# MULTIPLICITY FORMULAS FOR DISCRETE SERIES OF $Spin(1,2m)$ AND $SU(1,n)$

Dedicated to Professor Yozo Matsushima on his sixtieth birthday

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**Introduction.** Let $G$ be a connected semisimple Lie group with finite center. Let $\Gamma$ be a discrete subgroup of $G$ such that $\Gamma\backslash G$ is compact. Assume that $\Gamma$ has no elements with finite order other than the identity. Fix a Haar measure $dg$ on $G$. Then $dg$ induces the $G$-invariant measure on $\Gamma\backslash G$ and we can construct the right regular representation $\pi_\Gamma$ of $G$ on $L^2(\Gamma\backslash G)$. It is well-known that $\pi_\Gamma$ decomposes into the direct sum of irreducible unitary representations with finite multiplicity, up to unitarily equivalence. Let $\hat{G}$ be the set of all equivalence classes of irreducible unitary representations of $G$. For $U\in\hat{G}$, denote by $N_\Gamma(U)$ the multiplicity of $U$ in $\pi_\Gamma$.

Let $\hat{G}_d\subset\hat{G}$ be the discrete series of $G$. Assume that $\hat{G}_d$ is not empty. In this paper, we shall consider the multiplicity $N_\Gamma(U)$ of every class $U$ in $\hat{G}_d$. Langlands [8] showed that, if $U\in\hat{G}_d$ is integrable, we have the generic formula

$$N_\Gamma(U) = d(U)\mathrm{vol}(\Gamma\backslash G)$$

where $d(U)$ is the formal degree of $U$ and $\mathrm{vol}(\Gamma\backslash G)$ is the volume of $\Gamma\backslash G$. On the other hand, it has been known that there are examples of non-integrable classes in $\hat{G}_d$ for which the above Langlands' formula breaks down. Also Hotta-Parthasarathy [6] obtained a sufficient condition for $U$ to satisfy Langlands' formula.

Now, in [3], DeGeorge-Wallach proved a certain formula about $N_\Gamma(U)$ $(U\in\hat{G}_d)$ which seems to explain the reason why Langlands' formula breaks down for non-generic classes (c.f. Theorem 3.1.1). In fact, the formula of DeGeorge-Wallach shows that, in the case of non-generic classes, there can appear the terms of "trash" representations in the sense of Wallach [15].

The purpose of this paper is to apply the formula of DeGeorge-Wallach to the special cases of $G=Spin(1, 2m)$ $(m\geq 2)$ and $G=SU(1, n)$ $(n\geq 2)$ and provide the concrete formulas about $N_\Gamma(U)$ $(U\in\hat{G}_d)$. The most part of our work is devoted to finding all classes in $\hat{G}$ with given infinitesimal character. In [1], Borel and Wallach make the same arguments for the special infinitesimal charac-

ter.   Their method depends on Langlands' classification of admissible represen-
tations of reductive groups.   In this paper, we depend on detailed results by
Thieleker (c.f. [12], [13], [14]) and Kraljevic (c.f. [7]).

Main results are Theorem 7.6.1 and Theorem 8.7.1.   It will be interesting
which classes in $\hat{G}$ appear as "trash" representations.   Wallach [15] and Ragozin-
Warner [11] have obtained the similar formulas in the low dimensional cases.

## 1.   The discrete series

In this section, we shall recall some known results about the discrete series
of semisimple Lie groups.

1.1.   Let $G$ be a connected noncompact semisimple Lie group with finite
center.   Throughout this paper, we assume that $G$ is a real form of a simply
connected complex semisimple Lie group.   We fix a Haar measure $dg$ on $G$, once
and for all.

Let $\hat{G}$ be the set of all equivalence classes of irreducible unitary represen-
tations of $G$.   We call $U \in \hat{G}$ a *discrete* class if a representation in the class $U$
is equivalent to a subrepresentation of the regular representation of $G$ on $L^2(G)$.
Let us denote by $\hat{G}_d$ the set of all discrete classes in $\hat{G}$.   We call $\hat{G}_d$ *the discrete
series* of $G$.

Let $K$ be a maximal compact subgroup of $G$.   It is known that $G$ has a
discrete class if and only if there exists a Cartan subgroup $H$ of $G$ which is con-
tained in $K$ (c.f. [17], II, p. 401).   Hereafter we always assume that $G$ has
such a subgroup $H$.

1.2.   Let $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{h}_0$ be the Lie algebras of $G$, $K$ and $H$, respectively.
Let $\mathfrak{p}_0$ be the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to the Killing
form of $\mathfrak{g}_0$.   For any subalgebra $\mathfrak{u}_0$ of $\mathfrak{g}_0$, we shall denote by $\mathfrak{u}$ the complexifica-
tion of $\mathfrak{u}_0$.

Now $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ which is contained in $\mathfrak{k}$.   Let $\mathfrak{h}_R^*$ be the
dual space of $\mathfrak{h}_R = \sqrt{-1}\mathfrak{h}_0$ and let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$.   Let $\Delta$ be the root
system of $(\mathfrak{g}, \mathfrak{h})$.   For $\alpha \in \Delta$, denote by $\mathfrak{g}_\alpha$ the corresponding root space.   A root
$\alpha$ is called to be *compact* (resp. *noncompact*) if $\mathfrak{g}_\alpha$ is contained in $\mathfrak{k}$ (resp. $\mathfrak{p}$).
Denote by $\Delta_k$ (resp. $\Delta_n$) the set of all compact (resp. noncompact) roots.   The
Killing form of $\mathfrak{g}_0$ induces the inner product $\langle \ , \ \rangle$ on $\mathfrak{h}_R^*$.   Denote by $\mathfrak{F}$ the
set of all integral linear forms on $\mathfrak{h}_R$; i.e.

$$\mathfrak{F} = \left\{ \lambda \in \mathfrak{h}_R^* \ \middle| \ \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in Z \quad (\alpha \in \Delta) \right\}$$

and define

$$\mathfrak{F}_0 = \{ \lambda \in \mathfrak{F} \mid \langle \lambda, \alpha \rangle \neq 0 \quad (\alpha \in \Delta) \} .$$

An element $\lambda \in \mathfrak{F}_0$ is called *a regular integral form* on $\mathfrak{h}_R$.

Let $W$ be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Let $W_G$ be the Weyl group of $(G, H)$; i.e. the quotient group of the normalizer of $H$ in $G$ modulo $H$. The group $W$ and $W_G$ act on $\mathfrak{h}_R^*$ and we can identify $W_G$ with the subgroup of $W$ generated by reflections with respect to compact roots.

When we choose an ordering in $\mathfrak{h}_R^*$, put

$$\Delta^+ = \{\alpha \in \Delta \mid \alpha > 0\}, \quad \Delta_k^+ = \Delta^+ \cap \Delta_k, \quad \Delta_n^+ = \Delta^+ \cap \Delta_n,$$

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha, \qquad \rho_k = \frac{1}{2} \sum_{\alpha \in \Delta_k^+} \alpha, \quad \rho_n = \frac{1}{2} \sum_{\alpha \in \Delta_n^+} \alpha,$$

$$\mathfrak{F}^+ = \{\lambda \in \mathfrak{F} \mid \langle \lambda, \alpha \rangle \geq 0 \quad (\alpha \in \Delta^+)\},$$

$$\mathfrak{F}_0^+ = \mathfrak{F}_0 \cap \mathfrak{F}^+.$$

If an integral form $\lambda$ belongs to $\mathfrak{F}^+$, we say that $\lambda$ is *dominant* with respect to $\Delta^+$. Also define the subset $W^1$ of $W$ by

$$W^1 = \{s \in W \mid s\Delta^+ \supset \Delta_k^+\}.$$

Then the mapping $W_G \times W^1 \ni (s_1, s_2) \to s_1 s_2 \in W$ is a bijection.

Harish-Chandra showed that there is a distinguished surjection $D; \mathfrak{F}_0 \to \hat{G}_d$ (c.f. [17], II, p. 407). For $\Lambda \in \mathfrak{F}_0$, denote by $D_\Lambda$ the image of $\Lambda$ by $D$. Then, for $\Lambda, \Lambda' \in \mathfrak{F}_0$, we have $D_\Lambda = D_{\Lambda'}$ if and only if there exists an element $s \in W_G$ such that $\Lambda' = s\Lambda$ (c.f. [17], II, p. 407). Hence, if we choose a positive root system $\Delta^+$ of $\Delta$, $\hat{G}_d$ corresponds bijectively to the subset $\bigcup_{s \in W^1} s\mathfrak{F}_0^+$ of $\mathfrak{F}_0$.

1.3. Denote by $\hat{K}$ the set of all equivalence classes of irreducible $K$-modules. Let $\pi$ be a continuous representation of $G$ on a Hilbert space $E$. For convenience' sake, we shall often denote by the pair $(\pi, E)$ the representation $\pi$ on $E$. We say that $\pi$ is *K-finite* if it satisfies the following two conditions;

(1). The restriction $\pi|_K$ of $\pi$ to $K$ is a unitary representation of $K$.

(2). Each $\tau \in \hat{K}$ occurs with finite multiplicity in $\pi|_K$.

Let $(\pi, E)$ be a $K$-finite representation of $G$. Let $\tau \in \hat{K}$. Denote by $m(\pi; \tau)$ the multiplicity with which $\tau$ occurs in $\pi|_K$. Let $\Phi(\pi)$ be the set of all $\tau \in \hat{K}$ with $m(\pi; \tau) \neq 0$, which is repeated as many times as its multiplicity $m(\pi; \tau)$. We call $\Phi(\pi)$ *the K-spectrum* of $\pi$.

When we choose a positive root system $\Delta_k^+$ of $\Delta_k$, each $\tau \in \hat{K}$ is characterized by its highest weight. Put

$$\mathfrak{F}_k^+ = \left\{\lambda \in \mathfrak{h}_R^* \left| \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z} \text{ and } \geq 0 \quad (\alpha \in \Delta_k^+) \right.\right\}.$$

For $\gamma \in \mathfrak{F} \cap \mathfrak{F}_k^+$, denote by $\tau_\gamma$ the irreducible $K$-module with highest weight $\gamma$ and also denote by $\tau_\gamma$ the equivalence class of $\tau_\gamma$. Similarly, for $\gamma \in \mathfrak{F}_k^+$, we denote by $\tau_\gamma$ the irreducible $\mathfrak{k}$-module with highest weight $\gamma$.

1.4. Now every irreducible unitary representation $\pi$ of $G$ is $K$-finite (c.f. [17], I, p. 318). In particular, if $\pi$ belongs to a discrete class, the $K$-spectrum of $\pi$ is known under the assumption that $G$ is a linear group. We shall review it precisely.

Let $\Lambda \in \mathfrak{F}_0$ and $\pi_\Lambda$ be a representation of $G$ in the class $D_\Lambda$. Choose a positive root system $\Delta^+$ with respect to which $\Lambda$ is dominant. For $\lambda \in \mathfrak{h}_R^*$, let $Q(\lambda)$ be the number of distinct ways in which $\lambda$ can be written as a sum of positive noncompact roots. Then we have the following theorem.

**Theorem 1.4.1** (Hecht-Schmid, [5]). *Assume that $G$ has a faithful finite dimensional representation. Let $\gamma \in \mathfrak{F} \cap \mathfrak{F}_k^+$. Then we have*

$$m(\pi_\Lambda; \tau_\gamma) = \sum_{s \in W_G} \mathcal{E}(s) Q(s(\gamma + \rho_k) - \Lambda - \rho_n),$$

*where $\mathcal{E}(s)$ is the sign of $s \in W_G$.*

1.5. Finally, we refer to formal degrees of the discrete series. Let $(\pi, E)$ be an irreducible unitary representation of $G$. If $\pi$ belongs to a discrete class, there is a positive real number $d(\pi)$ such that

$$\int_G |(\pi(g)u, v)|^2 dg = d(\pi)^{-1}(u, u)(v, v) \qquad (u, v \in E)$$

where $( , )$ is the inner product in $E$ (c.f. [17], I, p. 351). We call $d(\pi)$ *the formal degree* of $\pi$.

For $\Lambda \in \mathfrak{F}_0$, denote by $d(D_\Lambda)$ the formal degree of a representation in the class $D_\Lambda \in \hat{G}_d$. If we choose a positive root system $\Delta^+$ as in 1.4., we have

$$(1.1) \qquad\qquad d(D_\Lambda) = c \left| \prod_{\alpha \in \Delta^+} \langle \Lambda, \alpha \rangle \right|$$

where $c$ is the positive constant depending on the measure $dg$ (c.f. [17], II, p. 407).

## 2. Infinitesimal characters

In this section, we shall review certain infinitesimal properties of $K$-finite representations of $G$. Let $G$ be as in §1.

2.1. Let $(\pi, E)$ be a $K$-finite representation of $G$. An element $u \in E$ is called $K$-*finite* if $\pi(K)u$ is contained in a finite dimensional subspace of $E$. Denote by $E_f$ the set of all $K$-finite elements of $E$. For $u \in E_f$ and $X \in \mathfrak{g}_0$, we can define

$$\pi_f(X)(u) = \lim_{t \to 0} \frac{\pi(\exp tX)u - u}{t}$$

and $\pi_f(X)(u)$ belongs to $E_f$ (c.f. [17], I, p. 326). In this way, we can induce the representation $\pi_f$ of $\mathfrak{g}_0$ on $E_f$. Denote by $U(\mathfrak{g})$ the universal enveloping

algebra of $\mathfrak{g}$. The representation $\pi_f$ of $\mathfrak{g}_0$ is canonically extended to the representation of $U(\mathfrak{g})$. This representation of $U(\mathfrak{g})$ is also denoted by $\pi_f$ and is called *the infinitesimal representation* of $\pi$.

Let $\pi^1$ and $\pi^2$ be $K$-finite representations of $G$. If $\pi_f^1$ is equivalent to $\pi_f^2$ as representations of $U(\mathfrak{g})$, we say that $\pi^1$ is *infinitesimally equivalent* to $\pi^2$. In particular, let $\pi^1$ and $\pi^2$ be irreducible unitary representations. Then $\pi^1$ is unitarily equivalent to $\pi^2$ if and only if $\pi^1$ is infinitesimally equivalent to $\pi^2$ (c.f. [17], I, p. 329).

Let $(\pi, E)$ be a $K$-finite representation of $G$. We say that $\pi$ is *infinitesimally unitary* if $E_f$ has an inner product with respect to which $\pi_f(\sqrt{-1}X)$ is a symmetric operator for each $X \in \mathfrak{g}_0$. If $\pi$ is unitary, $\pi$ is clearly infinitesimally unitary. Conversely, if $G$ is simply connected, an irreducible infinitesimally unitary representation of $G$ is infinitesimally equivalent to a unique irreducible unitary representation of $G$, up to unitarily equivalence (c.f. [17], I, p. 331).

2.2. Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. A $K$-finite representation $(\pi, E)$ is said to be *quasisimple* if there exists a homomorphism $\chi$ of $Z(\mathfrak{g})$ into $C$ such that $\pi_f(z)u = \chi(z)u$ for all $z \in Z(\mathfrak{g})$ and $u \in E_f$. Here $\chi$ is called *the infinitesimal character* of $\pi$. As it is well-known, an irreducible unitary representation of $G$ is quasisimple (c.f. [17], I, p. 318). For $U \in \hat{G}$, denote by $\chi_U$ the infinitesimal character of a representation in the class $U$.

We shall state some preliminary facts about infinitesimal characters. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ in § 1 and $U(\mathfrak{h})$ the universal enveloping algebra of $\mathfrak{h}$. Since $U(\mathfrak{h})$ is the symmetric tensor algebra of $\mathfrak{h}$, $U(\mathfrak{h})$ is naturally identified with the algebra of complex-valued polynomial functions on $\mathfrak{h}^*$. Choose a positive root system $\Delta^+$ of $\Delta$ and put $\mathfrak{n}^+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. Then $Z(\mathfrak{g})$ is contained in the direct sum $U(\mathfrak{h}) \oplus \cdot U(\mathfrak{g})\mathfrak{n}^+$. Let $\varphi$ be the projection of $Z(\mathfrak{g})$ into $U(\mathfrak{h})$. For $\lambda \in \mathfrak{h}^*$, define the homomorphism $\chi_\lambda; Z(\mathfrak{g}) \to C$ by

$$\chi_\lambda(z) = \varphi(z)(\lambda - \rho) \qquad (z \in Z(\mathfrak{g})).$$

Then it turns out that $\chi_\lambda$ does not depend on the choice of $\Delta^+$ (c.f. [4], p. 231). By a well-known result of Harish-Chandra, every homomorphism $\chi; Z(\mathfrak{g}) \to C$ is of the form $\chi_\lambda$ for some $\lambda \in \mathfrak{h}^*$. Moreover, for $\lambda_1, \lambda_2 \in \mathfrak{h}^*$, we have $\chi_{\lambda_1} = \chi_{\lambda_2}$ if and only if $\lambda_2 = s\lambda_1$ for some $s \in W$ (c.f. [4], p. 232). In particular, if $\lambda \in \mathfrak{h}^*$ is a dominant regular integral form with respect to $\Delta^+$, $\chi_\lambda$ is the central character of the irreducible $\mathfrak{g}$-module with highest weight $\lambda - \rho$ (c.f. [4], p. 230).

2.3. Now let $\mathfrak{h}'$ be another Cartan subalgebra of $\mathfrak{g}$. For $\lambda' \in (\mathfrak{h}')^*$, let $\chi_{\lambda'}'; Z(\mathfrak{g}) \to C$ be the homomorphism constructed from $\lambda'$ in the same way as in 2.2. We shall study the relation between $\chi_\lambda$ and $\chi_{\lambda'}'$. Let $\theta$ be an automorphism of $\mathfrak{g}$ such that $\theta(\mathfrak{h}') = \mathfrak{h}$ and let $\theta^*; \mathfrak{h}^* \to (\mathfrak{h}')^*$ be the linear isomorphism induced by $\theta; \mathfrak{h}' \to \mathfrak{h}$.

