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SOME RELATIONS AMONG VARIOUS NUMERICAL INVARIANTS FOR LINKS

TETSUO SHIBUYA

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Introduction. Throughout this paper, "a link \( l \) of \( \mu(l) \) components" means disjoint union of \( \mu(l) \) oriented 1-spheres in \( \mathbb{R}^3 \).

In §1, we study some 3-dimensional numerical invariants of links, that is, \( g(l) \) (genus of \( l \)), \( u(l) \) (see Definition 1) and \( c(l) \) (see Definition 3) will be defined and we will have some relations among them as follows.

Theorem 1. For any link \( l \), \( g(l) \leq c(l) \) and \( u(l) \leq c(l) \).

In §2, the 4-dimensional numerical invariants \( g^*(l) \), \( g^*_y(l) \) (see Definition 4), \( u^*(l) \), \( u^*_y(l) \) (see Definition 5), \( c^*(l) \) and \( c^*_y(l) \) (see Definition 6) will be defined and the main theorem will be proved.

Theorem 2. For any link \( l \), we obtain

\[
\begin{align*}
g^*(l) & \leq g^*_y(l) \leq g(l) \\
\forall l & \forall l & \forall l \\
c^*(l) & \leq c^*_y(l) \leq c(l) \\
\forall l & \forall l & \forall l \\
u^*(l) & \leq u^*_y(l) \leq u(l)
\end{align*}
\]

As is usual, two links \( l \) and \( l' \) are said to be of the same type or isotopic, denoted by \( l \simeq l' \), if there exists an orientation preserving homeomorphism \( f \) of \( \mathbb{R}^3 \) onto itself such that \( f(l) = l' \).

\( \partial X \), \( \text{Int } X \) and \( \text{cl } X \) represents the boundary, the interior and the closure of \( X \) respectively.

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1. 3-dimensional numerical invariants

Let \( l \) be a link of \( \mu(l) \) components in \( \mathbb{R}^3 \). It is known in [9], [11] that \( l \) always bounds an orientable connected surface \( F \) in \( \mathbb{R} \). The minimum genus of these surfaces is called the genus of the link \( l \) and is denoted by \( g(l) \). Note that \( g(F) \) denotes the usual genus of a surface \( F \).
Let \( L \) be a diagram of \( l \), i.e. \( L = p(l) \), where \( p \) is a regular projection of \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \) \([2]\). \( L \) has in general at least one double point if \( l \) is not a trivial link (unknotted and unlinked). A link can be deformed into a trivial link by employing a finite number of unlinking operation \((\Gamma)\) defined as follows.

\((\Gamma)\) \textit{Change an underpass into an overpass at a double point.}

\textbf{Definition 1.} The minimum number of unlinking operations required to deform a given link \( l \) into a trivial link is called the \textit{unlinking number} of \( l \) (in the 3-dimensional sense) and is denoted by \( u(l) \).

\textbf{Definition 2.} Let \( F_0 \) be a surface which may not be connected and \( f \) be an immersion of \( F_0 \) into \( \mathbb{R}^3 \). Put \( F = f(F_0) \). Suppose that \( F \) has a finite number of simple double lines and these double lines do not intersect each other. Each double line \( J \) is one of the following three types (see \([4]\))

1. a closed curve whose antecedents are closed curve \( J' \) and \( J'' \) that lie in \( \text{Int } F_0 \),
2. an arc whose antecedents are an arc \( J' \) that spans \( \partial F_0 \) and an arc \( J'' \) that lies entirely in \( \text{Int } F_0 \),
3. an arc whose antecedents are arcs \( J' \) and \( J'' \) each of which has an end point on \( \partial F_0 \), and the other one lies in \( \text{Int } F_0 \).

We call \( J \) a \textit{singularity} of \( F \). The singularities satisfying the condition (1), (2), (3) will be called \textit{(simple) loop}, \textit{ribbon} and \textit{clasp} singularities respectively. \([4]\)

We call \( F \) a \textit{non-singular} surface if \( f \) is an embedding.

Then, to define the \textit{clasp number} \( c(l) \) of a link \( l \) we need to prove the following lemma.

