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TENSOR PRODUCT AND GENERALIZED OTT-SCHAEFFER PLANES

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1. Introduction

The Ott-Schaeffer planes (O - S) are translation planes of even order q^2 with kernel $K \cong GF(q)$ which admit a collineation group \mathcal{Q} isomorphic to $SL(2, q)$ where the involutions are Baer. Furthermore, if S_2 is a Sylow 2-subgroup of \mathcal{Q} then no two nontrivial elements of \mathcal{Q} fix the same Baer subplane pointwise.

The O - S planes are also derivable and a plane may be defined for each automorphism α of $GF(q)$, $q=2^{2r+1}$ which has fixed field equal to $GF(2)$. Hence, the number of such translation planes of each order is $\Phi(2r+1)$ (the number of integers $\neq 1$ relatively prime to $2r+1$).

Further, the O - S planes may be defined by the tensor product of $SL(2, q)$ by a twisted version of the same (by an automorphism $\alpha \in \text{Fix } \alpha = GF(2)$).

Note that $GL(2, q) = SL(2, q) \times \mathcal{Z}(GL(2, q))$ (center) when q is even so as the kernel is $GF(q)$, the O - S planes also admit $GL(2, q)$.

Conversely, in [7], for arbitrary kernel we have

Theorem (Johnson [7]). *Let π be a translation plane of even order $q^2 > 16$ that admits $GL(2, q)$ as a collineation group in the translation complement where the 2-groups are Baer and no two nontrivial elements fix the same Baer subplane pointwise. Then π is an Ott-Schaeffer plane.*

DEFINITION 1.1. Tensor Product Plane.

A translation plane π of order q^2 , q even or odd, kernel $K \cong GF(q)$ that admits the collineation group

$$T = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} \mid a \in K \right\}, \sigma \in \text{Aut } K$$

where $\pi = \{(x_1, x_2, y_1, y_2) \mid x_i, y_i \in K, i=1, 2\}$, $x=(x_1, x_2)$, $y=(y_1, y_2)$ and $x=0$, $y=0$, $y=x$ are components in this representation is called a *tensor product plane*.

DEFINITION 1.2. Generalized Ott-Schaeffer Plane.

A translation plane π of order q^2 , q even or odd, $q=p^r$ for p a prime, kernel

$K \cong GF(q)$, that admits a p -group \mathcal{B} of order q in the translation complement such that each element of \mathcal{B} is Baer and no two nontrivial subgroups of \mathcal{B} can fix the same Baer subplane pointwise is called a *generalized Ott-Schaeffer plane*.

In section 2, we consider the basic structure of tensor product planes and of generalized Ott-Schaeffer planes. In section 3, we consider translation planes (T - P and O - S) which admit groups of order $q(q-1)$ in the translation complement.

Our main results *completely* classify both tensor product planes of even order q^2 admitting a tensor group of order $q(q-1)$ (see (3.4)) and generalized Ott-Schaeffer planes admitting groups of order $q(q-1)$ in the translation complement with prescribed Sylow 2 subgroups (see (3.21)).

2. The fundamental structure

NOTES 2.1. For q odd $q=p^r$, $p \geq 3$, there is no tensor product plane. If $p \geq 5$ there is no generalized Ott-Schaeffer plane of order q^2 .

Proof. Consider

$$\tau_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a^\sigma & a & a^{\sigma+1} \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & a^\sigma \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$\tau_a (a \neq 0)$ fixes $\pi_a = \{(0, x_2, -x_2 a^{1-\sigma}, y_2) \mid (x_2, y_2) \mid x_2, y_2 \in K\}$ pointwise. Furthermore, $\{\tau_a \mid a \in K \cong GF(q)\} = \mathcal{S}_p$ is elementary abelian. By Foulser [4], \mathcal{S}_p must fix some Baer subplane pointwise, which cannot be the case. Hence, there are no tensor product planes of characteristic ≥ 5 .

Now assume $p=3$. The components $x=0, y=xM$ of π_a , for $M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$, must satisfy $(0, x_2, x_2 m_3, x_2 m_4) = (0, x_2, -x_2 a^{1-\sigma}, y_2)$ so that $m_3 = -a^{1-\sigma}$.

On the other hand, in order that $y=xM$ is fixed by τ_a , we must have

$$\begin{aligned} & \left[x \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, x \left[\begin{bmatrix} a & a^{\sigma+1} \\ 0 & a \end{bmatrix} + M \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \right] \right] \in y = xM \\ & \Leftrightarrow \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \\ -a^{1-\sigma} & m_4 \end{bmatrix} = \begin{bmatrix} a & a^{\sigma+1} \\ 0 & a \end{bmatrix} + \begin{bmatrix} m_1 & m_2 \\ -a^{1-\sigma} & m_4 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \\ & \Leftrightarrow m_1 - a = a + m_1 \Leftrightarrow -a = +a. \end{aligned}$$

Hence, $p \neq 3$. This argument, which is also valid for arbitrary odd order planes, was pointed out to the author by Rolando Pomareda. Now assume that π is a generalized Ott-Schaeffer plane of odd order q^2 , $q=p^r$. Each element of \mathcal{B} is Baer and for $p > 3$, Foulser [4] has shown that the Baer subplanes involved must be disjoint. That is, since $|\mathcal{B}|=q$, \mathcal{B} must fix a 1-dimensional subspace

pointwise, which cannot be the case.

If $p=3$, it is possible that there are generalized Ott-Schaeffer planes of order 3^{2r} . However, hereafter, in this section, we shall consider only *even* order planes.

Theorem 2.2. *Any tensor product plane π of even order q^2 is derivable. The derivable net is a regulus and the derived plane $\bar{\pi}$ is a tensor product plane defined by the inverse σ^{-1} of the automorphism σ used in the definition of π . Moreover, Fix $\sigma=GF(2)$ and the spread may be represented by $x=0$,*

$$y = x \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \quad a \in K$$

$$y = x \begin{bmatrix} u & m(u, a) \\ a^{1-\sigma} & u+a \end{bmatrix}, \quad m: K \times K \rightarrow K, \quad u, a \in K, a \neq 0.$$

Proof. Consider the notation of (2.1) with

$$\tau_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix}.$$

Then $y=0 \xrightarrow{\tau_a} y = x \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & a^{\sigma+1} \\ 0 & a \end{bmatrix} = \left[y = x \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \right]$. But, $y=0$, $y = x \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for $a \in K \cong GF(q)$ is the vector form of a regulus in $PG(3, K)$.

