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GEVREY REGULARIZING EFFECT FOR NONLINEAR SCHRÖDINGER EQUATIONS

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1. Introduction

We consider the following Cauchy problem of nonlinear Schrödinger equations in n space dimensions,

$$(1.1) \quad \begin{cases} Lu \equiv i\partial_t u + \Delta u = f(t, x, u), \\ u(0, x) = \phi(x), \end{cases}$$

where $\Delta = \sum_{j=1}^n \partial^2 / \partial x_j^2$ and $f(t, x, u)$ is a complex valued function of Gevrey class in $(t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{C}$. We study the regularizing effect for (1.1). In what follows, we show that if the initial data ϕ is in some Gevrey class of order s with respect to $x \cdot \nabla_x$, then the solution u is in Gevrey class of order $\max(s/2, 1)$ with respect to x .

Concerning the regularizing effect for dispersive equations, many works have been done ([1], [2], [5], [6], [7], [8], [9]). All the above works treat regularizing effects with respect to Sobolev spaces. In [4], N. Hayashi and one of the authors treat regularity in time for nonlinear Schrödinger equations. They have shown that if the initial data is in Gevrey class of order s (≥ 1) with respect to $x \cdot \nabla$ and ∇ , then the solution is in Gevrey class of order s in space-time variables for $t \neq 0$. In [3], A. de Bouard, N. Hayashi and one of the authors treat Gevrey regularizing effect for nonlinear Schrödinger equations in one space dimension and Korteweg-de Vries equation. They have shown that if the initial data is in Gevrey class of order s (≥ 1) with respect to $x \cdot \nabla$ and ∇ , then the solution is in Gevrey class of order $\max(1, s/2)$ (or $\max(1, s/3)$ for KdV) with respect to the space variable for $t \neq 0$. We extend their results to the case that the nonlinear term is not polynomial, and for the local property, we extend their results to the case of higher space dimensions.

We introduce some notation and some function spaces to state the result precisely. Let $H^m(\Omega)$ denote Sobolev space of order m with respect to L^2 for an open set Ω in \mathbb{R}^n . For simplicity we write $H^m = H^m(\mathbb{R}^n)$. For a vector field Q with analytic coefficients and for a positive number M , we define a function space of Gevrey class $G_M^s(Q; H^m)$ in \mathbb{R}^n as follows:

$$G_M^s(Q; H^m) = \{g \in H^m; \|g\|_{G_M^s(Q; H^m)} = \|g\|_{H^m} + \sum_{j=1}^{\infty} M^{j-1} \|Q^j g\|_{H^m} / j!(j-1)!^{s-1} < \infty\}.$$

We also define a function space of Gevrey class $C([0, T]; G_M^s(Q; H^m))$ as follows:

$$\begin{aligned} C([0, T]; G_M^s(Q; H^m)) \\ = \{g \in C([0, T]; H^m); \|g\|_{C([0, T]; G_M^s(Q; H^m))} = \sup_{t \in [0, T]} \|g\|_{G_M^s(Q; H^m)} < \infty\}. \end{aligned}$$

We write $P = 2t\partial_t + x \cdot \nabla_x$ with $\nabla_x = (\partial_{x_1}, \dots, \partial_{x_n})$ and set $m = [n/2] + 1$ throughout this paper.

ASSUMPTION 1.1. The nonlinear term f satisfies

$$\|(P^l f)(t, x, 0)\|_{H^m} \leq C_1 A_1^l l^s$$

for all integers l and for some constants C_1 and A_1 .

ASSUMPTION 1.2. For every positive number K , there exist constants $C = C(K)$ and $A = A(K)$ such that

$$|(\partial_x^l P^l \partial_{\bar{u}}^k \partial_{\bar{u}}^{k'} f)| \leq C A^{l+k+k'} l^s k!^s k'!^s \quad \text{for } x \in R^n, |u| \leq K, |\gamma| \leq m$$

for all integers l, k and k' with $k+k' \geq 1$, where $\partial_{\bar{u}}$ is the differentiation with respect to the complex conjugate of u .

ASSUMPTION 1.3. For every positive numbers K and R , there exist positive constants $C = C(K, R)$ and $A = A(K, R)$ such that

$$|(\partial_t^l \partial_x^\alpha \partial_{\bar{u}}^k \partial_{\bar{u}}^{k'} f)(t, x, u)| \leq C A^{l+|\alpha|+k+k'} l^s \alpha!^\sigma k!^\sigma k'!^\sigma \quad \text{for } |x| \leq R, |u| \leq K$$

for all nonnegative integers l, k and k' and for some real number σ satisfying $\max(s/2, 1) \leq \sigma \leq s$.

We state our main results.

Theorem 1.1. *We assume that Assumptions 1.1 and 1.2 are valid. Suppose that the initial data ϕ is in $G_{M_1}^s(x \cdot \nabla_x; H^m)$ for some positive constant M_1 . Then there exist positive constants T and M such that the Cauchy problem (1.1) has a unique solution $u(t, x)$ in $C([0, T]; H^m) \cap C^1([0, T]; H^{m-2})$ and that the solution satisfies $u \in C([0, T]; G_M^s(P; H^m))$.*

Theorem 1.2. *We assume that Assumptions 1.1, 1.2 and 1.3 are valid. Suppose that the initial data ϕ is in $G_{M_1}^s(x \cdot \nabla_x; H^m)$ for some positive constant M_1 . Then*

the solution u to (1.1) constructed in Theorem 1.1 satisfies the following property: For any positive number R there exist constants $C=C(R)$ and $A=A(R)$ such that

$$(1.2) \quad \|\partial_x^\alpha u(t,x)\|_{H^m(B_R)} \leq CA^{|\alpha|} t^{-|\alpha|\sigma} \quad \text{for } t \in [0, T] \text{ with } t \neq 0,$$

where σ is a real number with $\max(\sigma/2, 1) \leq \sigma \leq s$ appearing in Assumption 1.3 and B_R is a ball with radius R .

REMARK 1.1. If $s=2$ and $\sigma=1$, the solution is analytic with respect to the space variables for $t \in (0, T]$, in spite of the fact that the initial value $\phi(x)$ belongs to only the Gevrey class of order 2.

We give several examples of nonlinear terms which satisfy Assumptions 1.1–1.3 and several examples of initial data which satisfy the assumption of the theorems.

EXAMPLE 1.1 (Examples of nonlinear terms). (1) A polynomial $F(u, \bar{u})$ of u and \bar{u} with $F(0,0)=0$.

$$(2) \quad f(t,x,u) = \frac{a(x)}{1+|u|^2},$$

where $a(x) \in G_M^s(x \cdot \nabla_x; H^m)$ and $a(x)$ is locally in Gevrey class of order σ .

$$(3) \quad f(t,x,u) = \frac{F(u, \bar{u})}{1+|u|^2},$$

where $F(u, \bar{u})$ is a polynomial of u and \bar{u} with $F(0,0)=0$.

