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GEVREY REGULARIZING EFFECT FOR NONLINEAR SCHRODINGER EQUATIONS

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1. Introduction

We consider the following Cauchy problem of nonlinear Schrödinger equations in n space dimensions,

$$\text{(1.1)} \qquad \begin{cases} Lu \equiv i\partial_t u + \Delta u = f(t,x,u), \\ u(0,x) = \phi(x), \end{cases}$$

where $\Delta = \Sigma_{j=1}^{n} \partial^2 / \partial x_j{}^2$ and $f(t,x,u)$ is a complex valued function of Gevrey class in $(t,x,u) \in R \times R^n \times C$. We study the regularizing effect for (1.1). In what follows, we show that if the initial data ϕ is in some Gevrey class of order s with respect to $x \cdot \nabla_x$, then the solution u is in Gevrey class of order $\max(s/2, 1)$ with respect to x.

Concering the regularizing effect for dispersive equations, many works have been done ([1], [2], [5], [6], [7], [8], [9]). All the above works treat regularizing effects with respect to Sobolev spaces. In [4], N. Hayashi and one of the authors treat regularity in time for nonlinear Schrödinger equations. They have shown that if the initial data is in Gevrey class of order s (≥ 1) with respect to $x \cdot \nabla$ and ∇, then the solution is in Gevrey class of order s in space-time variables for $t \neq 0$. In [3], A. de Bouard, N. Hayashi and one of the authors treat Gevrey regularizing effect for nonlinear Schrödinger equatons in one space dimension and Korteweg-de Vries equation. They have shown that if the initial data is in Gevrey class of order s (≥ 1) with respect to $x \cdot \nabla$ and ∇, then the solution is in Gevrey class of order $\max(1, s/2)$ (or $\max(1, s/3)$ for KdV) with respect to the space variable for $t \neq 0$. We extend their results to the case that the nonlinear term is not polynomial, and for the local property, we extend their results to the case of higher space dimensions.

We introduce some notation and some function spaces to state the result precisely. Let $H^m(\Omega)$ denote Sobolev space of order m with respect to L^2 for an open set Ω in R^n. For simplicity we write $H^m = H^m(R^n)$. For a vector field Q with analytic coefficients and for a positive number M, we define a function space of Gevrey class $G^s_M(Q; H^m)$ in R^n as follows:

$$G^s_M(Q;H^m)=\{g\in H^m;\ \|g\|_{G^s_M(Q;H^m)}=\|g\|_{H^m}+\sum_{j=1}^{\infty}M^{j-1}\|Q^jg\|_{H^m}/j!(j-1)!^{s-1}<\infty\}.$$

We also define a function space of Gevrey class $C([0,T];G^s_M(Q;H^m))$ as follows:

$$C([0,T];G^s_M(Q;H^m))$$
$$=\{g\in C([0,T];H^m);\ |||g|||_{C([0,T];G^s_M(Q;H^m))}=\sup_{t\in[0,T]}\|g\|_{G^s_M(Q;H^m)}<\infty\}.$$

We write $P=2t\partial_t+x\cdot\nabla_x$ with $\nabla_x=(\partial_{x_1},\cdots,\partial_{x_n})$ and set $m=[n/2]+1$ throughout this paper.

ASSUMPTION 1.1. The nonlinear term f satisfies

$$\|(P^lf)(t,x,0)\|_{H^m}\le C_1A_1^l l!^s$$

for all integers l and for some constants C_1 and A_1.

ASSUMPTION 1.2. For every positive number K, there exist constants $C=C(K)$ and $A=A(K)$ such that

$$|(\partial_x^\gamma P^l\partial_u^k\partial_{\bar{u}}^{k'}f)|\le CA^{l+k+k'}l!^sk!^sk'!^s \qquad \text{for } x\in R^n,\ |u|\le K,\ |\gamma|\le m$$

for all integers l, k and k' with $k+k'\ge 1$, where $\partial_{\bar{u}}$ is the differentiationn with respect to the complex conjugate of u.