Now derive π using this "regulus" partial spread. Recall, if

$$(x_1, x_2, y_1, y_2) \xrightarrow{\tau_a} (x_1, x_1 a^\sigma + x_2, x_1 a + y_1, x_1 a^{\sigma+1} + x_2 a + y_1 a^\sigma + y_2)$$

represents τ_a in π by the standard representation of coordinates in $\bar{\pi}$ by (x_1, y_1, x_2, y_2) , we obtain:

$$(x_1, y_1, x_2, y_2) \xrightarrow{\bar{\tau}_a} (x_1, x_1 a + y_1, x_1 a^\sigma + x_2, x_1 a^{\sigma+1} + x_2 a + y_1 a^\sigma + y_2),$$

$\bar{\tau}_a$ representing τ_a in $\bar{\pi}$ (see, e.g., Jha-Johnson [6]). Hence,

$$\begin{aligned} \bar{\tau}_a &= \begin{bmatrix} 1 & a & a^\sigma & a^{\sigma+1} \\ 0 & 1 & 0 & a^\sigma \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & (a^\sigma)^{\sigma^{-1}} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b^{\sigma^{-1}} \\ 0 & 1 \end{bmatrix} \end{aligned}$$

for $b=a^\sigma$. Hence, $\left\{ \tau_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} \right\}$ in π is $\left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b^{\sigma^{-1}} \\ 0 & 1 \end{bmatrix} \right\}$ in $\bar{\pi}$.

We now consider the components $y=xM$, $M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$ of $\pi_a = \{(0, x_2, x_2 a^{1-\sigma},$

$y_2) | x_2, y_2 \in K\}$, $a \neq 0$ (Baer subplane of π). τ_a fixes

$$\begin{aligned} y = xM &\Leftrightarrow \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} M = \begin{bmatrix} a & a^{\sigma+1} \\ 0 & a \end{bmatrix} + M \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} \\ &\Leftrightarrow \begin{bmatrix} a+m_1, & a^{\sigma+1}+m_1a^\sigma+m_2 \\ m_3, & a+m_3a^\sigma+m_4 \end{bmatrix} = \begin{bmatrix} m_1+m_3a^\sigma, & m_2+a^\sigma m_4 \\ m_3, & m_4 \end{bmatrix} \\ &\Leftrightarrow m_3 = a^{1-\sigma}, \end{aligned}$$

and $a^{\sigma+1}+m_1a^\sigma=a^\sigma m_4$ or rather that $a+m_1=m_4$.

Hence, $M = \begin{bmatrix} u & m(u, a) \\ a^{1-\sigma} & u+a \end{bmatrix}$ for all u where m is a function from $K \times K$ to K .

Note that if $[a^{1-\sigma}, u+a]$ does not take on all q^2 values then τ_a and τ_b for $a \neq b$ ($a, b \neq 0$) fix the same component $y=xM \neq 0$ and hence $\langle \tau_1, \tau_2 \rangle$ must fix a 1-space pointwise on both $y=xM$ and $x=0$. Hence, $\tau_a = \tau_b$. Thus, $[a^{1-\sigma}, u] = [c^{1-\sigma}, u] \Leftrightarrow a=c \Leftrightarrow a^{1-\sigma}=c^{1-\sigma} \Leftrightarrow (ac^{-1})^\sigma = (ac^{-1}) \Leftrightarrow \text{Fix } \sigma = GF(2)$.

Now we consider the general structure of a generalized Ott-Schaeffer plane π . Let \mathcal{Q} be a collineation group of π of order q in the linear translation complement and such that each involution in π is Baer. By Johnson and Ostrom [8], \mathcal{Q} is elementary abelian. Further, assume no two involutions in \mathcal{Q} fix the same Baer subplane pointwise. Then

Lemma 2.3. *The $q-1$ Baer subplanes corresponding to the involutions of \mathcal{Q} lie across $(q-1)q+1$ components. The remaining q components are in an orbit under \mathcal{Q} .*

Proof. \mathcal{Q} fixes a component which we call $x=0$, $x=(x_1, x_2)$. If $g, h \in \mathcal{Q} - \langle 1 \rangle$ and $\text{Fix } g$ and $\text{Fix } h$ share a component $\mathcal{L} \neq (x=0)$ then $\langle g, h \rangle$ has fixed points on both \mathcal{L} and $(x=0)$. Since \mathcal{Q} is linear, it follows that $\langle g, h \rangle$ is Baer—a contradiction. Hence, $\text{Fix } g$ and $\text{Fix } h$ cannot share a component $\neq (x=0)$. Thus, this accounts for $q(q-1)+1$ components. As the Baer subplanes corresponding to the involutions in \mathcal{Q} do not intersect the remaining set Γ of q components, Γ must be a \mathcal{Q} -orbit.

Now choose $(y=0) \in \Gamma$. Then

Lemma 2.4. *We may choose a basis so that*

$$\mathcal{Q} = \left\{ \tau_a = \left(\begin{array}{cc|cc} 1 & a & f(a) & g(a) \\ 0 & 1 & 0 & f(a) \\ \hline & & 1 & a \\ & & 0 & 1 \end{array} \right) \mid a \in K \cong GF(q) \right\},$$

f, g functions $K \rightarrow K$, where f is a 1-1 additive and $g(a+b) = g(a) + g(b) + af(b) + bf(a)$ for all $a, b \in K$.

Proof. Let \mathcal{Q} fix $x=0$. Choose a basis so one of the involutions fixes $\{(0, x_2, 0, y_2) | x_2, y_2 \in K\} = \pi_0$ pointwise and π_0 shares the components $x=0, y=$

$0, y=x$. Then one of the involutions has the form $\tau = \left(\begin{array}{cc|cc} 1 & d & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{array} \right)$. Now $\rho \in \mathcal{Q}$ fixes π_0 and $x=0$, so $\rho = \left(\begin{array}{cccc} a_1 & a_2 & b_1 & b_2 \\ 0 & a_4 & 0 & b_4 \\ \hline 0 & 0 & c_1 & c_2 \\ 0 & 0 & 0 & c_4 \end{array} \right)$. Since $|\rho|=2$, we must have $a_1^2=a_4^2=c_1^2=c_4^2=1$ so $a_1=a_4=c_1=c_4=1$. Now $\rho\tau=\tau\rho$ so

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

and hence $b_1=b_4$ ($d \neq 0$). We assert that $a_2=c_2$,

$$\rho^2 = \begin{bmatrix} I \begin{bmatrix} 1 & a_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} 1 & c_2 \\ 0 & 1 \end{bmatrix} \\ 0 \qquad \qquad \qquad I \end{bmatrix}$$

so $a_2 b_1 = b_1 c_2$.

