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ON THE PERTURBATION THEORY FOR FREDHOLM OPERATORS

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Introduction

In the classical theory of linear Fredholm operators a fundamental role is played by compact operators. In fact, a condition for an operator to be Fredholm is given in terms of compact operators, and it is known that the class of Fredholm operators is invariant under compact perturbations. The object of this paper is to generalize these results introducing a new class of operators containing the class of compact operators.

The organization of the paper is as follows: in the section 1 we introduce some basic definitions, propositions and examples; in the section 2 we establish the aimed results and in the section 3 we present an application of Theorem 2.2.

1. Let X and Y be Banach spaces.

DEFINITION 1.1. An operator $T: D(T)\subset X\to X$ is said to be demicompact if for every bounded sequence $\{x_n\}$ in $D(T)$ such that $x_n-Tx_n\to x_0$, for some x_0 in X, as $n\to\infty$, then there is a convergent subsequence of $\{x_n\}$.

Here $D(T)$ denotes the domain of T.

Examples of demicompact operators.

a) Compact operators $T: D(T)\subset X\to X$ are demicompact.

If X is a Hilbert space,

b) Operators $T: D(T)\subset X\to X$ which satisfy either the condition

$$\mathrm{Re}(Tx-Ty,\ x-y)\leq a\|x-y\|^2,\ a<1 \tag{1}$$

or the condition

$$\mathrm{Re}(Tx-Ty,\ x-y)\leq a\|Tx-Ty\|^2,\ a<1 \tag{2}$$

are demicompact.

c) Operators $T: D(T)\subset X\to X$ for which $(I-T)^{-1}$ exists and is continuous on its range $R(I-T)$ (and, in particular, demicontinuous operators T for which

[1)] This work was supported by CNPq and by FINEP.

(1) is valid with $a<1$ or for which the inequality

$$|(Tx-Ty,\ x-y)|\leq b\|x-y\|^2,\ 0<b<1$$

is valid for all x and y in X) are demicompact.

DEFINITION 1.2 ([1]). Let D be a bounded subset of X. We define $\gamma(D)$, the Kuratowski measure of noncompactness of D, to be $\inf\{d>0 \mid D$ can be covered by a finite number of sets of diameter minor than or equal to $d\}$.

Properties of $\gamma(D)$. If D, Q and D_i are bounded sets in X we have:
a) $\gamma(D)=0$ iff $\bar{D}$ is compact, where $\bar{D}$ is the closure of D.
b) $\gamma(\bar{D})=\gamma(D)$; $\gamma(\lambda D)=|\lambda|\gamma(D)$; $D\subset Q\Rightarrow\gamma(D)\leq\gamma(Q)$, where λ is a real or complex value.
c) If $D_i=\bar{D}_i$, $D_{i+1}\subset D_i$ and $\lim_{i\to\infty}\gamma(D_i)=0$, then $D_\infty=\bigcap_{i>1}D_i\neq\emptyset$ and $\gamma(D_\infty)=0$.
d) $\gamma(D\cup Q)=\max\{\gamma(D),\ \gamma(Q)\}$.
e) $\gamma(D+Q)\leq\gamma(D)+\gamma(Q)$, where, $D+Q=\{x+y;\ x\in D,\ y\in Q\}$.

Let $k\geq 0$ be a given real number, $T\colon D(T)\subset X\to Y$ a continuous operator and γ_1 and γ_2 respectively Kuratowski measures of noncompactness in X and Y.

DEFINITION 1.3. ([2]). *T is said to be k-set-contractive if, for any bounded subset B of $D(T)$, $T(B)$ is a bounded subset of Y and $\gamma_2(T(B))\leq k\gamma_1(B)$.*

Examples of k-set-contractive operators
a) T is continuous and compact iff T is 0-set-contractive.
b) If T is L-lipschitzian, then T is L-set-contractive.
c) If T is semicontractive type operator with constant $k\leq 1$, then T is k-set-contractive.

We recall that T is of semicontractive type with constant $k\leq 1$ if there exists a continuous operator $V\colon D(T)\times D(T)\to X$ such that $Tx=V(x,x)$ for all x in $D(T)$ and

$$\|V(x,z)-V(y,z)\|\leq k\|x-y\|\ (x,y,z\in D(T)),$$

and the operator $x\to V(\cdot, x)$ is compact from $D(T)$ into the space of operators from $D(T)$ into X with uniform metric.