EXAMPLE 1.2 (Examples of initial data). (1) $|x|^a(1+|x|^2)^{-b}$ with $2b-n/2 > a > m-n/2$ is in $G_M^a(x \cdot \nabla; H^m(\mathbb{R}^n))$. If a is not even integer, $|x|^a(1+|x|^2)^{-b}$ has a singularity at the origin.

(2) $\psi(x-a)\psi(b-x)$ with $a < b$ is in $G_M^2(x \cdot \nabla; H^1(\mathbb{R}))$, where $\psi(x) = \exp(-1/x)$ ($x > 0$), $= 0$ ($x \leq 0$).

2. Preliminaries

In this section, we prepare several propositions to prove the main theorems. We write $\|\cdot\|_m = \|\cdot\|_{H^m}$ for abbreviation.

Proposition 2.1. Let $m = [n/2] + 1$. If u, v are in $H^m(\mathbb{R}^n)$ then uv is also in $H^m(\mathbb{R}^n)$ with

$$\|uv\|_m \leq C_2 \|u\|_m \|v\|_m,$$

where C_2 is a positive constant depending only on n .

Proposition 2.2. *There exists a constant C_3 without depending on l such that*

$$\sum_{l'+l''=l} \frac{1}{(l'+1)^2(l''+1)^2} \leq C_3 \frac{1}{(l+1)^2}.$$

Proposition 2.3. *Suppose that $u_j \in H^{m_j}$ ($j=1, \dots, N$) with $0 < m_j < n/2$ and $\sum_{j=1}^N m_j = n/2$. Then $\prod_{j=1}^N u_j \in L^2$ with*

$$(2.1) \quad \left\| \prod_{j=1}^N u_j \right\|_{L^2} \leq C_4^{N-1} \prod_{j=1}^N \|u_j\|_{m_j},$$

where C_4 is a constant depending only on n .

Proof. We can prove the proposition by using Sobolev’s imbedding theorem. □

Proposition 2.4. *Suppose that u is in $C^\infty([0, T] \times R^n; C)$ and $f(\cdot)$ is in $C^\infty(C; C)$. We have*

$$(2.2) \quad P^l f(u(t, x)) = \sum_{1 \leq k+k' \leq l} \frac{l!}{k!k'} \partial_u^k \partial_u^{k'} f(u) \sum_{\substack{l_1 + \dots + l_{k+k'} = l \\ l_j \geq 1}} \prod_{j=1}^k \frac{1}{l_j!} P^{l_j} u \prod_{j=k+1}^{k+k'} \frac{1}{l_j!} P^{l_j} \bar{u}.$$

Lemma 2.1. *Suppose that $g(x, u) \in C^\infty(R^n \times C; C)$ satisfies $|\partial_x^\gamma \partial_u^k \partial_u^{k'} g(x, u)| \leq M_K$ for $k+k'+|\gamma| \leq m$, $x \in R^n$, $|u| \leq K$, and $u, v \in H^m$. Then $g(x, u)v \in H^m$ with*

$$(2.3) \quad \|g(x, u)v\|_m \leq C_5 M_K G(\|u\|_m) \|v\|_m,$$

where $G(\cdot)$ is a polynomial of order m and C_5 is a positive constant depending only on n and m .

Proof. We can prove this lemma by using Proposition 2.1 and Proposition 2.3. □

Lemma 2.2. *Suppose that $u \in H^m$ and that $f \in C^\infty([0, T] \times R^n \times C; C)$ satisfies Assumptions 1.1 and 1.2. Then there exist constants C_6 and A_2 such that*

$$(2.4) \quad \|[P^l f](t, x, u)\|_m \leq C_6 A_2 l!^s,$$

for all $l \in N$. Here C_6 and A_2 depends only on $\|u\|_m$.

Proof. Since we can write

$$P^l f(t, x, u) = P^l f(t, x, 0) + \int_0^1 \nabla_{u, \bar{u}} P^l(t, x, \theta u) d\theta \cdot (u, \bar{u})$$

with $\nabla_{u, \bar{u}} = (\partial_u, \partial_{\bar{u}})$, we have

$$\| [P^l f](t, x, u) \|_m \leq \| [P^l f](t, x, 0) \|_m + \left\| \int_0^1 \nabla_{u, \bar{u}} P^l f(t, x, \theta u) d\theta \cdot (u, \bar{u}) \right\|_m.$$

Applying Lemma 2.1 to the second term, we have for constants C_7 and A_3 depending on $\|u\|_m$

$$\begin{aligned} \| P^l f(t, x, u) \|_m &\leq C_1 A_1^l l!^s + 2C_5 C_7 A_3^l G(\|u\|_m) \|u\|_m l!^s \\ &\leq C_6 A_2^l l!^s, \end{aligned}$$

where $C_6 = C_1 + 2C_5 C_7 G(\|u\|_m) \|u\|_m$ and $A_2 = \max(A_1, A_3)$. □

In the following, we write

$$(2.5) \quad \|g\|_{X(M,P)} = \|g\|_{G_M^s(P;H^m)} - \|g\|_m.$$

Lemma 2.3. *Suppose that u is in $G_M^s(P;H^m)$ for some constant $M > 0$ and that $f \in C^\infty(R \times R^n \times C; C)$ satisfies Assumptions 1.1 and 1.2. If we take a positive number $M' (\leq M)$ small enough, we have*

$$(2.6) \quad \|f(t, x, u)\|_{X(M',P)} \leq C_8 + \frac{C_9 \|u\|_{X(M',P)}}{(1 - C_{10} M' \|u\|_{X(M',P)})^2},$$

where C_8, C_9 and C_{10} are positive constants depending only on $f, \|u\|_m, m$ and n .

REMARK 2.1. We note that $\|u\|_{X(M',P)} \leq \|u\|_{X(M,P)}$ if $M' \leq M$.