**Lemma 2.3.1.** *If $\lambda \in \mathfrak{h}^*$ is a regular integral form, then we have $\mathcal{X}_\lambda = \mathcal{X}'_{\theta^*\lambda}$ as homomorphisms of $Z(\mathfrak{g})$ into $C$.*

Proof. We take a positive root system $\Delta^+$ with respect to which $\lambda$ is dominant. Let $V_{\lambda-\rho}$ be the irreducible $\mathfrak{g}$-module with highest weight $\lambda - \rho$. Then $\mathcal{X}_\lambda$ is the central character of $V_{\lambda-\rho}$. On the other hand, if we take $\theta^*(\Delta^+)$ as a positive root system of $(\mathfrak{g}, \mathfrak{h}')$, the highest weight of $V_{\lambda-\rho}$ relative to $\theta^*(\Delta^+)$ is $\theta^*(\lambda-\rho) = \theta^*\lambda - \theta^*\rho \in (\mathfrak{h}')^*$. Hence $\mathcal{X}'_{\theta^*\lambda}$ is also the central character of $V_{\lambda-\rho}$. Thus the assertion follows.

## 3. Basic multiplicity formulas

In this section, we shall review multiplicity formulas obtained by DeGeorge and Wallach. It plays a basic role in this paper.

3.1. Let $G$ be as in §1. Let $\Gamma$ be a discrete subgroup of $G$ such that $\Gamma \backslash G$ is compact. In this paper, we always assume that $\Gamma$ has no elements with finite order other than the identity. We fix the $G$-invariant measure $d\dot{g}$ on $\Gamma \backslash G$ induced by $dg$. For $U \in \hat{G}$, we denote by $N_\Gamma(U)$ the multiplicity with which $U$ occurs in the right regular representation of $G$ on $L^2(\Gamma \backslash G)$.

Now we take the Cartan subalgebra $\mathfrak{h}$ as in §1. Consider a special linear form $\Lambda \in \mathfrak{F}_0$ and the discrete class $D_\Lambda \in \hat{G}_d$. Let $\mathcal{X}_\Lambda$ be the homomorphism of $Z(\mathfrak{g})$ into $C$ constructed from $\Lambda$ as in 2.2. Denote by $\hat{G}_\Lambda$ the set of all $U \in \hat{G}$ such that its infinitesimal character is $\mathcal{X}_\Lambda$; i.e.

$$\hat{G}_\Lambda = \{U \in \hat{G} \mid \mathcal{X}_U = \mathcal{X}_\Lambda\}.$$

Clearly we have $\hat{G}_\Lambda = \hat{G}_{s\Lambda}$ for all $s \in W$. Also it is known that $D_\Lambda \in \hat{G}_\Lambda$ and the number of elements in $\hat{G}_\Lambda$ is finite (c.f. [3], p. 141).

In [3], DeGeorge and Wallach obtained a formula describing the relation among numbers in $\{N_\Gamma(U) \mid U \in \hat{G}_\Lambda\}$. We shall precisely review it. Choose and fix a positive root system $\Delta^+$ with respect to which $\Lambda$ is dominant. Since rank $\mathfrak{g}_0 = $ rank $\mathfrak{k}_0$, we can construct the half-spin representations of $\mathfrak{k}_0$ as in [6], p. 144. Denote by $L^+$ the half-spin representation of $\mathfrak{k}_0$ with highest weight $\rho_n$ and denote by $L^-$ the other one. Note that $L^\pm$ depends on the choice of $\Delta^+$. Let $\tau_{\Lambda-\rho_k}$ be the irreducible $\mathfrak{k}$-module with highest weight $\Lambda - \rho_k$. Then the tensor product $\tau_{\Lambda-\rho_k} \otimes L^\pm$ integrates to the representation of $K$ which is also denoted by $\tau_{\Lambda-\rho_k} \otimes L^\pm$ (c.f. [3], p. 139). For $K$-modules $\tau_1$ and $\tau_2$, denote by $\dim \mathrm{Hom}_K(\tau_1, \tau_2)$ the intertwining number of $\tau_1$ and $\tau_2$. Let $U \in \hat{G}$ and let $\pi_U$ be a representation in the class $U$. Then define

$$\dim \mathrm{Hom}_K(\tau_{\Lambda-\rho_k} \otimes L^\pm, U) = \dim \mathrm{Hom}_K(\tau_{\Lambda-\lambda_k} \otimes L^\pm, \pi_U|_K).$$

**Theorem 3.1.1** (DeGeorge-Wallach, [3]). *Let $\Lambda \in \mathfrak{F}_0$. Then we have*

$$(3.1) \quad d(D_\Lambda) \text{vol}(\Gamma \backslash G) = N_\Gamma(D_\Lambda) + \sum_{U \in \hat{G}_\Lambda - \hat{G}_d} N_\Gamma(U) \{\dim \text{Hom}_K(\tau_{\Lambda - \rho_k} \otimes L^+, U)$$
$$- \dim \text{Hom}_K(\tau_{\Lambda - \rho_k} \otimes L^-, U)\}$$

*where $d(D_\Lambda)$ is the formal degree of $D_\Lambda$ and $\text{vol}(\Gamma \backslash G)$ is the volume of $\Gamma \backslash G$.*

Our purpose in this paper is to obtain concrete formulas about $N_\Gamma(D_\Lambda)$ by using Theorem 3.1.1. We divide our study into two steps as follows;

**Step 1;** To find out all elements in $\hat{G}_\Lambda - \hat{G}_d$.

**Step 2;** To examine $\dim \text{Hom}_K(\tau_{\Lambda - \rho_k} \otimes L^\pm, U)$ for every $U$ in $\hat{G}_\Lambda - \hat{G}_d$.

## 4. The nonunitary principal series

In order to find out irreducible unitary representations of $G$ with given infinitesimal character, we need the family of representations of $G$ which is called the nonunitary principal series. In this section, we shall make preparations about this family. Let $G$ be as in §1.

4.1. First we shall state some notations. Let $G = KAN$ be an Iwasawa decomposition of $G$. Let $\mathfrak{a}_0$ and $\mathfrak{n}_0$ be the Lie algebras of $A$ and $N$. We write $g = k(g) \exp(H(g)) n(g)$ for $g \in G$, where $k(g) \in K$, $H(g) \in \mathfrak{a}_0$ and $n(g) \in N$.

Let $\Sigma$ be the restricted root system of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$. Choose a positive root system $\Sigma^+$ of $\Sigma$ which is associated with the subalgebra $\mathfrak{n}_0$ (c.f. [16], p. 164). Define $\rho_a = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$.

Let $M$ be the centralizer of $A$ in $K$ and let $\mathfrak{m}_0$ be the Lie algebra of $M$. Let $\mathfrak{t}_0$ be a maximal abelian subalgebra of $\mathfrak{m}_0$ and let $\mathfrak{t}_R^*$ be the dual space of $\mathfrak{t}_R = \sqrt{-1}\mathfrak{t}_0$. Denote by $\Delta_m$ the root system of $(\mathfrak{m}, \mathfrak{t})$. Choose a positive root system $\Delta_m^+$ of $\Delta_m$ and put $\rho_m = \frac{1}{2} \sum_{\alpha \in \Delta_m^+} \alpha$. Also define

$$\mathfrak{F}_m^+ = \left\{ \mu \in \mathfrak{t}_R^* \,\middle|\, \frac{2\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in Z \text{ and } \geq 0 \quad (\alpha \in \Delta_m^+) \right\}$$

Let $\hat{M}$ be the set of all equivalence classes of irreducible $M$-modules. For $\xi \in \hat{M}$, denote by $\mu(\xi)$ the highest weight of $\xi$. Then $\mu(\xi)$ belongs to $\mathfrak{F}_m^+$.

If we set $\mathfrak{h}_0' = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}$. In this section, we use this Cartan subalgebra $\mathfrak{h}'$. Let $(\mathfrak{h}_R')^*$ be the dual space of $\mathfrak{h}_R' = \sqrt{-1}\mathfrak{t}_0 \oplus \mathfrak{a}_0$. Canonically, $\mathfrak{t}^*$ and $\mathfrak{a}^*$ can be considered as subsets of the dual space $(\mathfrak{h}')^*$ of $\mathfrak{h}'$. Denote by $\Delta'$ the root system of $(\mathfrak{g}, \mathfrak{h}')$. Then there exists a positive root system $\Delta'^+$ of $\Delta'$ such that $\Delta_m^+ \subset \Delta'^+$ and $\{\alpha \in \Delta' \mid \alpha|_{\mathfrak{a}_0} \in \Sigma^+\} \subset \Delta'^+$ (c.f. [16], p. 169). This ordering of $(\mathfrak{h}_R')^*$ is called a *compatible* ordering with respect to $\Sigma^+$. Throughout this section, we fix this ordering in $(\mathfrak{h}_R')^*$. Set $\rho' = \frac{1}{2} \sum_{\alpha \in \Delta'^+} \alpha$. Then we have $\rho' = \rho_m + \rho_a$.

4.2.  Let $\xi \in \hat{M}$ and let $\xi$ also be a representation of $M$ on the vector space $V_\xi$ which belongs to the class $\xi$.  Let $C_\xi$ be the set of all continuous mappings $f; K \to V_\xi$ such that $f(km)=\xi(m)^{-1}f(k)$ for all $k \in K$ and $m \in M$.  For $f_1, f_2 \in C_\xi$, define

$$(f_1, f_2) = \int_K (f_1(k), f_2(k))dk$$

where $dk$ is the normalized Haar measure on $K$ such that $\int_K dk=1$.  Denote by $E_\xi$ the completion of $C_\xi$ with respect to this inner product ( , ).

Let $\nu \in \mathfrak{a}^*$.  We define the representation $\pi_{\xi,\nu}$ of $G$ on the Hilbert space $E_\xi$ as follows;

$$(\pi_{\xi,\nu}(g)f)(k) = e^{-(\nu+\rho_a(H(g^{-1}k)))}f(k(g^{-1}k))$$

for $g \in G$, $k \in K$ and $f \in C_\xi$.  Then $\pi_{\xi,\nu}$ is not necessarily unitary, but it is $K$-finite (c.f. [16], p. 232) and quasisimple (c.f. [9], p. 29).  The family of representations $\{\pi_{\xi,\nu} | \xi \in \hat{M}, \nu \in \mathfrak{a}^*\}$ is called *the nonunitary principal series*.

4.3.  We shall review how the infinitesimal character of each representation in the nonunitary principal series is given.  Let $\pi_{\xi,\nu}$ $(\xi \in \hat{M}, \nu \in \mathfrak{a}^*)$ be a representation in the nonunitary principal series.  For $\lambda' \in (\mathfrak{h}')^*$, denote by $\chi_{\lambda'}$ the homomorphism of $Z(\mathfrak{g})$ into $C$ constructed from $\lambda'$ as in 2.2..  Then we have the following lemma.

**Lemma 4.3.1** (Lepowsky [9]).  *The infinitesimal character $\chi_{\pi_{\xi,\nu}}$ of $\pi_{\xi,\nu}$ is equal to $\chi'_{\mu(\xi)+\rho_m+\nu}$, where $\mu(\xi)$, $\rho_m$ and $\nu$ are regarded as elements in $(\mathfrak{h}')^*$.*

Proof.  In [9], $\chi_{\pi_{\xi,\nu}}$ was given explicitly.  We shall translate the result in [9] into our desirable form.  In Proposition 8.9 of [9], take $-\Delta_m^+=\{-\alpha | \alpha \in \Delta_m^+\}$ as a positive root system of $\Delta_m$.  Note that $\mu(\xi)$ is the lowest weight of $\xi$ with respect to $-\Delta_m^+$.  Denote by $\Delta_1'^+$ the positive root system of $\Delta'$ determined by $\Sigma^+$ and $-\Delta_m^+$.  Let $\mathfrak{n}_1'^+$ (resp. $\mathfrak{n}_1'^-$) be the sum of positive (resp. negative) root spaces.  Then we have

$$Z(\mathfrak{g}) \subset U(\mathfrak{h}')+\mathfrak{n}_1'^+ U(\mathfrak{g}),$$
$$Z(\mathfrak{g}) \subset U(\mathfrak{h}')+U(\mathfrak{g})\mathfrak{n}_1'^- .$$

In the above two direct sums, the projection mappings of $Z(\mathfrak{g})$ into $U(\mathfrak{h}')$ are the same (c.f. [4], p. 230).  Denote by $\varphi'$ this projection mapping.  Then Proposition 8.9 of [9] implies that

$$\chi_{\pi_{\xi,\nu}}(z) = \varphi'(z)\,(\mu(\xi)+\nu+\rho_a)$$

for $z \in Z(\mathfrak{g})$.  On the other hand, use a positive root system $-\Delta_1'^+$ of $\Delta'$ in the construction of $\chi_{\lambda'}$ $(\lambda' \in (\mathfrak{h}')^*)$.  Then we have

$$\chi'_{\mu(\xi)+\rho_m+\nu}(z) = \varphi'(z)((\mu(\xi)+\rho_m+\nu)-(\rho_m-\rho_a))$$
$$= \varphi'(z)(\mu(\xi)+\nu+\rho_a).$$

Hence we have $\chi_{\pi_{\xi,\nu}}(z)=\chi'_{\mu(\xi)+\rho_m+\nu}(z)$.

4.4 Finally we shall prepare some facts about the $K$-spectrum of $\pi_{\xi,\nu}$. Let $\xi \in \hat{M}$ and $\tau \in \hat{K}$. Denote by $m(\tau;\xi)$ the multiplicity with which $\xi$ occurs in the restriction of $\tau$ to $M$. If $m(\tau;\xi)\neq 0$, $\tau$ is called to be $\xi$-*admissible*. Denote by $\hat{K}(\xi)$ the set of all $\xi$-admissible elements in $\hat{K}$.

Let $\nu \in \mathfrak{a}^*$. From the definition of $\pi_{\xi,\nu}$, the restriction of $\pi_{\xi,\nu}$ ot $K$ is the representation induced from the representation $\xi$ of $M$. Hence, by Frobenius' reciprocity theorem, we have $m(\pi_{\xi,\nu};\tau)=m(\tau;\xi)$ for $\tau \in \hat{K}$. Thus the $K$-spectrum of $\pi_{\xi,\nu}$ is the set of all $\tau \in \hat{K}(\xi)$ repeated as many times as $m(\tau;\xi)$.

For $\tau \in \hat{K}$, let $E_\xi(\tau) \subset E_\xi$ be the subspace of all vectors transformed by $\pi_{\xi,\nu}|_K$ according to $\tau$. When $\Phi$ is a subset of $\hat{K}(\xi)$, define $E_\xi(\Phi)= \bigoplus_{\tau \in \Phi} E_\xi(\tau)$, where $\bigoplus$ means the orthogonal direct sum. Note that $E_\xi(\Phi)$ is a $K$-invariant subspace of $E_\xi$.

## 5. The subquotient theorem

In this section, for the sake of **Step 1,** we shall review the subquotient theorem of Harish-Chandra. It will give a method to find out the representations in question.

5.1. Let $(\pi, E)$ be a continuous representation of $G$. Let $E$ have a finite sequence of closed invariant subspaces

$$E = E_0 \supset E_1 \supset \cdots \supset E_{r-1} \supset E_r = \{0\}$$

such that, on each quotient space $E_{i-1}/E_i$, $\pi$ induces an irreducible representation of $G$. Then $\pi$ is said to *have a finite composition series* and each quotient representation $(\pi, E_{i-1}/E_i)$ is called *an irreducible subquotient of $\pi$*.

It is known that every member of the nonunitary principal series has a finite composition series (c.f. [9], p. 33). The following theorem gives a method of realizing our exploring representations.

**Theorem 5.1.1** (Harish-Chandra; c.f. [9], p. 29). *Every irreducible unitary representation of $G$ is infinitesimally equivalent to an irreducible subquotient of some member of the nonunitary principal series.*

5.2. Now, let $\mathfrak{h}$ be the Cartan subalgebra in § 1. We consider a special linear form $\Lambda \in \mathfrak{F}_0$. Let $\Delta^+$ be the positive root system with respect to which $\Lambda$ is dominant. On the other hand, let $\mathfrak{h}'$ and $\Delta'^+$ be as in 4.1. There is an automorphism $\theta$ of $\mathfrak{g}$ such that $\theta(\mathfrak{h}')=\mathfrak{h}$ and $\theta^*(\Delta^+)=\Delta'^+$. Identify $\mathfrak{h}^*$ with $(\mathfrak{h}')^*$ by this isomorphism. From Lemma 2.3.1, if $\lambda \in \mathfrak{h}^*$ is a regular integral

form, we have $\mathcal{X}_\lambda = \mathcal{X}'_\lambda$. Hence, for $\xi \in \hat{M}$ and $\nu \in \mathfrak{a}^*$, we have $\mathcal{X}'_{\mu(\xi)+\rho_m+\nu} = \mathcal{X}_\Lambda$ if and only if there exists an element $s \in W$ such that $\mu(\xi)+\rho_m+\nu = s(\Lambda)$. Let $R(\Lambda)$ be the set of all pair $(\xi, \nu)$ $(\xi \in \hat{M}, \nu \in \mathfrak{a}^*)$ such that $\mu(\xi)+\rho_m+\nu$ and $\Lambda$ are in the same $W$-orbit. From Lemma 4.4.1 and Theorem 5.1.1, we obtain the following proposition.