ASSUMPTION 1.3. For every positive numbers K and R, there exist positive constants $C=C(K,R)$ and $A=A(K,R)$ such that

$$|(\partial_t^l\partial_x^\alpha\partial_u^k\partial_{\bar{u}}^{k'}f)(t,x,u)|\le CA^{l+|\alpha|+k+k'}l!^s\alpha!^\sigma k!^\sigma k'!^\sigma \qquad \text{for } |x|\le R,\ |u|\le K$$

for all nonnegative integers l, k and k' and for some real number σ satisfying $\max(s/2,\ 1)\le\sigma\le s$.

We state our main results.

Theorem 1.1. *We assume that Assumtions* 1.1 *and* 1.2 *are valid. Suppose that the initial data* ϕ *is in* $G^s_{M_1}(x\cdot\nabla_x;\ H^m)$ *for some positive constant* M_1. *Then there exist positive constants* T *and* M *such that the Cauchy problem* (1.1) *has a unique solution* $u(t,x)$ *in* $C([0,T];H^m)\cap C^1([0,T];H^{m-2})$ *and that the solution satisfies* $u\in C([0,T];G^s_M(P;H^m))$.

Theorem 1.2. *We assume that Assumptions* 1.1, 1.2 *and* 1.3 *are valid. Suppose that the initial data* ϕ *is in* $G^s_{M_1}(x\cdot\nabla_x;\ H^m)$ *for some positive constant* M_1. *Then* *the solution u to* (1.1) *constructed in Theorem* 1.1 *satisfies the following property*: *For any positive number R there exist constans $C=C(R)$ and $A=A(R)$ such that*

$$\|\partial_x^\alpha u(t,x)\|_{H^m(B_R)} \leq CA^{|\alpha|}t^{-|\alpha|}\alpha!^\sigma \quad \textit{for} \quad t\in[0,T] \ \textit{with} \ t\neq 0, \tag{1.2}$$

where σ is a real number with $\max(\sigma/2,1)\leq\sigma\leq s$ *appearing in Assumption* 1.3 *and B_R is a ball with radius R.*

REMARK 1.1. If $s=2$ and $\sigma=1$, the solution is analytic with respect to the space variables for $t\in(0,T]$, in spite of the fact that the initial value $\phi(x)$ belongs to only the Gevrey class of order 2.

We give several examples of nonlinear terms which satisfy Assumptions 1.1–1.3 and several examples of initial data which satisfy the assumption of the theorems.

EXAMPLE 1.1 (Examples of nonlinear terms). (1) A polynomial $F(u,\bar{u})$ of u and $\bar{u}$ with $F(0,0)=0$.

(2)
$$f(t,x,u)=\frac{a(x)}{1+|u|^2},$$

where $a(x)\in G_M^s(x\cdot\nabla_x;H^m)$ and $a(x)$ is locally in Gevrey class of order σ.

(3)
$$f(t,x,u)=\frac{F(u,\bar{u})}{1+|u|^2},$$

where $F(u,\bar{u})$ is a polynomial of u and $\bar{u}$ with $F(0,0)=0$.

EXAMPLE 1.2 (Examples of initial data). (1) $|x|^a(1+|x|^2)^{-b}$ with $2b-n/2>a>m-n/2$ is in $G_M^\sigma(x\cdot\nabla;\ H^m(R^n))$. If a is not even integer, $|x|^a(1+|x|^2)^{-b}$ has a singularity at the origin.
(2) $\psi(x-a)\psi(b-x)$ with $a<b$ is in $G_M^2(x\cdot\nabla;\ H^1(R))$, where $\psi(x)=\exp(-1/x)$ $(x>0)$, $=0$ $(x\leq 0)$.

2. Preliminaries

In this section, we prepare several propositions to prove the main theorems. We write $\|\cdot\|_m=\|\cdot\|_{H^m}$ for abbreviation.

Proposition 2.1. *Let $m=[n/2]+1$. If u, v are in $H^m(R^n)$ then uv is also in $H^m(R^n)$ with*

$$\|uv\|_m\leq C_2\|u\|_m\|v\|_m,$$

where C_2 *is a positive constant depending only on* n.

Proposition 2.2. *There exists a constant* C_3 *without depending on* l *such that*

$$\sum_{l'+l''=l} \frac{1}{(l'+1)^2(l''+1)^2} \le C_3 \frac{1}{(l+1)^2} .$$

Proposition 2.3. *Suppose that* $u_j \in H^{m_j}$ $(j=1,\cdots,N)$ *with* $0<m_j<n/2$ *and* $\Sigma_{j=1}^N m_j = n/2$. *Then* $\Pi_{j=1}^N u_j \in L^2$ *with*

$$\|\prod_{j=1}^N u_j\|_{L^2} \le C_4^{N-1} \prod_{j=1}^N \|u_j\|_{m_j}, \tag{2.1}$$

where C_4 *is a constant depending only on* n.

