. **14**(1983), 560–595.

[2] V. Barbu: Nonlinear semigroup and differential equations in Banach spaces, Noordhoff International, 1976.

[3] K. Maruo: *Existence of solutions of some nonlinear wave equations*, Osaka J. Math. **22**(1985), 21–30.
[4] M. Schatzman: *A hyperbolic problem of second Order with Unilateral Constrains: The Vibrating String with a Concave String with a Concave Obstacle*, J. Math. Anal. Appl. **73**, (1980), 138–191.
[5] H. Cabannes and C. Citrini (editors): Vibrations with unilateral constraints, EUROMECH 209, Como (Italy), 1986.

Department of Mathematical Sciences
Faculty of Engineering
Osaka University
Suita, Osaka 565,
Japan

Current Address
Department of Mathematics
Kobe University of
Mercantile Marine
Higashinada, Kobe
Japan