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SOME RELATIONS AMONG VARIOUS NUMERICAL INVARIANTS FOR LINKS

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Introduction. Throughout this paper, "a link l of \( \mu(l) \) components" means disjoint union of \( \mu(l) \) oriented 1-spheres in \( \mathbb{R}^3 \).

In §1, we study some 3-dimensional numerical invariants of links, that is, \( g(l) \) (genus of \( l \)), \( u(l) \) (see Definition 1) and \( c(l) \) (see Definition 3) will be defined and we will have some relations among them as follows.

**Theorem 1.** For any link \( l \), \( g(l) \leq c(l) \) and \( u(l) \leq c(l) \).

In §2, the 4-dimensional numerical invariants \( g^*(l) \), \( u^*(l) \) (see Definition 4), \( u^*_l(l) \) (see Definition 5), \( c^*(l) \) and \( c^*_l(l) \) (see Definition 6) will be defined and the main theorem will be proved.

**Theorem 2.** For any link \( l \), we obtain

\[
\begin{align*}
g^*(l) & \leq g^*_l(l) \leq g(l) \\
u^*(l) & \leq u^*_l(l) \leq u(l)
\end{align*}
\]

As is usual, two links \( l \) and \( l' \) are said to be of the same type or isotopic, denoted by \( l \approx l' \), if there exists an orientation preserving homeomorphism \( f \) of \( \mathbb{R}^3 \) onto itself such that \( f(l) = l' \).

\( \partial X \), \( \text{Int } X \) and \( \text{cl } X \) represents the boundary, the interior and the closure of \( X \) respectively.

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1. 3-dimensional numerical invariants

Let \( l \) be a link of \( \mu(l) \) components in \( \mathbb{R}^3 \). It is known in [9], [11] that \( l \) always bounds an orientable connected surface \( F \) in \( \mathbb{R} \). The minimum genus of these surfaces is called the genus of the link \( l \) and is denoted by \( g(l) \). Note that \( g(F) \) denotes the usual genus of a surface \( F \).
Let $L$ be a diagram of $l$, i.e. $L = p(l)$, where $p$ is a regular projection of $\mathbb{R}^3$ to $\mathbb{R}^2$ ([2]). $L$ has in general at least one double point if $l$ is not a trivial link (unknotted and unlinked). A link can be deformed into a trivial link by employing a finite number of unlinking operation $(\Gamma)$ defined as follows.

(\Gamma) Change an underpass into an overpass at a double point.

**Definition 1.** The minimum number of unlinking operations required to deform a given link $l$ into a trivial link is called the *unlinking number* of $l$ (in the 3-dimensional sense) and is denoted by $u(l)$.

**Definition 2.** Let $F_0$ be a surface which may not be connected and $f$ be an immersion of $F_0$ into $\mathbb{R}^3$. Put $F = f(F_0)$. Suppose that $F$ has a finite number of simple double lines and these double lines do not intersect each other. Each double line $J$ is one of the following three types (see [4])

1. a closed curve whose antecedents are closed curve $J'$ and $J''$ that lie in $\text{Int } F_0$,
2. an arc whose antecedents are an arc $J'$ that spans $\partial F_0$ and an arc $J''$ that lies entirely in $\text{Int } F_0$,
3. an arc whose antecedents are arcs $J'$ and $J''$ each of which has an end point on $\partial F_0$ and the other one lies in $\text{Int } F_0$.

We call $J$ a *singularity* of $F$. The singularities satisfying the condition (1), (2), (3) will be called *simple) loop, ribbon and clasp* singularities respectively. [4]

We call $F$ a *non-singular* surface if $f$ is an embedding.

Then, to define the *clasp number* $c(l)$ of a link $l$ we need to prove the following lemma.

**Lemma 1.** Any link $l$ spans $\mu(l)$ singular disks whose singularities are only clasps and the number of these clasps is finite.