Proof. We can prove the proposition by using Sobolev's imbedding theorem. □

Proposition 2.4. *Suppose that* u *is in* $C^\infty([0,T]\times R^n; C)$ *and* $f(\cdot)$ *is in* $C^\infty(C; C)$. *We have*

$$P^l f(u(t,x)) = \sum_{1 \le k+k' \le l} \frac{l!}{k!k'!} \partial_u^k \partial_{\bar{u}}^{k} f(u) \sum_{\substack{l_1+\cdots+l_{k+k'}=l \\ l_j \ge 1}} \prod_{j=1}^{k} \frac{1}{l_j!} P^{l_j} u \prod_{j=k+1}^{k+k'} \frac{1}{l_j!} P^{l_j} \bar{u}. \tag{2.2}$$

Lemma 2.1. *Suppose that* $g(x,u) \in C^\infty(R^n \times C; C)$ *satisfies* $|\partial_x^\gamma \partial_u^k \partial_{\bar{u}}^{k'} g(x,u)| \le M_K$ *for* $k+k'+|\gamma| \le m$, $x \in R^n$, $|u| \le K$, *and* $u,v \in H^m$. *Then* $g(x,u)v \in H^m$ *with*

$$\|g(x,u)v\|_m \le C_5 M_K G(\|u\|_m) \|v\|_m , \tag{2.3}$$

where $G(\cdot)$ *is a polynomial of order* m *and* C_5 *is a positive constant depending only on* n *and* m.

Proof. We can prove this lemma by using Proposition 2.1 and Proposition 2.3. □

Lemma 2.2. *Suppose that* $u \in H^m$ *and that* $f \in C^\infty([0,T] \times R^n \times C; C)$ *satisfies Assumptions* 1.1 *and* 1.2. *Then there exist constants* C_6 *and* A_2 *such that*

$$\|[P^l f](t,x,u)\|_m \le C_6 A_2{}^l l!^s, \tag{2.4}$$

for all $l \in N$. *Here* C_6 *and* A_2 *depends only on* $\|u\|_m$.

Proof. Since we can write

$$P^l f(t,x,u)=P^l f(t,x,0)+\int_0^1 \nabla_{u,\bar{u}}P^l(t,x,\theta u)d\theta\cdot(u,\bar{u})$$

with $\nabla_{u,\bar{u}}=(\partial_u,\partial_{\bar{u}})$, we have

$$\|[P^l f](t,x,u)\|_m\le\|[P^l f](t,x,0)\|_m+\|\int_0^1 \nabla_{u,\bar{u}}P^l f(t,x,\theta u)d\theta\cdot(u,\bar{u})\|_m.$$

Applying Lemma 2.1 to the second term, we have for constants C_7 and A_3 depending on $\|u\|_m$

$$\begin{aligned}\|P^l f(t,x,u)\|_m&\le C_1A_1^l l!^s+2C_5C_7A_3^l G(\|u\|_m)\|u\|_m l!^s\\&\le C_6A_2^l l!^s,\end{aligned}$$

where $C_6=C_1+2C_5C_7G(\|u\|_m)\|u\|_m$ and $A_2=\max(A_1,A_3)$. □

In the following, we write

$$\|g\|_{X(M,P)}=\|g\|_{G^s_M(P;H^m)}-\|g\|_m. \tag{2.5}$$

Lemma 2.3. *Suppose that u is in $G^s_M(P;H^m)$ for some constant $M>0$ and that $f\in C^\infty(R\times R^n\times C;C)$ satisfies Assumptions* 1.1 *and* 1.2. *If we take a positive number $M'(\le M)$ small enough, we have*

$$\|f(t,x,u)\|_{X(M',P)}\le C_8+\frac{C_9\|u\|_{X(M',P)}}{(1-C_{10}M'\|u\|_{X(M',P)})^2}, \tag{2.6}$$

where C_8, C_9 and C_{10} are positive constants depending only on f, $\|u\|_m$, m and n.

