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SOME RELATIONS AMONG VARIOUS NUMERICAL INVARIANTS FOR LINKS

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Introduction. Throughout this paper, "a link l of $\mu(l)$ components" means disjoint union of $\mu(l)$ oriented 1-spheres in $R^3$.

In §1, we study some 3-dimensional numerical invariants of links, that is, $g(l)$ (genus of $l$), $u(l)$ (see Definition 1) and $c(l)$ (see Definition 3) will be defined and we will have some relations among them as follows.

**Theorem 1.** For any link $l$, $g(l) \leq c(l)$ and $u(l) \leq c(l)$.

In §2, the 4-dimensional numerical invariants $g^*(l)$, $g^*_x(l)$ (see Definition 4), $u^*(l)$, $u^*_x(l)$ (see Definition 5), $c^*(l)$ and $c^*_x(l)$ (see Definition 6) will be defined and the main theorem will be proved.

**Theorem 2.** For any link $l$, we obtain

$$g^*(l) \leq g^*_x(l) \leq g(l)$$

$$c^*(l) \leq c^*_x(l) \leq c(l)$$

$$u^*(l) \leq u^*_x(l) \leq u(l)$$

As is usual, two links $l$ and $l'$ are said to be of the same type or isotopic, denoted by $l \approx l'$, if there exists an orientation preserving homeomorphism $f$ of $R^3$ onto itself such that $f(l) = l'$.

$\partial X$, $Int X$ and $cl X$ represents the boundary, the interior and the closure of $X$ respectively.

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1. 3-dimensional numerical invariants

Let $l$ be a link of $\mu(l)$ components in $R^3$. It is known in [9], [11] that $l$ always bounds an orientable connected surface $F$ in $R$. The minimum genus of these surfaces is called the genus of the link $l$ and is denoted by $g(l)$. Note that $g(F)$ denotes the usual genus of a surface $F$. 
Let $L$ be a diagram of $l$, i.e., $L = p(l)$, where $p$ is a regular projection of $R^3$ to $R^2$ ([2]). $L$ has in general at least one double point if $l$ is not a trivial link (unknotted and unlinked). A link can be deformed into a trivial link by employing a finite number of unlinking operation $(\Gamma)$ defined as follows.

$(\Gamma)$ Change an underpass into an overpass at a double point.

**Definition 1.** The minimum number of unlinking operations required to deform a given link $l$ into a trivial link is called the **unlinking number** of $l$ (in the 3-dimensional sense) and is denoted by $u(l)$.

**Definition 2.** Let $F_0$ be a surface which may not be connected and $f$ be an immersion of $F_0$ into $R^3$. Put $F = f(F_0)$. Suppose that $F$ has a finite number of simple double lines and these double lines do not intersect each other. Each double line $J$ is one of the following three types (see [4])

1. a closed curve whose antecedents are closed curve $J'$ and $J''$ that lie in $\text{Int } F_0$,
2. an arc whose antecedents are an arc $J'$ that spans $\partial F_0$ and an arc $J''$ that lies entirely in $\text{Int } F_0$,
3. an arc whose antecedents are arcs $J'$ and $J''$ each of which has an end point on $\partial F_0$ and the other one lies in $\text{Int } F_0$.

We call $J$ a singularity of $F$. The singularities satisfying the condition (1), (2), (3) will be called (simple) loop, ribbon and clasp singularities respectively. [4]

We call $F$ a non-singular surface if $f$ is an embedding.

Then, to define the **clasp number** $c(l)$ of a link $l$ we need to prove the following lemma.

**Lemma 1.** Any link $l$ spans $\mu(l)$ singular disks whose singularities are only clasps and the number of these clasps is finite.