\textbf{Lemma 1.} Any link \( l \) spans \( \mu(l) \) singular disks whose singularities are only clasps and the number of these clasps is finite.

Proof. Let \( n \) be the unlinking number of \( l \) and \( p \) be a regular projection of \( l \) such that there exist \( n \) double points \( p_1, \ldots, p_n \) in \( p(l) \) and \( l \) becomes a trivial link by \((\Gamma)\)-operation along these points. We may make oriented small unknotted circles \( c_i, i=1, \ldots, 2n, \) near to \( p_i \) linking with \( l \) as shown Fig. 1 such that \( L(l, c_i) = -1 \) or \( -1 \) according as the orientation of \( l \), where \( p_i \) is a point of \( p^{-1}(p_i) \cap l \) and \( L(l, c) \) denotes the linking number of \( l \) and \( c \). Then there exist mutually disjoint bands \( B_i, i=1, \ldots, 2n, \) with \( B_i \cap l = \partial B_i \cap l \) an arc and

\[
\begin{align*}
\text{L}_+ & \partial (\bigcup_{i=1}^n B_i) + (\bigcup_{i=1}^n c_i) \approx O^\mu \\
\text{L}_+ & \partial (\bigcup_{i=1}^{2n} B_i) + (\bigcup_{i=1}^{2n} c_i) \approx l
\end{align*}
\]

where \( O^\mu \) is a trivial link of \( \mu = \mu(l) \) components and \( + \) means addition in the homology sense. Let \( E = \bigcup_{i=1}^{\mu} E_i \) be a union of mutually disjoint spanning disks
of $O^n$ and $B = \bigcup_{i=0}^{2n} B_i$. By a slight modification of $E$, $B$ and $D = \bigcup_{i=0}^{2n} D_i$, where $D$ is oriented mutually disjoint disks with $\partial D_i = c_i$, we have $B \cap D = \partial B \cap \partial D$, $B \cap E = (\partial B \cap \partial E) \cup$ (ribbon singularities), $D \cap E = (\text{clasp singularities})$ and $\partial (B \cup D \cup E) \approx l$. For each ribbon singularity $J$ we draw a simple area $\alpha_i$ on $E$ to connect a point of $dE$ and that of $\text{Int } J$ and put $E = \text{cl}(E - \bigcup_{i=1}^{r} N_i)$, where $r$ is the number of ribbon types on $E$ and $N_i$ is a regular neighborhood of $\alpha_i$ in $E$. Then clearly $\partial (B \cup D \cup \bar{E}) \approx l$ and the singularities of $\mu$ singular disks $B \cup D \cup \bar{E}$ are only clasps and of course the clasp number of $B \cup D \cup \bar{E}$ is finite. So the proof is complete.

**Definition 3.** For any link $l$, there is a singular disk with only clasps which spans $l$ by Lemma 1. The minimum number of the clasps is called the **clasp number** of $l$, denoted by $c(l)$.

Then we have,

**Theorem 1.** For any link $l$, $c(l) \geq u(l)$, $c(l) \geq g(l)$.

Proof. $c(l) \geq u(l)$ is obvious from the definitions of these numbers. So we have to prove $c(l) \leq g(l)$. Let $D$ be singular disks such that $c(D) = c(l)$ and $\partial D = l$, where $c(D)$ is the number of clasps of $D$. Making use of orientation preserving cuts ([4], [8]) along all clasps, we get an orientable surface $F$ of genus $c(l)$ such that $\partial F = \partial D = l$. So $c(l) \leq g(l)$, which completes the proof.

**Remark.** These inequalities can not be replaced by equalities. For example for the knot $6_2$, $6_2$ is alternating, so $g(6_2) = 2$ ([1]) and $c(6_2) = 2$ by using Theorem 1 but $u(6_2) = 1$, and for the link $\otimes$, $c(\otimes) = u(\otimes) = 1$ but $g(\otimes) = 0$. 
2. 4-dimensional numerical invariants

Let \( l \) be a link in \( R^3[0] \), where \( R^3[a]=\{(x, y, z) \in R^4 | t=a\} \). Since \( l \) bounds an orientable connected surface \( F \) in \( R^3[0] \), \( l \) always bounds an orientable locally flat connected surface in \( R^3[0, t_o]=\{(x, y, z, t) \in R^4 | 0 \leq t < t_o\} \). The minimum genus of these surfaces is an invariant of the link type ([3], [7]). It is denoted by \( g^*(l) \) (in the 4-dimensional sense).