DEFINITION 1.4 ([4]). *T is said to be condensing if for any bounded subset B of $D(T)$, $T(B)$ is a bounded subset in Y and $\gamma_2(T(B))<\gamma_1(B)$, whenever $\gamma_1(B)>0$.*

REMARK. Clearly, every k-set-contractive operator with $k<1$ is a condensing one. The converse is not true; however, every condensing operator is 1-set-contractive. If $X=Y$ then T condensing implies T demicompact.

DEFINITION 1.5 ([11]). Let W be a linear subspace of X. We say that a linear operator T: $W \to Y$ is invertible modulo compact operator if there is a linear operator L: $Y \to X$ such that $I-TL$ and $I-LT$ are compact, where I is the identity operator. We call L an inverse of T modulo compact operator.

REMARK. Clearly, if T_1 and T_2 are the inverses of T modulo compact operator, then there exists a compact operator K such that $T_1=T_2+K$. (See [11]).

DEFINITION 1.6 ([10]). Let W be a linear subspace of X. A linear operator T: $W \to Y$ is said to be Fredholm if:
a) T is densely defined in X, i.e., $\bar{W}=X$.
b) T is a closed operator.
c) $\alpha(T)<\infty$, where $\alpha(T)$ is the dimension of kernel of T.
d) $R(T)$ is closed in Y.
e) $\beta(T)<\infty$, where $\beta(T)$ is the codimension of $R(T)$.
The index of a Fredholm operator T is defined by

$$i(T) = \alpha(T)-\beta(T).$$

We denote by $\emptyset(X, Y)$ the set of all Fredholm operators T from $D(T)\subset X$ into Y.

If $T\in\emptyset(X, Y)$, we have $X=\ker T\oplus X_0$, where X_0 is a closed subspace of X. Since $\mathrm{Ker}T\subset D(T)$, this gives $D(T)=\ker T\oplus[X_0\cap D(T)]$. Also we have $\ker T'=R(T)^0$, where T' is the adjoint of T; hence $Y=R(T)\oplus Y_0$, where Y_0 is a subspace of Y of dimension $\beta(T)$. The restriction of T to $X_0\cap D(T)$ has a closed inverse defined everywhere on $R(T)$ (which is a Banach space) and hence, the inverse is bounded. This gives $||x||\leq C||Tx||$, $x\in X_0\cap D(T)$. Here $\ker T$ denotes the kernel of T; $R(T)^0$ denotes the set of all annihilators of $R(T)$.

Thus, we have

Proposition 1.1 ([10]). *If $T\in\emptyset(X, Y)$, then there exist an operator $T_0\in B(Y, X)$ such that*
a) $\ker T_0=Y_0$.
b) $R(T_0)=Y_0\cap D(T)$,
c) $T_0T=I$ on $X_0\cap D(T)$,
d) $TT_0=I$ *on* $R(T)$

and operators $F_1\in B(X)$, $F_2\in B(Y)$ such that
e) $T_0T=I-F_1$ *on* $D(T)$,
f) $TT_0=I-F_2$ *on* Y,
g) $R(F_1)=\ker T$, $\ker F_1=X_0$,
h) $R(F_2)=Y_0$, $\ker F_2=R(T)$.

Here $B(Y, X)$ denotes the set of all bounded linear operators T from $D(T)\subset Y$ to X; if $Y=X$ we simply write $B(X)$.

REMARK. Since $\alpha(T)<\infty$ and $\beta(T)<\infty$, F_1 and F_2 are operators of finite rank; therefore, F_1 and F_2 are compact operators and T_0 is an inverse of T modulo compact operator.

Proposition 1.2 ([10]). *Suppose that $X=N\oplus X_0$, where X_0 is a closed linear subspace and N is a finite dimensional subspace. If X_1 is a linear subspace of X containing X_0, then X_1 is closed.*

DEFINITION 1.7. Let W be a normed space. W is said to be continuously embedded in X if there is a one-to-one bounded linear operator P: $W\rightarrow X$.