**Proof.** Let $n$ be the unlinking number of $l$ and $p$ be a regular projection of $l$ such that there exist $n$ double points $p_1, \ldots, p_n$ in $p(l)$ and $l$ becomes a trivial link by $(\Gamma)$-operation along these points. We may make oriented small unknotted circles $c_i$, $i = 1, \ldots, 2n$, near to $p_i$, linking with $l$ as shown Fig. 1 such that $L(l, c_i) = -L(l, c_{n+i}) = 1$ or $-1$ according as the orientation of $l$, where $p_i$ is a point of $p^{-1}(p_i) \cap l$ and $L(l, c)$ denotes the linking number of $l$ and $c$. Then there exist mutually disjoint bands $B_i$, $i = 1, \ldots, 2n$, with $B_i \cap l = \partial B_i \cap l$ an arc and

$$l + \partial \left( \bigcup_{i=1}^{n} B_i \right) + \left( \bigcup_{i=1}^{n} c_i \right) \approx O^\mu$$

$$l + \partial \left( \bigcup_{i=1}^{2n} B_i \right) + \left( \bigcup_{i=1}^{2n} c_i \right) \approx l$$

where $O^\mu$ is a trivial link of $\mu = \mu(l)$ components and $+$ means addition in the homology sense. Let $E = \bigcup_{i=1}^{\mu} E_i$ be a union of mutually disjoint spanning disks
of $O^a$ and $B = \bigcup_{i=0}^{2a} B_i$. By a slight modification of $E$, $B$ and $D = \bigcup_{i=0}^{2a} D_i$, where $D$ is oriented mutually disjoint disks with $\partial D_i = e_i$, we have $B \cap D = \partial B \cap \partial D$, $B \cap E = (\partial B \cap \partial E) \cup$ (ribbon singularities), $D \cap E = \text{(clasp singularities)}$ and $\partial (B \cup D \cup E) \approx l$. For each ribbon singularity $J$ we draw a simple area, on $E$ to connect a point of $dE$ and that of $\text{Int } J$ and put $E = \text{cl}(E - \bigcup_{i=1}^{r} N_i)$, where $r$ is the number of ribbon types on $E$ and $N_i$ is a regular neighborhood of $\alpha_i$ in $E$. Then clearly $\partial(B \cup D \cup E) \approx l$ and the singularities of $\mu$ singular disks $B \cup D \cup E$ are only clasps and of course the clasp number of $B \cup D \cup E$ is finite. So the proof is complete.

**Definition 3.** For any link $l$, there is a singular disk with only clasps which spans $l$ by Lemma 1. The minimum number of the clasps is called the **clasp number** of $l$, denoted by $c(l)$.

Then we have,

**Theorem 1.** For any link $l$, $c(l) \geq u(l)$, $c(l) \geq g(l)$.

**Proof.** $c(l) \geq u(l)$ is obvious from the definitions of these numbers. So we have to prove $c(l) \geq g(l)$. Let $D$ be singular disks such that $c(D) = c(l)$ and $\partial D = l$, where $c(D)$ is the number of clasps of $D$. Making use of orientation preserving cuts ([4], [8]) along all clasps, we get an orientable surface $F$ of genus $c(l)$ such that $\partial F = \partial D = l$. So $c(l) \geq g(l)$, which completes the proof.

**Remark.** These inequalities can not be replaced by equalities. For example for the knot $6_2$, $6_2$ is alternating, so $g(6_2) = 2$ ([1]) and $c(6_2) = 2$ by using Theorem 1 but $u(6_2) = 1$, and for the link $\otimes$, $c(\otimes) = u(\otimes) = 1$ but $g(\otimes) = 0$. 

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**Fig. 1**

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2. 4-dimensional numerical invariants

Let $l$ be a link in $\mathbb{R}^4[0]$, where $\mathbb{R}^4[a]=\{(x, y, z, t)\in\mathbb{R}^4| t=a\}$. Since $l$ bounds an orientable connected surface $F$ in $\mathbb{R}^4[0]$, $l$ always bounds an orientable locally flat connected surface in $\mathbb{R}^4[0, t_0]=\{(x, y, z, t)\in\mathbb{R}^4| 0\leq t< t_0\}$. The minimum genus of these surfaces is an invariant of the link type ([3], [7]). It is denoted by $g^*(l)$ (in the 4-dimensional sense).