**Proposition 5.2.2.** *Let* $\Lambda \in \mathfrak{F}_0$ *and* $\pi$ *be an irreducible unitary representation of $G$ such that* $\mathcal{X}_\pi = \mathcal{X}_\Lambda$. *Then* $\pi$ *is infinitesimally equivalent to an irreducible subquotient of* $\pi_{\xi,\nu}$ *for some* $(\xi, \nu)$ *in* $R(\Lambda)$.

## 6. The decomposition of $\mathfrak{k}$-modules

In this section, we shall describe the direct sum decomposition of a tensor product of two $\mathfrak{k}$-modules. Also we shall review a known result about the $\mathfrak{k}$-module $L^\pm$. These facts will be used in studying **Step 2**.

6.1. Let $\mathfrak{k}_0$ and $\mathfrak{h}_0$ be as in § 1. Let $\mathfrak{c}_0$ be the center of $\mathfrak{k}_0$ and let $\mathfrak{k}_0^1$ be the semisimple part $[\mathfrak{k}_0, \mathfrak{k}_0]$ of $\mathfrak{k}_0$. Then we have

$$\mathfrak{k}_0 = \mathfrak{c}_0 \oplus \mathfrak{k}_0^1$$
$$\mathfrak{h}_0 = \mathfrak{c}_0 \oplus \mathfrak{h}_0^1$$

where $\mathfrak{h}_0^1$ is the maximal abelian subalgebra of $\mathfrak{k}_0^1$. We always regard $(\mathfrak{h}^1)^*$ and $\mathfrak{c}^*$ as subsets of $\mathfrak{h}^*$.

First we shall examine the decomposition as $\mathfrak{k}^1$-modules. As the positive root system of $(\mathfrak{k}^1, \mathfrak{h}^1)$, we take $\Delta_k^+$ in § 1. For $\gamma \in \mathfrak{F}_k^+ \cap (\mathfrak{h}^1)^*$, denote by $\tau_\gamma^1$ the irreducible $\mathfrak{k}^1$-module with highest weight $\gamma$. The following lemma is due to [2] (c.f. [2], Chapter VIII, § 9, Proposition 2).

**Lemma 6.1.1.** *Let* $\beta$, $\gamma$ *and* $\delta$ *be in* $\mathfrak{F}_k^+ \cap (\mathfrak{h}^1)^*$. *Then the multiplicity of* $\tau_\beta^1$ *in the tensor product* $\tau_\gamma^1 \otimes \tau_\delta^1$ *is given by*

$$\sum_{s \in W_G} \mathcal{E}(s) M_{\tau_\delta^1}(\beta + \rho_k - s(\gamma + \rho_k))$$

*where* $M_{\tau_\delta^1}(\lambda)$ $(\lambda \in (\mathfrak{h}^1)^*)$ *is the multiplicity of a weight $\lambda$ in* $\tau_\delta^1$.

Once we know the decomposition as $\mathfrak{k}^1$-modules, we can easily obtain the decomposition as $\mathfrak{k}$-modules.

**Proposition 6.1.2.** *Let* $\tau$ *be a $\mathfrak{k}$-module. Let* $\beta$ *and* $\gamma$ *be in* $\mathfrak{F}_k^+$ *and let* $\tau_\beta$ *and* $\tau_\gamma$ *be the irreducible $\mathfrak{k}$-modules with highest weight $\beta$ and $\gamma$ respectively. Then the multiplicity of* $\tau_\beta$ *in* $\tau_\gamma \otimes \tau$ *is given by*

$$\sum_{s \in W_G} \mathcal{E}(s) M_\tau(\beta + \rho_k - s(\gamma + \rho_k))$$

*where* $M_\tau(\lambda)$ $(\lambda \in \mathfrak{h}^*)$ *is the multiplicity of a weight $\lambda$ in* $\tau$.

Proof. It is sufficient to prove the statement in the case that $\tau$ is an irreducible $\mathfrak{k}$-module $\tau_\delta$ $(\delta\in\mathfrak{F}_k^+)$. Let $\beta$ be in $\mathfrak{F}_k^+$. If $\tau_\beta$ occurs in $\tau_\gamma\otimes\tau_\delta$ with nonzero multiplicity, the restriction of $\beta|_{\mathfrak{c}}$ to $\mathfrak{c}$ is equal to $(\gamma+\delta)|_{\mathfrak{c}}$. We put

$$\mathfrak{F}_k^+(\gamma,\ \delta) = \{\beta\in\mathfrak{F}_k^+|\ \beta|_{\mathfrak{c}} = (\gamma+\delta)|_{\mathfrak{c}}\}\ .$$

Assume that $\beta\in\mathfrak{F}_k^+(\gamma,\ \delta)$. Then $\tau_\beta$ occurs in $\tau_\gamma\otimes\tau_\delta$ with multiplicity $m$ if and only if $\tau_\beta|_{\mathfrak{h}^1}$ occurs in $\tau_\gamma|_{\mathfrak{h}^1}\otimes\tau_\delta|_{\mathfrak{h}^1}$ with multiplicity $m$. Hence, by Lemma 6.1.1, the multiplicity of $\tau_\beta$ in $\tau_\gamma\otimes\tau_\delta$ is given by

$$\sum_{s\in W_G} \mathcal{E}(s)M\tau_{\delta^1}^1(\beta^1+\rho_k-s(\gamma^1+\rho_k))$$

where $\beta^1$, $\gamma^1$ and $\delta^1$ are the restrictions of $\beta$, $\gamma$ and $\delta$ to $\mathfrak{h}^1$, respectively. Since, for $\lambda\in\mathfrak{h}^*$ and $s\in W_G$, we have $s(\lambda)|_{\mathfrak{h}^1}=s(\lambda|_{\mathfrak{h}^1})$ and $s(\lambda)|_{\mathfrak{c}}=\lambda|_{\mathfrak{c}}$, it follows that

$$\{\beta+\rho_k-s(\gamma+\rho_k)\}|_{\mathfrak{h}^1} = \beta^1+\rho_k-s(\gamma^1+\rho_k)$$
$$\{\beta+\rho_k-s(\gamma+\rho_k)\}|_{\mathfrak{c}} = (\beta-\gamma)|_{\mathfrak{c}}$$
$$= \delta|_{\mathfrak{c}}\ .$$

Hence, $\beta+\rho_k-s(\gamma+\rho_k)$ is a weight of $\tau_\delta$ if and only if $\beta^1+\rho_k-s(\gamma^1+\rho_k)$ is a weight of $\tau_{\delta^1}^1$. Therefore we have

$$M\tau_\delta(\beta+\rho_k-s(\gamma+\rho_k)) = M\tau_{\delta^1}^1(\beta^1+\rho_k-s(\gamma^1+\rho_k))\ ,$$

and the statement is proved for $\tau_\beta$ with $\beta\in\mathfrak{F}_k^+(\gamma,\ \delta)$.

Next, assume that $\beta\notin\mathfrak{F}_k^+(\gamma,\ \delta)$. Clearly the multiplicity of $\tau_\beta$ in $\tau_\gamma\otimes\tau_\delta$ is equal to 0. On the other hand, since we have $\{\beta+\rho_k-s(\gamma+\rho_k)\}|_{\mathfrak{c}}\neq\delta|_{\mathfrak{c}}$, we obtain that

$$M\tau_\delta(\beta+\rho_k-s(\gamma+\rho_k)) = 0 \qquad (s\in W_G)\ .$$

The proposition is proved.

6.2. Finally we shall state how the set of weights of $L^\pm$ is given. Choose an ordering in $\mathfrak{h}_R^*$ and appoint $L^\pm$ with respect to this ordering. Let $Q$ be a subset of $\Delta_n^+$. Denote by $\langle Q\rangle$ the sum of all elements in $Q$ and denote by $|Q|$ the number of elements in $Q$. The following lemma is known (c.f. [6], p. 144).

**Lemma 6.2.1.** *The set of all weights of the $\mathfrak{k}$-module $L^+$ (resp. $L^-$) is given by*

$$\{\rho_n-\langle Q\rangle\ |\ Q\subset\Delta_n^+,\ (-1)^{|Q|} = +1\ (\text{resp. } -1)\}$$

*and the multiplicity of each weight is the number of ways in which it can be expressed in the above form.*

## 7.  The case of $G=Spin(1, 2m)$

In this section, we shall study **Step 1** and **Step 2** in the case of $G=$ $Spin(1, 2m)$ and we shall obtain concrete multiplicity formulas.   In [12], [13] and [14], Thieleker classified all irreducible unitary representations of $Spin(1, 2m)$. We shall often use his results.

7.1.   Let $m$ be an integer such that $m \geq 2$.   Let $G$ be the group $Spin(1, 2m)$; i.e. the universal covering group of the identity component of $SO(1, 2m)$.   The subgroup $K$ is isomorphic to the universal covering group $Spin(2m)$ of $SO(2m)$. As usual, we realize $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{p}_0$ as Lie algebras of matrices;

$$\mathfrak{g}_0 = \left\{ \left( \begin{array}{c|c} 0 & u \\ \hline {}^t u & X \end{array} \right) \middle| \begin{array}{l} u \in \mathbf{R}^{2m} \\ X \in \mathfrak{so}(2m, \mathbf{R}) \end{array} \right\}$$

$$\mathfrak{k}_0 = \left\{ K(X) = \left( \begin{array}{c|c} 0 & 0 \\ \hline 0 & X \end{array} \right) \middle| X \in \mathfrak{so}(2m, \mathbf{R}) \right\}$$

$$\mathfrak{p}_0 = \left\{ P(u) = \left( \begin{array}{c|c} 0 & u \\ \hline {}^t u & 0 \end{array} \right) \middle| u = (u_1, u_2, \cdots, u_{2m}) \in \mathbf{R}^{2m} \right\} .$$

Take $\mathfrak{h}_0$ as follows;

$$\mathfrak{h}_0 = \left\{ H(h) = \left( \begin{array}{c|ccc|ccc} 0 & & & & & & 0 \\ \hline & & & & -h_1 & & 0 \\ & & 0 & & & \ddots & \\ 0 & & & & 0 & & -h_m \\ \hline & h_1 & & 0 & & & \\ & & \ddots & & & 0 & \\ & 0 & & h_m & & & \end{array} \right) \middle| h = (h_1, h_2, \cdots, h_m) \in \mathbf{R}^m \right\}$$

For $1 \leq i \leq m$, define a linear form $e_i$ on $\mathfrak{h}$ by

$$e_i(H(h_1, \cdots, h_m)) = \sqrt{-1}\, h_i .$$

Then $\mathfrak{h}_R^*$ is spanned by $\{e_1, \cdots, e_m\}$ over $\mathbf{R}$ and the inner product in $\mathfrak{h}_R^*$ is given by

$$\langle e_i, e_j \rangle = \frac{1}{2(2m-1)} \delta_{ij} \qquad (1 \leq i, j \leq m) .$$

Choose and fix a lexicographic ordering in $\mathfrak{h}_R^*$ with respect to the basis $\{e_1, \cdots, e_m\}$. Under these situations, the following facts are easily seen;

$$\Delta = \{\pm e_i \ (1 \leq i \leq m), \quad \pm(e_i + e_j) \ (1 \leq i < j \leq m)\}$$
$$\Delta^+ = \{e_i \ (1 \leq i \leq m), \quad e_i \pm e_j \ (1 \leq i < j \leq m)\}$$

$$\Delta_k^+ = \{e_i \pm e_j \ (1 \leqq i < j \leqq m)\}$$

$$\Delta_n^+ = \{e_i \ (1 \leqq i \leqq m)\}$$

$$\rho = \frac{1}{2} \sum_{i=1}^{m} (2m-2i+1)e_i$$

$$\rho_k = \sum_{i=1}^{m} (m-i)e_i$$

$$\rho_n = \frac{1}{2} \sum_{i=1}^{m} e_i$$

$$\mathfrak{F} = \{\lambda = \sum_{i=1}^{m} \lambda_i e_i \mid \lambda_i - \lambda_{i+1} \in \mathbf{Z} \ (1 \leqq i \leqq m-1), \ 2\lambda_m \in \mathbf{Z}\}$$

$$\mathfrak{F}_0 = \{\lambda \in \mathfrak{F} \mid \lambda_1, \cdots, \lambda_m \neq 0, \ \lambda_i \neq \lambda_j \ (i \neq j)\}$$

$$\mathfrak{F}_0^+ = \{\lambda \in \mathfrak{F} \mid \lambda_1 > \lambda_2 > \cdots > \lambda_m > 0\}$$

$$\mathfrak{F}_k^+ = \{\lambda \in \mathfrak{F} \mid \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_{m-1} \geqq |\lambda_m|\} \ .$$

Note that, if $\lambda = \sum_{i=1}^{m} \lambda_i e_i \in \mathfrak{F}$, $\{\lambda_1, \cdots, \lambda_m\}$ are either all integers or all half odd integers.

Let $\mathfrak{S}_m$ be the symmetric group of degree $m$ and let $\sigma \in \mathfrak{S}_m$. Let $\mathcal{E} = \{\varepsilon_i\}$ be a set of $m$ signs; i.e. $\varepsilon_i = 1$ or $\varepsilon_i = -1$ for $1 \leqq i \leqq m$. Define the orthogonal transformation $s_{\sigma, \varepsilon}$ of $\mathfrak{h}_R^*$ by

$$s_{\sigma, \varepsilon}(e_i) = \varepsilon_i e_{\sigma(i)} \qquad (1 \leqq i \leqq m) \ .$$

The Weyl groups $W$ and $W_G$ are given as follows;

$$W = \{s_{\sigma, \varepsilon} \mid \sigma \in \mathfrak{S}_m, \ \varepsilon_i = 1 \text{ or } -1 \ (1 \leqq i \leqq m)\}$$

$$W_G = \{s_{\sigma, \varepsilon} \mid \sigma \in \mathfrak{S}_m, \ \varepsilon_i = 1 \text{ or } -1 \ (1 \leqq i \leqq m), \ \prod_{i=1}^{m} \varepsilon_i = 1\}$$

Define $s_0 \in W$ by $s_0(e_i) = e_i$ for $1 \leqq i \leqq m-1$ and $s_0(e_m) = -e_m$. Then we have $W^1 = \{1, s_0\}$, where $1$ denotes the identity element of $W$. Hence the mapping $D; \mathfrak{F}_0^+ \cup s_0\mathfrak{F}_0^+ \to \hat{G}_d$ is bijective.

Take a maximal abelian subalgebra $\mathfrak{a}_0$ of $\mathfrak{p}_0$ defined by

$$\mathfrak{a}_0 = \{P(a, 0, \cdots, 0) \mid a \in \mathbf{R}\} \ .$$

Then $M$ is isomorphic to $Spin(2m-1)$ and we have

$$\mathfrak{m}_0 = \left\{ K(X) \ \middle| \ X = \left( \begin{array}{c|c} 0 & 0 \\ \hline 0 & Y \end{array} \right) Y \in \mathfrak{so}(2m-1, \mathbf{R}) \right\} \ .$$

Also take a maximal abelian subalgebra $\mathfrak{t}_0$ of $\mathfrak{m}_0$ defined by

$$\mathfrak{t}_0 = \{H(0, h_2, \cdots, h_m) \mid h_i \in \mathbf{R} \qquad (2 \leqq i \leqq m)\} \ .$$

Two Cartan subalgebras $\mathfrak{h}'=\mathfrak{t}\oplus\mathfrak{a}$ and $\mathfrak{h}$ can be identified by the following isomorphism $\theta$;

$$\theta(P(a,0,\cdots,0)+H(0,h_2,\cdots,h_m)) = H(a,h_2,\cdots,h_m).$$

Moreover, by this identification, the above ordering in $\mathfrak{h}_R^*$ corresponds to a compatible ordering of $(\mathfrak{h}_R')^*$. In this way, we always identify $(\mathfrak{h}_R')^*$ with $\mathfrak{h}_R^*$, together with their orderings.

As for $(\mathfrak{m},\mathfrak{t})$, $\mathfrak{t}_R^*$ is spanned by $\{e_2,\cdots,e_m\}$ and, in $\mathfrak{t}_R^*$, the lexicographic ordering with respect to this basis is given. Hence $\Delta_m^+$, $\rho_m$ and $\mathfrak{F}_m^+$ are given as follows;

$$\Delta_m^+ = \{\ e_i\ \ (2\leq i\leq m),\ \ \ \ e_i+e_j\ \ (2\leq i<j\leq m)\}$$

$$\rho_m = \sum_{i=2}^m \frac{2m-2i+1}{2}e_i$$

$$\mathfrak{F}_m^+ = \left\{\mu = \sum_{i=2}^m \mu_i e_i \ \middle|\ \begin{array}{l} \mu_i-\mu_{i+1}\in\mathbf{Z} \text{ and } \geq 0 \ \ (2\leq i\leq m-1) \\ 2\mu_m\in\mathbf{Z} \text{ and } \geq 0 \end{array}\right\}$$

Since $M$ is simply connected and semisimple, the mapping $\mu$; $\hat{M}\in\xi\to\mu(\xi)\in\mathfrak{F}_m^+$ is bijective. Also we have $\mathfrak{a}^*=\mathbf{C}e_1\subset\mathfrak{h}^*$. We shall identify $\nu e_1\in\mathfrak{a}^*$ with the complex number $\nu\in\mathbf{C}$.