Proof. Let $n$ be the unlinking number of $l$ and $p$ be a regular projection of $l$ such that there exist $n$ double points $p_1, \ldots, p_n$ in $p(l)$ and $l$ becomes a trivial link by $(\Gamma)$-operation along these points. We may make oriented small unknotted circles $c_i, i=1, \ldots, 2n$, near to $p_i$, linking with $l$ as shown Fig. 1 such that $L(l, c_i) = -L(l, c_{n+i}) = 1$ or $-1$ according as the orientation of $l$, where $p_i$ is a point of $p^{-1}(p_i) \cap l$ and $L(l, c)$ denotes the linking number of $l$ and $c$. Then there exist mutually disjoint bands $B_i, i=1, \ldots, 2n$, with $B_i \cap l = \partial B_i \cap l$ an arc and

\[
l + \partial \left( \bigcup_{i=1}^{n} B_i \right) + \left( \bigcup_{i=1}^{n} c_i \right) \approx O^\mu
\]

\[
l + \partial \left( \bigcup_{i=1}^{2n} B_i \right) + \left( \bigcup_{i=1}^{2n} c_i \right) \approx l
\]

where $O^\mu$ is a trivial link of $\mu = \mu(l)$ components and $+$ means addition in the homology sense. Let $E = \bigcup_{i=1}^{\mu} E_i$ be a union of mutually disjoint spanning disks
of $O^a$ and $B = \bigcup_{i=n+1}^{2n} B_i$. By a slight modification of $E$, $B$ and $D = \bigcup_{i=n+1}^{2n} D_i$, where $D$ is oriented mutually disjoint disks with $\partial D_i = c_i$, we have $B \cap D = \partial B \cap \partial D$, $B \cap E = (\partial B \cap \partial E) \cup$ (ribbon singularities), $D \cap E = $ (clasp singularities) and $\partial (B \cup D \cup E) \approx l$. For each ribbon singularity $J$ we draw a simple area $\alpha_i$ on $E$ to connect a point of $\partial E$ and that of $\text{Int} J$ and put $\overline{E} = \text{cl}(E - \bigcup_{i=1}^r N_i)$, where $r$ is the number of ribbon types on $E$ and $N_i$ is a regular neighborhood of $\alpha_i$ in $E$. Then clearly $\partial (B \cup D \cup \overline{E}) \approx l$ and the singularities of $\mu$ singular disks $B \cup D \cup \overline{E}$ are only clasps and of course the clasp number of $B \cup D \cup \overline{E}$ is finite. So the proof is complete.

**Definition 3.** For any link $l$, there is a singular disk with only clasps which spans $l$ by Lemma 1. The minimum number of the clasps is called the **clasp number** of $l$, denoted by $c(l)$.

Then we have,

**Theorem 1.** For any link $l$, $c(l) \geq u(l)$, $c(l) \geq g(l)$.

Proof. $c(l) \geq u(l)$ is obvious from the definitions of these numbers. So we have to prove $c(l) \geq g(l)$. Let $D$ be singular disks such that $c(D) = c(l)$ and $\partial D = l$, where $c(D)$ is the number of clasps of $D$. Making use of orientation preserving cuts ([4], [8]) along all clasps, we get an orientable surface $F$ of genus $c(l)$ such that $\partial F = \partial D = l$. So $c(l) \geq g(l)$, which completes the proof.

**Remark.** These inequalities can not be replaced by equalities. For example for the knot $6_2$, $6_2$ is alternating, so $g(6_2) = 2$ ([1]) and $c(6_2) = 2$ by using Theorem 1 but $u(6_2) = 1$, and for the link $\otimes$, $c(\otimes) = u(\otimes) = 1$ but $g(\otimes) = 0$.  

![Fig. 1](image-url)
2. 4-dimensional numerical invariants

Let \( l \) be a link in \( \mathbb{R}^3 \), where \( \mathbb{R}^4 = \{ (x, y, z, t) \in \mathbb{R}^4 | t = a \} \). Since \( l \) bounds an orientable connected surface \( F \) in \( \mathbb{R}^4 \), \( l \) always bounds an orientable locally flat connected surface in \( \mathbb{R}^3 \times [0, 1] = \{ (x, y, z, t) \in \mathbb{R}^4 | 0 \leq t < t_0 \} \). The minimum genus of these surfaces is an invariant of the link type ([3], [7]). It is denoted by \( g^*(l) \) (in the 4-dimensional sense).