**Definition 4.** Especially for any link $l$ we may span an orientable locally flat surface $F$ in $\mathbb{R}^3[0, t_0)$ which has no minimum points with $dF=l$ in $\mathbb{R}^3[0, t_0)$, the minimum genus of these surfaces is called the ribbon type genus of $l$ and is denoted by $g^*(l)$. It is clearly that $g^*(l)$ is an invariant of the link type of $l$.

**Lemma 2.** For any link $l$, $g^*(l)\leq g^*(l)\leq g(l)$.

A link $l$ will be called split into two components $l_1$ and $l_2$ if there is a 3-ball $B^3$ such that $l_1\cap B^3, l_2\subset \mathbb{R}^3-\text{Int } B^3$. Then $l$ is denoted by $l=l_1\cdot l_2$. Then

**Lemma 3.** For any link $l$, there is a number $\mu$ such that $g^*(l)=g^*(l\cdot O^\mu)$ for some trivial link $O^\mu$ of $\mu$ components.

Proof. Let $F$ be a locally flat orientable surface in $\mathbb{R}^4[0, 1)$ with $\partial F=l$ in $\mathbb{R}^4[0]$ and $g(F)=g^*(l)$. Let $p_1, \ldots, p_\mu$ be the minimum points of $F$. We may take $\mu$ distinct points $q_1, \ldots, q_\mu$ in $\mathbb{R}^3[-1]$ and disjoint simple arcs $\alpha_1, \ldots, \alpha_\mu$ and $\alpha_i$ connects $p_i$ with $q_i$ and $\alpha_i\cap \mathbb{R}^4[t]$ is at most one point for each $i, 1\leq i\leq \mu$ and $t, 0\leq t<1$. Then we can deform $F$ to a surface $F'$ by an isotopy along $\alpha_i$. The minimum points of $F'$ are $q_i$ and $F'\cap \mathbb{R}^3[0, 1)$ has no minimum points. Of course, $F'\cap \mathbb{R}^4[0, 1)\approx l\cdot O^\mu$, so $g^*(l\cdot O^\mu)\leq g^*(l)$ ([5], [10]).

Conversely, let $F_0$ be a locally flat surface in $\mathbb{R}^4[0, 1)$ with $F_0\cap \mathbb{R}^3[0]=l\cdot O^\mu$ which has no minimum points and $g^*(l\cdot O^\mu)=g(F_0)$. In $\mathbb{R}^3[-1, 0]$ we make $l\times [-1, 0]$. As $O^\mu$ is splitted from $l$, $O^\mu$ bounds mutually disjoint disks $D_i, i=1, \ldots, \mu$, in $\mathbb{R}^3[-1, 0]$ which do not intersect with $l\times [-1, 0]$. So $F=F_0\cup l\times [-1, 0] \cup (\cup D_i)$ is a locally flat orientable surface with boundary $l$ and $g(F)=g(F_0)=g^*(l\cdot O^\mu)$. Therefore $g^*(l)\leq g(l\cdot O^\mu)$, which completes the proof.

Lemma 4 is essential to prove the main theorem.

**Lemma 4.** Let $F$ be a locally flat orientable surface which has no minimum and maximum points and $F\cap \mathbb{R}^4[0]=l, F\cap \mathbb{R}^3[1]=l'$. Then there is a locally flat orientable surface $F'$ properly embedded in $\mathbb{R}^4[0, 1]$ and isotopic to $F$ in $\mathbb{R}^4[0, 1]$ ($F'\cap \mathbb{R}^4[0]\approx l$ in $\mathbb{R}^4[0], F'\cap \mathbb{R}^3[1]\approx l'$ in $\mathbb{R}^3[1]$ respectively). Furthermore there exist some disjoint 3-balls $B^3_i, i=1, \ldots, n$, in $\mathbb{R}^4[0]$. Such that
\[
\text{cl}(F' - \bigcup_{i=1}^{n} B_i^3 \times [0, 1]) = \text{cl}(F' \cap R^3[0] - \bigcup_{i=1}^{n} B_i^3) \times [0, 1].
\]