7.2. Now we shall set about **Step 1.** Consider and fix a special linear form $\Lambda=\sum_{i=1}^m \Lambda_i e_i\in\mathfrak{F}_0^+$. Then the numbers $\{\Lambda_1,\cdots,\Lambda_m\}$ are either all integers or all half odd integers and satisfy the inequalities

$$(7.1) \qquad\qquad \Lambda_1>\Lambda_2>\cdots>\Lambda_m>0.$$

Depending on Proposition 5.2.2, we shall first determine the set $R(\Lambda)\subset \hat{M}\times\mathfrak{a}^*$. For $\lambda\in\mathfrak{h}^*$, denote by $\lambda|_{\mathfrak{t}}$ and $\lambda|_{\mathfrak{a}}$ the restrictions of $\lambda$ to $\mathfrak{t}$ and to $\mathfrak{a}$, respectively. If $(\xi,\nu)\in R(\Lambda)$, we have $\mu(\xi)=s(\Lambda)|_{\mathfrak{t}}-\rho_m$ and $\nu=s(\Lambda)|_{\mathfrak{a}}$ for some $s\in W$. Hence it is necessary to find out all $s\in W$ such that $s(\Lambda)|_{\mathfrak{t}}-\rho_m$ belongs to $\mathfrak{F}_m^+$. Let $j$ be an integer such that $1\leq j\leq m$. Define $\sigma_j\in\mathfrak{S}_m$ and $\varepsilon_j^{\pm}$ by

$$\sigma_j = (1,2)(2,3)\cdots(j-1,j)$$

$$\varepsilon_j^{\pm} = \{1,\cdots,\underset{\underset{j}{\wedge}}{\pm 1},\cdots,1\}$$

and put $s_j^{\pm}=s_{\sigma_j,\varepsilon_j^{\pm}}\in W$.

**Lemma 7.2.1.** *Let $s\in W$. Then we have $s(\Lambda)|_{\mathfrak{t}}-\rho_m\in\mathfrak{F}_m^+$ if and only if $s$ is either $s_j^+$ or $s_j^-$ for some $1\leq j\leq m$.*

Proof. Let $\sigma\in\mathfrak{S}_m$ and $\varepsilon=\{\varepsilon_i\}$. We have

$$s_{\sigma,\varepsilon}(\Lambda)|_{\mathfrak{t}}-\rho_m = \sum_{i=2}^m\left(\varepsilon_{\sigma^{-1}(i)}\Lambda_{\sigma^{-1}(i)} - \frac{2m-2i+1}{2}\right)e_i.$$

Hence $s_{\sigma,\varepsilon}(\Lambda)|_t - \rho_m \in \mathfrak{F}_m^+$ if and only if

$$(7.2) \quad \begin{cases} \varepsilon_{\sigma^{-1}(i)}\Lambda_{\sigma^{-1}(i)} \geqq \varepsilon_{\sigma^{-1}(i+1)}\Lambda_{\sigma^{-1}(i+1)} + 1 & (2 \leqq i \leqq m-1) \\[2mm] \varepsilon_{\sigma^{-1}(m)}\Lambda_{\sigma^{-1}(m)} \geqq \dfrac{1}{2}. \end{cases}$$

Clearly, $\sigma_j$ and $\varepsilon_j^{\pm}$ satisfy (7.2). Conversely assume that $s_{\sigma,\varepsilon}(\Lambda)|_t - \rho_m \in \mathfrak{F}_m^+$. Then the left hand sides of (7.2) are positive. Since $\Lambda_{\sigma^{-1}(i)}$ is positive for all $i$, $\varepsilon_{\sigma^{-1}(i)}$ must be 1 for $2 \leqq i \leqq m$. Moreover, from (7.2), we have $\Lambda_{\sigma^{-1}(i)} > \Lambda_{\sigma^{-1}(i+1)}$ for $2 \leqq i \leqq m-1$. Hence, from (7.1), we obtain $\sigma^{-1}(2) < \sigma^{-1}(3) < \cdots < \sigma^{-1}(m)$. If we put $\sigma^{-1}(1) = j$, we have necessarily $\sigma = (1, 2)(2, 3) \cdots (j-1, j)$. The lemma is proved.

Denote by $\xi_j$ the element of $\hat{M}$ with highest weight $s_j^{\pm}(\Lambda)|_t - \rho_m$; i.e. the highest weight $\mu(\xi_j)$ of $\xi_j$ is given by

$$\begin{cases} \mu(\xi_1) = \displaystyle\sum_{i=2}^m \left(\Lambda_i - \frac{2m-2i+1}{2}\right)e_i \\[3mm] \mu(\xi_j) = \displaystyle\sum_{i=2}^j \left(\Lambda_{i-1} - \frac{2m-2i+1}{2}\right)e_i + \sum_{i=j+1}^m \left(\Lambda_i - \frac{2m-2i+1}{2}\right)e_i & (2 \leqq j \leqq m-1) \\[3mm] \mu(\xi_m) = \displaystyle\sum_{i=2}^m \left(\Lambda_{i-1} - \frac{2m-2i+1}{2}\right)e_i \end{cases}$$

Also note that $s_j^{\pm}(\Lambda)|_a = \pm\Lambda_j$. After all, we obtain that

$$R(\Lambda) = \{ (\xi_j, \Lambda_j), (\xi_j, -\Lambda_j) \mid 1 \leqq j \leqq m \}.$$

Now consider the representations $\pi_{\xi_j, \Lambda_j}$ and $\pi_{\xi_j, -\Lambda_j}$ ($1 \leqq j \leqq m$). From Proposition 5.2.2, an irreducible unitary representation $\pi$ such that $\chi_\pi = \chi_\Lambda$ is infinitesimally equivalent to an irreducible subquotient of $\pi_{\xi_j, \Lambda_j}$ or $\pi_{\xi_j, -\Lambda_j}$ for some $1 \leqq j \leqq m$.

For convenience' sake, we put $\Lambda - \rho = \sum_{i=1}^m \bar{\Lambda}_i e_i$; i.e.

$$\bar{\Lambda}_i = \Lambda_i - \frac{2m-2i+1}{2} \quad (1 \leqq i \leqq m).$$

Hence we have

$$(7.3) \qquad \qquad \bar{\Lambda}_1 \geqq \bar{\Lambda}_2 \geqq \cdots \geqq \bar{\Lambda}_m \geqq 0,$$

and we can write

$$(7.4) \quad \begin{cases} \mu(\xi_1) = \displaystyle\sum_{i=2}^m \bar{\Lambda}_i e_i \\[3mm] \mu(\xi_j) = \displaystyle\sum_{i=2}^j (\bar{\Lambda}_{i-1}+1)e_i + \sum_{i=j+1}^m \bar{\Lambda}_i e_i & (2 \leqq j \leqq m-1) \\[3mm] \mu(\xi_m) = \displaystyle\sum_{i=2}^m (\bar{\Lambda}_{i-1}+1)e_i \end{cases}$$

7.3.   Next we shall find out all irreducible subquotients of $\pi_{\xi_j, \pm \Lambda_j}$ $(1 \le j \le m)$, up to infinitesimally equivalence.   In [13], Thieleker determined all irreducible subquotients of every representation in the nonunitary principal series.   Let us rely on his results.

Let $\xi$ be in $\hat{M}$ and $\tau$ in $\hat{K}$.   The highest weight $\mu(\xi)$ of $\xi$ and the highest weight $\gamma(\tau)$ of $\tau$ can be written as follows;

$$\mu(\xi) = \sum_{i=2}^{m} \mu(\xi)_i e_i$$

$$\gamma(\tau) = \sum_{i=1}^{m} \gamma(\tau)_i e_i \,.$$

By Lemma 4 in §5 of [12], $\tau$ is $\xi$-admissible if and only if $\gamma(\tau)$ satisfies the following conditions;

(7.5)
$$\begin{cases} \mu(\xi)_i - \gamma(\tau)_i \in \mathbf{Z} \qquad (2 \le i \le m) \\ \gamma(\tau)_1 \ge \mu(\xi)_2 \ge \gamma(\tau)_2 \ge \mu(\xi)_3 \ge \cdots \ge \mu(\xi)_m \ge |\gamma(\tau)_m| \,. \end{cases}$$

Moreover, for every $\xi$-admissible $\tau$, we have $m(\tau; \xi) = 1$.   Hence, by the statement in 4.4, we have

$$\begin{aligned} \Phi(\pi_{\xi, \nu}) &= \hat{K}(\xi) \\ &= \{\tau \in \hat{K} \,|\, \gamma(\tau) \text{ satisfies } (7.5)\} \,. \end{aligned}$$

Also, note that every $K$-invariant subspace of $E_\xi$, and hence every $G$-invariant subspace of $E_\xi$, is of the form $E_\xi(\Phi)$ for a certain subset $\Phi$ of $\hat{K}(\xi)$.

Following Thieleker, we shall define some $K$-invariant subspaces of $E_{\xi_j}$ $(1 \le j \le m)$.   In the case that $1 \le j \le m-1$, we have

(7.6)   $$\hat{K}(\xi_j) = \left\{ \tau_\gamma \in \hat{K} \,\middle|\, \begin{array}{l} \gamma_i - \bar{\Lambda}_i \in \mathbf{Z} \qquad (1 \le i \le m) \\ \gamma_1 \ge \bar{\Lambda}_1 + 1 \ge \gamma_2 \ge \cdots \ge \gamma_{j-1} \ge \bar{\Lambda}_{j-1} + 1 \\ \qquad \ge \gamma_j \ge \bar{\Lambda}_{j+1} \ge \cdots \ge \bar{\Lambda}_m \ge |\gamma_m| \end{array} \right\},$$

where $\gamma = \sum_{i=1}^{m} \gamma_i e_i \in \mathfrak{F} \cap \mathfrak{F}_k^+$.   Let $\Phi_j^+$ and $\Phi_j^-$ be subsets of $\hat{K}(\xi_j)$ defined as follows;

(7.7)                    $$\Phi_j^+ = \{\tau_\gamma \in \hat{K}(\xi_j) \,|\, \gamma_j \ge \bar{\Lambda}_j + 1\}$$

(7.8)                    $$\Phi_j^- = \{\tau_\gamma \in \hat{K}(\xi_j) \,|\, \gamma_j < \bar{\Lambda}_j + 1\} \,,$$

and put $E_j^+ = E_{\xi_j}(\Phi_j^+)$ and $E_j^- = E_{\xi_j}(\Phi_j^-)$.   In the case that $j = m$, we have

$$\hat{K}(\xi_m) = \left\{ \tau_\gamma \in \hat{K} \,\middle|\, \begin{array}{l} \gamma_i - \bar{\Lambda}_i \in \mathbf{Z} \qquad (1 \le i \le m) \\ \gamma_1 \ge \bar{\Lambda}_1 + 1 \ge \gamma_2 \ge \cdots \ge \gamma_{m-1} \ge \bar{\Lambda}_{m-1} + 1 \ge |\gamma_m| \end{array} \right\} \,.$$

Let $\Phi_m^+$, $\Phi_m^-$ and $\Phi_m^F$ be subsets of $\hat{K}(\xi_m)$ defined as follows;

$$\Phi_m^+ = \{\tau_\gamma \in \hat{K}(\xi_m) \mid \gamma_m \geq \bar{\Lambda}_m + 1\}$$

$$\Phi_m^- = \{\tau_\gamma \in \hat{K}(\xi_m) \mid \gamma_m \leq -(\bar{\Lambda}_m + 1)\}$$

$$\Phi_m^F = \{\tau_\gamma \in \hat{K}(\xi_m) \mid -(\bar{\Lambda}_m + 1) < \gamma_m < \bar{\Lambda}_m + 1\} \, ,$$

and put $E_m^+ = E_{\xi_m}(\Phi_m^+)$, $E_m^- = E_{\xi_m}(\Phi_m^-)$ and $E_m^F = E_{\xi_m}(\Phi_m^F)$.

REMARK 7.3.1. In [14], our representation $(\pi_{\xi,\nu}, E_\xi)$ is denoted by $(\pi_\nu, L_\xi^2(K))$. Hence $E_j^+$ $(1 \leq j \leq m)$ corresponds to $D_j^+(\Lambda_j, \xi_j)$ in [14] and $E_m^F$ corresponds ot $D_m^F(\Lambda_m, \xi_m)$.

Under these notations, apply Theorem 2 and Theorem 3 in [14] to our representations $\pi_{\xi_j, \pm \Lambda_j}(1 \leq j \leq m)$. For $1 \leq j \leq m-1$, $E_j^+$ (resp. $E_j^-$) is an irreducible $G$-invariant subspace of $E_{\xi_j}$ under the action $\pi_{\xi_j, \Lambda_j}$ (resp. $\pi_{\xi_j, -\Lambda_j}$). Also $E_m^+$ and $E_m^-$ are irreducible $G$-invariant subspaces of $E_{\xi_m}$ under the action $\pi_{\xi_m, \Lambda_m}$ and $E_m^F$ is an irreducible $G$-invariant subspace of $E_{\xi_m}$ under the action $\pi_{\xi_m, -\Lambda_m}$. There are no more irreducible $G$-invariant subspaces of $E_{\xi_j}$ $(1 \leq j \leq m)$ than the above.

Among these subrepresentations and quotient representations by them, there are following infinitesimal equivalences;

$$(\pi_{\xi_j, \Lambda_j}, E_j^+) \sim (\pi_{\xi_{j+1}, -\Lambda_{j+1}}, E_{j+1}^-) \qquad (1 \leq j \leq m-2)$$

$$(\pi_{\xi_{m-1}, \Lambda_{m-1}}, E_{m-1}^+) \sim (\pi_{\xi_m, -\Lambda_m}, E_m^F)$$

$$(\pi_{\xi_j, \Lambda_j}, E_{\xi_j}/E_j^+) \sim (\pi_{\xi_j, -\Lambda_j}, E_j^-) \qquad (1 \leq j \leq m-1)$$

$$(\pi_{\xi_j, -\Lambda_j}, E_{\xi_j}/E_j^-) \sim (\pi_{\xi_j, \Lambda_j}, E_j^+) \qquad (1 \leq j \leq m-1)$$

$$(\pi_{\xi_m, \Lambda_m}, E_{\xi_m}/E_m^+ + E_m^-) \sim (\pi_{\xi_m, -\Lambda_m}, E_m^F)$$

$$(\pi_{\xi_m, -\Lambda_m}, E_{\xi_m}/E_m^F) \sim (\pi_{\xi_m, \Lambda_m}, E_m^+ \oplus E_m^-)$$

where the symbol $\sim$ denotes the infinitesimal equivalence. Moreover, there is no more infinitesimal equivalences than the above. For the simplification, denote by $\pi_j^\mp(\Lambda)$ $(1 \leq j \leq m-1)$ the representation $(\pi_{\xi_j, \pm \Lambda_j}, E_j^\mp)$ and denote by $\pi_m^\pm(\Lambda)$ the representation $(\pi_{\xi_m, \Lambda_m}, E_m^\pm)$. After all, we have the following proposition.

**Proposition 7.3.2.** *Up to infinitesimal equivalence, all irreducible subquotients of the nonunitary principal series whose infinitesimal characters are equal to $\chi_\Lambda$ are given by*

$$\{\pi_1^-(\Lambda), \pi_1^+(\Lambda), \pi_2^+(\Lambda), \cdots, \pi_{m-1}^+(\Lambda), \pi_m^+(\Lambda), \pi_m^-(\Lambda)\} \, .$$

*Moreover these representations are not infinitesimally equivalent to one another.*

7.4. We shall examine which subquotients in Proposition 7.3.2 are infinitesimally unitary. Using Theorem 4 in [14], we have the following proposition.

**Proposition 7.4.1.**

(1). *The representation $\pi_{\bar{1}}(\Lambda)$ is infinitesimally unitary if and only if $\Lambda = \rho$. Moreover, $\pi_{\bar{1}}(\rho)$ is the trivial representation $1_G$ of $G$.*

(2). *For $1 \leq j \leq m-1$, $\pi_j^+(\Lambda)$ is infinitesimally unitary if and only if $\bar{\Lambda}_{j+1} = \bar{\Lambda}_{j+2} = \cdots = \bar{\Lambda}_m = 0$.*

(3). *The representation $\pi_m^\pm(\Lambda)$ is always infinitesimally unitary.*

Proof. By (3) of Theorem 4 in [14], $\pi_{\bar{1}}(\Lambda)$ is infinitesimally unitary if and only if $\bar{\Lambda}_2 = 0$ and $\Lambda_1 = \dfrac{2m-1}{2}$. If $\Lambda_1 = \dfrac{2m-1}{2}$, we have $\bar{\Lambda}_1 = 0$ and hence, by (7.3), $\bar{\Lambda}_1 = \bar{\Lambda}_2 = \cdots = \bar{\Lambda}_m = 0$. This implies $\Lambda = \rho$. Also, by the definition of $E_{\bar{1}}^-$, $\pi_{\bar{1}}(\rho)$ is clearly the one-dimensional trivial representation of $G$. Hence the assertion (1) follows.

The assertion (2) follows immediately by (3) of Theorem 4 in [1].

In the case of $j = m$, $\pi_m^\pm(\Lambda)$ is infinitesimally unitary if and only if $\bar{\Lambda}_{m-1} + 1 > 0$. This inequality is always satisfied. Thus the assertion (3) is proved. The proof is completed.

Now, if $\pi$ is an irreducible infinitesimally unitary representation of $G$, $\pi$ is infinitesimally equivalent to a unique irreducible unitary representation of $G$, up to unitarily equivalence. Hence $\pi$ determines a class in $\hat{G}$. When $\Lambda$ satisfies the condition that $\bar{\Lambda}_{j+1} = \cdots = \bar{\Lambda}_m = 0$, denote by $U_j(\Lambda) \in \hat{G}$ the class which is determined by $\pi_j^+(\Lambda)$. Also denote by $U_m^\pm(\Lambda)$ the class which is determined by $\pi_m^\pm(\Lambda)$.

By Theorem 5 in [14], $U_m^\pm(\Lambda)$ is a discrete class. Note that, by Theorem 6 in [14], $U_j(\Lambda)$ $(1 \leq j \leq m-1)$ is not a discrete class. In effect, we have the following corollary and complete **Step 1** for $\Lambda \in \mathfrak{F}_0^+$.