**Definition 4.** Especially for any link \( l \) we may span an orientable locally flat surface \( F \) in \( \mathbb{R}^3 \times [0, 1) \) which has no minimum points with \( \partial F = l \) in \( \mathbb{R}^3 \). The minimum genus of these surfaces is called the ribbon type genus of \( l \) and is denoted by \( g^?(l) \).

It is clearly that \( g^*(l) \) is an invariant of the link type of \( l \).

**Lemma 2.** For any link \( l \), \( g^*(l) \leq g^?(l) \leq g(l) \).

A link \( l \) will be called split into two components \( l_1 \) and \( l_2 \) if there is a 3-ball \( B^3 \) such that \( \partial B^3 \cap \mathbb{R}^3 = l \). Then \( l \) is denoted by \( l = l_1 \cup l_2 \). Then

**Lemma 3.** For any link \( l \), there is a number \( \mu \) such that \( g^*(l) = g^?((l \circ O^\mu)) \) for some trivial link \( O^\mu \) of \( \mu \) components.

Proof. Let \( F \) be a locally flat orientable surface in \( \mathbb{R}^3 \times [0, 1) \) with \( \partial F = l \) in \( \mathbb{R}^3 \) and \( g(F) = g^*(l) \). Let \( p_1, \ldots, p_\mu \) be the minimum points of \( F \). We may take \( \mu \) distinct points \( q_1, \ldots, q_\mu \) in \( \mathbb{R}^3 \times [0, 1] \) and disjoint simple arcs \( \alpha_1, \ldots, \alpha_\mu \) and \( \alpha_i \) connects \( p_i \) with \( q_i \) and \( \alpha_i \cap \mathbb{R}^3 \{t\} \) is at most one point for each \( i \), \( 1 \leq i \leq \mu \) and \( t, 0 \leq t < 1 \). Then we can deform \( F \) to a surface \( F' \) by an isotopy along \( \alpha_i \). The minimum points of \( F' \) are \( q_i \) and \( F' \cap \mathbb{R}^3 \{t\} \) has no minimum points. Of course, \( F' \cap \mathbb{R}^3 \approx l \circ O^\mu \), so \( g^?((l \circ O^\mu)) \leq g^*(l) \). ([5], [10]).

Conversely, let \( F_0 \) be a locally flat surface in \( \mathbb{R}^3 \times [0, 1) \) with \( F_0 \cap \mathbb{R}^3 = l \circ O^\mu \) which has no minimum points and \( g^?((l \circ O^\mu)) = g(F_0) \). In \( \mathbb{R}^3 \times [-1, 0) \) we make \( l \times [-1, 0] \). As \( O^\mu \) is splitted from \( l \), \( O^\mu \) bounds mutually disjoint disks \( D_i, i=1, \ldots, \mu \) in \( \mathbb{R}^3 \times [-1, 0] \) which do not intersect with \( l \times [-1, 0] \). So \( F = F_0 \cup \cup D_i \) is a locally flat orientable surface with boundary \( l \) and \( g(F) = g(F_0) = g^?((l \circ O^\mu)) \). Therefore \( g^*(l) \leq g(l \circ O^\mu) \), which completes the proof.

Lemma 4 is essential to prove the main theorem.

**Lemma 4.** Let \( F \) be a locally flat orientable surface which has no minimum and maximum points and \( F \cap \mathbb{R}^3 = l \). Then there is a locally flat orientable surface \( F' \) properly embedded in \( \mathbb{R}^3 \) and isotopic to \( F \) in \( \mathbb{R}^3 \) \( (F' \cap \mathbb{R}^3 \approx l \cap \mathbb{R}^3, F' \cap \mathbb{R}^3 \approx l' \) in \( \mathbb{R}^3 \) respectively). Furthermore there exist some disjoint 3-balls \( B_i, i=1, \ldots, n \), in \( \mathbb{R}^3 \) such that...
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\[ cl(F' - \bigcup_{i=1}^{n} B_i^3 \times [0, 1]) = cl(F' \cap R^3[0] - \bigcup_{i=1}^{n} B_i^3) \times [0, 1]. \]