**Definition 4.** Especially for any link \( l \) we may span an orientable locally flat surface \( F \) in \( R^3[0, t_o) \) which has no minimum points with \( dF=l \) in \( R^3[0) \). The minimum genus of these surfaces is called the ribbon type genus of \( l \) and is denoted by \( g^*(l) \).

It is clearly that \( g^*(l) \) is an invariant of the link type of \( l \).

Then from the definition of \( g^*(l) \), \( g^*(l) \) and \( g(l) \), we have

**Lemma 2.** For any link \( l \), \( g^*(l) \leq g^*(l) \leq g(l) \).

A link \( l \) will be called split into two components \( l_1 \) and \( l_2 \) if there is a 3-ball \( B^3 \) such that \( l \cap B^3 \neq \emptyset \). Then \( l \) is denoted by \( l=l_1 \cup l_2 \). Then

**Lemma 3.** For any link \( l \), there is a number \( \mu \) such that \( g^*(l)=g^*(l \cup \mu) \) for some trivial link \( l \cup \mu \) of \( \mu \) components.

Proof. Let \( F \) be a locally flat orientable surface in \( R^3[0, 1) \) with \( \partial F=l \) in \( R^3[0] \) and \( g(F)=g^*(l) \). Let \( p_1, \cdots, p_\mu \) be the minimum points of \( F \). We may take \( \mu \) distinct points \( q_1, \cdots, q_\mu \) in \( R^3[-1, 1] \) and disjoint simple arcs \( \alpha_1, \cdots, \alpha_\mu \) and \( \alpha_i \) connects \( p_1 \) with \( q_1 \) and \( \alpha_i \cap R^3[t] \) is at most one point for each \( i, 1 \leq i \leq \mu \) and \( t, 0 \leq t < 1 \). Then we can deform \( F \) to a surface \( F' \) by an isotopy along \( \alpha_i \). The minimum points of \( F' \) are \( q_i \) and \( F' \cap R^3[0, 1) \) has no minimum points. Of course, \( F' \cap R^3[0]=l \cup \mu \), so \( g^*(l \cup \mu) \leq g^*(l) \). ([5], [10]).

Conversely, let \( F_\mu \) be a locally flat surface in \( R^3[0, 1] \) with \( F_\mu \cap R^3[0]=l \cup \mu \) which has no minimum points and \( g^*(l \cup \mu)=g(F_\mu) \). In \( R^3[-1, 0] \) we make \( l \cup [-1, 0] \). As \( O^\mu \) is splitted from \( l \), \( O^\mu \) bounds mutually disjoint disks \( D_i, i=1, \cdots, \mu \) in \( R^3[-1, 0] \) which do not intersect with \( l \cup [-1, 0] \). So \( F=F_\mu \cup l \cup [-1, 0] \cup \bigcup D_i \) is a locally flat orientable surface with boundary \( l \) and \( g(F)=g(F_\mu)=g^*(l \cup \mu) \). Therefore \( g^*(l) \leq g(l \cup \mu) \), which completes the proof.

Lemma 4 is essential to prove the main theorem.

**Lemma 4.** Let \( F \) be a locally flat orientable surface which has no minimum and maximum points and \( F \cap R^3[0]=l, F \cap R^3[1]=l' \). Then there is a locally flat orientable surface \( F' \) properly embedded in \( R^3[0, 1] \) and isotopic to \( F \) in \( R^3[0, 1] \) \( (F' \cap R^3[0]=l \) in \( R^3[0], F' \cap R^3[1]=l' \) in \( R^3[1] \) respectively). Furthermore there exist some disjoint 3-balls \( B_i^3, i=1, \cdots, n \) in \( R^3[0] \) such that
\[ \text{cl}(F' - \bigcup_{i=1}^{n} B_{i}^{+3} \times [0, 1]) = \text{cl}(F' \cap R^{3}[0] - \bigcup_{i=1}^{n} B_{i}^{3}) \times [0, 1]. \]