**Corollary 7.4.2.** *Let $\Lambda \in \mathfrak{F}_0^+$. The subset $\hat{G}_\Lambda - \hat{G}_d$ of $\hat{G}$ is given as follows;*

(1). *If $\bar{\Lambda}_m \neq 0$, we have $\hat{G}_\Lambda - \hat{G}_d = \phi$.*

(2). *If $\bar{\Lambda}_{j+1} = \cdots = \bar{\Lambda}_m = 0$ and $\bar{\Lambda}_j \neq 0$ for some $1 \leq j \leq m-1$, we have $\hat{G}_\Lambda - \hat{G}_d = \{U_j(\Lambda), U_{j+1}(\Lambda), \cdots, U_{m-1}(\Lambda)\}$.*

(3). *If $\bar{\Lambda}_1 = \bar{\Lambda}_2 = \cdots = \bar{\Lambda}_m = 0$ i.e. $\Lambda = \rho$, we have $\hat{G}_\Lambda - \hat{G}_d = \{1_G, U_1(\Lambda), U_2(\Lambda), \cdots, U_{m-1}(\Lambda)\}$, where $1_G$ is the class of the trivial representation of $G$.*

REMARK 7.4.3. In fact, we have $U_m^+(\Lambda) = D_\Lambda \in \hat{G}_d$ and $U_m^-(\Lambda) = D_{s_0\Lambda} \in \hat{G}_d$. This is shown by comparing $K$-spectra of these representations.

7.5. We shall go forward **Step 2**. As before, let $\Lambda$ be in $\mathfrak{F}_0^+$. Our purpose is to compute $\dim \text{Hom}_K(\tau_{\Lambda - \rho_k} \otimes L^\pm, U_j(\Lambda))$ $(1 \leq j \leq m-1)$ and $\dim \text{Hom}_K(\tau_{\rho_n} \otimes L^\pm, 1_G)$.

Let $j$ be an integer such that $1 \leq j \leq m-1$. When $\Lambda$ satisfies that $\bar{\Lambda}_{j+1} = \bar{\Lambda}_{j+2} = \cdots = \bar{\Lambda}_m = 0$, a representation in the class $U_j(\Lambda)$ has the same $K$-spectrum as $\pi_j^+(\Lambda)$. Hence we shall examine $\dim \text{Hom}_K(\tau_{\Lambda - \rho_k} \otimes L^\pm, \pi_j^+(\Lambda)|_K)$ and

$\dim \operatorname{Hom}_K(\tau_{\rho_n} \otimes L^{\pm}, 1_G|_K)$. Recall that the $K$-spectrum of $\pi_j^+(\Lambda)$ is given by

$$(7.9) \quad \Phi(\pi_j^+(\Lambda)) = \Phi_j^+$$

$$= \left\{ \tau_\gamma \in \hat{K} \,\middle|\, \begin{array}{l} \gamma_i - \bar{\Lambda}_i \in \mathbf{Z} \qquad (1 \leqq i \leqq m) \\ \gamma_1 \geqq \bar{\Lambda}_1 + 1 \geqq \gamma_2 \geqq \bar{\Lambda}_2 + 1 \geqq \cdots \geqq \gamma_j \geqq \bar{\Lambda}_j + 1 \\ \bar{\Lambda}_{j+1} \geqq \gamma_{j+1} \geqq \bar{\Lambda}_{j+2} \geqq \cdots \geqq \gamma_{m-1} \geqq \bar{\Lambda}_m \geqq |\gamma_m| \end{array} \right\}.$$

Also the $K$-spectrum of $1_G$ is given by $\Phi(1_G) = \{1_K\}$, where $1_K$ is the trivial representation of $K$. Thus it is sufficient to compute the multiplicities of $\tau \in \Phi(\pi_j^+(\Lambda))$ in $\tau_{\Lambda - \rho_k} \otimes L^{\pm}$ and the multiplicity of $1_K$ in $\tau_{\rho_n} \otimes L^{\pm}$.

In order to use Proposition 6.1.2, we shall prepare the following lemma. Put $L = L^+ \oplus L^-$.

**Lemma 7.5.1.** *Let $\beta$ and $\gamma$ be in $\mathfrak{F}_k^+$.*

(1). *Let $s \in W_G$. If $\beta + \rho_k - s(\gamma + \rho_k)$ is a weight of $L$, $s$ is the identity.*

(2). *The multiplicity of $\tau_\beta$ in $\tau_\gamma \otimes L$ is equal to $M_L(\beta - \gamma)$, where $M_L(\lambda)$ $(\lambda \in \mathfrak{h}_R^*)$ is the multiplicity of the weight $\lambda$ in $L$.*

Proof. Note that, by Lemma 6.2.1, the set of weights of $L^+$ (resp. $L^-$) is given by

$$(7.10) \quad \left\{ \frac{1}{2} \sum_{i=1}^m \varepsilon_i e_i \,\middle|\, \begin{array}{l} \varepsilon_i = 1 \text{ or } -1 \qquad (1 \leqq i \leqq m) \\ \prod_{i=1}^m \varepsilon_i = +1 \text{ (resp. } -1) \end{array} \right\},$$

and the multiplicity of each weight is equal to 1. If $\beta + \rho_k - s(\gamma + \rho_k)$ is a weight of $L$, we have

$$\beta + \rho_k - s(\gamma + \rho_k) = \frac{1}{2} \sum_{i=1}^m \varepsilon_i e_i$$

where $\varepsilon_i = 1$ or $-1$ for $1 \leqq i \leqq m$. Assume that $s$ is not the identity. Then there is an element $\alpha$ of $\Delta_k^+$ such that $s^{-1}(\alpha) \in -\Delta_k^+$. Since $\beta + \rho_k$ and $\gamma + \rho_k$ are dominant regular integral forms with respect to $\Delta_k^+$, we have

$$\frac{2\langle \beta + \rho_k - s(\gamma + \rho_k), \alpha \rangle}{\langle \alpha, \alpha \rangle} = \frac{2\langle \beta + \rho_k, \alpha \rangle}{\langle \alpha, \alpha \rangle} + \frac{2\langle \gamma + \rho_k, -s^{-1}(\alpha) \rangle}{\langle \alpha, \alpha \rangle} \geqq 2.$$

On the other hand, if we set $\alpha = e_i \pm e_j$ for some $1 \leqq i < j \leqq m$, we have

$$\frac{2\left\langle \frac{1}{2} \sum_{i=1}^m \varepsilon_i e_i, \alpha \right\rangle}{\langle \alpha, \alpha \rangle} = \frac{\varepsilon_i \pm \varepsilon_j}{2} \leqq 1.$$

This is the contradiction. Thus the assertion (1) is proved. The assertion (2) follows immediately from Proposition 6.1.2 and the assertion (1). The lemma is proved.

The following proposition completes **Step 2** for $\Lambda \in \mathfrak{F}_0^+$.

**Proposition 7.5.2.**

(1). *Let $j$ be an integer such that $1 \leq j \leq m-1$. Then we have*

$$\dim \operatorname{Hom}_K(\tau_{\Lambda-\rho_k} \otimes L^+, \pi_j^+(\Lambda)|_K) = \begin{cases} 1 & \text{if } m-j \text{ is even}, \\ 0 & \text{if } m-j \text{ is odd}, \end{cases}$$

$$\dim \operatorname{Hom}_K(\tau_{\Lambda-\rho_k} \otimes L^-, \pi_j^+(\Lambda)|_K) = \begin{cases} 0 & \text{if } m-j \text{ is even}, \\ 1 & \text{if } m-j \text{ is odd}. \end{cases}$$

(2). *We have*

$$\dim \operatorname{Hom}_K(\tau_{\rho_n} \otimes L^+, 1_G|_K) = \begin{cases} 1 & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}, \end{cases}$$

$$\dim \operatorname{Hom}_K(\tau_{\rho_n} \otimes L^-, 1_G|_K) = \begin{cases} 0 & \text{if } m \text{ is even}, \\ 1 & \text{if } m \text{ is odd}. \end{cases}$$

Proof. Let $\tau_\gamma \in \hat{K}$. Assume that $\tau_\gamma \in \Phi(\pi_j^+(\Lambda))$ and the multiplicity of $\tau_\gamma$ in $\tau_{\Lambda-\rho_k} \otimes L$ is not equal to 0. By (2) of Lemma 7.5.1, we have $M_L(\gamma - \Lambda + \rho_k)$ $\neq 0$. Hence, by (7.10), we have $\gamma - \Lambda + \rho_k = \frac{1}{2} \sum_{i=1}^m \varepsilon_i e_i$ for some $\{\varepsilon_1, \cdots, \varepsilon_m\}$ such that $\varepsilon_i = 1$ or $-1$ $(1 \leq i \leq m)$; i.e.

$$\gamma_i = \Lambda_i - m + i + \frac{\varepsilon_i}{2}$$

$$= \bar{\Lambda}_i + \frac{\varepsilon_i + 1}{2} \qquad (1 \leq i \leq m).$$

On the other hand, $\{\gamma_1, \cdots, \gamma_m\}$ satisfy the inequalities (7.9). Therefore we have necessarily

$$\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_j = 1 \quad \text{and} \quad \varepsilon_{j+1} = \varepsilon_{j+2} = \cdots = \varepsilon_m = -1,$$

and hence we have

(7.11) $$\gamma = \sum_{i=1}^j (\bar{\Lambda}_i + 1) e_i + \sum_{i=j+1}^m \bar{\Lambda}_i e_i.$$

Conversely, if $\gamma$ is given by (7.11), then we have clearly that $\tau_\gamma \in \Phi(\pi_j^+(\Lambda))$ and $\gamma - \Lambda + \rho_k = \frac{1}{2}(e_1 + \cdots + e_j - e_{j+1} - \cdots - e_m)$. Then, by (2) of Lemma 7.5.1, the multiplicity of $\tau_\gamma$ in $\tau_{\Lambda+\rho_k} \otimes L$ is equal to 1. Whether $\tau_\gamma$ occurs in $\tau_{\Lambda-\rho_k} \otimes L^+$ or $\tau_{\Lambda-\rho_k} \otimes L^-$ is determined by the sign of $(-1)^{m-j}$. Since each $\tau_\gamma \in \Phi(\pi_j^+(\Lambda))$ occurs with multiplicity 1 in $\pi_j^+(\Lambda)|_K$, the assertion (1) is proved.

Next, the highest weight of $1_K$ is 0. Since $-\rho_n$ is a weight of $L$, by (2) of

Lemma 7.5.1, the multiplicity of $1_K$ in $\tau_{\rho_n} \otimes L$ is equal to 1. Whether $1_K$ occurs in $\tau_{\rho_n} \otimes L^+$ or $\tau_{\rho_n} \otimes L^-$ depends on the sign of $(-1)^m$. The assertion (2) immediately follows. The proof is completed.

7.6. Now, we have completed **Step 1** and **Step 2**. From the basic formula in Theorem 3.1.1, we shall obtain the main theorem in the case of $G=$ $Spin(1, 2m)$ $(m \geq 2)$.

Let $\Gamma$ be a torsion free discrete subgroup of $G$ such that $\Gamma \backslash G$ is compact. Then we have the following theorem.

**Theorem 7.6.1.** *Let* $\Lambda$ *be in* $\mathfrak{F}_0^+$ *and set* $\Lambda - \rho = \sum\limits_{i=1}^{m} \bar{\Lambda}_i e_i$. *Then we have the following formulas about the multiplicities* $N_\Gamma(D_\Lambda)$ *and* $N_\Gamma(D_{s_0\Lambda})$;

(I). $N_\Gamma(D_\Lambda) = N_\Gamma(D_{s_0\Lambda})$.

(II). (1). *If* $\bar{\Lambda}_m \neq 0$,

$$N_\Gamma(D_\Lambda) = d(D_\Lambda) \operatorname{vol}(\Gamma \backslash G).$$

(2). *If* $\bar{\Lambda}_{j+1} = \cdots = \bar{\Lambda}_m = 0$ *and* $\bar{\Lambda}_j \neq 0$ *for some* $1 \leq j \leq m-1$,

$$N_\Gamma(D_\Lambda) = d(D_\Lambda) \operatorname{vol}(\Gamma \backslash G) + N_\Gamma(U_{m-1}(\Lambda)) - N_\Gamma(U_{m-2}(\Lambda)) + \cdots$$
$$\cdots + (-1)^{m-j-1} N_\Gamma(U_j(\Lambda)).$$

(3). *If* $\bar{\Lambda}_1 = \bar{\Lambda}_2 = \cdots = \bar{\Lambda}_m = 0$, *i.e.* $\Lambda = \rho$,

$$N_\Gamma(D_\rho) = d(D_\rho) \operatorname{vol}(\Gamma \backslash G) + N_\Gamma(U_{m-1}(\rho)) - N_\Gamma(U_{m-2}(\rho)) + \cdots$$
$$\cdots + (-1)^m N_\Gamma(U_1(\rho)) + (-1)^{m+1}.$$

Proof. The assertion (II) follows directly from Theorem 3.1.1, Corollary 7.4.2 and Proposition 7.5.2.

We shall prove the assertion (I). To begin with, note that the positive root system of $\Delta$ with respect to which $s_0\Lambda$ is dominant is given by $s_0\Delta^+$ and we have $s_0\Delta^+ \cap \Delta_k = \Delta_k^+$. Compare the multiplicity formula (3.1) for $D_\Lambda$ with that for $D_{s_0\Lambda}$. From the formula (1.1), we have $d(D_\Lambda) = d(D_{s_0\Lambda})$. Since $\hat{G}_\Lambda = \hat{G}_{s_0\Lambda}$, it is sufficient to prove that

$$\dim \operatorname{Hom}_K(\tau_{\Lambda - \rho_k} \otimes L^\pm, U) = \dim \operatorname{Hom}_K(\tau_{s_0\Lambda - \rho_k} \otimes L^\pm, U)$$

for $U \in \hat{G}_\Lambda - \hat{G}_d$, where $L^\pm$ in the right hand side is appointed with respect to the positive root system $s_0\Delta^+$.

Since $s_0\Delta^+ \cap \Delta_n = s_0\Delta_n^+$, the set of weights of $L^\pm$ associated with $s_0\Delta^+$ is given by

$$\{s_0\langle Q \rangle - s_0\rho_n \mid Q \subset \Delta_n^+, (-1)^{m-|Q|} = \pm 1\}.$$

Let $\gamma$ be in $\mathfrak{F}_k^+$. Then $s_0\gamma$ is also in $\mathfrak{F}_k^+$. For $s \in W_G$ and $Q \subset \Delta_n^+$, $\gamma + \rho_k - s\Lambda = \langle Q \rangle - \rho_n$ if and only if $s_0\gamma + \rho_k - (s_0 s s_0^{-1})(s_0\Lambda) = s_0\langle Q \rangle - s_0\rho_n$. Also the mapping;

$W_G \in s \to s_0 s s_0^{-1} \in W$ is a bijection of $W_G$ onto $W_G$ and we have $\varepsilon(s) = \varepsilon(s_0 s s_0^{-1})$. Hence, from Proposition 6.1.2, the multiplicity of $\tau_\gamma$ in $\tau_{\Lambda-\rho_k} \otimes L^\pm$ is equal to the multiplicity of $\tau_{s_0\gamma}$ in $\tau_{s_0\Lambda-\rho_k} \otimes L^\pm$. On the other hand, from (7.6), $\tau_\gamma$ belongs to the $K$-spectrum of $\pi_j^+(\Lambda)$ $(1 \leq j \leq m-1)$ if and only if $\tau_{s_0\gamma}$ belongs to it. Also we have clearly that $\tau_\gamma = 1_K$ if and only if $\tau_{s_0\gamma} = 1_K$. After all, we obtain the following equalities;

$$\dim \mathrm{Hom}_K(\tau_{\Lambda-\rho_k} \otimes L^\pm, \pi_j^+(\Lambda)) = \dim \mathrm{Hom}_K(\tau_{s_0\Lambda-\rho_k} \otimes L^+, \pi_j^+(\Lambda)) \quad (1 \leq j \leq m-1),$$

$$\dim \mathrm{Hom}_K(\tau_{\Lambda-\rho_k} \otimes L^\pm, 1_G) = \dim \mathrm{Hom}_K(\tau_{s_0\Lambda-\rho_k} \otimes L^\pm, 1_G).$$

Hence the assertion (I) follows. Thus the theorem is proved.

REMARK 7.7.2.

(1). In the above theorem, the condition on $\Lambda$ in (1) of (II) agree with the condition (i) of Theorem in the introduction of [6] (c.f. [6], p. 176).

(2). In the case that $m=2$, the above theorem gives the formula in Theorem 4.5 of [11] (c.f. [11], p. 306).

(3). The set of classes $\{1_G, U_1(\rho), U_2(\rho), \cdots, U_{m-1}(\rho), D_\rho, D_{s_0\rho}\}$ corresponds with $\Pi^0(G) = \{J_0, J_1, \cdots, J_{m-1}, D_1, D_2\}$ in [1], Chapter VI, §4, 4.5. To be more precise, $U_j(\rho)$ $(1 \leq j \leq m-1)$ corresponds with $J_j$ and $1_G$ with $J_0$. Also, these classes contribute to Mutsushima's formula about Betti numbers of the manifold $\Gamma \backslash G / K$ (c.f. [15], p. 174).

## 8. The case of $G = SU(1, n)$

In this section, we shall pursue the same argument as in §7 in the case of $G = SU(1, n)$ $(n \geq 2)$. It is more complicated than that in §7, but the methods are the same. We will depend largely on the classification of all irreducible unitary representations of $SU(1, n)$ obtained by Kraljevic (c.f. [7]).