Proposition 1.3 ([10]). *Let W be a normed space continuously embedded in X. If $T\in\emptyset(X, Y)$ and $\overline{D(T)}=W$, then $T\in\emptyset(W, Y)$ with* $\ker T$ *and* $R(T)$ *remaining the same, where $\overline{D(T)}$ means the closure of $D(T)$ in W.*

Proposition 1.4 ([10]). *If $A\in\emptyset(X, Y)$ and $B\in\emptyset(Y, Z)$, then $BA\in\emptyset(X, Z)$ and $i(BA)=i(B)+i(A)$, where Z is another Banach space.*

2. Let X and Y be Banach spaces. Using the results of Petryshyn [8] (Theorem 10′) and Proposition 1.2 we first generalize in Theorem 2.1 the classical result on a sufficient condition for an operator to be Fredholm with demicompact 1-set-contractive, hence in particular condensing, operators in place of compact operators, and then apply this result to obtain a new result concerning perturbation theory for Fredholm operators in Theorem 2.2.

Theorem 2.1. *Let T: $D(T)\subset X\rightarrow Y$ be a densely defined closed linear operator. Suppose that there are linear bounded operators T_1: $Y\rightarrow X$; T_2: $Y\rightarrow X$; A_1: $X\rightarrow X$ and A_2: $Y\rightarrow Y$ with A_1 demicompact and A_2 demicompact* 1-set-con*tractive such that*

a) $T_1T=I-A_1$ *on* $D(T)$,

b) $TT_2=I-A_2$ *on* Y.

Then, T is a Fredholm operator.

Proof. By hypothesis T is a densely defined and closed linear operator in X. So we have only to prove conditions (c), (d) and (e) of Definition 1.6.

Proof of (c): Since $\ker T\subset\ker T_1T$, we have $\alpha(T)\leq\alpha(T_1T)=\alpha(I-A_1)$. But, $\alpha(I-A_1)<\infty$. In fact, we shall consider $S=\{x\in\ker(I-A_1);\ \|x\|=1\}$ and prove that S is a compact set in $\ker(I-A_1)$. Let $\{x_n\}$ be any sequence in S; then, $\{x_n\}\subset\ker(I-A_1)$ and $x_n-A_1x_n=0$ for each n. Since A_1 is demicompact, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\rightarrow x_0\in X$ as $n_j\rightarrow\infty$.

Clearly, $||x_0||=1$, i.e., $x_0 \in S$. This proves that S is a compact set in $\ker(I-A_1)$ and, consequently $\alpha(I-A_1)<\infty$. Thus $\alpha(T)<\infty$.

Proof of (d) and (e): First note that $R(T)\supset R(TT_2)=R(I-A_2)$. Now, $(I-A_2)$ is Fredholm (see [8]-theorem 10′). Hence by Proposition 1.2, $R(T)$ is closed and of finite codimension. Q.E.D.

Theorem 2.2. *Let T: $D(T)\subset X\rightarrow Y$ be a Fredholm operator. Suppose that F: $X\rightarrow Y$ is any bounded linear operator such that*

$-GF$ and $-FG$ are demicompact 1-set-contractive for some operator G which is an inverse of T modulo compact operator. (A)

Then $T+F$ is a Fredholm operator with $i(T+F)=i(T)$.

Proof. First note that, by remark of Definition 1.5, assumption (A) implies that $-T_0F$ and $-FT_0$ are demicompact 1-set-contractive for all T_0 which is an inverse of T modulo compact operator.

Since T is a Fredholm operator, by Proposition 1.1, there is a bounded linear operator T_0: $Y\rightarrow X$ such that

$$T_0T = I-F_1 \text{ on } D(T)$$

and

$$TT_0 = I-F_2 \text{ on } Y,$$

where F_1 and F_2 are compact operators. This implies that T_0 is an inverse of T modulo compact operator.

Now,

$$T_0(T+F) = I-F_1+T_0F = I-L_1 \text{ on } D(T+F)$$

and

$$(T+F)T_0 = I-F_2+FT_0 = I-L_2 \text{ on } Y$$

where

$$L_1 = F_1-T_0F \text{ and } L_2 = F_2-FT_0 .$$

Since $-T_0F$ and $-FT_0$ are demicompact 1-set-contractive and F_1 and F_2 are compact, we have that L_1 and L_2 are demicompact 1-set-contractive operators. Clearly, $D(T+F)$ is dense in X. Therefore, by Theorem 2.1, $T+F$ is a Fredholm operator.