If $b_1=0$ then ρ fixes π_0 pointwise. Hence, if $\rho \neq \tau$, $a_2=c_2$. Now there exists a component $y=xT$ in the orbit of length q for $T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix}$. Change bases by $\begin{bmatrix} I & T \\ 0 & I \end{bmatrix}$. Then $x=0 \rightarrow x=0$,

$$y = 0 \leftrightarrow y = xT$$

$$y = x \rightarrow y = x(I+T).$$

(That is, after the basis change, $y=x$ may not be an equation of a line.) Then

for $\rho = \begin{pmatrix} 1 & a & b_1 & b_2 \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$ we obtain

$$\begin{bmatrix} I & T \\ 0 & I \end{bmatrix} \rho \begin{bmatrix} I & T \\ 0 & I \end{bmatrix} = \begin{pmatrix} 1 & a & (b_1+at_3) & b_2+a(t_1+t_4) \\ 0 & 1 & 0 & (b_1+at_3) \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Letting

$$f(a) = b_1 + at_3$$

$$g(a) = b_2 + a(t_1 + t_4)$$

(note $\begin{pmatrix} 1 & a & c_1 & c_2 \\ 0 & 1 & 0 & c_1 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & a & \bar{c}_1 & \bar{c}_2 \\ 0 & 1 & 0 & \bar{c}_1 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{Q}$ implies $c_1 = \bar{c}_1$, $c_2 = \bar{c}_2$ since there are no elations in \mathcal{Q}) where $f, g: K \rightarrow K$, we have the proof to (2.4) since \mathcal{Q} is elementary abelian. Note

$$\begin{aligned} y = 0 &\xrightarrow{\mathcal{Q}} y = x \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f(a) & g(a) \\ 0 & f(a) \end{bmatrix} \\ &\equiv y = x \begin{bmatrix} f(a) & g(a) + af(a) \\ 0 & f(a) \end{bmatrix}. \end{aligned}$$

If $f(a) = 0$ for $a \neq 0$, then

$y = x \begin{bmatrix} 0 & g(a) \\ 0 & 0 \end{bmatrix} = \{(x_1, x_2, 0, x_1 g(a)) \mid x_1, x_2 \in K\} \cap \{(x_1, x_2, 0, 0) \mid x_1, x_2 \in K\}$ is $\{(0, x_2, 0, 0) \mid x_2 \in K\}$ and since both equations represent components, we have a contradiction. $\therefore f$ is 1-1. And, we have:

Lemma 2.5. *The \mathcal{Q} -orbit of length q may be represented by $y = x \begin{bmatrix} f(a) \\ g(a) + af(a) \\ f(a) \end{bmatrix}$ where $a \in K$.*

At this point, let the components be $x=0, y=0, y=xM, M \in \mathcal{M}$ but I may not be in \mathcal{M} . Let $\tau_a = \begin{pmatrix} 1 & a & f(a) & g(a) \\ 0 & 1 & 0 & f(a) \\ & & 1 & a \\ & & 0 & 1 \end{pmatrix}$ for $a \neq 0$.

(2.6). τ_a fixes $y = xM \Leftrightarrow M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$, $m_3 = a^{-1}f(a)$, $m_4 = m_1 + a^{-1}g(a)$.

Proof. $(x, xM) \xrightarrow{\tau_a} \left(x \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, x \left(\begin{bmatrix} f(a) & g(a) \\ 0 & f(a) \end{bmatrix} + M \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \right) \right) \in y = xM \Leftrightarrow \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$
 $M = \begin{bmatrix} f(a) & g(a) \\ 0 & f(a) \end{bmatrix} + M \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$

Hence we have the following set of components

$$y = x \begin{bmatrix} u & , & m(u, a) \\ a^{-1}f(a) & , & u + a^{-1}g(a) \end{bmatrix}$$

for all $a \neq 0, u \in K, m: K \times K \rightarrow K$,

$$\begin{aligned} x &= 0 \\ y &= x \begin{bmatrix} u & h(u) \\ 0 & u \end{bmatrix} \end{aligned}$$

where $h(u) = g(f^{-1}(u)) + uf^{-1}(u)$ for all $u \in K$. So, we obtain the following theorem,

Theorem 2.7. *Let π be a generalized Ott-Schaeffer plane of even order q^2 , kernel $K \cong GF(q)$. Let \mathcal{Q} be a collineation group of order q in the linear translation complement such that each involution of \mathcal{Q} is Baer and no two involutions fix the same Baer subplane pointwise. Then π and \mathcal{Q} may be represented in the following form:*

$$\mathcal{Q} = \left\{ \tau_a = \begin{pmatrix} 1 & a & f(a) & g(a) \\ 0 & 1 & 0 & f(a) \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a \in K, f \text{ 1-1 and additive} \right. \\ \left. g(a+b) = g(a) + g(b) + bf(a) + af(b) \right\}.$$

The components for π are $x=0, y=0, y=x \begin{bmatrix} u & h(u) \\ 0 & u \end{bmatrix} \forall u \in K, h(u) = uf^{-1}(u) + g(f^{-1}(u)), h(0)=0$, and $y=x \begin{bmatrix} u & m(u, a) \\ a^{-1}f(a) & u+a^{-1}g(a) \end{bmatrix}$ for some function $m: K \times K \rightarrow K$.

Proof. Note $y=x$ may not represent a component.

3. Groups of order $q(q-1)$

We first assume that π is a tensor product plane of even order q^2 that admits a group \mathcal{QH} of order $q(q-1)$. Further, we assume $\mathcal{Q} = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} \mid a \in K \right\} \cong GF(q)$ as in section 2 and $\mathcal{H} = \left\{ \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \otimes \begin{bmatrix} a & \\ & a^{-\sigma} \end{bmatrix} \mid a \in K \right\}$ for some $\sigma \in \text{Aut } K$.

Recall from (2.2) that a spread for π may be represented in the form $x=0, y=x \begin{bmatrix} u & m(u, a) \\ a^{1-\sigma} & u+a \end{bmatrix}$ for all $u, a \in K, m$ a function from $K \times K \rightarrow K$.

$$\text{Let } \tau_a = \begin{pmatrix} 1 & a^\sigma & a & a^{\sigma+1} \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & a^\sigma \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \rho_a = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \otimes \begin{bmatrix} a^\sigma & 0 \\ 0 & a^{-\sigma} \end{bmatrix} = \begin{pmatrix} a^{\sigma+1} & 0 & 0 & 0 \\ 0 & a^{1-\sigma} & 0 & 0 \\ 0 & 0 & a^{\sigma-1} & 0 \\ 0 & 0 & 0 & a^{-\sigma-1} \end{pmatrix}.$$

Consider the images of

$$y = x \begin{bmatrix} u & m(u, b) \\ b^{1-\sigma} & u+b \end{bmatrix} \\ \xrightarrow{\tau_a} \left[x \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix}, x \begin{bmatrix} a & a^{\sigma+1} \\ 0 & a \end{bmatrix} + \begin{bmatrix} u & m(u, b) \\ b^{1-\sigma} & u+b \end{bmatrix} \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} \right]$$

$$\begin{aligned} \in y &= x \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a+u, & a^{\sigma+1}+ua^\sigma+m(u, b) \\ b^{1-\sigma}, & a+b^{1-\sigma}a^\sigma+u+b \end{bmatrix} \\ &= \left[y = x \begin{bmatrix} (a+u+a^\sigma b^{1-\sigma}), & m(u, b)+a^\sigma(b^{1-\sigma}a^\sigma+b) \\ b^{1-\sigma}, & (a+u+a^\sigma b^{1-\sigma})+b \end{bmatrix} \right]. \end{aligned}$$

So,

Lemma 3.1.