Proof. It may be assumed that \( F \) has \( n \) critical points and \( R^n[0, 1] \) contains only one critical point for \( t_i, 0 < t_i < \cdots < t_n < 1 \). A critical point \( p_i \) may be changed by a critical band \( B_i^3 \) for each \( i \) (see [6]). We may deform \( F \) by an isotopy of \( R^n[0, 1] \) carrying \( B_i^3 \) into \( R^3\left[\frac{1}{2}\right] \) so that maximum and minimum points do not appear in the resulting surface. We will write the resulting surface and the band \( F \) and \( B_i^3 \) again. Since \( F \cap R^3\left[\frac{1}{2}, 1\right] \) is a locally flat orientable surface which has no maximum, minimum points and critical bands,

\[
\left( F \cap R^3\left[\frac{1}{2}, 1\right] - \partial(\bigcup_{i=1}^{n} B_i^3) \right) \cup (\bigcup_{i=1}^{n} \alpha_i \cup \alpha_i) \approx (F \cap R^3[1]) \times \left[\frac{1}{2}, 1\right]
\]

in \( R^3\left[\frac{1}{2}, 1\right] \) (for \( \alpha_i \) and \( \bar{\alpha}_i \), see Fig. 2). Then using the same argument as in [10] we may assume that the critical bands do not intersect with each other. Put \( F_1 = F \cap R^3[1] \times \left[\frac{1}{2}, 1\right] \). Because \( F \cap R^3\left[0, \frac{1}{2}\right] \) has no minimum, maximum points and critical bands, we see

\[
\left( F \cap R^3\left[0, \frac{1}{2}\right] - \partial(\bigcup_{i=1}^{n} B_i^3) \right) \cup (\bigcup_{i=1}^{n} \beta_i \cup \beta_i) \approx \left( F_1 \cap R^3\left[\frac{1}{2}, 1\right] - \partial(\bigcup_{i=1}^{n} B_i^3) \right) \cup (\bigcup_{i=1}^{n} \beta_i \cup \beta_i) \times \left[0, \frac{1}{2}\right] \text{ in } R^3\left[0, \frac{1}{2}\right]
\]

\[ R^3\left[\frac{1}{2} + \epsilon\right] \]

\[ R^3\left[\frac{1}{2}\right] \]

\[ R^3\left[\frac{1}{2} - \epsilon\right] \]

Fig. 2
Then we project mutually disjoint bands $\bigcup_{i=1}^n B_i^3$ in $R^3\left[ \frac{1}{2} \right]$ to $R^3[0]$ by a natural projection $p$ i.e. for any points $(x, y, z, t) \in R^4$, $p(x, y, z, t) = (x, y, z, 0) \in R^3[0]$. Then we can take mutually disjoint 3-balls $B_i^3$ each of which contains only one band properly, i.e. $\text{Int } B_i^3 \supset \text{Int } p(B_i^3)$ and $\partial B_i^3 \supset \partial(p(B_i^3))$. So we may easily determine the surface $F'$ to be a required one. This completes the proof.

Let $l$ be a link in $R^3$ (or $R^3[0]$). $l$ is called a **weak ribbon link** if $l$ bounds a singular surface $F$ in $R^3$ of genus 0 with $\partial F = l$ and mutually disjoint ribbon singularities. And $l$ is called a **weak slice link** if $l$ bounds a non-singular locally flat orientable surface $F$ of genus 0 in $R^3[0, \infty)$ with $\partial F = l$. ([3], [4]).

Then if $l$ is a weak ribbon link $l$ is also a weak slice link (see Lemma 5).

**Lemma 5.** $l$ is a weak ribbon link if and only if $l$ bounds a non-singular locally flat orientable surface $F$ in $R^3[0, 1]$ of genus 0 with $\partial F = l$ which has no minimum points.