It remains to prove that $i(T+F)=i(T)$. Since T is closed, one can make $D(T)$ into a Banach space W by equipping it with the graph norm

$$||x||_W = ||x||+||Tx|| .$$

Moreover, W is continuously embedded in X and $\overline{D(T)}=W$. Hence, $T\in\emptyset(W, Y)$ by Proposition 1.3. So, there is a bounded linear operator U:

$Y\to W$ such that

$$UT=I-K_1 \text{ on } D(T)$$

and

$$TU=I-K_2 \text{ on } Y$$

where K_1 and K_2 are compact with $R(K_1)=\ker T$ (see Proposition 1.1).

In addition, $U\in\emptyset(Y, W)$. Thus, applying Proposition 1.4 we have that

$$i[U(T+F)]=i(U)+i(T+F).$$

Let T_1 be the operator K_1-UF. If we consider T_1 as an operator from X into X, T_1 is demicompact and 1-set-contractive. Then, by [8]-Theorem 10′ we conclude that $I-T_1\in\emptyset(X, X)$ with $i(I-T_1)=0$.

Now,

$$i[U(T+F)]=i(I-T_1). \tag{1}$$

Assuming this for the moment, we see that

$$i(T+F)=-(U).$$

Proposition 1.4 still yields

$$i(UT)=i(U)+i(T).$$

Since

$$i(UT)=i(I-K_1)=0,$$

we have

$$i(T)=-i(U).$$

Then

$$i(T+F)=i(T).$$

Therefore, it remains only to prove (1).

Since $R(K_1)=\ker T\subset D(T)$ and UF is an operator from X into $D(T)$, we have that

$$R(T_1)\subset D(T). \tag{2}$$

It is clear that $\ker[U(T+F)]\subset\ker(I-T_1)$. Conversely, if $x\in\ker(I-T_1)$, then $x=T_1x\in D(T)$ by (2), and hence,

$$\ker[U(T+F)]=\ker(I-T_1). \tag{3}$$

Since $I-T_1\in\emptyset(X, X)$ and W is dense in X, there is a finite dimensional subspace X_1 of X such that

$$X = R(I - T_1) \oplus X_1,\ X_1 \subset D(T). \tag{4}$$

Hence

$$W = [R(I - T_1) \cap W] \oplus X_1. \tag{5}$$

It is clear that $R[U(T+F)] \subset R(I-T_1) \cap W$.

Conversely, if $z \in R(I-T_1) \cap W$, then $z = x - T_1 x \in W$, for some $x \in X$. In view of (2), $x \in D(T)$. Hence $z = U(T+F)x$.

Thus,

$$R(I-T_1) \cap W = R[U(T+F)].$$

Combining this with (5),

$$W = R[U(T+F)] \oplus X_1. \tag{6}$$

It follows from (4) and (6) that

$$\beta[U(T+F)] = \dim X_1 = \beta(I-T_1). \tag{7}$$

(3) and (7) imply (1). Q.E.D.

REMARK. It is important to observe that, Theorem 2.2 is valid when GF and FG are condensing operators because condensing operators are demicompact and 1-set-contractive (see remark of Definition 1.4).

When F is compact, Theorem 2.2 is also valid (this is the classic result of perturbation theory of Fredholm operators).

When T is the identity operator, we need $-F$ to be demicompact 1-set-contractive in order to guarantee the validity of Theorem 2.2.

As a consequence of Theorem 2.2 we deduce the following classic result whose perturbator operator is not necessarily compact.

Corollary 2.2.1. *For $T \in \emptyset(X, Y)$ there is an $\eta > 0$ such that for every linear operator A: $X \to Y$ satisfying $\|A\| < \eta$ one has $(T+A) \in \emptyset(X, Y)$ and $i(T+A) = i(T)$.*

Proof. Since $T \in \emptyset(X, Y)$ there is a bounded linear operator T_0: $Y \to X$ that T_0 is an inverse of T modulo compact operator. We take $\eta = \|T_0\|^{-1}$. then

$$\|T_0 A\| \leq \|T_0\|\ \|A\| < 1$$

and similarly,

$$\|AT_0\| < 1.$$

Since a bounded linear operator L is $\|L\|$-set-contractive, $T_0 A$ and AT_0 are k-set-contractive with $k<1$ and, consequentely, $T_0 A$ and AT_0 are condens-

ing. Then, by the above remark, A satisfies the hypothesis of Theorem 2.2.
Q.E.D.