REMARK 2.1. We note that $\|u\|_{X(M',P)}\le\|u\|_{X(M,P)}$ if $M'\le M$.

Proof. Using Proposition 2.4, we have

$$\begin{aligned}P^l[f(t,x,u)]=P^l f(t,x,u)+\sum_{\substack{l'+l''=l\\ l'\ge1}}\sum_{1\le k+k'\le l'}\frac{l!}{l''!k!k'!}P^{l''}\partial_u^k\partial_{\bar{u}}^{k'}f(t,x,u)\\ \times\sum_{\substack{l_1+\cdots+l_{k+k'}=l'\\ l_j\ge1}}\prod_{j=1}^{k}\frac{1}{l_j!}P^{l_j}u\prod_{j=k+1}^{k+k'}\frac{1}{l_j!}P^{l_j}\bar{u}.\end{aligned} \tag{2.7}$$

Taking H^m-norm of the both sides, we have from Assumptions 1.1–1.2, Proposition 2.3 and Lemma 2.1,

$$\|P^l[f(t,x,u)]\|_m\le\|P^l f(t,x,u)\|_m+\sum_{\substack{l'+l''=l\\ l'\ge1}}\sum_{1\le k+k'\le l'}\sum_{\substack{l_1+\cdots+l_{k+k'}=l'\\ l_j\ge1}}\frac{l!}{l''!k!k'!} \tag{2.8}$$

$$\times \|P^{l''}\partial_u^k\partial_{\bar{u}}^{k'}f(t,x,u)\prod_{j=1}^{k}\frac{1}{l_j!}P^{l_j}u\prod_{j=k+1}^{k+k'}\frac{1}{l_j!}P^{l_j}\bar{u}\|_m$$

$$\leq \|P^l f(t,x,u)\|_m$$

$$+\sum_{\substack{l'+l''=l\\ l'\geq 1}}\sum_{1\leq k+k'\leq l'}\sum_{\substack{l_1+\cdots+l_{k+k'}=l'\\ l_j\geq 1}}\frac{l!}{l''!k!k'!}C_{11}A_4^{l''+k+k'}l''!^s k!^s k'!^s C_4^{k+k'-1}\prod_{j=1}^{k+k'}\frac{1}{l_j!}\|P^{l_j}u\|_m,$$

where C_{11} and A_4 are positive constants which are independent of $\|P^j u\|_m$ $(j\geq 1)$. Multiplying $M'^{l-1}/l!(l-1)!^{s-1}$ to the above quantity and making a summation with respect to l, we have

$$\|f(t,x,u)\|_{X(M',P)}\leq I_1+I_2, \tag{2.9}$$

where

$$I_1=\sum_{l=1}^{\infty}M'^{l-1}l^{s-1}C_6A_2^l$$

and

$$I_2=\sum_{l''=0}^{\infty}\sum_{l'=1}^{\infty}\sum_{1\leq k+k'\leq l'}\frac{M'^{l'+l''-1}}{(l'+l'')!(l'+l''-1)!^{s-1}}\frac{(l'+l'')!}{l''!k!k'!}C_{11}A_4^{l''+k+k'}l''!^s k!^s k'!^s$$

$$\times\sum_{\substack{l_1+\cdots+l_{k+k'}=l'\\ l_j\geq 1}}C_4^{k+k'-1}\prod_{j=1}^{k+k'}\frac{1}{l_j!}\|P^{l_j}u\|_m.$$

First we estimate I_1. If we take C_{12} and A_4 sufficiently large with $l^{s-1}C_6A_6^l\leq C_{12}A_4^l$ and we take M' so small that $M'A_4<1$, we have

$$I_1\leq\sum_{l=1}^{\infty}M'^{l-1}l^{s-1}C_6A_2^l\leq C_{12}\sum_{l=1}^{\infty}(M'A_4)^{l-1}=\frac{C_{12}}{1-M'A_4}. \tag{2.10}$$

Next we estimate I_2.