$m(a+u+a^\sigma b^{1-\sigma}, b)=m(u, b)+a^\sigma(b^{1-\sigma}a^\sigma+b)$ for all $a, u, b, b \neq 0$ of K .

Applying ρ_a we obtain

Lemma 3.2.

$$\begin{aligned} y &= x \begin{bmatrix} u, & m(u, b) \\ b^{1-\sigma}, & u+b \end{bmatrix} \\ \xrightarrow{\rho_a} y &= x \begin{bmatrix} a^{-(\sigma+1)} & \\ & a^{-(1-\sigma)} \end{bmatrix} \begin{bmatrix} u, & m(u, b) \\ b^{1-\sigma}, & u+b \end{bmatrix} \begin{bmatrix} a^{\sigma-1} & 0 \\ 0 & a^{-\sigma-1} \end{bmatrix} \\ &= \left[y = x \begin{bmatrix} ua^{-2}, & (m(u, b)a^{-2(\sigma+1)}) \\ (ba^{-2})^{1-\sigma}, & ua^{-2}+ba^{-2} \end{bmatrix} \right]. \end{aligned}$$

So, $m(ua^{-2}, ba^{-2})=m(u, b)a^{-2(\sigma+1)}$ for all u, a, b in $K, a \neq 0, b \neq 0$.

Hence,

Lemma 3.3.

$y=x \begin{bmatrix} b, & m(b, b) \\ b^{1-\sigma}, & 0 \end{bmatrix}$ has $\frac{q(q-1)}{2}$ images under \mathcal{GH} and this orbit includes all components with a zero in the $(2, 2)$ -entry of the image matrix.

In particular, we have the orbits of length $q\left(\frac{q-1}{2}\right)$ defined by the images of $y=x \begin{bmatrix} 1, & m(1, 1) \\ 1, & 0 \end{bmatrix}$ and $y=x \begin{bmatrix} 0, & m(0, 1) \\ 1, & 1 \end{bmatrix}$ (by analogy). By (3.1) and (3.2), the orbit of $y=x \begin{bmatrix} 1, & m(1, 1)=m_1 \\ 1, & 0 \end{bmatrix}$ is:

$$\begin{aligned} y &= x \begin{bmatrix} 1 & m_1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\tau_a} y = x \begin{bmatrix} a+1+a^\sigma, & m_1+a^\sigma(a^\sigma+1) \\ 1, & a+a^\sigma \end{bmatrix} \\ \xrightarrow{\rho_b} y &= x \begin{bmatrix} (a+1+a^\sigma)b^{-2}, & (m_1+a^\sigma(a^\sigma+1))b^{-2(\sigma+1)} \\ (b^{-2})^{1-\sigma}, & (a+1+a^\sigma)b^{-2}+b^{-2} \end{bmatrix} \\ &\equiv \left[y = x \begin{bmatrix} (a+1+a^\sigma)c, & (m_1+a^\sigma(a^\sigma+1))c^{1+\sigma} \\ c^{1-\sigma}, & (a+a^\sigma)c \end{bmatrix} \right] \end{aligned}$$

for all $a, c \neq 0$ in K .

Similarly, the orbit of $\begin{bmatrix} 0, & m(0, 1)=m_0 \\ 1, & 1 \end{bmatrix}$ is

$$\left\{ y = x \begin{bmatrix} (a+a^\sigma)c, & (m_0+a^\sigma(a^\sigma+1))c^{\sigma+1} \\ c^{1-\sigma}, & (a+a^\sigma+1)c \end{bmatrix} \right\}.$$

Therefore, we obtain the following theorem:

Theorem 3.4. *A translation plane π of even order q^2 and kernel $k \cong GF(q)$ admits the group $H = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} \mid a \in K \right\} \cdot \left\{ \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \otimes \begin{bmatrix} a^\sigma & \\ & a^{-\sigma} \end{bmatrix} \mid a \in K - \{0\} \right\}$ with components $x=0, y=0, y=x \Leftrightarrow$ there exists constants $m_0, m_1 \in K$ such that the spread for π may be represented by the matrix spread set:*

$$\begin{aligned} x &= 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}, \\ y &= x \begin{bmatrix} (a^\sigma+a+1)c, & (m_1+a^\sigma(a^\sigma+1))c^{\sigma+1} \\ c^{1-\sigma}, & (a+a^\sigma)c \end{bmatrix}, \\ y &= x \begin{bmatrix} (a^\sigma+a)c, & (m_0+a^\sigma(a^\sigma+1))c^{\sigma+1} \\ c^{1-\sigma}, & (a+a^\sigma+1)c \end{bmatrix} \end{aligned}$$

for all $u, a, c \in K, c \neq 0$. Also, the fixed field of $\sigma = GF(2)$, $q = 2^r$, and r is odd.

Proof. It remains to prove that r is odd. We see that $\begin{bmatrix} 0 & m_0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & m_1 \\ 1 & 1 \end{bmatrix}$ are in distinct H -orbits. Hence $\begin{bmatrix} 0 & m_1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\tau_a} \begin{bmatrix} a+a^\sigma, & m_0+a^\sigma(a^\sigma+1) \\ 1 & a+a^\sigma+1 \end{bmatrix}$, so that $a+a^\sigma \neq 1$. Suppose $b^2=b+1$ for some $b \in K$. Then if $q=2^r$, assume r even, and $\sigma=2^s$ for s odd. Then $(b^2)^{\sigma/2}=b^{\sigma/2}+1=b+1 \Leftrightarrow b^{\sigma/2}=b \Leftrightarrow b^\sigma=b^2 \Leftrightarrow b^{\sigma-2}=1 \Leftrightarrow$ (for $b \neq 1$), $\left[\frac{\sigma-2}{2}, 2^r-1 \right] \neq 1$. Since $\frac{\sigma-2}{2}=2^{s-1}-1$ and $(s-1, r)=2 \cdot t$, we have that $GF(4)$ cannot be a subfield of $GF(q)$. That is, r is odd.