Proof. If $l$ is a weak ribbon link, there is a singular surface $F_o$ in $R^3[0]$ of genus 0 with $\partial F_o = l$ and just ribbon singularities. Now we take small disks $D_i$, $i=1, \ldots, n$, on $F_o$ along the singularities such that $\text{cl}(F_o - \bigcup_{i=1}^n D_i)$ is a non-singular surface and $l \cap (\bigcup_{i=1}^n D_i) = \emptyset$. As $\partial(\bigcup_{i=1}^n D_i)$ is a trivial link, we may construct mutually disjoint cones $p_i^+ \partial D_i$ in $R^3\left[ 0, \frac{1}{2} \right]$, where $p_1, \ldots, p_n$ are different points in $R^3\left[ \frac{1}{2} \right]$. Then $(F_o - \bigcup_{i=1}^n D_i) \cup (\bigcup_{i=1}^n p_i^+ \partial D_i)$ is a required surface $F$.

Conversely, let $F$ be a locally flat orientable surface of genus 0 with $\partial F = l$ which has no minimum points and is embedded in $R^3[0, 1)$. We can bring the maximum points of $F$ to $R^3[2]$ by the same technique we used to prove Lemma 3 without making new maximum and minimum points and with $\partial F$ fixed. Put the deformed surface $F'$. Clearly $F' \cap R^3[1] \approx O^*$ and $F' \cap R^3[0, 1]$ has no minimum and maximum points. So by Lemma 4, we may construct a proper surface $F''$ in $R^3[0, 1]$ which is isotopic to $F' \cap R^3[0, 1]$ and there exist mutually disjoint 3-balls $B_i^3$, $i=1, \ldots, p$, in $R^3[0]$ such that

$$\text{cl}(F'' - \bigcup_{i=1}^p B_i^3 \times [0, 1]) \approx \text{cl}(F'' \cap R^3[0] - \bigcup_{i=1}^p B_i^3) \times [0, 1]$$

and the mutually disjoint bands $B_i^3$ are properly embedded in $B_i^3 \times \left[ \frac{1}{2} \right]$. Let $D_i$, $i=1, \ldots, n$, be mutually disjoint disks in $R^3[1]$ with boundary $O^n$. Then we project $\tilde{F}' = F'' \cup (\bigcup_{i=1}^n D_i)$ on $R^3[0]$ by a natural projection $\tilde{p}$. Then we may easily prove that $\partial p(\tilde{F}) \approx l$ and the singularities of $p(\tilde{F})$ are only ribbon singularities by
an easy modification of disks and bands. Now the proof is complete.

Remark. From this Lemma, \( l \) is a weak ribbon link if and only if \( g^*(l) = 0 \) (Clearly \( l \) is a weak slice link if and only if \( g^*(l) = 0 \)).

Definition 5. The minimum number of unlinking (\( \Gamma \)) operations required to deform a given link \( l \) into a weak slice link, a weak ribbon link are called the \textit{unlinking number} of \( l \) (in the 4-dimensional sense), denoted by \( u^*(l) \), \( u^*_s(l) \) respectively. We may easily prove the following.

Lemma 6. For any link \( l \), \( u^*(l) \leq u^*_s(l) \leq u(l) \).

By Lemma 1 any link \( l \) in \( R^4[0] \) may span \( \mu(l) \) singular disks \( D \) whose only singularities are finite clasps. Let \( \alpha_i, \ldots, \alpha_n \) be all the clasps on \( D \) and take mutually disjoint regular neighborhoods \( \bigcup N(\alpha_i; R^4[0]) \). Then \( \partial(N(\alpha_i; R^4[0]) \cap D) \approx \varnothing \). Let \( p_i, \ldots, p_n \) be different points in \( R^4[1] \) and make a cone \( \bar{D}_i = p_i \ast (\partial(N(\alpha_i; R^4[0]) \cap D)) \) for each \( i \) and we may construct these cones not to intersect with each other. Then \( \bar{D} = (\bigcup_{i=1}^n N(\alpha_i; R^4[0])) \cup (\bigcup_{i=1}^n \bar{D}_i) \) is a locally flat \( \mu(l) \) disks with singularities \( p_i, \ldots, p_n \) such that \( \partial(N(p_i; R^4[1/2, 3/2])) \cap \bar{D} \approx \varnothing \), \( \partial \bar{D} = l \) and \( \bar{D} \) has no minimum points. So we may define the clasp number of a link (in the 4-dimensional sense) as follows.