$$I_2\leq\sum_{l''=0}^{\infty}\sum_{l'=1}^{\infty}\sum_{1\leq k+k'\leq l'}\sum_{\substack{l_1+\cdots+l_{k+k'}=l'\\ l_j\geq 1}}\left\{\frac{(l'+l''-1)!}{(\prod_{j=1}^{k+k'}(l_j-1)!)(k+k'-1)!l''!}\right\}^{1-s}$$

$$\times\frac{M'^{l''+k+k'-1}}{l''!^s k!k'!(k+k'-1)!^{s-1}}C_{11}A_4^{l''+k+k'}l''!^s k!^s k'!^s\prod_{j=1}^{k+k'}\frac{C_4M'^{l_j-1}}{l_j!(l_j-1)!^{s-1}}\|P^{l_j}u\|_m$$

$$\leq C_{11}\sum_{l''=0}^{\infty}\sum_{l'=1}^{\infty}\sum_{1\leq k+k'\leq l'}A_4^{l''}M'^{l''}\binom{k+k'}{k}^{1-s}(k+k')^{s-1}M'^{k+k'-1}A_4^{k+k'}$$

$$\times\sum_{\substack{l_1+\cdots+l_{k+k'}=l'\\ l_j\geq 1}}C_4^{k+k'-1}\prod_{j=1}^{k+k'}\frac{M'^{l_j-1}}{l_j!(l_j-1)!^{s-1}}\|P^{l_j}u\|_m.$$

If we take C_{13} and A_5 so large that $A_4 \le A_5$ and $(k+k')^{s-1}C_{11}A_4^{k+k'} \le C_{13}A_5^{k+k'}$, we have

$$
(2.11) \qquad I_2 \le C_{13}\left(\sum_{l''=0}^{\infty} (M'A_5)^{l''}\right) A_5 \sum_{k+k' \ge 1} (M'A_5)^{k+k'-1}
\times \sum_{l'=l}^{\infty} \sum_{\substack{l_1+\cdots+l_{k+k'}=l' \\ l_j \ge 1}} C_4^{k+k'-1} \prod_{j=1}^{k+k'} \frac{M'^{l_j-1}}{l_j!(l_j-1)!^{s-1}} \|P^{l_j}u\|_m
$$

If we take M' so small that $M'A_5 < 1$, we have

$$
\begin{aligned}
(2.12) \qquad I_2 &\le C_{13}\frac{A_5}{1-M'A_5}\sum_{k+k' \ge 1}(C_4M'A_5)^{k+k'-1}\|u\|_{X(M',P)}{}^{k+k'} \\
&= C_{13}\frac{A_5}{1-M'A_5}\left(\sum_{k'=1}^{\infty}(C_4M'A_5)^{k'-1}\|u\|_{X(M',P)}{}^{k'}\right. \\
&\qquad \left. + \sum_{k=1}^{\infty}\sum_{k'=0}^{\infty}(C_4M'A_5)^{k+k'-1}\|u\|_{X(M',P)}{}^{k+k'}\right)
\end{aligned}
$$

If we take M' so small that $C_4M'A_5\|u\|_{X(M',P)} < 1$, we have

$$
\begin{aligned}
(2.13) \quad I_2 &\le C_{13}\frac{A_5}{1-M'A_5}\left(\frac{\|u\|_{X(M',P)}}{1-C_4M'A_5\|u\|_{X(M',P)}} + \frac{\|u\|_{X(M',P)}}{(1-C_4M'A_5\|u\|_{X(M',P)})^2}\right) \\
&\le C_{13}\frac{A_5}{1-M'A_5}\frac{2\|u\|_{X(M',P)}}{(1-C_4M'A_5\|u\|_{X(M,P)})^2}.
\end{aligned}
$$

From (2.10) and (2.13), we have the lemma with $C_8 = C_{12}/(1-M'A_5)$, $C_9 = C_{13}C_4A_5/(1-M'A_5)$ and $C_{10} = C_4A_5$. □

Proposition 2.5. *Let α be a multi-index and let v and l be integers satisfying $v \le |\alpha| + l$. Then, we have*

$$
(2.14) \qquad \sum_{|\alpha'|+l'=v} \binom{\alpha}{\alpha'}\binom{l}{l'} = \binom{|\alpha|+l}{v}.
$$

The lemma is derived from the caluculation of the coefficients of the term t^v in the both sides of $(1+t)^\alpha(1+t)^l = (1+t)^{|\alpha|+l}$.