Proof. Using Proposition 2.4, we have

$$(2.7) \quad \begin{aligned} P^l [f(t, x, u)] &= P^l f(t, x, u) + \sum_{\substack{l'+l''=l \\ l' \geq 1}} \sum_{1 \leq k+k' \leq l'} \frac{l!}{l''!k!k'} P^{l''} \partial_u^k \partial_{\bar{u}}^{k'} f(t, x, u) \\ &\quad \times \sum_{\substack{l_1 + \dots + l_{k+k'} = l' \\ l_j \geq 1}} \prod_{j=1}^k \frac{1}{l_j!} P^{l_j} u \prod_{j=k+1}^{k+k'} \frac{1}{l_j!} P^{l_j} \bar{u}. \end{aligned}$$

Taking H^m -norm of the both sides, we have from Assumptions 1.1–1.2, Proposition 2.3 and Lemma 2.1,

$$(2.8) \quad \|P^l [f(t, x, u)]\|_m \leq \|P^l f(t, x, u)\|_m + \sum_{\substack{l'+l''=l \\ l' \geq 1}} \sum_{1 \leq k+k' \leq l'} \sum_{\substack{l_1 + \dots + l_{k+k'} = l' \\ l_j \geq 1}} \frac{l!}{l''!k!k'!}$$

$$\begin{aligned} & \times \|P^{l''} \partial_u^k \partial_{\bar{u}}^{k'} f(t, x, u) \prod_{j=1}^k \frac{1}{l_j!} P^{l_j} u \prod_{j=k+1}^{k+k'} \frac{1}{l_j!} P^{l_j} \bar{u}\|_m \\ & \leq \|P^l f(t, x, u)\|_m \\ & + \sum_{\substack{l'+l''=l \\ l' \geq 1}} \sum_{1 \leq k+k' \leq l'} \sum_{\substack{l_1+\dots+l_{k+k'}=l' \\ l_j \geq 1}} \frac{l!}{l''!k!k'} C_{11} A_4^{l''+k+k'} l''!s!k'!s C_4^{k+k'-1} \prod_{j=1}^{k+k'} \frac{1}{l_j!} \|P^{l_j} u\|_m, \end{aligned}$$

where C_{11} and A_4 are positive constants which are independent of $\|P^j u\|_m$ ($j \geq 1$). Multiplying $M^{l-1}/l!(l-1)^{s-1}$ to the above quantity and making a summation with respect to l , we have

$$(2.9) \quad \|f(t, x, u)\|_{X(M', P)} \leq I_1 + I_2,$$

where

$$I_1 = \sum_{l=1}^{\infty} M^{l-1} l^{s-1} C_6 A_2^l$$

and

$$\begin{aligned} I_2 = & \sum_{l''=0}^{\infty} \sum_{l'=1}^{\infty} \sum_{1 \leq k+k' \leq l'} \frac{M^{l'+l''-1}}{(l'+l'')!(l'+l''-1)^{s-1}} \frac{(l'+l'')!}{l''!k!k'} C_{11} A_4^{l''+k+k'} l''!s!k'!s \\ & \times \sum_{\substack{l_1+\dots+l_{k+k'}=l' \\ l_j \geq 1}} C_4^{k+k'-1} \prod_{j=1}^{k+k'} \frac{1}{l_j!} \|P^{l_j} u\|_m. \end{aligned}$$

First we estimate I_1 . If we take C_{12} and A_4 sufficiently large with $l^{s-1} C_6 A_4^l \leq C_{12} A_4^l$ and we take M' so small that $M' A_4 < 1$, we have

$$(2.10) \quad I_1 \leq \sum_{l=1}^{\infty} M^{l-1} l^{s-1} C_6 A_4^l \leq C_{12} \sum_{l=1}^{\infty} (M' A_4)^{l-1} = \frac{C_{12}}{1 - M' A_4}.$$

Next we estimate I_2 .

$$\begin{aligned} I_2 \leq & \sum_{l''=0}^{\infty} \sum_{l'=1}^{\infty} \sum_{1 \leq k+k' \leq l'} \sum_{\substack{l_1+\dots+l_{k+k'}=l' \\ l_j \geq 1}} \left\{ \frac{(l'+l''-1)!}{(\prod_{j=1}^{k+k'} (l_j-1)!)(k+k'-1)!l''!} \right\}^{1-s} \\ & \times \frac{M^{l''+k+k'-1}}{l''!s!k'!(k+k'-1)^{s-1}} C_{11} A_4^{l''+k+k'} l''!s!k'!s \prod_{j=1}^{k+k'} \frac{C_4 M^{l_j-1}}{l_j!(l_j-1)^{s-1}} \|P^{l_j} u\|_m \\ \leq & C_{11} \sum_{l''=0}^{\infty} \sum_{l'=1}^{\infty} \sum_{1 \leq k+k' \leq l'} A_4^{l''} M^{l''} \binom{k+k'}{k}^{1-s} (k+k')^{s-1} M^{k+k'-1} A_4^{k+k'} \\ & \times \sum_{\substack{l_1+\dots+l_{k+k'}=l' \\ l_j \geq 1}} C_4^{k+k'-1} \prod_{j=1}^{k+k'} \frac{M^{l_j-1}}{l_j!(l_j-1)^{s-1}} \|P^{l_j} u\|_m. \end{aligned}$$

If we take C_{13} and A_5 so large that $A_4 \leq A_5$ and $(k+k')^{s-1} C_{11} A_4^{k+k'} \leq C_{13} A_5^{k+k'}$, we have

$$(2.11) \quad I_2 \leq C_{13} \left(\sum_{l''=0}^{\infty} (M'A_5)^{l''} \right) A_5 \sum_{k+k' \geq 1} (M'A_5)^{k+k'-1} \\ \times \sum_{l'=l}^{\infty} \sum_{\substack{l_1+\dots+l_{k+k'}=l' \\ l_j \geq 1}} C_4^{k+k'-1} \prod_{j=1}^{k+k'} \frac{M'^{l_j-1}}{l_j!(l_j-1)!^{s-1}} \|P^{l_j}u\|_m$$

If we take M' so small that $M'A_5 < 1$, we have

$$(2.12) \quad I_2 \leq C_{13} \frac{A_5}{1-M'A_5} \sum_{k+k' \geq 1} (C_4 M'A_5)^{k+k'-1} \|u\|_{X(M',P)}^{k+k'} \\ = C_{13} \frac{A_5}{1-M'A_5} \left(\sum_{k'=1}^{\infty} (C_4 M'A_5)^{k'-1} \|u\|_{X(M',P)}^{k'} \right. \\ \left. + \sum_{k=1}^{\infty} \sum_{k'=0}^{\infty} (C_4 M'A_5)^{k+k'-1} \|u\|_{X(M',P)}^{k+k'} \right)$$

If we take M' so small that $C_4 M'A_5 \|u\|_{X(M',P)} < 1$, we have

$$(2.13) \quad I_2 \leq C_{13} \frac{A_5}{1-M'A_5} \left(\frac{\|u\|_{X(M',P)}}{1-C_4 M'A_5 \|u\|_{X(M',P)}} + \frac{\|u\|_{X(M',P)}}{(1-C_4 M'A_5 \|u\|_{X(M',P)})^2} \right) \\ \leq C_{13} \frac{A_5}{1-M'A_5} \frac{2\|u\|_{X(M',P)}}{(1-C_4 M'A_5 \|u\|_{X(M',P)})^2}.$$

From (2.10) and (2.13), we have the lemma with $C_8 = C_{12}/(1-M'A_5)$, $C_9 = C_{13}C_4A_5/(1-M'A_5)$ and $C_{10} = C_4A_5$. □

Proposition 2.5. *Let α be a multi-index and let v and l be integers satisfying $v \leq |\alpha| + l$. Then, we have*

$$(2.14) \quad \sum_{|\alpha'|+l'=v} \binom{\alpha}{\alpha'} \binom{l}{l'} = \binom{|\alpha|+l}{v}.$$

The lemma is derived from the calculation of the coefficients of the term t^v in the both sides of $(1+t)^\alpha(1+t)^l = (1+t)^{|\alpha|+l}$.