Proof. It may be assumed that \( F \) has \( n \) critical points and \( R^3[t_i] \) contains only one critical point for \( t_i, 0 < t_i < \cdots < t_n < 1 \). A critical point \( p_i \) may be changed by a critical band \( B_i^+ \) for each \( i \) (see [6]). We may deform \( F \) by an isotopy of \( R^3[0, 1] \) carrying \( B_i^+ \) into \( R^3[\frac{1}{2}] \) so that maximum and minimum points do not appear in the resulting surface. We will write the resulting surface and the band \( F \) and \( B_i^+ \) again. Since \( F \cap R^3(\frac{1}{2}, 1) \) is a locally flat orientable surface which has no maximum, minimum points and critical bands,

\[
\left( F \cap R^3\left[ \frac{1}{2}, 1 \right] - \partial(\bigcup_{i=1}^{n} B_i^{+3}) \right) \cup \left( \bigcup_{i=1}^{n} \alpha_i \cup \alpha_i \right) \cong (F \cap R^3[1]) \times \left[ \frac{1}{2}, 1 \right]
\]

in \( R^3[\frac{1}{2}, 1] \) (for \( \alpha_i \) and \( \alpha_i \), see Fig. 2). Then using the same argument as in [10] we may assume that the critical bands do not intersect with each other. Put \( F = F \cap R^3[1] \times \left[ \frac{1}{2}, 1 \right] \). Because \( F \cap R^3\left[ 0, \frac{1}{2} \right) \) has no minimum, maximum points and critical bands, we see

\[
\left( F \cap R^3\left[ 0, \frac{1}{2} \right] - \partial(\bigcup_{i=1}^{n} B_i^{+3}) \right) \cup \left( \bigcup_{i=1}^{n} \beta_i \cup \beta_i \right) \cong (F \cap R^3[1]) \times \left[ 0, \frac{1}{2} \right] \text{ in } R^3\left[ 0, \frac{1}{2} \right]
\]
Then we project mutually disjoint bands \( \bigcup_{i=1}^{n} B_i^2 \) in \( \mathbb{R}^3 \left[ \frac{1}{2} \right] \) to \( \mathbb{R}^2[0] \) by a natural projection \( p \) i.e. for any points \((x, y, z, t) \in \mathbb{R}^4, p(x, y, z, t) = (x, y, z, 0) \in \mathbb{R}^2[0] \).

Then we can take mutually disjoint 3-balls \( B_i^3 \) each of which contains only one band properly, i.e. \( \text{Int} \ B_i^3 \supset \text{Int} \ p(B_i^3) \) and \( \partial B_i^3 \supset \partial(p(B_i^3)) \). So we may easily determine the surface \( F' \) to be a required one. This completes the proof.

Let \( l \) be a link in \( \mathbb{R}^3 \) (or \( \mathbb{R}^3[0] \)). \( l \) is called a **weak ribbon link** if \( l \) bounds a singular surface \( F \) in \( \mathbb{R}^3 \) of genus 0 with \( \partial F = l \) and mutually disjoint ribbon singularities. And \( l \) is called a **weak slice link** if \( l \) bounds a non-singular locally flat orientable surface \( F \) of genus 0 in \( \mathbb{R}^3[0, \infty) \) with \( \partial F = l \) \(([3], [4])\).

Then if \( l \) is a weak ribbon link \( l \) is also a weak slice link (see Lemma 5).

**Lemma 5.** \( l \) is a weak ribbon link if and only if \( l \) bounds a non-singular locally flat orientable surface \( F \) in \( \mathbb{R}^3[0, 1) \) of genus 0 with \( \partial F = l \) which has no minimum points.