Proposition 2.6. *For a multi-index α and an integer l we assume that the integers v_j (≥ 1) $(j=1,\cdots,k)$ satisfy $v_1 + \cdots + v_k = |\alpha| + l$. Then, we have*

$$\alpha!l! \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ l_1+\cdots+l_k=l \\ |\alpha_j|+l_j=\nu_j}} \prod_{j=1}^{k} \frac{(|\alpha_j|+l_j)!}{\alpha_j! l_j!} = (|\alpha|+l)!. \tag{2.15}$$

Proof. First, we consider the case $k=2$. Using (2.14), we have

$$\begin{aligned}
&\alpha!l! \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ l_1+l_2=l \\ |\alpha_j|+l_j=\nu_j}} \frac{(|\alpha_1|+l_1)!}{\alpha_1! l_1!} \frac{(|\alpha_2|+l_2)!}{\alpha_2! l_2!} \\
&= \nu_1!\nu_2! \sum_{|\alpha_1|+l_1=\nu_1} \binom{\alpha}{\alpha_1}\binom{l}{l_1} \\
&= \nu_1!\nu_2! \binom{|\alpha|+l}{\nu_1} = (|\alpha|+l)!.
\end{aligned}$$

This proves (2.15) for $k=2$. In the general case, we can prove (2.15) by the induction on k. $\square$

3. Proof of Theorem 1.1

We prove Theorem 1.1 by the contraction principle.

Proof of Theorem 1.1. We consider the following linearized equation with respect to (1.1),

$$\begin{cases} Lu \equiv i\partial_t u + \Delta u = f(t,x,v), \\ u(0,x) = \phi(x). \end{cases} \tag{3.1}$$

We denote the mapping which corresponds v to u by S. We write $W(M) = C([0,T]; G^s_M(P; H^m))$ with norm $|||\cdot|||_{W(M)}$ and we denote $W(M,\rho) = \{f \in W(M); |||f|||_{W(M)} \leq \rho\}$ for $\rho > 0$. We let $\rho = 2\|\phi\|_{G^s_M(x\cdot\nabla; H^m)}$.

First we show that S maps $W(M,\rho)$ to itself if we take T and M sufficiently small. The associate integral equation to the Cauchy problem (3.1) is

$$u = e^{it\Delta}\phi + i\int_0^t e^{i(t-s)\Delta} f(s,x,v(s))ds, \tag{3.2}$$

where $e^{it\Delta}$ is an evolution operator for $i\partial_t - \Delta$. Since $[L,P] = 2L$, we have the linearized equation for $P^l u$,

$$\begin{cases} L(P^l u) = (P+2)^l f(t,x,v), \\ P^l u(t,x)|_{t=0} = (x\cdot\nabla)^l \phi(x). \end{cases} \tag{3.3}$$

The associate integral equation to the above Cauchy problem is

$$P^l u = e^{it\Delta}[(x\cdot\nabla)^l\phi] + i\int_0^t e^{i(t-s)\Delta}[(P+2)^l\{f(s,x,v(s))\}]ds. \tag{3.4}$$

Taking H^m norm of the both sides of the above equation, we have

$$\|P^l u\|_m \leq \|(x\cdot\nabla)^l\phi\|_m + i\int_0^T \|(P+2)^l\{f(s,x,v(s))\}\|_m ds. \tag{3.5}$$

Let $M_1(\leq M)$ be a positive number to be determined later. Multiplying $M_1^{l-1}/(l!(l-1)!^{s-1})$ to the both sides of the above for $l\geq 1$ and making a summation with respect to l, we have

$$\|u\|_{G^s_{M_1}(P;H^m)} \leq \|\phi\|_{G^s_{M_1}(x\cdot\nabla;H^m)} + \int_0^T \|f(s,x,v(s))\|_{G^s_{M_1}(P+2;H^m)}ds \tag{3.6}$$

$$\leq \|\phi\|_{G^s_{M_1}(x\cdot\nabla;H^m)} + e^{2M_1}T|||f(t,x,v(t))|||_{W(M_1)}. \tag{3.7}$$

Taking suprimun with respect to t in $[0,T]$ of the both sides of the above inequlity, we have

$$|||u|||_{W(M_1)} \leq \|\phi\|_{G^s_{M_1}(x\cdot\nabla;H^m)} + e^{2M_1}T|||f(t,x,v(t))|||_{W(M_1)}. \tag{3.8}$$