Let \( F \) be an orientable surface of genus 0 with \( \mu \) boundaries. Suppose that \( f \) is a locally flat immersion of \( F \) in \( R^4[0, \infty) \) such that \( f(\partial F) = l \) is a given link \( l \) in \( R^4[0] \), \( f(\text{Int} F) \subset R^4(0, \infty) \) and the singularities of \( f(\text{Int} F) \) are finite points \( p_i, \ldots, p_n \) with \( \partial B^4(p_i) \cap f(\text{Int} F) \approx \varnothing \).

Definition 6. For all the locally flat immersions satisfying the above condition, the minimum number of these singularities is called the \textit{clasp number} of \( l \) and is denoted by \( c^*(l) \). Especially when we restrict Definition 6 only for the locally flat immersions which has no minimum points, the minimum number of these singularities is denoted by \( c^*_s(l) \).

Then the next Lemma is trivial from the definition and the explanation above Definition 6.

Lemma 7. For any link \( l \), \( c^*(l) \leq c^*_s(l) \leq c(l) \).

Modifying the technique we used to prove Lemma 3, we obtain

Lemma 8. For any link \( l \), there is a number \( \mu \) such that \( c^*(l) = c^*_s(l \circ O^\mu) \) for some trivial link \( O^\mu \).

Now we will examine the relation between \( g^*(l) \), \( c^*(l) \), \( u^*(l) \) and \( g^*_s(l) \), \( c^*_s(l) \), \( u^*_s(l) \).
Lemma 9. For any link, \( g^*(l) \leq c^*(l) \), \( g^*_\tau(l) \leq c^*_\tau(l) \).

Proof. Let \( F \) be a locally flat non-singular surface except \( c^*(l) \) points \( p_1, \ldots, p_n \), where \( n = c^*(l) \), with \( \partial F = l \) and \( l_i = \partial N(p_i; R^3[0, \infty)) \cap F \approx \emptyset \). Then \( l_i \) may span an orientable surface \( F_i \) of genus 0 in \( \partial N(p_i; R^3[0, \infty)) \). So
\[
\widetilde{F} = (F - \bigcup_{i=1}^n N(p_i; R^3[0, \infty))) \cup \left( \bigcup_{i=1}^n F_i \right)
\]
is a non-singular locally flat orientable surface of genus \( n \) with \( \partial \widetilde{F} = l \). Thus \( g^*(l) \leq c^*(l) \). We can prove \( g^*_\tau(l) \leq c^*_\tau(l) \) by using the technique to prove the first half of Lemma 9. Now the proof is complete.

Lemma 10. For any link \( l \), \( c^*(l) \leq u^*(l) \) and \( c^*_\tau(l) \leq u^*_\tau(l) \).

Proof. Let \( l \) be a link in \( R^3[0] \). Now we perform \( u^*(l) \)-times (or \( u^*_\tau(l) \)-times) \((\Gamma)\) operation to \( l \) in \( R^3 \) so that \( V \) in \( R^3[1] \) is a weak slice (or weak ribbon) link. Then there exist proper annuli \( F_a \) in \( R^3[0, 1] \) with \( \partial F_a = l \cup (-l') \) and \( F_a \) has no minimum and maximum points and singularities are finite points \( p_1, \ldots, p_n \) in \( \text{Int} \ F_a \) where \( n = u^*(l) \) (or \( u^*_\tau(l) \)), such that \( \partial N(p_i; R^3[0, \infty)) \cap F_a \approx \emptyset \). As \( l' \) is a weak slice (or a weak ribbon) link, we may span a locally flat orientable surface \( F \) in \( R^3[1, \infty) \) with \( \partial F = l' \) (if \( l' \) is a weak ribbon link, \( l' \) has no minimum points). By Lemma 5. Then there is a singular surface \( F_0 \cup F_1 \) of genus 0 whose boundary is \( l \). Thus \( c^*(l) \leq u^*(l) \) (or \( c^*_\tau(l) \leq u^*_\tau(l) \)). This completes the proof of Lemma 10.