Notes 3.5. In the Ott-Schaeffer planes $m_0=m_1=1$. Here, at least it is possible that there are other translation planes distinct from the O - S planes and admitting the same group of order $q(q-1)$ that the O - S planes admit.

We now further consider generalized Ott-Schaeffer planes π of order q^2 and kernel $K \cong GF(q)$. We may use the representation given in (2.7). Assume there is a linear collineation group H such that $HK^*/K^* \cong H(K^*=K-\{0\})$ and $|H|=q(q-1)$. Note that we use the notation HK^* to refer to the product of H by the kernel homology group of order $q-1$.

Lemma 3.6. *A Sylow 2-subgroup $S_2 \trianglelefteq H$ or π is Ott-Schaeffer.*

Proof. Let S_2 fix the component \mathcal{L} . Since the involutions of H (see (2.7)) are all Baer, it follows from Johnson-Ostrom [8] (3.27) that if π is not Ott-Schaeffer then the group R generated by the Sylow 2-subgroups of H is reducible and solvable and by the argument to [8] (3.27), R must be a 2-group. That is, $S_2 \trianglelefteq H$.

Lemma 3.7. $H = S_2 \cdot C$ where C is a 2-complement of S_2 . Then C fixes two components.

Proof. Clearly, S_2 is a Hall normal subgroup so let C be a 2-complement of order $q-1$.

C fixes \mathcal{L} and by Maschke's Theorem, decompose $\pi = \mathcal{L} \oplus \mathcal{W}$ where \mathcal{W} is a C -invariant 2-space. Either \mathcal{W} is a component and (3.7) is finished or \mathcal{W} is a C -invariant Baer subplane. Further, \mathcal{W} is Desarguesian and $C|_{\mathcal{W}} \leq GL(2, q)$ acting on \mathcal{W} . Hence, C must fix two components of \mathcal{W} which are 1-spaces of π . Hence, C fixes the components of π which contain the C -invariant components of \mathcal{W} .

Lemma 3.8. H acts faithfully on \mathcal{L} .

Proof. If $h \in H$ fixes \mathcal{L} pointwise then h is a homology and by the orbit structure of π (see section 2), it must be that the coaxis of h is moved by S_2 . That is, there must be elations in H by André [1]. Hence, we have the proof to (3.8).

Lemma 3.9. C acts regularly on $S_2 - \langle 1 \rangle$ by conjugation.

Proof. Represent S_2 as in (2.7), then let

$$h = \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right] \in C.$$

So $A, C \in N_{GL(2, q)} \left[\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in K \right\} \right]$ so that

$$A = \begin{bmatrix} a_1 & a_2 \\ 0 & a_4 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix}.$$

Let $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$. If $\tau_a^h = \tau_a$ for some $a \neq 0$ (see (2.7)) then clearly $a_1 = a_4$, $c_1 = c_4$ and so $\tau_b^h = \tau_b \forall b \neq 0$. This implies

$$[A+C] \begin{bmatrix} f(b) & g(b) \\ 0 & f(b) \end{bmatrix} = B \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} B = \begin{bmatrix} bb_3 & b(b_1+b_4) \\ 0 & bb_3 \end{bmatrix}$$

for all $b \neq 0$.

Hence

$$\begin{bmatrix} a_1+c_1 & a_2+c_2 \\ 0 & a_4+c_4 \end{bmatrix} \begin{bmatrix} f(b) & g(b) \\ 0 & f(b) \end{bmatrix} = \begin{bmatrix} bb_3 & b(b_1+b_4) \\ 0 & bb_3 \end{bmatrix}.$$

So $(a_2+c_2)f(b)+(a_1+c_1)g(b)=b(b_1+b_4)$. That is, if $a_1 \neq c_1$ then $g(b)=bK_1+f(b)K_2$ for

$$K_1 = \frac{b_1+b_4}{a_1+c_1} \quad \text{and} \quad K_2 = \frac{a_2+c_2}{a_1+c_1}.$$

Thus, if $a_1 \neq c_1$ then g is additive.

However, $g(b+t)=g(b)+g(t)+bf(t)+tf(b)$ so that $bf(t)=tf(b)$ for all $b, t \in K$. Hence $f(t)=tf(1)$.

But the components include the $q(q-1)$ elements

$$y = x \begin{bmatrix} u & m(u, a) \\ a^{-1}f(a) & u+a^{-1}g(a) \end{bmatrix} \quad \text{for } u, a \in K, a \neq 0$$

(see (2.7)) so that the $(2, 1)$ -entries are always $a^{-1}af(1)=f(1)$. That is,

$$y = x \begin{bmatrix} u & m(u, a) \\ f(1) & u+a^{-1}g(a) \end{bmatrix}$$

for $f(1)$ a constant, represents $q(q-1)$ components which clearly is a contradiction

Hence, $a_1+c_1=0$ so that $a_1=a_4=c_1=c_4$ and we have

$$(a_2+c_2f(b)) = b(b_1+b_4).$$

If $a_2+c_2 \neq 0$ then $f(b)=b \cdot a$ for a some constant. Then we still have $y=x \begin{bmatrix} u & m(u, a) \\ f(1) & u+a^{-1}g(a) \end{bmatrix}$ to represent $q(q-1)$ components—a contradiction.

Thus, $a_2=c_2$ and $b_1=b_4$. So the element in \mathcal{C} has the form

$$\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ 0 & a_1 & 0 & b_1 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_1 \end{pmatrix}. \quad \text{Multiplying by } a_1^{-1}I_4 \text{ we obtain } \rho = \begin{pmatrix} 1 & a_2a_1^{-1} & b_1a_1^{-1} & b_2a_1^{-1} \\ 0 & 1 & 0 & b_1a_1^{-1} \\ 0 & 0 & 1 & a_2a_1^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad \text{Let}$$

$$a_2a_1^{-1}=e. \quad \text{We have } \tau_e \in \mathcal{S}_2, \tau_e = \begin{pmatrix} 1 & e & f(e) & g(e) \\ 0 & 1 & 0 & f(e) \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad \text{Since } \mathcal{S}_2 \trianglelefteq HK^* \text{ we must have}$$

that $\rho=\tau_e$. In other words, the element is in \mathcal{S}_2K^* . However, we assumed that $HK^*/K^* \cong H$. This proves (3.9).

Thus, \mathcal{C} must fix one of the components in the orbit of \mathcal{S}_2 of length q . That

is, \mathcal{C} must fix a component of the form $y=x \begin{bmatrix} u & h(u) \\ 0 & u \end{bmatrix}$ for some $u \in K$.

So, we may choose a 2-complement \mathcal{C} for \mathcal{S}_2 so that \mathcal{C} fixes $y=0$.