8.1. Let $n$ be an integer such that $n \geq 2$. Let $G$ be the group $SU(1, n)$ and $K$ the group $U(n)$ imbedded in $SU(1, n)$ as follows;

$$K = \left\{ \begin{pmatrix} b & 0 \\ \hline 0 & X \end{pmatrix} \,\middle|\, X \in U(n), \quad b = \overline{\det X} \right\}.$$

Then the Lie algebras $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are given by

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} b & 0 \\ \hline 0 & X \end{pmatrix} \,\middle|\, X \in \mathfrak{u}(n), \quad b = -\mathrm{tr}\, X \right\},$$

$$\mathfrak{p}_0 = \left\{ P(u) = \left( \begin{array}{c|c} 0 & u \\ \hline {}^t\bar{u} & 0 \end{array} \right) \middle| u \in \mathbf{C}^n \right\}.$$

Take a Cartan subalgebra $\mathfrak{h}_0$ as follows;

$$\mathfrak{h}_0 = \left\{ H(h_0, \cdots, h_n) = \begin{pmatrix} h_0 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} \middle| h_i \in \sqrt{-1}\mathbf{R}, \ \sum_{i=0}^n h_i = 0 \right\}.$$

The semisimple part $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$ of $\mathfrak{k}$ is isomorphic to $\mathfrak{sl}(n, \mathbf{C})$. If we set

$$\mathfrak{c}_k = \left\{ c \begin{pmatrix} n & & 0 \\ & -1 & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix} \middle| c \in \mathbf{C} \right\},$$

$\mathfrak{c}_k$ is the center of $\mathfrak{k}$ and we have

$$\mathfrak{k} = \mathfrak{c}_k \oplus \mathfrak{k}_1, \quad \mathfrak{h} = \mathfrak{c}_k \oplus \mathfrak{h}_1$$

where $\mathfrak{h}_1$ is the Cartan subalgebra of $\mathfrak{k}_1$ which is given by

$$\mathfrak{h}_1 = \{ H(h_0, \cdots, h_n) \in \mathfrak{h} \mid h_0 = 0 \}.$$

Define a linear form $e_i$ $(0 \leq i \leq n)$ on $\mathfrak{h}$ by

$$e_i(H(h_0, \cdots, h_n)) = h_i.$$

Then $\mathfrak{h}_R^*$ is the hyperplane of the $(n+1)$-dimensional real vector space $V$ spanned by $\{e_0, \cdots, e_n\}$ which is given by

$$\mathfrak{h}_R^* = \{ \sum_{i=0}^n \lambda_i e_i \mid \lambda_i \in \mathbf{R}, \ \sum_{i=0}^n \lambda_i = 0 \}.$$

Also $\mathfrak{h}^*$ and $\mathfrak{h}_1^*$ are given as follows;

$$\mathfrak{h}^* = \{ \sum_{i=0}^n \lambda_i e_i \mid \lambda_i \in \mathbf{C}, \ \sum_{i=0}^n \lambda_i = 0 \}$$

$$\mathfrak{h}_1^* = \{ \sum_{i=0}^n \lambda_i e_i \in \mathfrak{h}^* \mid \lambda_0 = 0 \}.$$

The inner product in $\mathfrak{h}_R^*$ is given by

$$\langle e_i, e_j \rangle = \frac{1}{2(n+1)} \delta_{ij} \qquad (0 \leq i, j \leq n).$$

Take a lexicographic ordering in $V$ relative to the basis $\{e_0, \cdots, e_n\}$. This ordering in $V$ induces an ordering in $\mathfrak{h}_R^*$. Fix this ordering in $\mathfrak{h}_R^*$. Then the following facts are easily seen;

$$\Delta = \{e_i - e_j \,|\, 0 \leq i, j \leq n, \qquad i \neq j\}$$

$$\Delta^+ = \{e_i - e_j \,|\, 0 \leq i < j \leq n\}$$

$$\Delta_k^+ = \{e_i - e_j \,|\, 1 \leq i < j \leq n\}$$

$$\Delta_n^+ = \{e_0 - e_i \,|\, 1 \leq i \leq n\}$$

$$\rho \ = \frac{1}{2} \sum_{i=0}^{n} (n - 2i) e_i$$

$$\rho_k = \frac{1}{2} \sum_{i=1}^{n} (n - 2i + 1) e_i$$

$$\rho_n = \frac{1}{2} \left( n e_0 - \sum_{i=1}^{n} e_i \right)$$

$$\mathfrak{F} = \{\lambda = \sum_{i=0}^{n} \lambda_i e_i \in \mathfrak{h}_R^* \,|\, \lambda_i - \lambda_j \in \mathbf{Z} \qquad (0 \leq i, j \leq n)\}$$

$$\mathfrak{F}_0 = \{\lambda \in \mathfrak{F} \,|\, \lambda_i - \lambda_j \neq 0 \qquad (0 \leq i, j \leq n, \ i \neq j)\}$$

$$\mathfrak{F}_0^+ = \{\lambda \in \mathfrak{F} \,|\, \lambda_0 > \lambda_1 > \cdots > \lambda_{n-1} > \lambda_n\}$$

$$\mathfrak{F}_k^+ = \{\lambda \in \mathfrak{h}_R^* \,|\, \lambda_i - \lambda_j \in \mathbf{Z} \qquad (1 \leq i, j \leq n), \ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\} \ .$$

Note that, if $\lambda = \sum_{i=0}^{n} \lambda_i e_i \in \mathfrak{F}$, we have

$$(n+1) \lambda_i \in \mathbf{Z}; \text{ i.e. } \lambda_i \in \frac{1}{n+1} \mathbf{Z} \qquad (0 \leq i \leq n) \ .$$

Now we need the Weyl group $W$ and $W_G$. Let $\mathfrak{S}_{n+1}$ be the group of permutations of the set $\{0, 1, 2, \cdots, n\}$. For $\sigma \in \mathfrak{S}_{n+1}$, define an orthogonal transformation $s_\sigma$ of $\mathfrak{h}_R^*$ by

$$s_\sigma(e_i) = e_{\sigma(i)} \qquad (0 \leq i \leq n) \ .$$

Then $W$ and $W_G$ are given as follows;

$$W = \{s_\sigma \,|\, \sigma \in \mathfrak{S}_{n+1}\}$$

$$W_G = \{s_\sigma \,|\, \sigma \in \mathfrak{S}_{n+1}, \quad \sigma(0) = 0\} \ .$$

In the following, we shall always identify $s_\sigma$ with $\sigma$. Then we have

$$W^1 = \{1 = \sigma_0, \sigma_1, \cdots, \sigma_n\} \ ,$$

where $\sigma_l \ (0 \leq l \leq n)$ is the permutation $(0, 1, \cdots, l)$. Therefore, there is the bijection between $\hat{G}_d$ and $\bigcup_{l=0}^{n} \sigma_l \mathfrak{F}_0^+$.

Next, we take a maximal abelian subalgebra $\mathfrak{a}_0$ of $\mathfrak{p}_0$ as follows;

$$\mathfrak{a}_0 = \{P(0, \cdots, 0, a) \,|\, a \in \mathbf{R}\} \ .$$

Then $M$ and $\mathfrak{m}_0$ are given by

$$M = \left\{ \left( \begin{array}{c|c|c} b & 0 & 0 \\ \hline 0 & Y & 0 \\ \hline 0 & 0 & b \end{array} \right) \middle| Y \in U(n-1), \quad b \in C, \quad \det Y = \bar{b}^2 \right\}$$

$$\mathfrak{m}_0 = \left\{ \left( \begin{array}{c|c|c} b & 0 & 0 \\ \hline 0 & Y & 0 \\ \hline 0 & 0 & b \end{array} \right) \middle| Y \in \mathfrak{u}(n-1, C), \quad b \in \sqrt{-1}R, \quad 2b = -\operatorname{tr} Y \right\}$$

$$\cong \mathfrak{su}(n-1, C) \oplus R.$$

The semisimple part $\mathfrak{m}_1 = [\mathfrak{m}, \mathfrak{m}]$ of $\mathfrak{m}$ is isomorphic to $\mathfrak{sl}(n-1, C)$. If we put

$$\mathfrak{c}_m = \left\{ c \cdot H\left( \frac{n-1}{2}, -1, \cdots, -1, \frac{n-1}{2} \right) \middle| c \in C \right\},$$

$\mathfrak{c}_m$ is the center of $\mathfrak{m}$ and we have $\mathfrak{m} = \mathfrak{c}_m + \mathfrak{m}_1$. Take a maximal abelian abelian subalgebra $\mathfrak{t}_0$ of $\mathfrak{m}_0$ as follows;

$$\mathfrak{t}_0 = \{ H(b, h_1, \cdots, h_{n-1}, b) \mid h_i, b \in \sqrt{-1}R, \; \textstyle\sum_{i=1}^{n-1} h_i + 2b = 0 \}.$$

Then we have $\mathfrak{t} = \mathfrak{c}_m + \mathfrak{t}_1$, where $\mathfrak{t}_1$ is a Cartan subalgebra of $\mathfrak{m}_1$ defined by

$$\mathfrak{t}_1 = \{ H(0, h_1, \cdots, h_{n-1}, 0) \mid h_i \in C, \; \textstyle\sum_{i=1}^{n-1} h_i = 0 \}.$$

Two Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}' = \mathfrak{a} + \mathfrak{t}$ are identified by the following isomorphism $\theta$;

$$\theta(P(0, \cdots, 0, a) + H(b, h_1, \cdots, h_{n-1}, b)) = H(b+a, h_1, \cdots, h_{n-1}, b-a).$$

Also, by $\theta$, our ordering in $\mathfrak{h}^*$ corresponds to a compatible ordering of $(\mathfrak{h}'_R)^*$ and induces an ordering in $\mathfrak{t}_R^*$. In this way, we always identify $(\mathfrak{h}')^*$ and $\mathfrak{h}^*$. Regarding $\mathfrak{t}^*$, $\mathfrak{t}_1^*$ and $\mathfrak{a}^*$ as subsets of $\mathfrak{h}^*$, we have

$$\mathfrak{t}^* = \{ \mu = \textstyle\sum_{i=0}^{n} \mu_i e_i \in \mathfrak{h}^* \mid \mu_0 = \mu_n \}$$

$$\mathfrak{t}_1^* = \{ \mu \in \mathfrak{t}^* \mid \mu_0 = \mu_n = 0 \}$$

$$\mathfrak{a}^* = \left\{ \nu \frac{e_0 - e_n}{2} \middle| \nu \in C \right\}.$$

We shall identify $\nu \cdot \dfrac{e_0 - e_n}{2} \in \mathfrak{a}^*$ with the complex number $\nu$ and hence identify $\mathfrak{a}^*$ with $C$.

As for the Lie algebra $\mathfrak{m}$, we have

$$\Delta_m = \{ e_i - e_j \mid 1 \leq i, j \leq n-1, \quad i \neq j \}$$
$$\Delta_m^+ = \{ e_i - e_j \mid 1 \leq i < j \leq n-1 \}$$

$$\rho_m = \frac{1}{2}\sum_{1=i}^{n-1}(n-2i)e_i$$

(8.1)    $$\mathfrak{F}_m^+ = \left\{\mu = \sum_{i=0}^{n}\mu_i e_i \in \mathfrak{h}_R^* \;\middle|\; \begin{array}{l} \mu_0 = \mu_n,\;\; \mu_i - \mu_j \in \mathbf{Z} \quad (0 \leq i,j \leq n) \\ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \end{array}\right\}.$$

8.2.  Following Kraljevic [7], we shall introduce some notations.  Let $r$ be a positive integer.  We define the subset $\mathbf{R}_>^r$ of $\mathbf{R}^r$ by

$$\mathbf{R}_>^r = \left\{p = (p_1, \cdots, p_r) \in \mathbf{R}^r \;\middle|\; \begin{array}{l} p_i - p_j \in \mathbf{Z} \quad (1 \leq i,j \leq r) \\ p_1 \geq p_2 \geq \cdots \geq p_r \end{array}\right\}.$$

For $p = (p_1, \cdots, p_r) \in \mathbf{R}_>^r$ and $q = (q_1, \cdots, q_{r+1}) \in \mathbf{R}_>^{r+1}$, we write $p < q$ if $p$ and $q$ satisfy that

(8.2)    $$\begin{cases} q_i - p_i \in \mathbf{Z} \quad (1 \leq i \leq r) \\ q_1 \geq p_1 \geq q_2 \geq p_2 \geq \cdots \geq q_r \geq p_r \geq q_{r+1}. \end{cases}$$

Also, for $p, q \in \mathbf{R}_>^r$, we write $(p_1, \cdots, p_r) < (\infty, q_1, \cdots, q_r)$ (resp. $(p_1, \cdots, p_r) < (q_1, \cdots, q_r, -\infty)$) if $(p_1, \cdots, p_r) < (p_1, q_1, \cdots, q_r)$ (resp. $(p_1, \cdots, p_r) < (q_1, \cdots, q_r, p_r)$).

Let $\gamma = \sum_{i=0}^{n}\gamma_i e_i \in \mathfrak{h}_R^*$.  Then $\gamma$ belongs to $\mathfrak{F} \cap \mathfrak{F}_k^+$ if and only if $\{\gamma_1, \cdots, \gamma_n\}$ satisfy the conditions

$$\begin{cases} \gamma_i \in \dfrac{1}{n+1}\mathbf{Z} \quad (1 \leq i \leq n) \\ \gamma_i - \gamma_j \in \mathbf{Z} \quad (1 \leq i,j \leq n) \\ \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \end{cases}$$

If we put

$$\left(\frac{1}{n+1}\mathbf{Z}\right)_>^n = \left\{p \in \mathbf{R}_>^n \;\middle|\; p_i \in \frac{1}{n+1}\mathbf{Z} \quad (1 \leq i \leq n)\right\},$$

the mapping; $\mathfrak{F} \cap \mathfrak{F}_k^+ \in \gamma \to (\gamma_1, \cdots, \gamma_n) \in \left(\dfrac{1}{n+1}\mathbf{Z}\right)_>^n$ is a bijection.  Thus we shall always identify an element $\gamma$ of $\mathfrak{F} \cup \mathfrak{F}_k^+$ with an element $\gamma = (\gamma_1, \cdots, \gamma_n)$ of $\left(\dfrac{1}{n+1}\mathbf{Z}\right)_>^n$.  Similarly, $\mu = \sum_{i=0}^{n}\mu_i e_i \in \mathfrak{F} \cap \mathfrak{F}_m^+$ can be identified with an element $\mu = (\mu_1, \cdots, \mu_{n-1})$ of $\left(\dfrac{1}{n+1}\mathbf{Z}\right)_>^{n-1}$.  Hence $\hat{K}$ is isomorphic to the set $\left(\dfrac{1}{n+1}\mathbf{Z}\right)_>^n$ and $\hat{M}$ is isomorphic to the set $\left(\dfrac{1}{n+1}\mathbf{Z}\right)_>^{n-1}$ (c.f. [7], p. 35).

8.3.  Now, we shall start off with **Step 1.**  Once and for all, take a special element $\Lambda = \sum_{i=0}^{n}\Lambda_i e_i \in \mathfrak{F}_0^+$ and fix.  Then we have

$$(8.3) \quad \begin{cases} \Lambda_i \in \dfrac{1}{n+1}\mathbf{Z} \quad (0 \le i \le n) \\[2mm] \Lambda_0 > \Lambda_1 > \cdots > \Lambda_n \,. \end{cases}$$

**Lemma 8.3.1.** *Let $h$ and $k$ be integers such that $0 \le h < k \le n$. Define $\sigma_{h,k}^{\pm} \in \mathfrak{S}_{n+1}$ by*

$$\sigma_{h,k}^{+} = \begin{pmatrix} 0, & 1, & \cdots, & h-1, & h, & h+1, & \cdots, & k-1, & k, & k+1, & \cdots, & n \\ 1, & 2, & \cdots, & h, & 0, & h+1, & \cdots, & k-1, & n, & k, & \cdots, & n-1 \end{pmatrix}$$

$$\sigma_{h,k}^{-} = \begin{pmatrix} 0, & 1, & \cdots, & h-1, & h, & h+1, & \cdots, & k-1, & k, & k+1, & \cdots, & n \\ 1, & 2, & \cdots, & h, & n, & h+1, & \cdots, & k-1, & 0, & k, & \cdots, & n-1 \end{pmatrix}$$

*Then, an element $\sigma \in W$ satisfies that $\sigma\Lambda|_t - \rho_m \in \mathfrak{F}_m^{+}$ if and only if $\sigma$ is either $\sigma_{h,k}^{+}$ or $\sigma_{h,k}^{-}$ for some $0 \le h < k \le n$.*

Proof. For $\sigma \in \mathfrak{S}_{n+1}$, we have

$$\sigma\Lambda|_t - \rho_m = \sum_{i=1}^{n-1} \left( \Lambda_{\sigma^{-1}(i)} - \frac{n-2i}{2} \right) e_i + b(e_0 + e_n)$$

where $b \in \mathbf{R}$ is given by $2b + \sum_{i=1}^{n-1} \Lambda_{\sigma^{-1}(i)} = 0$. Assume that $\sigma\Lambda|_t - \rho_m$ belongs to $\mathfrak{F}_m^{+}$. From (8.1), we have

$$\Lambda_{\sigma^{-1}(i)} - \frac{n-2i}{2} \ge \Lambda_{\sigma^{-1}(i+1)} - \frac{n-2(i+1)}{2} \quad (1 \le i \le n-2),$$

and hence we have

$$\Lambda_{\sigma^{-1}(i)} - \Lambda_{\sigma^{-1}(i+1)} \ge 1 \quad (1 \le i \le n-2).$$

Since $\Lambda$ satisfies (8.3), it follows that

$$\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n-1).$$

Therefore, if we put $h =$ the minimum of $\{\sigma^{-1}(0), \sigma^{-1}(n)\}$ and $k =$ the maximum of $\{\sigma^{-1}(0), \sigma^{-1}(n)\}$, $\sigma$ is necessarily either $\sigma_{h,k}^{+}$ or $\sigma_{h,k}^{-}$. The lemma is proved.