Proposition 2.6. *For a multi-index α and an integer l we assume that the integers $v_j (\geq 1) (j=1, \dots, k)$ satisfy $v_1 + \dots + v_k = |\alpha| + l$. Then, we have*

$$(2.15) \quad \alpha!l! \sum_{\substack{\alpha_1 + \dots + \alpha_k = \alpha \\ l_1 + \dots + l_k = l \\ |\alpha_j| + l_j = v_j}} \prod_{j=1}^k \frac{(|\alpha_j| + l_j)!}{\alpha_j! l_j!} = (|\alpha| + l)!.$$

Proof. First, we consider the case $k=2$. Using (2.14), we have

$$\begin{aligned} \alpha!l! \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ l_1 + l_2 = l \\ |\alpha_j| + l_j = v_j}} \frac{(|\alpha_1| + l_1)! (|\alpha_2| + l_2)!}{\alpha_1! l_1! \alpha_2! l_2!} \\ = v_1! v_2! \sum_{|\alpha_1| + l_1 = v_1} \binom{\alpha}{\alpha_1} \binom{l}{l_1} \\ = v_1! v_2! \binom{|\alpha| + l}{v_1} = (|\alpha| + l)! . \end{aligned}$$

This proves (2.15) for $k=2$. In the general case, we can prove (2.15) by the induction on k . □

3. Proof of Theorem 1.1

We prove Theorem 1.1 by the contraction principle.

Proof of Theorem 1.1. We consider the following linearized equation with respect to (1.1),

$$(3.1) \quad \begin{cases} Lu \equiv i\partial_t u + \Delta u = f(t, x, v), \\ u(0, x) = \phi(x). \end{cases}$$

We denote the mapping which corresponds v to u by S . We write $W(M) = C([0, T]; G_M^s(P; H^m))$ with norm $\|\cdot\|_{W(M)}$ and we denote $W(M, \rho) = \{f \in W(M); \|f\|_{W(M)} \leq \rho\}$ for $\rho > 0$. We let $\rho = 2\|\phi\|_{G_M^{s-\nu}(x \cdot \nabla; H^m)}$.

First we show that S maps $W(M, \rho)$ to itself if we take T and M sufficiently small. The associate integral equation to the Cauchy problem (3.1) is

$$(3.2) \quad u = e^{it\Delta} \phi + i \int_0^t e^{i(t-s)\Delta} f(s, x, v(s)) ds,$$

where $e^{it\Delta}$ is an evolution operator for $i\partial_t - \Delta$. Since $[L, P] = 2L$, we have the linearized equation for $P^l u$,

$$(3.3) \quad \begin{cases} L(P^l u) = (P+2)^l f(t, x, v), \\ P^l u(t, x)|_{t=0} = (x \cdot \nabla)^l \phi(x). \end{cases}$$

The associate integral equation to the above Cauchy problem is

$$(3.4) \quad P^l u = e^{it\Delta} [(x \cdot \nabla)^l \phi] + i \int_0^t e^{i(t-s)\Delta} [(P+2)^l \{f(s, x, v(s))\}] ds.$$

Taking H^m norm of the both sides of the above equation, we have

$$(3.5) \quad \|P^l u\|_m \leq \|(x \cdot \nabla)^l \phi\|_m + i \int_0^T \|(P+2)^l \{f(s, x, v(s))\}\|_m ds.$$

Let $M_1 (\leq M)$ be a positive number to be determined later. Multiplying $M_1^{l-1} / (l!(l-1)!^{s-1})$ to the both sides of the above for $l \geq 1$ and making a summation with respect to l , we have

$$(3.6) \quad \|u\|_{G_{M_1}^{s, H^m}(P; H^m)} \leq \|\phi\|_{G_{M_1}^{s, H^m}(x \cdot \nabla; H^m)} + \int_0^T \|f(s, x, v(s))\|_{G_{M_1}^{s, H^m}(P+2; H^m)} ds$$

$$(3.7) \quad \leq \|\phi\|_{G_{M_1}^{s, H^m}(x \cdot \nabla; H^m)} + e^{2M_1 T} \| \|f(t, x, v(t))\| \|_{W(M_1)}.$$

Taking suprimun with respect to t in $[0, T]$ of the both sides of the above inequality, we have

$$(3.8) \quad \| \|u\| \|_{W(M_1)} \leq \|\phi\|_{G_{M_1}^{s, H^m}(x \cdot \nabla; H^m)} + e^{2M_1 T} \| \|f(t, x, v(t))\| \|_{W(M_1)}.$$

From Lemma 2.3, we have with $M_1 \leq M'$

$$(3.9) \quad \| \|f(t, x, v)\| \|_{W(M_1)} \leq \|f(t, x, v)\|_m + C_8 + \sup_{t \in [0, T]} \frac{C_9 \|v\|_{X(M_1, P)}}{(1 - C_{10} M_1 \|v\|_{X(M_1, P)})^2}.$$

Since $\| \|f(t, x, v)\| \|_{X(M_1, P)} \leq \|v\|_{X(M, P)} \leq \rho$, the last term of the right hand side of (3.9) is estimated by

$$(3.10) \quad \frac{C_9 \rho}{(1 - C_{10} M_1 \rho)^2}.$$

On the other hand, we have by Lemma 2.1,

$$(3.11) \quad \|f(t, x, v)\|_m \leq C_{14},$$

where C_{14} is a constant depending on ρ . So we have

$$(3.12) \quad \| \|f(t, x, v)\| \|_{W(M_1)} \leq C_{15},$$

where $C_{15} = C_{14} + C_8 + C_9 \rho / (1 - C_{10} M_1 \rho)^2$. If we take T so small that $T \leq \rho / (2e^{2M_1} C_{15})$, we have from (3.8)

$$(3.13) \quad \| \|u\| \|_{W(M_1)} \leq \frac{\rho}{2} + e^{2M_1 T} C_{15} \leq \rho.$$

Next we prove S is a contraction mapping in $W(M_1, \rho)$ with sufficiently small M_1 . We show that $\|Su - Sv\|_{W(M_1)} \leq (1/2)\|u - v\|_{W(M_1)}$ for $u, v \in W(M_1)$. The associate integral equation for $u - v$ is

$$(3.14) \quad Su - Sv = i \int_0^t e^{i(t-s)\Delta} [f(s, x, u(s)) - f(s, x, v(s))] ds.$$