(3.10). Thus, the elements of \mathcal{C} have the form

$$\left(\begin{array}{c|c} \begin{bmatrix} a_1 & a_3 \\ 0 & a_4 \end{bmatrix} & 0 \\ \hline 0 & \begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix} \end{array} \right).$$

$$\left[x, x \begin{bmatrix} u & h(u) \\ 0 & u \end{bmatrix} \right] \rightarrow \left[x \begin{bmatrix} a_1 & a_2 \\ 0 & a_4 \end{bmatrix}, x \begin{bmatrix} u & h(u) \\ 0 & u \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix} \right]$$

so that

$$\begin{aligned} & \begin{bmatrix} a_1^{-1} & a_2 a_1^{-1} a_4^{-1} \\ 0 & a_4^{-1} \end{bmatrix} \begin{bmatrix} u & h(u) \\ 0 & u \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix} \\ &= \begin{bmatrix} u a_1^{-1} & a_1^{-1} h(u) + a_2 a_1^{-1} a_4^{-1} u \\ 0 & u a_4^{-1} \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix} \\ &= \begin{bmatrix} u a_1^{-1} c_1 & u a_1^{-1} c_2 + (a_1^{-1} h(u) + a_2 a_1^{-1} a_4^{-1} u) c_4 \\ 0 & u a_4^{-1} c_4 \end{bmatrix} \in \left\{ \begin{bmatrix} u & f(u) \\ 0 & u \end{bmatrix} \mid u \in K \right\}. \end{aligned}$$

(3.11). So $a_1^{-1} c_1 = a_4^{-1} c_4$.

Since $HK^*/K^* \cong H$, then $CK^*/K^* \cong \mathcal{C} \Leftrightarrow \mathcal{C} \cap K^* = \{1\}$. So multiplying a typical element by $\begin{bmatrix} a_1^{-1} & & \\ & a_1^{-1} & \\ & & a_1^{-1} \end{bmatrix}$ then in CK^* , there are $q-1$ elements of the general form $\begin{bmatrix} 1 & a_2 & \\ 0 & a_4 & \\ & c & c_2 \\ & 0 & a_4 c \end{bmatrix}$. Assume two such elements have equal (2,2)-entries.

Consider

$$\rho = \begin{bmatrix} 1 & a_2 & \\ 0 & a_4 & \\ & \bar{c} & \bar{c}_2 \\ & 0 & a_4 \bar{c} \end{bmatrix} \quad \text{and} \quad \chi = \begin{bmatrix} 1 & a_2 & \\ 0 & a_4 & \\ & c & c_2 \\ & 0 & a_4 c \end{bmatrix}$$

when $a_4 = a_4$. Then the product

$$\rho \chi^{-1} = \left(\begin{array}{c|c} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} & \\ \hline & \end{array} \right)$$

where $d = a_2 a_4^{-1} + a_2 a_4^{-1}$ and $(\rho \chi^{-1})^2$ must fix $x=0$ pointwise. By (3.8), $(\rho \chi^{-1})^2 =$

1. However, $|CK^*| \mid (q-1)^2$, so $\rho\chi^{-1}=1 \Leftrightarrow \rho=\chi$. Thus, there must exist a col-

lineation $\rho = \begin{bmatrix} 1 & a_2 \\ 0 & a \\ & c & c_2 \\ & 0 & ac \end{bmatrix}$ where $|a|=q-1$.

Hence, since $\mathcal{S}_2 \trianglelefteq HK^*$, we obtain

$$\begin{aligned} & \begin{bmatrix} A^{-1} & \\ & C^{-1} \end{bmatrix} \begin{bmatrix} 1 & b & f(b) & g(b) \\ 0 & 1 & 0 & f(b) \\ & 1 & b & \\ & 0 & 1 & \end{bmatrix} \begin{bmatrix} A & \\ & C \end{bmatrix} \\ &= \begin{bmatrix} A^{-1} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} A & A^{-1} \begin{bmatrix} f(b) & g(b) \\ 0 & f(b) \end{bmatrix} C \\ 0 & C^{-1} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} C \end{bmatrix} \in \mathcal{S}_2 \end{aligned}$$

where $A = \begin{bmatrix} 1 & a_2 \\ 0 & a \end{bmatrix}$, $C = \begin{bmatrix} c & c_2 \\ 0 & ac \end{bmatrix}$ so that

$$A^{-1} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & a_2 a^{-1} \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a_2 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 1 & ba \\ 0 & 1 \end{bmatrix} = C^{-1} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} C$$

and

$$\begin{aligned} A^{-1} \begin{bmatrix} f(b) & g(b) \\ 0 & f(b) \end{bmatrix} C &= \begin{bmatrix} 1 & a_2 a^{-1} \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} f(b) & g(b) \\ 0 & f(b) \end{bmatrix} \begin{bmatrix} c & c_2 \\ 0 & ac \end{bmatrix} \\ &= \begin{bmatrix} cf(b), f(b)(c_2 + a_2 c) + g(b)ac \\ 0, cf(b) \end{bmatrix}. \end{aligned}$$

Since $|a|=q-1$, $\exists j \ni a^j = c$, so that we obtain the elements

$$\begin{bmatrix} 1, ab & a^j f(b), f(b)(c_2 + a_2 a^j) + g(b) a^{j+1} \\ 0, 1 & 0, a^j f(b) \\ & 1 & ab \\ & 0 & 1 \end{bmatrix}$$

in \mathcal{S}_2 . Hence,

(3.12) $f(ab) = a^j f(b)$ for all b in K^* and

(3.13) $g(ab) = f(b)(c_2 + a_2 a^j) + g(b) a^{j+1}$ for all b in K^* .

Also,

$$\rho^i = \left(\begin{array}{c|c} 1 & a_2 \left[\frac{a^i - 1}{a - 1} \right] \\ 0 & a^i \\ \hline & a^{ij} & c_2 a^{(i-1)j} & \left[\frac{a^i - 1}{a - 1} \right] \\ & & & a^{i(j+1)} \end{array} \right).$$

The previous argument applied to ρ^i implies

$$(3.14) \quad f(a^i b) = a^{ij} f(b) \quad \text{for all } b \text{ in } K^* \text{ and}$$

$$(3.15) \quad g(a^i b) = f(b) \left(c_2 a^{(i-1)j} \left[\frac{a^i - 1}{a - 1} \right] + a_2 a^{ij} \left[\frac{a^i - 1}{a - 1} \right] \right) + g(b) a^{i(j+1)}$$

for all $i \geq 1, b \text{ in } K$.

Let $b=1$ in (3.14) and $a^i=c$ to obtain $f(c)=c^j f(1)$ and further

$$(3.16) \quad f(c) = c^\tau f \quad \text{where } c^j = c^\tau \text{ and } \tau \in \text{Aut } K, f = f(1).$$

Pf. f is additive and i is arbitrary as $|a|=q-1$.