Proof. If \( l \) is a weak ribbon link, there is a singular surface \( F_0 \) in \( \mathbb{R}^3[0] \) of genus 0 with \( \partial F_0 = l \) and just ribbon singularities. Now we take small disks \( D_i, i=1, \cdots, n, \) on \( F_0 \) along the singularities such that \( \text{cl}(F_0 - \bigcup_{i=1}^{n} D_i) \) is a non-singular surface and \( l \cap (\bigcup_{i=1}^{n} D_i) = \emptyset \). As \( \partial(\bigcup_{i=1}^{n} D_i) \) is a trivial link, we may construct mutually disjoint cones \( p_i^* \partial D_i \) in \( \mathbb{R}^3 \left[ 0, \frac{1}{2} \right] \), where \( p_1, \cdots, p_n \) are different points in \( \mathbb{R}^3 \left[ 0, \frac{1}{2} \right] \). Then \( (F_0 - \bigcup_{i=1}^{n} D_i) \cup (\bigcup_{i=1}^{n} p_i^* \partial D_i) \) is a required surface \( F \).

Conversely, let \( F \) be a locally flat orientable surface of genus 0 with \( \partial F = l \) which has no minimum points and is embedded in \( \mathbb{R}^3[0, 1) \). We can bring the maximum points of \( F \) to \( \mathbb{R}^3[2] \) by the same technique we used to prove Lemma 3 without making new maximum and minimum points and with \( \partial F \) fixed. Put the deformed surface \( F' \). Clearly \( F' \cap R^3[1] \approx O^n \) and \( F' \cap R^3[0, 1) \) has no minimum and maximum points. So by Lemma 4, we may construct a proper surface \( F'' \) in \( \mathbb{R}^3[0, 1] \) which is isotopic to \( F' \cap R^3[0, 1] \) and there exist mutually disjoint 3-balls \( B_i^3, i=1, \cdots, p, \) in \( \mathbb{R}^3[0] \) such that

\[
\text{cl}(F'' - \bigcup_{i=1}^{n} B_i^3 \times [0, 1]) \approx \text{cl}(F'' \cap \mathbb{R}^3[0] - \bigcup_{i=1}^{n} B_i^3) \times [0, 1]
\]

and the mutually disjoint bands \( B_i^3 \) are properly embedded in \( B_i^3 \times \left[ \frac{1}{2} \right] \). Let \( D, i=1, \cdots, n, \) be mutually disjoint disks in \( \mathbb{R}^3[1] \) with boundary \( O^n \). Then we project \( \tilde{F} = F'' \cup (\bigcup_{i=1}^{n} D_i) \) on \( \mathbb{R}^3[0] \) by a natural projection \( p \). Then we may easily prove that \( \partial p(\tilde{F}) \approx l \) and the singularities of \( p(\tilde{F}) \) are only ribbon singularities by
an easy modification of disks and bands. Now the proof is complete.

**Remark.** From this Lemma, \( l \) is a weak ribbon link if and only if \( g^* (l) = 0 \) (Clearly \( l \) is a weak slice link if and only if \( g^* (l) = 0 \)).

**Definition 5.** The minimum number of unlinking (\( \Gamma \)) operations required to deform a given link \( l \) into a weak slice link, a weak ribbon link are called the unlinking number of \( l \) (in the 4-dimensional sense), denoted by \( u^* (l) \), \( u^*_x (l) \) respectively. We may easily prove the following.

**Lemma 6.** For any link \( l \), \( u^* (l) \leq u^*_x (l) \leq u (l) \).

By Lemma 1 any link \( l \) in \( R[0] \) may span \( \mu (l) \) singular disks \( D \) whose only singularities are finite clasps. Let \( \alpha_1 , \ldots , \alpha_n \) be all the clasps on \( D \) and take mutually disjoint regular neighborhoods \( \bigcup \bigcap N (\alpha_i : R[0]) \). Then \( \partial (N (\alpha_i : R[0]) \cap D) \approx \emptyset \). Let \( p_1 , \ldots , p_n \) be different points in \( R[1] \) and make a cone \( D_i = \partial \overline{p}_i (\partial (N (\alpha_i : R[0]) \cap D)) \) for each \( i \) and we may construct these cones not to intersect with each other. Then \( D = (D - \bigcup_{i=1}^n N (\alpha_i : R[0])) \cup (\bigcup D_i) \) is a locally flat \( \mu (l) \) disks with singularities \( p_1 , \ldots , p_n \) such that \( \partial (N (p_i : R[\frac{1}{2}, \frac{3}{2}]) \cap D) \approx \emptyset \), \( \partial D = l \) and \( D \) has no minimum points. So we may define the clasp number of a link (in the 4-dimensional sense) as follows.