Taking $W(M_1)$ norm of the both sides, we have

$$(3.15) \quad \|Su - Sv\|_{W(M_1)} \leq e^{2M_1 T} \|f(t, x, u(t)) - f(t, x, v(t))\|_{W(M_1)}.$$

Since

$$f(t, x, u) - f(t, x, v) = \int_0^1 \nabla_{u, \bar{u}} f(t, x, v + \theta(u - v)) d\theta \cdot (u - v, \bar{u} - \bar{v}),$$

we have

$$(3.16) \quad \begin{aligned} & \|f(t, x, u) - f(t, x, v)\|_{W(M_1)} \\ & \leq C_2 \sup_{0 \leq \theta \leq 1} \|\nabla_{u, \bar{u}} f(t, x, v + \theta(u - v))\|_{W(M_1)} \|u - v\|_{W(M_1)}. \end{aligned}$$

From Lemma 2.3, we have

$$(3.17) \quad \begin{aligned} & \|\nabla_{u, \bar{u}} f(t, x, v + \theta(u - v))\|_{W(M_1)} \\ & \leq \|\nabla_{u, \bar{u}} f(t, x, v + \theta(u - v))\|_m \\ & \quad + C_8 + C_9 \sup_{t \in [0, T]} \frac{\|v + \theta(u - v)\|_{X(M_1, P)}}{(1 - C_{10} M_1 \|v + \theta(u - v)\|_{X(M_1, P)})^2}. \end{aligned}$$

Since $\|v + \theta(u - v)\|_{X(M_1, P)} \leq (1 - \theta)\|v\|_{X(M_1, P)} + \theta\|u\|_{X(M_1, P)} \leq \rho$, the last term of the right hand side of (3.17) is estimated by

$$(3.18) \quad C_9 \sup_{t \in [0, T]} \frac{\rho}{(1 - C_{10} M_1 \rho)^2} \leq C_9 \frac{\rho}{(1 - C_{10} M_1 \rho)^2}.$$

The same argument as in the proof that S maps $W(M_1, \rho)$ to $W(M_1, \rho)$, we have

$$(3.19) \quad \|f(t, x, u) - f(t, x, v)\|_{W(M_1)} \leq C_{16} T \|u - v\|_{W(M_1)}.$$

Taking T so small that $T \leq 1/(2e^{2M_1} C_{16})$, we have

$$(3.20) \quad \|Su - Sv\|_{W(M_1)} \leq \frac{1}{2} \|u - v\|_{W(M_1)}.$$

By the construction principle, there exists a unique solution u in $W(M_1)$ consequently. □

4. Local Gevrey Regularizing Property

In this section, we prove Theorem 1.2, which shows the local Gevrey regularizing property of the solution u . We take a positive constant R and take a C^∞ -function $r(x)$ with the property

$$\begin{cases} r(x) = 1 & \text{for } |x| \leq R, \\ r(x) = 0 & \text{for } |x| \geq R + 1. \end{cases}$$

We note that

$$(4.1) \quad \|\partial_x^\alpha(ru)\|_m \leq C \|\Delta(ru)\|_m,$$

for $u \in H^m$ and for a multi-index $|\alpha| \leq 2$.

Let $u(x)$ be a solution of (1.1) constructed in Theorem 1.1. Since $[L, P] = 2L$, we have

$$(4.2) \quad LP^l u = (P + 2)^l [f(t, x, u)].$$

and hence we have from $\partial_t = \frac{1}{2i}P - \frac{1}{2i}x \cdot \nabla$,

$$(4.3) \quad \begin{aligned} \Delta P^l u &= -i\partial_t P^l u + (P + 2)^l [f(t, x, u)] \\ &= -\frac{i}{2t} P^{l+1} u + \frac{i}{2t} x \cdot \nabla_x P^l u + (P + 2)^l [f(t, x, u)]. \end{aligned}$$

Using this equation, we can estimate $\Delta P^l u$ by at most second derivative of $P^l u$.

Lemma 4.1. *Let $u(x)$ be a solution of (1.1). There exist constants C_{17} and A_6 such that*

$$(4.4) \quad \|r(x)^{|\alpha|} \partial_x^\alpha P^l u\|_m \leq C_{17} A_6^l t^{-|\alpha|l!^s}$$

for all integer l and for a multi-index α with $|\alpha| \leq 2$.

Proof. From the fact that $u \in C([0, T]; G_M^s(P; H^m))$ and Lemma 2.3, the inequalities

$$(4.5) \quad \|P^l u\|_m \leq C_{18} A_7^l l!^s,$$

$$(4.6) \quad \|P^l [f(t, x, u)]\|_m \leq C_{19} A_7^l l!^s$$

hold for any $l \geq 1$. (4.5) is nothing but (4.4) for $\alpha=0$. Next we treat the case $|\alpha|=1$. Using (4.5)–(4.6), we have

$$\begin{aligned}
 (4.7) \quad \|r(x)\partial_x^\alpha P^l u\|_m &\leq \|\partial_x^\alpha r(x)P^l u\|_m + \|[r(x), \partial_x^\alpha]P^l u\|_m \\
 &\leq \|r(x)P^l u\|_{m+1} + C_{20}\|P^l u\|_m \\
 &\leq C_{21}\|\Delta r(x)P^l u\|_{m-1} + C_{20}C_{18}A_7^l l!^s \\
 &\leq C_{21}\|r(x)\Delta P^l u\|_{m-1} + C_{21}\|[\Delta, r(x)]P^l u\|_{m-1} + C_{20}C_{18}A_7^l l!^s \\
 &\leq C_{21}\{\|r(x)P^{l+1}u\|_{m-1}/2t + \|r(x)x \cdot \nabla_x P^l u\|_{m-1}/2t \\
 &\quad + \|r(x)(P+2)^l[f(t, x, u)]\|_{m-1}\} + C_{22}C_{18}A_7^l l!^s \\
 &\leq C_{21}\{\|P^{l+1}u\|_{m-1}/2t + C_{23}\|P^l u\|_m/2t \\
 &\quad + \|(P+2)^l[f(t, x, u)]\|_{m-1}\} + C_{22}C_{18}A_7^l l!^s \\
 &\leq C_{21}C_{18}A_7^{l+1}t^{-1}(l+1)!^s + \{C_{21}(C_{23}C_{18} + e^2C_{19})\}t^{-1}A_7^l l!^s + C_{22}C_{18}A_7^l l!^s \\
 &\leq \{C_{21}(C_{18}A_7e^s + C_{23}C_{18} + e^2C_{19} + C_{22}C_{18})\}(A_7e^s)^l t^{-1}l!^s.
 \end{aligned}$$

This yields

$$(4.8) \quad \|r\partial_x P^l u\|_m \leq C_{24}A_6^l t^{-1}l!^s.$$

Using (4.8) to estimate the term $\|r(x)x \cdot \nabla_x P^l u\|_m/2t$, we can prove (4.4) for $|\alpha|=2$. □

In the follow, we prove Theorem 1.2 by showing

$$(4.9) \quad \|r^{|\alpha|}\partial_x^\alpha P^l u\|_m \leq A_0^{|\alpha|+l-1}t^{-|\alpha|}(|\alpha|+l-2)!^\sigma l!^{s-\sigma} \quad \text{for all } l$$

for all $|\alpha| \geq 2$. We note that (4.9) for $|\alpha|=2$ hold from (4.4). So we assume (4.9) for $|\beta| < |\alpha|$ and prove (4.9) for a multi-index α satisfying $|\alpha| \geq 3$.