Let $h$ and $k$ be integers such that $0 \le h < k \le n$. We denote by $\xi_{h,k}$ the element of $\hat{M}$ with highest weight $\sigma_{h,k}^{\pm}\Lambda|_t - \rho_m$ and put $\nu_{h,k}^{\pm} = \sigma_{h,k}^{\pm}\Lambda|_a$; i.e. the highest weight of $\xi_{h,k}$ is given by

$$(8.4) \quad \mu(\xi_{h,k}) = \Big( \Lambda_0 - \frac{n-2}{2},\ \Lambda_1 - \frac{n-4}{2},\ \cdots,\ \Lambda_{h-1} - \frac{n-2h}{2}, $$

$$\Lambda_{h+1} - \frac{n-2(h+1)}{2},\ \cdots,\ \Lambda_{k-1} - \frac{n-2(k-1)}{2}, $$

$$\Lambda_{k+1} - \frac{n-2k}{2},\ \cdots,\ \Lambda_n - \frac{-n+2}{2} \Big).$$

and we have

$$\nu_{h,k}^{\pm} = \pm(\Lambda_h - \Lambda_k).$$

In the above formulas, if $h=0$, (8.4) means

$$\mu(\xi_{0,k}) = \left(\Lambda_1 - \frac{n-2}{2}, \Lambda_2 - \frac{n-4}{2}, \cdots, \Lambda_{k-1} - \frac{n-2(k-1)}{2}, \right.$$
$$\left. \Lambda_{k+1} - \frac{n-2k}{2}, \cdots, \Lambda_n - \frac{-n+2}{2}\right),$$

and if $k=n$, (8.4) means

$$\mu(\xi_{h,n}) = \left(\Lambda_0 - \frac{n-2}{2}, \Lambda_1 - \frac{n-4}{2}, \cdots, \Lambda_{h-1} - \frac{n-2h}{2}, \right.$$
$$\left. \Lambda_{h+1} - \frac{n-2(h+1)}{2}, \cdots, \Lambda_{n-1} - \frac{-n+2}{2}\right).$$

Throughout this section, the case of $h=0$ or $k=n$ should be suitably appreciated, even if it is not specially offered. Moreover, for convenience' sake, define $\bar{\Lambda}_0, \cdots, \bar{\Lambda}_n$ by

$$\Lambda - \rho = \sum_{i=0}^{n} \bar{\Lambda}_i e_i.$$

Then we have

$$\bar{\Lambda}_i = \Lambda_i - \frac{n-2i}{2} \qquad (0 \leq i \leq n)$$

$$\bar{\Lambda}_0 \geq \bar{\Lambda}_1 \geq \cdots \geq \bar{\Lambda}_n,$$

and we can write

(8.5)      $\mu(\xi_{h,k}) = (\bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{h-1}+1, \bar{\Lambda}_{h+1}, \cdots, \bar{\Lambda}_{k-1}, \bar{\Lambda}_{k+1}-1, \cdots, \bar{\Lambda}_n)$

Now consider the representations $\pi_{\xi_{h,k}, \nu_{h,k}^{\pm}}$ $(0 \leq h < k \leq n)$ in the nonunitary principal series. From Proposition 5.2.2, an irreducible unitary representation $\pi$ such that $\chi_\pi = \chi_\Lambda$ is infinitesimally equivalent to an irreducible subquotient of $\pi_{\xi_{h,k}, \nu_{h,k}^{+}}$ or $\pi_{\xi_{h,k}, \nu_{h,k}^{-}}$ for some $0 \leq h < k \leq n$.

8.4.  Using results of [7], we shall determine all irreducible subquotients of $\pi_{\xi_{h,k}, \nu_{h,k}^{\pm}}$ $(0 \leq h < k \leq n)$, up to infinitesimally equivalence.

Let $h$ and $k$ be integers such that $0 \leq h < k \leq n$. By Proposition 6.1 in [7] and (8.5), the $K$-spectrum of $\pi_{\xi_{h,k} \nu_{h,k}^{\pm}}$ is given by

$$\Phi(\pi_{\xi_{h,k}, \nu_{h,k}^{\pm}}) = \hat{K}(\xi_{h,k})$$
$$= \{\tau_\gamma \in \hat{K} \mid (\gamma_1, \cdots, \gamma_n) > (\bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{h-1}+1, $$
$$\bar{\Lambda}_{h+1}, \cdots, \bar{\Lambda}_{k-1}, \bar{\Lambda}_{k+1}-1, \cdots, \bar{\Lambda}_n-1)\}$$

where we use the notation $>$ in the sence of (8.2).

We shall apply Theorem 7.5 and Theorem 8.7 in [7] to our representation $\pi_{\xi_{h,k},\nu_{h,k}^\pm}$. Following Kraljevic, put

$$s_j(\xi) = 2\mu(\xi)_j + n - 2j + \sum_{j=1}^{n-1} \mu(\xi)_i \qquad (1 \leq j \leq n-1)$$

for $\xi \in \hat{M}$, where $\mu(\xi) = (\mu(\xi)_1, \cdots, \mu(\xi)_{n-1}) \in \mathfrak{F}_m^\pm \cap \mathfrak{F}$ is the highest weight of $\xi$ (c.f. [7], p. 48). In the case of $\xi = \xi_{h,k}$, we have

$$s_j(\xi_{h,k}) = \begin{cases} 2\Lambda_{j-1} - \Lambda_h - \Lambda_k & (1 \leq j \leq h) \\ 2\Lambda_j - \Lambda_h - \Lambda_k & (h+1 \leq j \leq k-1) \\ 2\Lambda_{j+1} - \Lambda_h - \Lambda_k & (k \leq j \leq n-1) . \end{cases}$$

Since $\nu_{h,k}^\pm = \pm(\Lambda_h - \Lambda_k)$, it follows that

$$s_j(\xi_{h,k}) - \nu_{h,k}^\pm \in 2\mathbf{Z} \qquad (1 \leq j \leq n-1)$$

$$s_h(\xi_{h,k}) > \nu_{h,k}^+ > s_{h+1}(\xi_{h,k})$$

$$s_{k-1}(\xi_{h,k}) > \nu_{h,k}^- = -\nu_{h,k}^+ > s_k(\xi_{h,k}) .$$

Hence, if $h+1=k$ (resp. $h+1<k$), $\xi_{h,k} \in \hat{M}$ and $\nu_{h,k}^\pm \in \mathbf{C}$ satisfy the condition of the case (c) (resp. (d)) in Theorem 7.5 in [7]. By Theorem 7.5 and Theorem 8.7 in [7], $K$-spectra of all irreducible subquotients of $\pi_{\xi_{h,k},\nu_{h,k}^\pm}$ are given as follows;

(1). If $h+1=k$, there can occur only the following three $K$-spectra;

$$\Phi_+(\xi_{h,h+1}) = \left\{ \tau_\gamma \in \hat{K} \;\middle|\; \begin{array}{l} (\gamma_1, \cdots, \gamma_{h+1}) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_h+1) \\ (\gamma_{h+2}, \cdots, \gamma_n) < (\bar{\Lambda}_{h+2}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right\}$$

$$\Phi_0(\xi_{h,h+1}) = \left\{ \tau_\gamma \in \hat{K} \;\middle|\; \begin{array}{l} (\gamma_1, \cdots, \gamma_h) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{h-1}+1) \\ (\gamma_{h+1}) < (\bar{\Lambda}_h, \bar{\Lambda}_{h+1}) \\ (\gamma_{h+2}, \cdots, \gamma_n) < (\bar{\Lambda}_{h+2}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right\}$$

$$\Phi_-(\xi_{h,h+1}) = \left\{ \tau_\gamma \in \hat{K} \;\middle|\; \begin{array}{l} (\gamma_1, \cdots, \gamma_h) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{h-1}+1) \\ (\gamma_{h+1}, \cdots, \gamma_n) < (\bar{\Lambda}_{h+1}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right\}$$

(2). If $h+1<k$, there can occur only the following four $K$-spectra;

$$\Phi_{++}(\xi_{h,k}) = \left\{ \tau_\gamma \in \hat{K} \;\middle|\; \begin{array}{l} (\gamma_1, \cdots, \gamma_{h+1}) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_h+1) \\ (\gamma_{h+2}, \cdots, \gamma_k) < (\bar{\Lambda}_{h+1}, \cdots, \bar{\Lambda}_k) \\ (\gamma_{k+1}, \cdots, \gamma_n) < (\bar{\Lambda}_{k+1}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right\}$$

$$\Phi_{+-}(\xi_{h,k}) = \left\{ \tau_\gamma \in \hat{K} \;\middle|\; \begin{array}{l} (\gamma_1, \cdots, \gamma_{h+1}) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_h+1) \\ (\gamma_{h+2}, \cdots, \gamma_{k-1}) < (\bar{\Lambda}_{h+1}, \cdots, \bar{\Lambda}_{k-1}) \\ (\gamma_k, \cdots, \gamma_n) < (\bar{\Lambda}_k-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right\}$$

$$\Phi_{-+}(\xi_{h,k}) = \left\{ \tau_\gamma \in \hat{K} \left| \begin{array}{l} (\gamma_1, \cdots, \gamma_h) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{h-1}+1) \\ (\gamma_{h+1}, \cdots, \gamma_k) < (\bar{\Lambda}_h, \cdots, \bar{\Lambda}_k) \\ (\gamma_{k+1}, \cdots, \gamma_n) < (\bar{\Lambda}_{k+1}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right. \right\}$$

$$\Phi_{--}(\xi_{h,k}) = \left\{ \tau_\gamma \in \hat{K} \left| \begin{array}{l} (\gamma_1, \cdots, \gamma_h) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{h-1}+1) \\ (\gamma_{h+1}, \cdots, \gamma_{k-1}) < (\bar{\Lambda}_h, \cdots, \bar{\Lambda}_{k-1}) \\ (\gamma_k, \cdots, \gamma_n) < (\bar{\Lambda}_k-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right. \right\}.$$

On the other hand, by Theorem 9.2 in [7], two irreducible subquotients of representations $\{\pi_{\xi_{h,k},\nu_{h,k}^\pm} | 0 \le h < k \le n\}$ are infinitesimally equivalent to each other if and only if they have the same $K$-spectrum. By (1) and (2) above, among $K$-spectra of all irreducible subquotients of $\{\pi_{\xi_{h,k},\nu_{h,k}^\pm} | 0 \le h < k \le n\}$, there are $\dfrac{(n+1)(n+2)}{2}$ kinds of $K$-spectra which are different from one another. They are given as follows;

(8.6)    $$\Phi_l(\Lambda) = \left\{ \tau_\gamma \in \hat{K} \left| \begin{array}{l} (\gamma_1, \cdots, \gamma_l) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{l-1}+1) \\ (\gamma_{l+1}, \cdots, \gamma_n) < (\bar{\Lambda}_{l+1}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right. \right\}$$

(8.7)    $$\Phi_{p,q}(\Lambda) = \left\{ \tau_\gamma \in \hat{K} \left| \begin{array}{l} (\gamma_1, \cdots, \gamma_p) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{p-1}+1) \\ (\gamma_{p+1}, \cdots, \gamma_q) < (\bar{\Lambda}_p, \cdots, \bar{\Lambda}_q) \\ (\gamma_{q+1}, \cdots, \gamma_n) < (\bar{\Lambda}_{q+1}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right. \right\}$$

where $l$ runs over $\{0, 1, \cdots, n\}$ and $(p, q)$ runs over the set $\{(p, q) | 0 \le p < q \le n\}$.

Denote by $\pi_l(\Lambda)$ the irreducible subquotient with $K$-spectrum $\Phi_l(\Lambda)$ and denote by $\pi_{p,q}(\Lambda)$ the one with $K$-spectrum $\Phi_{p,q}(\Lambda)$. After all we have the following proposition.

**Proposition 8.4.1.** *Up to infinitesimal equivalence, all irreducible subquotients of the nonunitary principal series whose infinitesimal characters are equal to $\chi_\Lambda$ are given as follows;*

$$\{\pi_l(\Lambda) \quad (0 \le l \le n), \qquad \pi_{p,q}(\Lambda) \quad (0 \le p < q \le n)\}.$$

*Moreover these representations are not infinitesimally equivalent to one another.*

8.5.   We shall examine which subquotients in Proposition 8.4.1 are infinitesimally unitary. Using Proposition 11.4 in [7], we have the following proposition.

**Proposition 8.5.1.**
(1).   *For $0 \le l \le n$, $\pi_l(\Lambda)$ is always infinitesimally unitary.*
(2).   *For $0 \le p < q \le n$, $\pi_{p,q}(\Lambda)$ is infinitesimally unitary if and only if $\Lambda_i - \Lambda_{i+1} = 1$ for all $p \le i \le q-1$.*

Proof. By easy computations, we can prove that $\Phi_l(\Lambda)$ satisfies the inequalities (6) in (ii) of Proposition 11.4 in [7]. As for $\pi_{p,q}(\Lambda)$, the condition in (iii) of Proposition 11.4 in [7] is clearly equivalent to our condition.

Now, let $\tilde{G}$ be the universal covering group of $G$. Canonically, we can regard $\pi_l(\Lambda)$ as a representation of $\tilde{G}$. Since $\pi_l(\Lambda)$ is infinitesimally unitary, there exists an irreducible unitary representation $\pi_l(\Lambda)^U$ of $\tilde{G}$ which is infinitesimally equivalent to $\pi_l(\Lambda)$. Then $\pi_l(\Lambda)^U$ turns out to be a representation of $G$ (c.f. Remark 8.5.2). Denote by $U_l(\Lambda) \in \hat{G}$ the class which contains $\pi_l(\Lambda)^U$. Similarly, when $\pi_{p,q}(\Lambda)$ is infinitesimally unitary, denote by $U_{p,q}(\Lambda) \in \hat{G}$ the class which is determined by $\pi_{p,q}(\Lambda)$ in the same way.

REMARK 8.5.2. Let $\psi$; $\tilde{G} \to G$ be the covering epimorphism and put $\tilde{K} = \psi^{-1}(K)$. Let $(\pi^1, E^1)$ and $(\pi^2, E^2)$ be $\tilde{K}$-finite representations of $\tilde{G}$. Assume that $\pi^1$ is infinitesimally equivalent to $\pi^2$. Then $\pi^1$ is a representation of $G$ if and only if $\pi^2$ is so. In fact, $\pi^1|_{\tilde{K}}$ is equivalent to $\pi^2|_{\tilde{K}}$ as representations of $\tilde{K}$ on the spaces $E_f^1$ and $E_f^2$, respectively. Also the kernel of $\psi$ is contained in $\tilde{K}$. Hence, for an element $\tilde{k}$ of the kernel of $\psi$, $\pi^1(\tilde{k})$ is the identity operator of $E^1$ if and only if $\pi^2(\tilde{k})$ is the identity operator of $E^2$.

**Proposition 8.5.3.** For $0 \leq l \leq n$, $U_l(\Lambda)$ is a discrete class. More precisely, we have $U_l(\Lambda) = D_{\sigma_l \Lambda} \in \hat{G}_d$.

Proof. It is sufficient to prove that the $K$-spectrum $\Phi_l(\Lambda)$ of $U_l(\Lambda)$ is the same as the $K$-spectrum of $D_{\sigma_l \Lambda}$. To begin with, we shall show that the $K$-spectrum of $D_{\sigma_l \Lambda}$ contains $\Phi_l(\Lambda)$.