Proof. It may be assumed that \(F\) has \(n\) critical points and \(R^3[t_i]\) contains only one critical point for \(t_i, 0 < t_i < \cdots < t_n < 1\). A critical point \(p_i\) may be changed by a critical band \(B_i^3\) for each \(i\) (see [6]). We may deform \(F\) by an isotopy of \(R^3[0, 1]\) carrying \(B_i^3\) into \(R^3[1/2]\) so that maximum and minimum points do not appear in the resulting surface. We will write the resulting surface and the band \(F\) and \(B_i^3\) again. Since \(F \cap R^3[1/2, 1]\) is a locally flat orientable surface which has no maximum, minimum points and critical bands,

\[
\left(F \cap R^3\left[\frac{1}{2}, 1\right] - \partial\left(\bigcup_{i=1}^{n} B_i^3\right)\right) \cup \left(\bigcup_{i=1}^{n} \alpha_i \cup \alpha_i\right) \approx \left(F \cap R^3[1]\right) \times \left[\frac{1}{2}, 1\right]
\]

in \(R^3[1/2, 1]\) (for \(\alpha_i\) and \(\bar{\alpha}_i\), see Fig. 2) Then using the same argument as in [10] we may assume that the critical bands do not intersect with each other. Put \(F_1 = F \cap R^3[1/2, 1]\). Because \(F \cap R^3[0, 1/2]\) has no minimum, maximum points and critical bands, we see

\[
\left(F \cap R^3\left[0, \frac{1}{2}\right] - \partial\left(\bigcup_{i=1}^{n} B_i^3\right)\right) \cup \left(\bigcup_{i=1}^{n} \beta_i \cup \beta_i\right) \\
\approx \left(F \cap R^3\left[\frac{1}{2}, 1\right] - \partial\left(\bigcup_{i=1}^{n} B_i^3\right)\right) \cup \left(\bigcup_{i=1}^{n} \beta_i \cup \beta_i\right) \times \left[0, \frac{1}{2}\right] \text{ in } R^3[0, 1/2]
\]

\[\text{Fig. 2}\]
Then we project mutually disjoint bands $\bigcup_{i=1}^{n} B^3_i$ in $R^3[0,1]_2$ to $R^3[0]$ by a natural projection $p$, i.e., for any points $(x, y, z, t) \in R^4$, $p(x, y, z, t) = (x, y, z) \in R^3[0]$. Then we can take mutually disjoint 3-balls $B^3_i$ each of which contains only one band properly, i.e., $Int B^3_i \supset p(B^3_i)$ and $\partial B^3_i \supset \partial(p(B^3_i))$. So we may easily determine the surface $F'$ to be a required one. This completes the proof.

Let $l$ be a link in $R^3$ (or $R^3[0]$). $l$ is called a weak ribbon link if $l$ bounds a singular surface $F$ in $R^3$ of genus 0 with $\partial F = l$ and mutually disjoint ribbon singularities. And $l$ is called a weak slice link if $l$ bounds a non-singular locally flat orientable surface $F$ of genus 0 in $R^3[0, \infty)$, with $\partial F = l$. ([3], [4]).

Then if $l$ is a weak ribbon link, $l$ is also a weak slice link (see Lemma 5).

**Lemma 5.** $l$ is a weak ribbon link if and only if $l$ bounds a non-singular locally flat orientable surface $F$ in $R^3[0, 1]$ of genus 0 with $\partial F = l$ which has no minimum points.