Let \( F \) be an orientable surface of genus 0 with \( \mu \) boundaries. Suppose that \( f \) is a locally flat immersion of \( F \) in \( R[0, \infty) \) such that \( f (\partial F) = l \) is a given link \( l \) in \( R[0] \), \( f (\text{Int} F) \subset R[0, \infty) \) and the singularities of \( f (\text{Int} F) \) are finite points \( p_1 , \ldots , p_n \) with \( \partial B^*(p_i) \cap f (\text{Int} F) \approx \emptyset \).

**Definition 6.** For all the locally flat immersions satisfying the above condition, the minimum number of these singularities is called the clasp number of \( l \) and is denoted by \( c^* (l) \). Especially when we restrict Definition 6 only for the locally flat immersions which has no minimum points, the minimum number of these singularities is denoted by \( c^*_x (l) \).

Then the next Lemma is trivial from the definition and the explanation above Definition 6.

**Lemma 7.** For any link \( l \), \( c^* (l) \leq c^*_x (l) \leq c (l) \).

Modifying the technique we used to prove Lemma 3, we obtain

**Lemma 8.** For any link \( l \), there is a number \( \mu \) such that \( c^* (l) = c^*_x (l \cdot O^\mu) \) for some trivial link \( O^\mu \).

Now we will examine the relation between \( g^* (l) \), \( c^* (l) \), \( u^* (l) \) and \( g^*_x (l) \), \( c^*_x (l) \), \( u^*_x (l) \).
Lemma 9. For any link, $g^*(l) \leq c^*(l)$, $g^*_e(l) \leq c^*_e(l)$.

Proof. Let $F$ be a locally flat non-singular surface except $c^*(l)$ points
$p_1, \ldots, p_m$ where $n = c^*(l)$, with $\partial F = l$ and $l_i = \partial N(p_i; R^3(0, \infty)) \cap F \approx \mathbb{S}^2$. Then
$l_i$ may span an orientable surface $F_i$ of genus 0 in $\partial N(p_i; R^3(0, \infty))$. So
\[
F = (F - \bigcup_{i=1}^n N(p_i; R^3(0, \infty))) \cup (\bigcup_{i=1}^m F_i)
\]
is a non-singular locally flat orientable surface of genus $n$ with $\partial F = l$. Thus $g^*(l) \leq c^*(l)$. We can prove $g^*_e(l) \leq c^*_e(l)$ by using the technique to prove the first half of Lemma 9. Now the proof is complete.

Lemma 10. For any link $l$, $c^*(l) \leq u^*(l)$ and $c^*_e(l) \leq u^*_e(l)$.

Proof. Let $l$ be a link in $R^3[0]$. Now we perform $u^*(l)$-times (or $u^*_e(l)$-times) $(\Gamma)$ operation to $l$ in $R^3[0, 1)$ so that $l'$ in $R^3[1]$ is a weak slice (or weak ribbon) link. Then there exist proper annuli $F_i$ in $R^3[0, 1]$ with $\partial F_i = l \cup (-l')$ and $F_i$ has no minimum and maximum points and singularities are finite points $p_1, \ldots, p_n$ in $\text{Int } F_i$, such that $\partial N(p_i; R^3(0, \infty)) \cap F_i \approx \mathbb{S}^2$. As $l'$ is a weak slice (or a weak ribbon) link, we may span a locally flat orientable surface $F$ in $R^3[1, \infty)$ with $\partial F = l$. Thus $c^*(l) \leq u^*(l)$ (or $c^*_e(l) \leq u^*_e(l)$). This completes the proof of Lemma 10.

And by Lemma 11, $c^*_e(l) = u^*_e(l)$ follows.

Lemma 11. For any link $l$, $u^*_e(l) \leq c^*_e(l)$.