From Lemma 2.3, we have with $M_1\leq M'$

$$|||f(t,x,v)|||_{W(M_1)} \leq \|f(t,x,v)\|_m + C_8 + \operatorname*{syp}_{t\in[0,T]} \frac{C_9\|v\|_{X(M_1,P)}}{(1-C_{10}M_1\|v\|_{X(M_1,P)})^2}. \tag{3.9}$$

Since $|||f(t,x,v)|||_{X(M_1,P)} \leq \|v\|_{X(M,P)} \leq \rho$, the last term of the right hand side of (3.9) is estimated by

$$\frac{C_9\rho}{(1-C_{10}M_1\rho)^2}. \tag{3.10}$$

On the other hand, we have by Lemma 2.1,

$$\|f(t,x,v)\|_m \leq C_{14}, \tag{3.11}$$

where C_{14} is a constant depending on ρ. So we have

$$|||f(t,x,v)|||_{W(M_1)} \leq C_{15}, \tag{3.12}$$

where $C_{15}=C_{14}+C_8+C_9\rho/(1-C_{10}M_1\rho)^2$. If we take T so small that $T \leq \rho/(2e^{2M_1}C_{15})$, we have from (3.8)

$$|||u|||_{W(M_1)} \leq \frac{\rho}{2} + e^{2M_1}TC_{15} \leq \rho. \tag{3.13}$$

Next we prove S is a contraction mapping in $W(M_1,\rho)$ with sufficiently small M_1. We show that $|||Su-Sv|||_{W(M_1)}\le(1/2)|||u-v|||_{W(M_1)}$ for $u,v\in W(M_1)$. The associate integral equation for $u-v$ is

$$Su-Sv=i\int_0^t e^{i(t-s)\Delta}[f(s,x,u(s))-f(s,x,v(s))]ds. \tag{3.14}$$

Taking $W(M_1)$ norm of the both sides, we have

$$|||Su-Sv|||_{W(M_1)}\le e^{2M_1}T|||f(t,x,u(t))-f(t,x,v(t))|||_{W(M_1)}. \tag{3.15}$$

Since

$$f(t,x,u)-f(t,x,v)=\int_0^1 \nabla_{u,\bar{u}}f(t,x,v+\theta(u-v))d\theta\cdot(u-v,\bar{u}-\bar{v}),$$

we have

$$\begin{aligned}&|||f(t,x,u)-f(t,x,v)|||_{W(M_1)}\\ &\quad\le C_2\sup_{0\le\theta\le1}|||\nabla_{u,\bar{u}}f(t,x,v+\theta(u-v))|||_{W(M_1)}|||u-v|||_{W(M_1)}.\end{aligned} \tag{3.16}$$

From Lemma 2.3, we have

$$\begin{aligned}&|||\nabla_{u,\bar{u}}f(t,x,v+\theta(u-v))|||_{W(M_1)}\\ &\le|||\nabla_{u,\bar{u}}f(t,x,v+\theta(u-v))|||_m\\ &\quad+C_8+C_9\sup_{t\in[0,T]}\frac{\|v+\theta(u-v)\|_{X(M_1,P)}}{(1-C_{10}M_1\|v+\theta(u-v)\|_{X(M_1,P)})^2}.\end{aligned} \tag{3.17}$$

Since $\|v+\theta(u-v)\|_{X(M_1,P)}\le(1-\theta)\|v\|_{X(M_1,P)}+\theta\|u\|_{X(M_1,P)}\le\rho$, the last term of the right hand side of (3.17) is estimated by

$$C_9\sup_{t\in[0,T]}\frac{\rho}{(1-C_{10}M_1\rho)^2}\le C_9\frac{\rho}{(1-C_{10}M_1\rho)^2}. \tag{3.18}$$

The same argument as in the proof that S maps $W(M_1,\rho)$ to $W(M_1,\rho)$, we have

$$|||f(t,x,u)-f(t,x,v)|||_{W(M_1)}\le C_{16}T|||u-v|||_{W(M_1)}. \tag{3.19}$$

Taking T so small that $T\le1/(2e^{2M_1}C_{16})$, we have

$$|||Su-Sv|||_{W(M_1)}\le\frac{1}{2}|||u-v|||_{W(M_1)}. \tag{3.20}$$

By the constraction principle, there exists a unique solution u in $W(M_1)$ consequently. □

4. Local Gevrey Regularizing Property

In this section, we prove Theorem 1.2, which shows the local Gevrey regularizing property of the solution u. We take a positive constant R and take a C^∞-function $r(x)$ with the property

$$\begin{cases} r(x)=1 & \text{for} \quad |x|\leq R, \\ r(x)=0 & \text{for} \quad |x|\geq R+1. \end{cases}$$

We note that

$$\|\partial_x^\alpha(ru)\|_m \leq C\|\Delta(ru)\|_m, \tag{4.1}$$

for $u\in H^m$ and for a multi-index $|\alpha|\leq 2$.