**Proof.** If $l$ is a weak ribbon link, there is a singular surface $F_o$ in $R^3[0]$ of genus 0 with $\partial F_o = l$ and just ribbon singularities. Now we take small disks $D_i$, $i=1, \ldots, n$, on $F_o$ along the singularities such that $cl(F_o - \bigcup_{i=1}^{n} D_i)$ is a non-singular surface and $l \cap \bigcup_{i=1}^{n} D_i = \emptyset$. As $\partial(\bigcup_{i=1}^{n} D_i)$ is a trivial link, we may construct mutually disjoint cones $p_i^* \partial D_i$ in $R^3[0, 1]_2$, where $p_1, \ldots, p_n$ are different points in $R^3[0, 1]_2$. Then $(F_o - \bigcup_{i=1}^{n} D_i) \cup (\bigcup_{i=1}^{n} p_i^* \partial D_i)$ is a required surface $F$.

Conversely, let $F$ be a locally flat orientable surface of genus 0 with $\partial F = l$ which has no minimum points and is embedded in $R^3[0, 1)$. We can bring the maximum points of $F$ to $R^3[2]$ by the same technique we used to prove Lemma 3 without making new maximum and minimum points and with $\partial F$ fixed. Put the deformed surface $F'$. Clearly $F' \cap R^3[1] \approx O^n$ and $F' \cap R^3[0, 1]$ has no minimum and maximum points. So by Lemma 4, we may construct a proper surface $F''$ in $R^3[0, 1]$ which is isotopic to $F' \cap R^3[0, 1]$ and there exist mutually disjoint 3-balls $B^3_i$, $i=1, \ldots, n$, in $R^3[0]$ such that

$$cl(F'' - \bigcup_{i=1}^{n} B^3_i \times [0, 1]) \approx cl(F'' \cap R^3[0] - \bigcup_{i=1}^{n} B^3_i) \times [0, 1]$$

and the mutually disjoint bands $B^3_i$ are properly embedded in $B^3_i \times [0, 1]$. Let $D_i$, $i=1, \ldots, n$, be mutually disjoint disks in $R^3$ with boundary $O^n$. Then we project $\bar{F} = F'' \cup (\bigcup_{i=1}^{n} D_i)$ on $R^3[0]$ by a natural projection $p$. Then we may easily prove that $\partial p(\bar{F}) \approx l$ and the singularities of $p(\bar{F})$ are only ribbon singularities by
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an easy modification of disks and bands. Now the proof is complete.

REMARK. From this Lemma, l is a weak ribbon link if and only if $g^*_l(l)=0$ (Clearly l is a weak slice link if and only if $g^*_l(l)=0$).

DEFINITION 5. The minimum number of unlinking (I') operations required to deform a given link l into a weak slice link, a weak ribbon link are called the unlinking number of l (in the 4-dimensional sense), denoted by $u^*(l)$, $u^*_l(l)$ respectively. We may easily prove the following.

Lemma 6. For any link $l$, $u^*(l) \leq u^*_l(l) \leq u(l)$.

By Lemma 1 any link $l$ in $R^4[0]$ may span $\mu(l)$ singular disks $D$ whose only singularities are finite clasps. Let $\alpha_1$, ..., $\alpha_\mu$ be all the clasps on $D$ and take mutually disjoint regular neighborhoods $\cup N(\alpha_i; R^4[0])$. Then $\partial(N(\alpha_i; R^4[0]) \cap D) \approx \varnothing$. Let $p_1$, ..., $p_\mu$ be different points in $R^4[1]$ and make a cone $\tilde{D}_i=p_i* (\partial(N(\alpha_i; R^4[0]) \cap D))$ for each $i$ and we may construct these cones not to intersect with each other. Then $\tilde{D}=D- \bigcup_{i=1}^\mu N(\alpha_i; R^4[0]) \cup (\bigcup \tilde{D}_i)$ is a locally flat $\mu(l)$ disks with singularities $p_1$, ..., $p_\mu$ such that $\partial\left(N\left(p_i; R^4\left(\frac{1}{2}, \frac{3}{2}\right)\right) \cap \tilde{D}\right) \approx \varnothing$, $\partial \tilde{D}=l$ and $\tilde{D}$ has no minimum points. So we may define the clasp number of a link (in the 4-dimensional sense) as follows.