Let γ be a multi-index with $|\gamma|=2$ and we put $\alpha' = \alpha - \gamma$. We estimate the each term of the right hand side of the identity,

$$(4.10) \quad r^{|\alpha|}\partial_x^\alpha P^l u = \partial_x^\gamma r^{|\alpha|}\partial_x^{\alpha'} P^l u + [r^{|\alpha|}, \partial_x^\gamma]\partial_x^{\alpha'} P^l u.$$

Lemma 4.2. *Assume that (4.9) holds for $|\beta| < |\alpha|$. Then we have*

$$(4.11) \quad \|[r^{|\alpha|}, \partial_x^\gamma]\partial_x^{\alpha'} P^l u\|_m \leq C_{25}A_0^{|\alpha|+l-1}t^{-|\alpha|-1}(|\alpha|+l)!^\sigma l!^{s-\sigma}.$$

Proof. Let $\partial_x^\gamma = \partial_j \partial_k$ with $\partial_j = \partial_{x_j}$ and set $r_j = \partial_j r$, $r_k = \partial_k r$. Then, since

$$\begin{aligned}
 [r^{|\alpha|}, \partial_x^\gamma] &= -\{|\alpha|r^{|\alpha|-1}r_j\partial_k + |\alpha|r^{|\alpha|-1}r_k\partial_j \\
 &\quad + |\alpha|(|\alpha|-1)r^{|\alpha|-2}r_jr_k + |\alpha|r^{|\alpha|-1}(\partial_x^\gamma r)\},
 \end{aligned}$$

we have

$$\begin{aligned} & \| [r^{|\alpha|}, \partial_x^y] \partial_x^{\alpha'} P^l u \|_m \\ & \leq |\alpha| \| r^{|\alpha|-1} r_j \partial_k \partial_x^{\alpha'} P^l u \|_m + |\alpha| \| r^{|\alpha|-1} r_k \partial_j \partial_x^{\alpha'} P^l u \|_m \\ & \quad + |\alpha| (|\alpha| - 1) \| r^{|\alpha|-2} \partial_x^{\alpha'} P^l u \|_m + |\alpha| \| (\partial_x^y r) \partial_x^{\alpha'} P^l u \|_m. \end{aligned}$$

First we treat the case $|\alpha'| \geq 2$. Then from (4.9) for $|\beta| < |\alpha|$, we have

$$\begin{aligned} & \| [r^{|\alpha|}, \partial_x^y] \partial_x^{\alpha'} P^l u \|_m \\ & \leq 2C_{26} |\alpha| A_0^{|\alpha'+l-t-|\alpha'|-1} (|\alpha'| + l - 1)! \sigma! s^{-\sigma} \\ & \quad + |\alpha| (|\alpha| - 1) A_0^{|\alpha'+l-1-t-|\alpha'|-1} (|\alpha'| + l - 2)! \sigma! s^{-\sigma} \\ & \quad + C_{27} |\alpha| A_0^{|\alpha'+l-1-t-|\alpha'|-1} (|\alpha'| + l - 2)! \sigma! s^{-\sigma} \\ & \leq C_{28} A_0^{|\alpha'+l-t-|\alpha'|-1} (|\alpha'| + l)! \sigma! s^{-\sigma}. \end{aligned}$$

Here, we used $|\alpha| \leq 3(|\alpha'| + l)$. Next we treat the case $|\alpha'| = 1$. Since $|\alpha| = 3$, we get (4.11) by (4.4). This proves (4.11). \square

In order to estimate the H^m -norm of the first term of the right hand side of (4.10) we use the estimate (4.1). Then, We have

$$\begin{aligned} (4.12) \quad & \| \partial_x^y r^{|\alpha|} \partial_x^{\alpha'} P^l u \|_m \leq \| r^{|\alpha|} \partial_x^{\alpha'} P^l u \|_{m+2} \\ & \leq C_{29} \Delta r^{|\alpha|} \partial_x^{\alpha'} P^l u \|_m \\ & \leq C_{29} \| r^{|\alpha|} \partial_x^{\alpha'} \Delta P^l u \|_m + \| [\Delta, r^{|\alpha|}] \partial_x^{\alpha'} P^l u \|_m \} \\ & \leq C_{29} \{ \| r^{|\alpha|} \partial_x^{\alpha'} P^{l+1} u \|_m / 2t + \| r^{|\alpha|} \partial_x^{\alpha'} x \cdot \nabla_x P^l u \|_m / 2t \\ & \quad + \| r^{|\alpha|} \partial_x^{\alpha'} (P + 2)^l [f(t, x, u)] \|_m \\ & \quad + \| [\Delta, r^{|\alpha|}] \partial_x^{\alpha'} P^l u \|_m \}. \end{aligned}$$

Now, we estimate the each term in the right hand side of (4.12).

Lemma 4.3. *Assume that (4.9) holds for $|\beta| < |\alpha|$. Then, we have*

$$(4.13) \quad \| [\Delta, r^{|\alpha|}] \partial_x^{\alpha'} P^l u \|_m \leq C_{30} A_0^{|\alpha'+l-t-|\alpha'|-1} (|\alpha'| + l)! \sigma! s^{-\sigma}.$$

We can prove this lemma by the same way as in the proof of Lemma 4.2.

Lemma 4.4. *Let σ be a positive number with $\sigma \geq s/2$. Assume that (4.9) holds for $|\beta| < |\alpha|$. Then the inequality*

$$(4.14) \quad \| r^{|\alpha|} \partial_x^{\alpha'} P^{l+1} u \|_m \leq C_{31} A_0^{|\alpha'+l-t-|\alpha'|-1} (|\alpha'| + l)! \sigma! s^{-\sigma}$$

holds with $\alpha' = \alpha - \gamma$ and $|\gamma| = 2$.