From (3.10),

$$y = x \begin{bmatrix} f(b), & bf(b)+g(b) \\ 0, & f(b) \end{bmatrix}$$

$$\xrightarrow{\rho} y = x \begin{bmatrix} f(b)a^\tau, & \{f(b)(c_2+a_2a^\tau)+f(b)ba^{\tau+1}+g(b)a^{\tau+1}\} \\ 0, & f(b)a^\tau \end{bmatrix}.$$

Using (3.16), $f(b)a^\tau=f(ab)$ so that the (2,2) entry of the preceding matrix is $df(d)+g(d)$ for $d=ab$ since $(ab)f(ab)=f(b)ba^{\tau+1}$, we obtain

$$(3.17) \quad g(ab) = g(b) a^{\tau+1} + b^\tau f(1) (c_2 + a_2 a^\tau) \quad \text{for all } b \text{ in } K^*.$$

Let $b=1$ in (3.15) to obtain:

$$(3.18) \quad g(a^i) = g(1) a^{i(\tau+1)} + f(1) (c_2 + a_2 a^\tau) \left[a^{(i-1)\tau} \left[\frac{a^i - 1}{a - 1} \right] \right] \quad \text{all } i \geq 1 \text{ in } \mathbb{Z}.$$

Since $|a|=q-1$, we obtain

$$g(c) = g(1) c^{\tau+1} + f(1) \frac{(c_2 + a_2 a^\tau)}{(a-1)} a^{-\tau} (c^\tau (c-1))$$

for all $c \neq 0$. Let $s = f(1) \frac{(c_2 + a_2 a^\tau)}{(a-1)} a^{-\tau}$, $g(1)=g$, $f(1)=f$ to obtain

$$(3.19) \quad g(c) = c^{\tau+1}(s+g) + c^\tau s$$

for all $c \neq 0$ in K .

Recall $f(d)=d^\tau f$ for all d and $g(e+d)=g(e)+g(d)+ef(d)+df(e)$ for all e, d in K . Thus,

$$\begin{aligned} g(e+d) &= (e+d)^{\tau+1}(s+g) + (e+d)^\tau s \\ &= (e^{\tau+1}(s+g) + e^\tau s) + (d^{\tau+1}(s+g) + d^\tau s) + ed^\tau f + de^\tau f \end{aligned}$$

so that

$$(3.20) \quad (ed^\tau + ed^\tau)(s+g+f) = 0$$

so that either $\tau=1$ or $s+g+f=0$. If $\tau=1$ then from (2.7), the components have the form

$$\begin{aligned} \left[y = x \begin{bmatrix} u & h(u) \\ 0 & u \end{bmatrix} \right], \left[y = x \begin{bmatrix} u & m(u, d) \\ d^{-1}f(d) & u + d^{-1}g(d) \end{bmatrix} \right] \\ \equiv \left[y = x \begin{bmatrix} u & m(u, d) \\ f & u + d^{-1}g(d) \end{bmatrix} \right]. \end{aligned}$$

But, the latter set for all $u, d \neq 0$ represents $q(q-1)$ components. For $d_1 \neq d_2$

$$\begin{bmatrix} u & m(u, d_1) \\ f & u + d^{-1}g(d_1) \end{bmatrix} - \begin{bmatrix} u & m(u, d_2) \\ f & u + d^{-1}g(d_2) \end{bmatrix}$$

is singular so $\tau \neq 1$.

Change bases by

$$\chi = \left(\begin{array}{cc|cc} 1 & t & & 0 \\ 0 & 1 & & 0 \\ \hline & & 1 & t \\ 0 & & 0 & 1 \end{array} \right)$$

where $t = \frac{a_2}{1+a}$.

Recall

$$\tau_d = \begin{pmatrix} 1 & d & f(d) & g(d) \\ 0 & 1 & 0 & f(d) \\ & & 1 & d \\ 0 & & 0 & 1 \end{pmatrix}$$

(from section 2) and χ commutes with τ_d . Recall

$$\rho = \left(\begin{array}{cc|cc} 1 & a_2 & & \\ 0 & 1 & & \\ \hline & & a^\tau & c_2 \\ & & 0 & a^{\tau+1} \end{array} \right)$$

so that

$$\bar{\rho} = \chi \rho \chi^{-1} = \left(\begin{array}{cc|cc} 1 & 0 & & \\ 0 & a & & \\ \hline & & a^\tau & \frac{a^\tau(a+1)s}{f} \\ 0 & & 0 & a^{\tau+1} \end{array} \right)$$

as

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a_2 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t+a_2+ta \\ 0 & a \end{bmatrix}$$

when $t = \frac{a_2}{1+a}$ so that $t+a_2+ta = \frac{a_2}{1+a} + a_2 + \frac{a_2}{1+a} a = 0$ and

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a^\tau & c_2 \\ 0 & a^{\tau+1} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a^\tau & a^\tau t + c_2 + ta^{\tau+1} \\ 0 & a^{\tau+1} \end{bmatrix}$$

where

$$a^\tau t + c_2 + ta^{\tau+1} = a^\tau \frac{a_2}{1+a} + \frac{a^\tau(a+1)}{f} s + a_2 a^\tau + \frac{a_2}{1+a} a^{\tau+1} = \frac{a^\tau(a+1)s}{f}$$

as

$$s = f \frac{(c_2 + a_2 a^\tau) a^{-\tau}}{(1+a)}.$$

We originally had the components in the form

$$y = x \begin{bmatrix} f(b) & g(b) + bf(b) \\ 0 & f(b) \end{bmatrix}$$

and

$$y = x \begin{bmatrix} u & , & m(u, c) \\ c^{-1}f(c) & , & u + c^{-1}g(c) \end{bmatrix}.$$

Apply \mathcal{X} , the forms become:

$$y = x \begin{bmatrix} f(b) & g(b) + bf(b) \\ 0 & f(b) \end{bmatrix}$$

and

$$\begin{aligned} & \left[y = x \begin{bmatrix} u + c^{\tau-1} f \cdot t, & \text{---} \\ c^{\tau-1} f & , & u + c^{\tau-1} f \cdot t + c^{-1} g(c) \end{bmatrix} \right] \\ & \equiv \left[y = x \begin{bmatrix} v & \text{---} \\ c^{\tau-1} f & , & v + c^{-1} g(c) \end{bmatrix} \right] \end{aligned}$$

for $v = u + c^{\tau-1} f \cdot t$. In other words the form is invariant under \mathcal{X} .

Now $g(b) + bf(b) = (b^{\tau+1}(s+g) + b^\tau s + b^{\tau+1}f)$ by (3.19). By (3.20) $s+g+f=0$ so that $g(b) + bf(b) = b^\tau s$.