Let $F$ be an orientable surface of genus 0 with $\mu$ boundaries. Suppose that $f$ is a locally flat immersion of $F$ in $R^4(0, \infty)$ such that $f(\partial F)=l$ is a given link $l$ in $R^4[0]$, $f(\text{Int } F)\subset R^4(0, \infty)$ and the singularities of $f(\text{Int } F)$ are finite points $p_1$, ..., $p_\mu$ with $\partial B^4(p_i) \cap f(\text{Int } F) \approx \varnothing$.

DEFINITION 6. For all the locally flat immersions satisfying the above condition, the minimum number of these singularities is called the clasp number of $l$ and is denoted by $c^*(l)$. Especially when we restrict Definition 6 only for the locally flat immersions which has no minimum points, the minimum number of these singularities is denoted by $c^*_l(l)$.

Then the next Lemma is trivial from the definition and the explanation above Definition 6.

Lemma 7. For any link $l$, $c^*(l) \leq c^*_l(l) \leq c(l)$

Modifying the technique we used to prove Lemma 3, we obtain

Lemma 8. For any link $l$, there is a number $\mu$ such that $c^*(l)=c^*_l(l\cdot O^\mu)$ for some trivial link $O^\mu$.

Now we will examine the relation between $g^*(l)$, $c^*(l)$, $u^*(l)$ and $g^*_l(l)$, $c^*_l(l)$, $u^*_l(l)$.
Lemma 9. For any link, \( g^*(l) \leq c^*(l) \), \( g^*_r(l) \leq c^*_r(l) \).

Proof. Let \( F \) be a locally flat non-singular surface except \( c^*(l) \) points \( p_1, \ldots, p_m \), where \( n = c^*(l) \), with \( \partial F = l \) and \( l_i = \partial N(p_i; R^3[0, \infty)) \cap F \approx \bigcirc \). Then \( l_i \) may span an orientable surface \( F_i \) of genus \( 0 \) in \( \partial N(p_i; R^3[0, \infty)) \). So

\[
\tilde{F} = (F - \bigcup_{i=1}^n N(p_i; R^3[0, \infty))) \cup \bigcup_{i=1}^n F_i
\]

is a non-singular locally flat orientable surface of genus \( n \) with \( \partial \tilde{F} = l \). Thus \( g^*(l) \leq c^*(l) \). We can prove \( g^*_r(l) \leq c^*_r(l) \) by using the technique to prove the first half of Lemma 9. Now the proof is complete.

Lemma 10. For any link \( l \), \( c^*(l) \leq u^*(l) \) and \( c^*_r(l) \leq u^*_r(l) \).

Proof. Let \( l \) be a link in \( R^3[0] \). Now we perform \( u^*(l) \)-times (or \( u^*_r(l) \)-times) \((\Gamma)\) operation to \( l \) in \( R^3[0, 1] \) so that \( V \in R^3[1] \) is a weak slice (or weak ribbon) link. Then there exist proper annuli \( F \in R^3[0, 1] \) with \( \partial F = l \) and \( \partial F \) has no minimum and maximum points and singularities are finite points \( p_1, \ldots, p_n \) in \( \text{Int} F \), where \( m = c^*(l) \) (or \( m = c^*_r(l) \)). As \( l' \) is a weak slice (or a weak ribbon) link, we may span a locally flat orientable surface \( F \) in \( R^3[1, \infty) \) with \( \partial F = l' \) (if \( l' \) is a weak ribbon link, \( F \) has no minimum points by Lemma 5). Then there is a singular surface \( F_0 \cup F_1 \) of genus \( 0 \) whose boundary is \( l \). Thus \( c^*(l) \leq u^*(l) \) (or \( c^*_r(l) \leq u^*_r(l) \)). This completes the proof of Lemma 10.

And by Lemma 11, \( c^*_r(l) = u^*_r(l) \) follows.

Lemma 11. For any link \( l \), \( u^*_r(l) \leq c^*_r(l) \).