Proof. Let $l$ be a link in $R^3[0]$ and $F$ be a surface in $R^3[0, 1]$ which has no minimum points with $\partial F = l$ and $c^*_e(l)$ be the number of clasps. $F$ has $m$ singular points $p_1, \ldots, p_m$ and $n$ maximum points $p_{m+1}, \ldots, p_{m+n}$, where $m = c^*_e(l)$. We may connect these points to distinct points $q_1, \ldots, q_{m+n}$ in $R^3[2]$ by disjoint arcs $\alpha_1, \ldots, \alpha_{m+n}$ such that $\alpha_i \cap F = \partial \alpha_i \cap F = p_i$ and $\alpha_i \cap R^3[2]$ is at most one point for each $i$, $0 < t < 2$. By an isotopy we may bring $p_i$ to $q_i$ along $\alpha_i$ with $\partial F$ fixed to make a new surface $F'$ which is isotopic to $F$ and $F'$ has no minimum and maximum points and singularities are finite points $p_1, \ldots, p_m$ in $R^3[0, 1]$ which is a component of $O^*$ is $m$. By Lemma 4, $F'$ is deformed to $F''$ which is a proper surface in $R^3[0, 1]$ and is isotopic to $F' \cap R^3[1] \approx \bigcup_{i=1}^m B^i \times [0, 1]$ for some mutually disjoint 3-balls $B^i$ in $R^3[0]$. Let $D^i, i = 1, \ldots, m$, be mutually disjoint 3-balls in $R^3[1]$ such that $D^i$ contains only one $O^*$ in its interior and $D^i \cap D^j = \emptyset$, where $D^i$ is a spanning disk of $O_i$ which is a component of $O^*$, for each $i, j, 1 \leq i \leq m, m+1 \leq j \leq m+n$. Then we may take a simple arc $\beta_i$ in $p(D^i) - \bigcup_{j=1}^i B^j$ to connect two points of $l$ as shown in
Fig. 3 (a) such that $\otimes$ becomes a trivial link by the $(\Gamma)$ operation along $p^{-1}(\beta_i) \cap R^3[1]$ in $D^3_i$ for each $i$, where $p$ is a natural projection of $R^3[0, 1]$ to $R^3[0]$. Then we determine $\beta_{i_0}, \beta_{i_1}$ as shown in Fig. 3 (b) which may be taken in the neighborhood of $\beta_i$ and $F''' = (F'' - (\bigcup_{i=1}^{n} \beta_{i_0} \times [0, 1])) \cup (\bigcup_{i=1}^{n} \beta_{i_1} \times [0, 1]) \cup (\bigcup_{j=m-1}^{n} D_j) \cup (\bigcup_{j=1}^{m} D_j)$, where $D_i$ and $D_{m+i}$ are disjoint disks in Int $D^3_i$. Then as $F'''$ has no minimum points, $\partial F''' \cap R^3[0] = l'$ is a weak ribbon link by Lemma 5 and $l$ is obtained from $l'$ by $c^*(l)$-times $(\Gamma)$ operation. So $u^*(l) \leq c^*(l)$ which completes the proof.

Let $\sigma(l)$ be the signature of a link (for the definition of $(l)$, see [7]), then it is known $1/2 (|\sigma(l)| - \mu(l) + 1) \leq g^*(l)$ by Theorem 9.1 [7].

Now we complete our researches.

**Theorem 2.** For any link $l$, we obtain $1/2 (|\sigma(l)| - \mu(l) + 1) \leq g^*(l)$ and

\[
\begin{align*}
g^*(l) & \leq g^2(l) \leq g(l) \\
c^*(l) & \leq c^*(l) \leq c(l) \\
u^*(l) & \leq u^*(l) \leq u(l)
\end{align*}
\]

**Remark.** If $l$ is a non-trivial weak ribbon link of 1 component, then $g^2(l) = c^*(l) = u^*(l) = 0$, but $g(l) \cdot c(l) \cdot u(l) \neq 0$.

**Question.** In the above diagram of 4-dimensional numerical invariants of links, which inequality can be replaced by an equality?
References