Change bases by

$$\gamma = \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} f^{-1} & sf^{-2} \\ 0 & f^{-1} \end{bmatrix} \end{bmatrix}$$

so that

$$\begin{aligned}
 y &= x \begin{bmatrix} b^\tau f, & b^\tau s \\ 0, & b^\tau f \end{bmatrix} \xrightarrow{\gamma} y = x \begin{bmatrix} b^\tau & 0 \\ 0 & b^\tau \end{bmatrix}, \\
 \left[y = x \begin{bmatrix} v, & \overline{} \\ c^{\tau-1} f, & v + c^{-1} g(c) \end{bmatrix} \right] &= \left[y = x \begin{bmatrix} v, & \overline{} \\ c^{\tau-1} f, & v + c^\tau(s+g) + c^{\tau-1} s \end{bmatrix} \right] \\
 \xrightarrow{\gamma} y &= x \begin{bmatrix} vf^{-1}, & \overline{\phantom{vf^{-1}}} \\ c^{\tau-1} f, & vf^{-1} + c^\tau \end{bmatrix}
 \end{aligned}$$

since $s+g+f=0$.

Furthermore,

$$\begin{aligned}
 \gamma^{-1} \bar{p} \gamma &= \left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & a \end{matrix} & \begin{bmatrix} 1 & sf^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a^\tau, & a^\tau(a+1)\frac{s}{f} \\ 0, & a^{\tau+1} \end{bmatrix} \begin{bmatrix} 1 & sf^{-1} \\ 0 & 1 \end{bmatrix} \end{array} \right) \\
 &= \left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & a \end{matrix} & \begin{matrix} a^\tau & 0 \\ 0 & a^{\tau+1} \end{matrix} \end{array} \right)
 \end{aligned}$$

and

$$\gamma^{-1} \tau_d \gamma = \left(\begin{array}{c|c} \begin{matrix} 1 & d \\ 0 & 1 \end{matrix} \begin{bmatrix} f(d) & g(d) \\ 0 & f(d) \end{bmatrix} \begin{bmatrix} f^{-1}, & sf^{-2} \\ 0, & f^{-1} \end{bmatrix} & \begin{matrix} 1 & d \\ 0 & 1 \end{matrix} \end{array} \right)$$

and

$$\begin{aligned}
 \begin{bmatrix} f(d) & g(d) \\ 0 & f(d) \end{bmatrix} \begin{bmatrix} f^{-1} & sf^{-2} \\ 0 & f^{-1} \end{bmatrix} &= \begin{bmatrix} d^\tau f & d^{\tau+1}(s+g) + d^\tau s \\ 0 & d^\tau f \end{bmatrix} \begin{bmatrix} f^{-1} & sf^{-2} \\ 0 & f^{-1} \end{bmatrix} \\
 &= \begin{bmatrix} d^\tau & d^{\tau+1} \frac{(s+g)}{f} \\ 0 & d^\tau \end{bmatrix} = \begin{bmatrix} d^\tau & d^{\tau+1} \\ 0 & d^\tau \end{bmatrix}.
 \end{aligned}$$

Thus

$$\gamma^{-1} \tau_d \gamma = \begin{pmatrix} 1 & d & d^\tau & d^{\tau+1} \\ 0 & 1 & 0 & d^\tau \\ 0 & 1 & d & \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Now let $\tau = \sigma^{-1}$. Then

$$\begin{pmatrix} 1 & b & b^\tau & b^{\tau+1} \\ 0 & 1 & 0 & b^\tau \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a^\sigma & a & a^{\sigma+1} \\ 0 & 1 & 0 & a \\ & & 1 & a^\sigma \\ 0 & 1 & & \end{pmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix}$$

for $b=a^\sigma$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & a \\ & a^\tau & 0 \\ & 0 & a^{\tau+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & d^{-2\sigma} \\ & d^{-2} & 0 \\ & 0 & d^{-2-2\sigma} \end{pmatrix}$$

for $a=d^{-2\sigma}$. Multiplying by

$$\begin{bmatrix} d^{\sigma+1} & & & \\ & d^{\sigma+1} & & \\ & & d^{\sigma+1} & \\ & & & d^{\sigma+1} \end{bmatrix}$$

of the kernel homology group, we obtain

$$\begin{bmatrix} d^{\sigma+1} & & & \\ & d^{1-\sigma} & & \\ & & d^{\sigma-1} & \\ & & & d^{-\sigma-1} \end{bmatrix} = \begin{bmatrix} d & \\ & d^{-1} \end{bmatrix} \otimes \begin{bmatrix} d^\sigma & \\ & d^{-\sigma} \end{bmatrix}.$$

Hence, we may apply the results on tensor product planes with groups of order $q(q-1)$.

Thus we obtain:

Theorem 3.21. *Let π be a translation plane of order q^2 and kernel $K \cong GF(q)$, q even. Let H be a group in the linear translation complement of order $q(q-1)$ and $HK^*/K^* \cong H$. Further, if S_2 is a Sylow 2-subgroup of H , assume the involutions are Baer and no two involutions fix the same subplane pointwise.*

Then π is a tensor product plane and the spread is completely determined. A matrix spread set may be represented as follows:

$$\begin{aligned} x &= 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}, \\ y &= x \begin{bmatrix} (a^\sigma + a + 1)c, (m_1 + a^\sigma(a^\sigma + 1))c^{\sigma+1} \\ c^{1-\sigma}, (a + a^\sigma)c \end{bmatrix}, \\ y &= x \begin{bmatrix} (a^\sigma + a)c, (m_0 + a^\sigma(a^\sigma + 1))c^{\sigma+1} \\ c^{1-\sigma}, (a + a^\sigma + 1)c \end{bmatrix} \end{aligned}$$

for all $u, a, c \neq 0$ of K , m_0, m_1 constants in K and $\sigma \in \text{Aut } K$ such that the fixed field of σ is $GF(2)$ and $q = 2^{2r+1}$.

NOTES 3.22. Several authors, [2], [3], [5], have recently studied translation planes of order q^2 that admit H groups of order $q(q-1)$ where H is an autotopism group. In this situation, there are many different classes of nonisomorphic translation planes. So, we see that the assumption on the nature of the Sylow p -subgroups for $p' = q$ is crucial in (3.21).

(3.23) Open Problems and Related Questions.

- 1) (a) What are the Tensor Product Planes?
 (b) Are there nontrivial generalized Ott-Schaeffer Planes?
- 2) Study translation planes of order q^2 , $p^r = q$ kernel $GF(q)$ that admit linear collineation groups of order $q(q-1)$: The Sylow p -subgroups are
 - (a) planar.
 - (b) quartic.
 - (a) 1) if planar the group is an autotopism group.
 - (a) 2) no two p -elements fix the same Baer subplane pointwise.

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