In order to apply Theorem 1.4.1 to $D_{\sigma_l \Lambda}$, take an ordering in $\mathfrak{h}_R^*$ with respect to which $\sigma_l \Lambda$ is dominant; i.e. take $\sigma_l(\Delta^+)$ as the positive root system of $\Delta$. Then the set of noncompact positive roots is given by

$$(8.8) \qquad \{e_i - e_0 \quad (1 \leq i \leq l), \qquad e_0 - e_j \quad (l+1 \leq j \leq n)\}$$

and we have

$$\rho_n = \frac{n-2l}{2} e_0 + \frac{1}{2}(e_1 + \cdots + e_l - e_{l+1} - \cdots - e_n).$$

Let $\lambda = \sum_{0=i}^{n} \lambda_i e_i$ be in $\mathfrak{h}_R^*$. In our case, $Q(\lambda)$ in Theorem 1.4.1 is equal to either 1 or 0 and $Q(\lambda) = 1$ if and only if

$$\begin{cases} \lambda_i \in \mathbf{Z} & (0 \leq i \leq n) \\ \lambda_i \geq 0 & \text{if } 1 \leq i \text{ and } i \leq l \\ \lambda_i \leq 0 & \text{if } l+1 \leq i \text{ and } i \leq n. \end{cases}$$

Let $\tau_\gamma$ be in $\Phi_l(\Lambda)$. From (8.6), $\gamma = \sum_{i=0}^{n} \gamma_i e_i$ satisfies the following inequalities;

$$
(8.9)
\begin{aligned}
&\Lambda_{i-2}-1 \geqq \gamma_i+\frac{n-2i}{2} \geqq \Lambda_{i-1} && \text{if } 1 \leqq i \text{ and } i \leqq l, \\
&\Lambda_i-1 \geqq \gamma_i+\frac{n-2i}{2} \geqq \Lambda_{i+1} && \text{if } l+1 \leqq i \text{ and } i \leqq n
\end{aligned}
$$

where $\Lambda_{-1}$ should be considered $\infty$ and $\Lambda_{n+1}$ should be considered $-\infty$. Let $\sigma$ be an element of $W_G$. Since $\sigma(0)=0$, we have

$$
(8.10) \quad \sigma(\gamma+\rho_k)-\sigma_l\Lambda-\rho_n = \left(\gamma_0-\Lambda_l-\frac{n-2l}{2}\right)e_0
$$
$$
+\sum_{i=1}^{l}\left(\gamma_{\sigma^{-1}(i)}+\frac{n-2\sigma^{-1}(i)}{2}-\Lambda_{i-1}\right)e_i+\sum_{i=l+1}^{n}\left(\gamma_{\sigma^{-1}(i)}+\frac{n-2\sigma^{-1}(i)}{2}-\Lambda_i+1\right)e_i.
$$

If there is an integer $i$ such that $1 \leqq i$, $i \leqq l$ and $i < \sigma^{-1}(i)$, we have $\Lambda_{i-1} \geqq \Lambda_{\sigma^{-1}(i)-2}$ and hence, from (8.9), we have

$$
\gamma_{\sigma^{-1}(i)}+\frac{n-2\sigma^{-1}(i)}{2}-\Lambda_{i-1} \leqq \Lambda_{\sigma^{-1}(i)-2}-1-\Lambda_{i-1} < 0.
$$

Also, if there is an integer $i$ such that $l+1 \leqq i$, $i \leqq n$ and $i > \sigma^{-1}(i)$, we have $\Lambda_i \leqq \Lambda_{\sigma^{-1}(i)+1}$ and hence, from (8.9), we have

$$
\gamma_{\sigma^{-1}(i)}+\frac{n-2\sigma^{-1}(i)}{2}-\Lambda_i+1 \geqq \Lambda_{\sigma^{-1}(i)+1}-\Lambda_i+1 > 0.
$$

Therefore, in these two cases, we have

$$
Q(\sigma(\gamma+\rho_k)-\sigma_l\Lambda-\rho_n) = 0.
$$

If otherwise, $\sigma$ satisfies the condition that

$$
\begin{aligned}
&\sigma^{-1}(i) \leqq i && \text{if } 1 \leqq i \text{ and } i \leqq l, \\
&\sigma^{-1}(i) \geqq i && \text{if } l+1 \leqq i \text{ and } i \leqq n.
\end{aligned}
$$

Hence $\sigma$ must be the identity. If we put $\sigma^{-1}(i)=i$ $(1 \leqq i \leqq n)$ in (8.10), we have clearly

$$
Q((\gamma+\rho_k)-\sigma_l\Lambda-\rho_n) = 1.
$$

After all, by Theorem 1.4.1, the multiplicity of $\tau_\gamma$ in a representation in the class $D_{\sigma_l\Lambda}$ is equal to 1. Therefore $\Phi_l(\Lambda)$ is contained in the $K$-spectrum of $D_{\sigma_l\Lambda}$.

Now, by Proposition 5.2.2, the $K$-spectrum of $D_{\sigma_l\Lambda}$ is one of $\{\Phi_{l'}(\Lambda)$ $(0 \leqq l' \leqq n)$, $\Phi_{p,q}(\Lambda)$ $(0 \leqq p < q \leqq n)\}$. Each $\Phi_{p,q}(\Lambda)$ does not contain $\Phi_l(\Lambda)$ and, if $l' \neq l$, $\Phi_{l'}(\Lambda)$ does not also contain $\Phi_l(\Lambda)$. Hence, $\Phi_l(\Lambda)$ is necessarily the $K$-spectrum of $D_{\sigma_l\Lambda}$. The proposition is proved.

From Proposition 8.4.1, 8.5.1 and 8.5.3, we obtain the following corollary and complete **Step 1** for $\Lambda \in \mathfrak{F}_0^+$.

**Corollary 8.5.4.** *For $\Lambda \in \mathfrak{F}_0^+$, the set $\hat{G}_\Lambda - \hat{G}_d$ is given as follows;*
(1). *If $\Lambda_i - \Lambda_{i+1} \geqq 2$ for all $0 \leqq i \leqq n-1$, we have $\hat{G}_\Lambda - \hat{G}_d = \phi$.*
(2). *If $\Lambda_i - \Lambda_{i+1} = 1$ for some $0 \leqq i \leqq n-1$, put*

$$p_\Lambda = \text{the minimum of } \{i \,|\, 0 \leqq i \leqq n-1, \ \Lambda_i - \Lambda_{i+1} = 1\}$$

$$q_\Lambda = \text{the maximum of } \{i \,|\, 1 \leqq i \leqq n, \ \Lambda_{i-1} - \Lambda_i = 1\} \,.$$

*The we have $\hat{G}_\Lambda - \hat{G}_d = \{U_{p,q}(\Lambda) \,|\, p_\Lambda \leqq p < q \leqq q_\Lambda\}$.*
(3). *In particular, if $\Lambda = \rho$, we have $\hat{G}_\Lambda - \hat{G}_d = \{U_{p,q}(\rho) \,|\, 0 \leqq p < q \leqq n\}$ where $U_{0,n}(\rho)$ is the class $1_G$.*

8.6. Let us go forward **Step 2**. Once and for all, we fix $\Lambda \in \mathfrak{F}_0^+$ and an integer $l$ such that $0 \leqq l \leqq n$. We shall study **Step 2** for $\sigma_l \Lambda \in \mathfrak{F}_0$. Our purpose is to compute

$$\dim \mathrm{Hom}_K(\tau_{\sigma_l \Lambda - \rho_k} \otimes L^\pm, \ \pi_{p,q}(\Lambda)|_K)$$

for all $0 \leqq p < q \leqq n$.

Take an ordering in $\mathfrak{h}_R^*$ as in the proof of Proposition 8.5.3 and fix. From (8.8) and Lemma 6.2.1, the set of weights of $L^+$ (resp. $L^-$) is given by

$$(8.11) \qquad \left\{ \frac{1}{2} \sum_{i=1}^n \varepsilon_i (e_0 - e_i) \,\middle|\, \begin{matrix} \varepsilon_i = 1 \text{ or } -1 \quad (1 \leqq i \leqq n) \\ \left(\prod_{i=1}^n \varepsilon_i\right) \cdot (-1)^l = +1 \text{ (resp. } -1) \end{matrix} \right\}.$$

and the multiplicity of each weight is equal to 1.

The following lemma is proved by the same arguments as in Lemma 7.5.1.

**Lemma 8.6.1.** *Let $\beta$ and $\gamma$ be in $\mathfrak{F}_k^+$.*
(1). *Let $\sigma \in W_G$. If $\beta + \rho_k - \sigma(\gamma + \rho_k)$ is a weight of $L$, $\sigma$ is the identity.*
(2). *The multiplicity of $\tau_\beta$ in $\tau_\gamma \otimes L$ is equal to $M_L(\beta - \gamma)$, where $M_L(\lambda)$ $(\lambda \in \mathfrak{h}_R^*)$ is the multiplicity of the weight $\lambda$ in $L$.*

Using this lemma, we can obtain the following proposition and complete **Step 2** for $\sigma_l \Lambda$ $(0 \leqq l \leqq n)$.

**Proposition 8.6.2.** *Let $l$, $p$ and $q$ be integers such that $0 \leqq l \leqq n$ abd $0 \leqq p < q \leqq n$.*
(1). *If $p \leqq l \leqq q$, we have*

$$\dim \mathrm{Hom}_K(\tau_{\sigma_l \Lambda - \rho_k} \otimes L^+, \ \pi_{p,q}(\Lambda)|_K) = \begin{cases} 1 & \text{if } p+q \text{ is even}, \\ 0 & \text{if } p+q \text{ is odd}, \end{cases}$$

$$\dim \operatorname{Hom}_K(\tau_{\sigma_l\Lambda-\rho_k}\otimes L^-, \pi_{p,q}(\Lambda)|_K) = \begin{cases} 0 & \textit{if } p+q \textit{ is even}, \\ 1 & \textit{if } p+q \textit{ is odd}, \end{cases}$$

(2). *If $l<p$ or $q<l$, we have*

$$\dim \operatorname{Hom}_K(\tau_{\sigma_l\Lambda-\rho_k}\otimes L^{\pm}, \pi_{p,q}(\Lambda)|_K) = 0.$$

Proof. Let $\tau_\gamma \in \hat{K}$. Assume that $\tau_\gamma \in \Phi_{p,q}(\Lambda)$ and the multiplicity of $\tau_\gamma$ in $\tau_{\sigma_l\Lambda-\rho_k}\otimes L$ is not equal to 0. By (2) of Lemma 8.6.1, we have $M_L(\gamma-\sigma_l\Lambda+\rho_k)\neq0$. Hence, by (8.11), we have

$$\gamma-\sigma_l\Lambda+\rho_k = \frac{1}{2}\sum_{i=1}^n \varepsilon_i(e_0-e_i)$$

for some $\{\varepsilon_1, \cdots, \varepsilon_n\}$ such that $\varepsilon_i=1$ or $-1$ $(1\leq i\leq n)$; i.e.

$$(8.12)\quad \begin{cases} \gamma_i = \bar{\Lambda}_{i-1}+\dfrac{1-\varepsilon_i}{2} & \text{if } 1\leq i \text{ and } i\leq l, \\[2mm] \gamma_i = \bar{\Lambda}_i+\dfrac{-1-\varepsilon_i}{2} & \text{if } l+1\leq i \text{ and } i\leq n. \end{cases}$$

On the other hand, $\{\gamma_1, \cdots, \gamma_n\}$ satisfy the condition (8.7). Hence we have

$$(8.13)\quad \begin{cases} \bar{\Lambda}_{i-2}+1\geq\gamma_i\geq\bar{\Lambda}_{i-1}+1 & \text{if } 1\leq i \text{ and } i\leq p \\ \bar{\Lambda}_{i-1}\geq\gamma_i\geq\bar{\Lambda}_i & \text{if } p+1\leq i \text{ and } i\leq q \\ \bar{\Lambda}_i-1\geq\gamma_i\geq\bar{\Lambda}_{i+1}-1 & \text{if } q+1\leq i \text{ and } i\leq n. \end{cases}$$

If $p>l$, there is an integer $i_0$ such that $l+1\leq i_0\leq p$. Then, by (8.12) and (8.13), we have

$$\bar{\Lambda}_{i_0}+\frac{-1-\varepsilon_{i_0}}{2}\geq\bar{\Lambda}_{i_0-1}+1$$

and hence we have $\bar{\Lambda}_{i_0}\leq\bar{\Lambda}_{i_0-1}+1$. This is inconsistent with the fact that $\bar{\Lambda}_{i_0}\leq\bar{\Lambda}_{i_0-1}$. Therefore we have $p\leq l$. Similarly, if $q<l$, we have a contradiction. Hence we have also $q\geq l$. Thus the assertion (2) is proved.

Now, assume that $p\leq l\leq q$. By (8.12) and (8.13), we have the following inqualities;

$$\begin{cases} \bar{\Lambda}_{i-2}+1\geq\bar{\Lambda}_{i-1}+\dfrac{1-\varepsilon_i}{2}\geq\bar{\Lambda}_{i-1}+1 & \text{if } 1\leq i \text{ and } i\leq p, \\[2mm] \bar{\Lambda}_{i-1}\geq\bar{\Lambda}_{i-1}+\dfrac{1-\varepsilon_i}{2}\geq\bar{\Lambda}_i & \text{if } p+1\leq i \text{ and } i\leq l, \\[2mm] \bar{\Lambda}_{i-1}\geq\bar{\Lambda}_i+\dfrac{-1-\varepsilon_i}{2}\geq\bar{\Lambda}_i & \text{if } l+1\leq i \text{ and } i\leq q, \\[2mm] \bar{\Lambda}_i-1\geq\bar{\Lambda}_i+\dfrac{-1-\varepsilon_i}{2}\geq\bar{\Lambda}_{i+1}-1 & \text{if } q+1\leq i \text{ and } i\leq n. \end{cases}$$

These inequalities imply that

$$(8.14) \quad \begin{cases} \varepsilon_i = -1 & \text{if } 1 \leq i \text{ and } i \leq p, \\ \varepsilon_i = 1 & \text{if } p+1 \leq i \text{ and } i \leq l, \\ \varepsilon_i = -1 & \text{if } l+1 \leq i \text{ and } i \leq q, \\ \varepsilon_i = 1 & \text{if } q+1 \leq i \text{ and } i \leq n. \end{cases}$$

After all, by (8.12), we have

$$(8.15) \quad \gamma = (\bar{\Lambda}_0+1, \bar{\Lambda}_1+1, \cdots, \bar{\Lambda}_{p-1}+1, \cdots, \bar{\Lambda}_p, \cdots, \bar{\Lambda}_{l-1}, \bar{\Lambda}_{l+1}, \cdots$$
$$\cdots, \bar{\Lambda}_q, \bar{\Lambda}_{q+1}-1, \cdots, \bar{\Lambda}_n-1)$$

$$(8.16) \quad \prod_{i=1}^{n} \varepsilon_i = (-1)^{p+q-l}.$$

Conversely, if $\gamma$ is given by (8.15), it is clear that $\tau_\gamma \in \Phi_{p,q}(\Lambda)$ and we have

$$\gamma - \sigma_l \Lambda + \rho_k = \frac{1}{2} \sum_{i=1}^{n} \varepsilon_i (e_0 - e_i)$$

where $\{\varepsilon_1, \cdots, \varepsilon_n\}$ are given by (8.14). Then, by (2) of Lemma 8.6.1, the multiplicity of $\tau_\gamma$ in $\tau_{\sigma_l \Lambda - \rho_k} \otimes L$ is equal to 1. Whether $\tau_\gamma$ occurs in $\tau_{\sigma_l \Lambda - \rho_k} \otimes L^+$ or $\tau_{\sigma_l \Lambda - \rho_k} \otimes L^-$ is determined by the sign of $\left( \prod_{i=1}^{n} \varepsilon_i \right) \cdot (-1)^l$. By (8.16) we have $\left( \prod_{i=1}^{n} \varepsilon_i \right) \cdot (-1)^l = (-1)^{p+q}$. Since $\tau_\gamma$ occurs in $\pi_{p,q}(\Lambda)$ with multiplicity 1, the assertion (2) follows. The proposition is proved.

8.7. Now, we have completed **Step 1** and **Step 2**. From the basic formula in Theorem 3.1.1, we shall obtain the main theorem in the case of $G = SU(1, n)$ $(n \geq 2)$.

Let $\Gamma$ be a torsion free discrete subgroup of $G$ such that $\Gamma \backslash G$ is compact. Then we have the following theorem.

**Theorem 8.7.1.** *Let $\Lambda$ be in $\mathfrak{F}_0^+$ and set $\Lambda = \sum_{i=0}^{n} \Lambda_i e_i$. Let $l$ be an integer such that $0 \leq l \leq n$. Then we have the following formulas for the multiplicity $N_\Gamma(D_{\sigma_l \Lambda})$.*

(1). *If $\Lambda_l - \Lambda_{l+1} \geq 2$ and $\Lambda_{l-1} - \Lambda_l \geq 2$, we have*

$$N_\Gamma(D_{\sigma_l \Lambda}) = d(D_{\sigma_l \Lambda}) \cdot \mathrm{vol}(\Gamma \backslash G).$$

(2). *If either $\Lambda_l - \Lambda_{l+1} = 1$ or $\Lambda_{l-1} - \Lambda_l = 1$, put*

$$p_\Lambda = \text{the minimum of } \{i \mid 0 \leq i \leq n-1, \Lambda_i - \Lambda_{i+1} = 1\}$$
$$q_\Lambda = \text{the maximum of } \{i \mid 1 \leq i \leq n, \Lambda_{i-1} - \Lambda_i = 1\},$$

*and define*

$$J_{l,\Lambda} = \{(p, q)\in \mathbf{Z}\times\mathbf{Z} \mid p_\Lambda \leqq p \leqq l,\ l \leqq q \leqq q_\Lambda,\ p \neq q\}\ .$$

*Then we have*

$$N_\Gamma(D_{\sigma_l\Lambda}) = d(D_{\sigma_l\Lambda})\ \mathrm{vol}\,(\Gamma\backslash G) + \sum_{(p,q)\in J_{l,\Lambda}} (-1)^{p+q-1}N_\Gamma(U_{p,q}(\Lambda))\ .$$

(3). *In particular, if* $\Lambda_i - \Lambda_{i+1} = 1$ *for all* $0 \leqq i \leqq n-1$, *that is,* $\Lambda = \rho$, *we have*

$$N_\Gamma(D_{\sigma_l\rho}) = d(D_{\sigma_l\rho})\ \mathrm{vol}\,(\Gamma\backslash G) + \sum_{\substack{0 \leqq p < q \leqq n \\ p \leqq l \leqq q}} (-1)^{p+q-1}N_\Gamma(U_{p,q}(\rho))$$

*where* $N_\Gamma(U_{0,n}(\rho)) = N_\Gamma(1_G) = 1$.

Proof.   All assertions immediately follow from Theorem 3.1,1, Corollary 8.5.4 and Proposition 8.6.2.

REMARK 8.7.2.

(1).   In the above theorem, the condition on $\Lambda$ in (1) agree with the condition (i) of Theorem in the introduction of [6] (c.f. [6], p. 176).

(2).   In the case that $n=2$, our theorem gives some formulas in [15] (c.f. [15], p. 192).

(3).   In the same way as the case of $G=Spin(1, 2m)$, the set of classes $\{U_{p,q}(\rho)\ (0 \leqq p < q \leqq n),\ D_{\sigma_l(\rho)}\ (0 \leqq l \leqq n)\}$ corresponds with $\Pi^0(G) = \{J_{ij} \mid i, j \geqq 0,\ i+j \leqq n-1\} \cup \{D_0, D_1, \cdots, D_n\}$ in [1], Chapter VI, §4, 4.8; that is, $U_{p,q}(\rho)$ corresponds with $J_{p,n-q}$ and $D_{\sigma_l\rho}$ with $D_l$. Also, these classes contribute to Matsushima's formula about Betti numbers of the manifold $\Gamma\backslash G/K$ (c.f. [15], p. 175, p. 191).

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