And by Lemma 11, \( c^*_\tau(l) = u^*_\tau(l) \) follows.

Lemma 11. For any link \( l \), \( u^*_\tau(l) \leq c^*_\tau(l) \).

Proof. Let \( l \) be a link in \( R^3[0] \) and \( F \) be a surface in \( R^3[0, 1] \) which has no minimum points with \( \partial F = l \) and \( c^*_\tau(l) \) be the number of clasps. \( F \) has \( m \) singular points \( p_1, \ldots, p_m \) and \( n \) maximum points \( p_{m+1}, \ldots, p_{m+n} \), where \( m = c^*_\tau(l) \). We may connect these points to distinct points \( q_1, \ldots, q_{m+n} \) in \( R^3[1] \) by disjoint arcs \( \alpha_1, \ldots, \alpha_{m+n} \) such that \( \alpha_i \cap F = \partial \alpha_i \cap F = p_i \) and \( \alpha_i \cap R^3[1] \) is at most one point for each \( i, 0 \leq t \leq 2 \). By an isotopy we may bring \( p_i \) to \( q_i \) along \( \alpha_i \) with \( \partial F \) fixed to make a new surface \( F' \) which is isotopic to \( F \) and \( F' \cap R^3[1] \approx \emptyset \cdots \emptyset \cup O^* \), where the number of \( \emptyset \) is \( m \). By Lemma 4, \( F' \) is deformed to \( F'' \) which is a proper surface in \( R^3[0, 1] \) and is isotopic to \( F'' \cap R^3[0, 1] \approx F' \cap R^3[0, 1] \approx F'' \cap R^3[0] \) in \( R^3[0] \) and \( F'' \cap R^3[1] \approx F' \cap R^3[1] \) in \( R^3[1] \), and \( cl(F'' \cap R^3[0] - \bigcup_{i=1}^m B_i \times [0, 1]) = cl(F'' \cap R^3[0] - \bigcup_{i=1}^m B_i \times [0, 1]) \) for some mutually disjoint 3-balls \( B_i \) in \( R^3[0] \).

Let \( D_i, i = 1, \ldots, m \), be mutually disjoint 3-balls in \( R^3[1] \) such that \( D_i \) contains only one \( \emptyset \) in its interior and \( D_i \cap D_j = \emptyset \), where \( D_i \) is a spanning disk of \( O_i \), which is a component of \( O^* \), for each \( i, j, 1 \leq i \leq m, m+1 \leq j \leq m+n \). Then we may take a simple arc \( \beta_i \) in \( p(D_i) - \bigcup_{j=1}^m B_j \) to connect two points of \( l \) as shown in
Let $\sigma(l)$ be the signature of a link (for the definition of $(l)$, see [7]), then it is known $\frac{1}{2}(|\sigma(l)| - \mu(l) + 1) \leq g^*(l)$ by Theorem 9.1 [7].

Now we complete our researches.

**Theorem 2.** For any link $l$, we obtain $\frac{1}{2}(|\sigma(l)| - \mu(l) + 1) \leq g^*(l)$ and

\[
\begin{align*}
g^*(l) & \leq g^x_\infty(l) \leq g(l) \\
c^*(l) & \leq c^x_\infty(l) \leq c(l) \\
u^*(l) & \leq u^x_\infty(l) \leq u(l)
\end{align*}
\]

**Remark.** If $l$ is a non-trivial weak ribbon link of 1 component, then $g^x_\infty(l) = c^x_\infty(l) = u^x_\infty(l) = 0$, but $g(l) \cdot c(l) \cdot u(l) \neq 0$.

**Question.** In the above diagram of 4-dimensional numerical invariants of links, which inequality can be replaced by an equality?
References