Proof. Since $\sigma \geq s/2$ and $|\alpha'| \geq 1$, we have $(l + 1)^{s-\sigma} / (|\alpha'| + l)^\sigma \leq 1$. Hence, if $|\alpha'| \geq 2$, we get from (4.9)

$$\begin{aligned} \|r^{|\alpha|} \partial_x^{\alpha'} P^{l+1} u\|_m &\leq \|r^{|\alpha|} \partial_x^{\alpha'} P^{l+1} u\|_m \\ &\leq A_0^{|\alpha'|+l} t^{-|\alpha'|} (|\alpha'| + l - 1)^\sigma (l + 1)^{s-\sigma} \\ &= A_0^{|\alpha'|+l} t^{-|\alpha'|} \{(l + 1)^{s-\sigma} / (|\alpha'| + l)\} (|\alpha'| + l)^\sigma l^{s-\sigma} \\ &\leq A_0^{|\alpha'|+l} t^{-|\alpha'|} (|\alpha'| + l)^\sigma l^{s-\sigma} \end{aligned}$$

and get (4.14). We also have (4.14) for $|\alpha'| = 1$ from (4.4). □

Lemma 4.5. Assume that (4.9) holds for $|\beta| < |\alpha|$. Then there exists a constant C_{32} such that

$$(4.15) \quad \|r^{|\alpha|} \partial_x^{\alpha'} x \cdot \nabla_x P^l u\|_m \leq C_{32} A_0^{|\alpha'|+l} t^{-|\alpha'|-1} (|\alpha'| + l)^\sigma l^{s-\sigma}$$

holds with $\alpha' = \alpha - \gamma$ and $|\gamma| = 2$.

Proof. Using the boundedness of $\text{supp } r(x)$, we have from (4.9) for $|\beta| = |\alpha| - 1$ and $\beta = \alpha'$

$$\begin{aligned} \|r^{|\alpha|} \partial_x^{\alpha'} x \cdot \nabla_x P^l u\|_m &\leq \sum_{j=1}^n \|r^{|\alpha|} x_j \partial_k \partial_x^{\alpha'} \partial_j P^l u\|_m + \sum_{j=1}^n \alpha'_j \|r^{|\alpha|} \partial_x^{\alpha'} P^l u\|_m \\ &\leq C_{32} A_0^{|\alpha'|+l} t^{-|\alpha'|-1} (|\alpha'| + l)^\sigma l^{s-\sigma} \end{aligned}$$

This proves (4.14). □

Lemma 4.6. Let $f(t, x, u)$ be a function satisfying Assumption 1.3. Assume that (4.9) holds for $|\beta| < |\alpha|$. Then, we have

$$(4.16) \quad \|r^{|\alpha|} \partial_x^{\alpha'} (P + 2)^l [f(t, x, u(t, x))]\|_m \leq C_{33} A_0^{|\alpha'|+l} t^{-|\alpha'|} (|\alpha'| + l)^\sigma l^{s-\sigma}.$$

Proof. We note that we have

$$\begin{aligned} (|\beta| + l - 2)! &= (|\beta| + l)! / \{(|\beta| + l - 1)(|\beta| + l)\} \\ &\leq C_{34} (|\beta| + l)! / (|\beta| + l + 1)^2, \end{aligned}$$

for $|\beta| + l \geq 2$, which and (4.9) for $|\beta| < |\alpha|$ yield

$$(4.17) \quad \|r^{|\beta|} \partial_x^\beta P^l u\|_m \leq C_{34} A_0^{|\beta|+l-1} t^{-|\beta|} (|\beta|+l)(|\beta|+l-1)!^{\sigma-1} \times l!^{\sigma-1} / (|\beta|+l+1)^2$$

for $2 \leq |\beta| < |\alpha|$. We note that, from (4.4), the above (4.17) holds also for $|\beta|+l \geq 1$ with $\beta=0$ or $|\beta|=1$. Moreover, from the Assumption 1.3 we have

$$(4.18) \quad \begin{aligned} & \| (r^{|\alpha|} \partial_x^{\alpha'} (P+2)^l f)(t, \cdot, u(t, \cdot)) \|_m \leq C_{35} A_8^{|\alpha'|+l} \alpha'!^{\sigma} l!^{\sigma}, \\ & |(\partial_x^{\gamma} \partial_u^j \partial_u^{j'} r^{|\alpha|} \partial_x^{\alpha'} (P+2)^l \partial^k u \partial_u^{k'} f)(t, x, u(t, x))| \\ & \leq C_{36} A_9^{|\alpha|+l+k+k'} \alpha'!^{\sigma} l!^{\sigma} k! k'! (k+k')!^{\sigma-1} \\ & \text{for } j+j'+|\gamma| \leq m \end{aligned}$$

Using these estimates we prove (4.16). Since (4.16) is trivial when $|\alpha'|+l=0$, we may assume $|\alpha'|+l \geq 1$. Then, from the differentiation of composite function and (4.17)–(4.18) we have

$$\begin{aligned} & \|r^{|\alpha|} \partial_x^{\alpha'} (P+2)^l [f(t, x, u(t, x))] \|_m \\ & \leq \| (r^{|\alpha|} \partial_x^{\alpha'} (P+2)^l f)(t, x, u(t, x)) \|_m \\ & \quad + C_{37} \sum_{\substack{\beta'+\beta''=\alpha' \\ l'+l''=l \\ |\beta'|+l' \neq 0}} \sum_{1 \leq k+k' \leq |\beta'|+l'} \frac{\alpha'! l!}{\beta''! l''! k! k'!} \\ & \quad \times \sup_{|x| \leq R+1, j+j'+|\gamma| \leq m} | \partial_x^{\gamma} \partial_u^j \partial_u^{j'} (r^{|\beta''|} \partial_x^{\beta''} (P+2)^{l''} \partial_u^k \partial_u^{k'} f) | \\ & \quad \times C_4^{k+k'-1} \sum_{\substack{\beta_1+\dots+\beta_{k+k'}=\beta' \\ l_1+\dots+l_{k+k'}=l' \\ |\beta_j|+l_j \neq 0}} \prod_{j=1}^k \frac{1}{\beta_j! l_j!} \| r^{|\beta_j|} \partial_x^{\beta_j} P^{l_j} u \|_m \\ & \quad \times \prod_{j=k+1}^{k+k'} \frac{1}{\beta_j! l_j!} \| r^{|\beta_j|} \partial_x^{\beta_j} P^{l_j} \bar{u} \|_m \\ & \leq C_{35} A_8^{|\alpha'|+l} \alpha'!^{\sigma} l!^{\sigma} \\ & \quad + C_{37} \sum_{\substack{v'+v''=|\alpha'|+l \\ v' \neq 0}} \sum_{\substack{\beta'+\beta''=\alpha' \\ l'+l''=l \\ |\beta'|+l' \neq v'}} \sum_{1 \leq k+k' \leq |\beta'|+l'} \binom{\alpha'}{\beta'} \binom{l}{l'} \\ & \quad \times C_{36} A_9^{|\beta''|+l''+k+k'} \beta''!^{\sigma} l''!^{\sigma} (k+k')!^{\sigma-1} \\ & \quad \times C_4^{k+k'-1} K_{\beta', l', k+k'}, \end{aligned}$$