3. An application of Theorem 2.2

Let C_ω and C'_ω be, respectively, the space of continuous ω-periodic functions x: $\boldsymbol{R}\to R^n$ and the space of continuously differentiable ω-periodic functions x: $\boldsymbol{R}\to R^n$. C_ω equipped with the maximum norm $||\cdot||_\infty$ and C'_ω with the norm $||\cdot||^1_\infty$ given by $||\cdot||^1_\infty=\max\{||u||_\infty, ||u'||_\infty\}$ for $u\in C'_\omega$ are Banach spaces.

Let us consider the following differential equation:

$$x'(t) = a(t)x'(t-h_1)+b(t)x(t-h_2)+f(t)\,.$$

Here, a and b are continuous ω-periodic $(n\times n)$-matrix function such that $||a(t)||\leq k$, $(-\infty<t<+\infty)$, where $k<\frac{1}{\omega}$ if $\omega>2$ or $k<\frac{1}{2}$ if $\omega\leq 2$; $f\in C_\omega$ is a given function and $x\in C'_\omega$ is an unknown function.

This equation can be rewritten in the operator from

$$Dx-Ax=f\,,$$

where D: $C'_\omega\to C_\omega$ is given by the formula

$$(Dx)\,(t) = x'(t)\,,$$

and the operator A: $C'_\omega\to C_\omega$ by the formula

$$(Ax)\,(t) = a(t)x'(t-h_1)+b(t)x(t-h_2)\,.$$

Clearly, D and A are bounded linear operators with $||D||=1$ and therefore, D is 1-set-contractive.

The kernel of the operator D is n-dimensional and it consists of constant functions. The range of D is readily seen to be the set of $x\in C_\omega$ that satisfy the condition

$$\int_0^\omega x(t)dt = 0\,.$$

This set is obviously closed and its codimension is n. Thus, D is a Fredholm operator with $i(D)=0$.

Let γ and γ_1 be respectively the Kuratowski measure of noncompacteness in C_ω and C'_ω.

We represent A as the sum $A=A_1+A_2$, where

$$(A_1x)\,(t) = a(t)x'(t-h_1),\ (A_2x)\,(t) = b(t)x(t-h_2)\,.$$

The operator A_2, which acts from C'_ω to C_ω, is obviously compact.

Let D_0: $C_\omega\to C'_\omega$ be an inverse of D modulo compact operator, that is,

$K_1=I-DD_0$ and $K_2=I-D_0D$ are compact operators respectively in C_ω and C'_ω, where I is the identity operator. Then we have

$$AD_0=(A_1+A_2)D_0=A_1D_0+A_2D_0$$

and

$$D_0A=D_0(A_1+A_2)=D_0A_1+D_0A_2$$

with A_2D_0, D_0A_2 compact. Therefore, for any bounded subset X of C_ω,

$$\gamma[AD_0(X)]\leq\gamma[A_1D_0(X)]+\gamma[A_2D_0(X)]=\gamma[A_1D_0(X)]\,.$$

Note that $A_1=A_3S_{h_1}D$, where the operators S_{h_1} and A_3 acting in C_ω, are given by the formula

$$(S_{h_1}x)\,(t)=x(t-h_1),\ (A_3x)\,(t)=a(t)x(t)\,.$$

Obviously S_{h_1} is 1-set-contractive and A_3 is k-set-contractive. Therefore,

$$\begin{aligned}\gamma[A_1D_0(X)]&=\gamma[A_3S_{h_1}DD_0(X)]=\gamma[A_3S_{h_1}(I-K_1)\,(X)]\\&\leq\gamma[A_3S_{h_1}(X)]+\gamma[A_3S_{h_1}K_1(X)]=\gamma[A_3S_{h_1}(X)]\leq k\gamma[S_{h_1}(X)]<k\gamma(X)\,.\end{aligned}$$

Since $k<1$, we have that AD_0 is condensing.

We can take D_0 as the function defined by

$$(D_0\,y)\,(t)=\int_0^t[y(s)-\frac{1}{\omega}\int_0^\omega y(r)dr]\,ds(-\infty<t<+\infty)$$

and it is obvious that $||D_0||\leq\omega$ if $\omega>2$ or $||D_0||\leq 2$ if $\omega\leq 2$. Hence, D_0 is ω-set-contractive if $\omega>2$ or 2-set-contractive if $\omega\leq 2$. With the same argument used in the case AD_0 also we prove that D_0A is condensing. Then, by Theorem 2.2. D–A is a Fredholm operator with $i(D-A)=0$.

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