Proof. Let \( l \) be a link in \( R^3[0] \) and \( F \) be a surface in \( R^3[0, 1] \) which has no minimum points with \( \partial F = l \) and \( c^*_r(l) \) be the number of clasps. \( F \) has \( m \) singular points \( p_1, \ldots, p_m \) and \( n \) maximum points \( p_{m+1}, \ldots, p_{m+n} \), where \( m = c^*_r(l) \). We may connect these points to disjoint arcs \( q_1, \ldots, q_m, q_{m+1, \ldots, m+n} \) in \( R^3[2] \) by disjoint arcs \( \alpha_1, \ldots, \alpha_m + n \) such that \( \alpha_i \cap F = \partial \alpha_i \cap F = p_i \) and \( \alpha_i \cap R^3[1] \) is at most one point for each \( i, 0 < t < 2 \). By an isotopy we may bring \( p_i \) to \( q_i \) along \( \alpha_i \) with \( \partial F \) fixed to make a new surface \( F' \) which is isotopic to \( F \) and \( F' \) has no minimum and maximum points and singularities are finite points \( p_1, \ldots, p_m \). Let \( D_i = \{ p_i \} \) be disjoint balls in \( R^3[1] \) which is a component of \( O^* \). By Lemma 4, \( F' \) is deformed to \( F'' \) which is a proper surface in \( R^3[0, 1] \) and is isotopic to \( F' \cap R^3[0, 1] \) (or \( F' \cap R^3[0] \)).

Let \( D_i^3, i = 1, \ldots, m, \) be mutually disjoint 3-balls in \( R^3[1] \) such that \( D_i^3 \) contains only one \( \bigcirc \) in its interior and \( D_i^3 \cap D_j^3 = \phi \), where \( D_i^3 \) is a spanning disk of \( O_i \) which is a component of \( O^* \). Then we may take a simple arc \( \beta_i \) in \( p(D_i^3) - \bigcup_{j \neq i} B_j^3 \) to connect two points of \( l \) as shown in
Fig. 3 (a) such that $\otimes$ becomes a trivial link by the $(\Gamma)$ operation along $p^{-1}(\beta_i) \cap R^3[1]$ in $D^3_i$ for each $i$, where $p$ is a natural projection of $R^3[0, 1]$ to $R^3[0]$. Then we determine $\beta_{i_0}, \beta_{i_1}$ as shown in Fig. 3 (b) which may be taken in the neighborhood of $\beta_i$ and $F''''= (F'' - (\bigcup_{i=1}^{m} \beta_{i_0} \times [0, 1])) \cup (\bigcup_{i=1}^{m} \beta_{i_1} \times [0, 1]) \cup (\bigcup_{j=m+1}^{2m} D_j) \cup (\bigcup_{l=1}^{2m} D_l)$, where $D_i$ and $D_{m+i}$ are disjoint disks in $\text{Int} D^3$. Then as $F''''$ has no minimum points, $\partial F'''' \cap R^3[0] = l'$ is a weak ribbon link by Lemma 5 and $l$ is obtained from $l'$ by $c^*(l)$-times $(\Gamma)$ operation. So $u^*(l) \leq c^*(l)$ which completes the proof.

Let $\sigma(l)$ be the signature of a link (for the definition of $(l)$, see [7]), then it is known $\frac{1}{2} (|\sigma(l)| - \mu(l) + 1) \leq g^*(l)$ by Theorem 9.1 [7].

Now we complete our researches.

**Theorem 2.** For any link $l$, we obtain $\frac{1}{2} (|\sigma(l)| - \mu(l) + 1) \leq g^*(l)$ and

$$g^*(l) \leq c^*(l) \leq g(l)$$

$$c^*(l) \leq c^*(l) \leq c(l)$$

$$u^*(l) \leq u^*(l) \leq u(l)$$

**Remark.** If $l$ is a non-trivial weak ribbon link of 1 component, then $g^*(l) = c^*(l) = u^*(l) = 0$, but $g(l) \cdot c(l) \cdot u(l) \neq 0$.

**Question.** In the above diagram of 4-dimensional numerical invariants of links, which inequality can be replaced by an equality?
References