where

$$\begin{aligned}
 K_{\beta', l', k+k'} &= \beta'! l'! \sum_{\substack{\beta_1 + \dots + \beta_{k+k'} = \beta' \\ l_1 + \dots + l_{k+k'} = l' \\ |\beta_j| + l_j \neq 0}} \prod_{j=1}^{k+k'} C_{34} A_0^{|\beta_j| + l_j - 1} \\
 &\quad \times t^{-|\beta_j|} \frac{(|\beta_j| + l_j)!}{\beta_j! l_j!} (|\beta_j| + l_j - 1)^{\sigma-1} l_j!^{s-\sigma} / (|\beta_j| + l_j + 1)^2.
 \end{aligned}$$

Now, we use Proposition 2.6 and Proposition 2.2. Then, we have

$$\begin{aligned}
 K_{\beta', l', k+k'} &\leq C_{34}^{k+k'} A_0^{|\beta'| + l' - k - k'} t^{-|\beta'|} (|\beta'| + l' - k - k')!^{\sigma-1} l'!^{s-\sigma} \\
 &\quad \times \sum_{v_1 + \dots + v_{k+k'} = |\beta'| + l'} \beta'! l'! \\
 &\quad \times \sum_{\substack{\beta_1 + \dots + \beta_{k+k'} = \beta' \\ l_1 + \dots + l_{k+k'} = l' \\ |\beta_j| + l_j = v_j}} \prod_{j=1}^{k+k'} \frac{(|\beta_j| + l_j)!}{\beta_j! l_j!} \frac{1}{(|\beta_j| + l_j + 1)^2} \\
 &\leq C_{34}^{k+k'} A_0^{|\beta'| + l' - k - k'} t^{-|\beta'|} (|\beta'| + l')! (|\beta'| + l' - k - k')!^{\sigma-1} l'!^{s-\sigma} \\
 &\quad \times \sum_{v_1 + \dots + v_{k+k'} = |\beta'| + l'} \prod_{j=1}^{k+k'} \frac{1}{(v_j + 1)^2} \\
 &\leq C_3^{k+k'-1} C_{34}^{k+k'} A_0^{|\beta'| + l' - k - k'} t^{-|\beta'|} (|\beta'| + l')! (|\beta'| + l' - k - k')!^{\sigma-1} \\
 &\quad \times l'!^{s-\sigma} / (|\beta'| + l' + 1)^2.
 \end{aligned}$$

Hence, using Proposition 2.5 now, we have from $|\beta'| + l' = v'$ and $|\beta''| + l'' = v''$

$$\begin{aligned}
 &\|r^{|\alpha|} \partial_x^{\alpha'} (P+2)^l [f(t, x, u(t, x))]\|_m \\
 &\leq C_{35} A_8^{|\alpha'| + l} \alpha'! \sigma! l!^s \\
 &\quad + C_{37} \sum_{\substack{v' + v'' = |\alpha'| + l \\ v' \neq 0}} \sum_{\substack{\beta' + \beta'' = \beta \\ l' + l'' = l \\ |\beta'| + l' = v'}} \sum_{1 \leq k+k' \leq v'} \binom{\alpha'}{\beta'} \binom{l}{l'} \\
 &\quad \times C_{36} A_9^{v'' + k+k'} \beta''! \sigma! l''!^s (k+k')!^{\sigma-1} \\
 &\quad \times C_4^{k+k'-1} C_3^{k+k'-1} C_{34}^{k+k'} A_0^{v' - k - k'} t^{-|\beta'|} v'! \\
 &\quad \times (v' - k - k')!^{\sigma-1} l'!^{s-\sigma} / (v' + 1)^2 \\
 &\leq C_{35} A_8^{|\alpha'| + l} \alpha'! \sigma! l!^s \\
 &\quad + (C_{37} C_{36} / C_4 C_3) A_0^{|\alpha'| + l} t^{-|\alpha'|} (|\alpha'| + l)!^{\sigma-1} l!^{s-\sigma} \\
 &\quad \times \sum_{v' + v'' = |\alpha'| + l} (A_9 / A_0)^{v''} / (v' + 1)^2 \\
 &\quad \times \{v'! v''! \sum_{|\beta'| + l' = v'} \binom{\alpha'}{\beta'} \binom{l}{l'}\}
 \end{aligned}$$

$$\times \left\{ \sum_{1 \leq k+k' \leq v'} (C_3 C_4 C_{34} A_9 / A_0)^{k+k'} \right\}.$$

Here, we used $\beta''! \sigma l''!^\sigma \leq v''!^\sigma$ and $\{v''!(k+k')!(v'-k-k')!\}^{\sigma-1} \leq (|\alpha'|+l)!^{\sigma-1}$. Hence, assumming $A_0 \geq 2A_9$ and $A_0 \geq 2C_3 C_4 C_{34} A_9$, we get

$$\begin{aligned} & \|r^{|\alpha|} \partial_x^{\alpha'} (P+2)^l [f(t,x,u(t,x))]\|_m \\ & \leq C_{35} A_0^{|\alpha'+l|} (|\alpha'+l)!^\sigma l^{s-\sigma} \\ & \quad + 16(C_{37} C_{36} / C_4 C_3) A_0^{|\alpha'+l|} t^{-|\alpha'|} (|\alpha'+l)!^\sigma l^{s-\sigma}. \end{aligned}$$

This proves (4.16). □

Now, we are prepared to prove Theorem 1.2.

Proof of Theorem 1.2. For any fixed positive constant R we take a C^∞ -function $r(x)$ satisfying (4). In order to prove (1.2), we have only to prove (4.9) for any α with $|\alpha| \geq 2$. Note that (4.9) for $|\alpha|=2$ holds from (4.4). So it suffices to show (4.9) for $|\alpha|=N \geq 3$ under the assumption that (4.9) holds for $|\alpha| < N$. Let γ be a multi-index with $|\gamma|=2$ and let $\alpha' = \alpha - \gamma$. From Lemmas 4.2–4.6, we have

$$\|r^\alpha \partial_x^\alpha P^l u\|_m \leq (C_{25} + C_{29}(C_{30} + C_{31} + C_{32} + C_{33})) A_0^{|\alpha'+l|} t^{-|\alpha'|-1} (|\alpha'+l)!^\sigma l^{s-\sigma}.$$

Retaking the constant A_0 so large that $A_0 \geq C_{25} + C_{29}(C_{30} + C_{31} + C_{32} + C_{33})$ we have the inequality (4.9). □

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