Let $u(x)$ be a solution of (1.1) constructed in Theorem 1.1. Since $[L,P]=2L$, we have

$$LP^l u=(P+2)^l[f(t,x,u)]. \tag{4.2}$$

and hence we have from $\partial_t=\frac{1}{2t}P-\frac{1}{2t}x\cdot\nabla$,

$$\begin{aligned} \Delta P^l u &= -i\partial_t P^l u+(P+2)^l[f(t,x,u)] \\ &= -\frac{i}{2t}P^{l+1}u+\frac{i}{2t}x\cdot\nabla_x P^l u+(P+2)^l[f(t,x,u)]. \end{aligned} \tag{4.3}$$

Using this equation, we can estimate $\Delta P^l u$ by at most second derivative of $P^l u$.

Lemma 4.1. *Let $u(x)$ be a solution of* (1.1). *There exist constants C_{17} and A_6 such that*

$$\|r(x)^{|\alpha|}\partial_x^\alpha P^l u\|_m \leq C_{17}A_6^l t^{-|\alpha|} l!^s \tag{4.4}$$

for all integer l and for a multi-index α with $|\alpha|\leq 2$.

Proof. From the fact that $u\in C([0,T];G_M^s(P;H^m))$ and Lemma 2.3, the inequalities

$$\|P^l u\|_m \leq C_{18}A_7^l l!^s, \tag{4.5}$$

$$\|P^l[f(t,x,u)]\|_m \leq C_{19}A_7^l l!^s \tag{4.6}$$

hold for any $l\geq 1$. (4.5) is nothing but (4.4) for $\alpha=0$. Next we treat the case $|\alpha|=1$. Using (4.5)–(4.6), we have

$$
\begin{aligned}
(4.7)\quad \|r(x)\partial_x^\alpha P^l u\|_m &\leq \|\partial_x^\alpha r(x)P^l u\|_m+\|[r(x),\partial_x^\alpha]P^l u\|_m \\
&\leq \|r(x)P^l u\|_{m+1}+C_{20}\|P^l u\|_m \\
&\leq C_{21}\|\Delta r(x)P^l u\|_{m-1}+C_{20}C_{18}A_7^l l!^s \\
&\leq C_{21}\|r(x)\Delta P^l u\|_{m-1}+C_{21}\|[\Delta,r(x)]P^l u\|_{m-1}+C_{20}C_{18}A_7^l l!^s \\
&\leq C_{21}\{\|r(x)P^{l+1}u\|_{m-1}/2t+\|r(x)x\cdot\nabla_x P^l u\|_{m-1}/2t \\
&\qquad +\|r(x)(P+2)^l[f(t,x,u)]\|_{m-1}\}+C_{22}C_{18}A_7^l l!^s \\
&\leq C_{21}\{\|P^{l+1}u\|_{m-1}/2t+C_{23}\|P^l u\|_m/2t \\
&\qquad +\|(P+2)^l[f(t,x,u)]\|_{m-1}\}+C_{22}C_{18}A_7^l l!^s \\
&\leq C_{21}C_{18}A_7^{l+1}t^{-1}(l+1)!^s+\{C_{21}(C_{23}C_{18}+e^2C_{19})\}t^{-1}A^l l!^s+C_{22}C_{18}A_7^l l!^s \\
&\leq \{C_{21}(C_{18}A_7e^s+C_{23}C_{18}+e^2C_{19}+C_{22}C_{18})\}(A_7e^s)^l t^{-1}l!^s.
\end{aligned}
$$

This yields

$$\|r\partial_x P^l u\|_m\leq C_{24}A_6^l t^{-1}l!^s. \tag{4.8}$$

Using (4.8) to estimate the term $\|r(x)x\cdot\nabla_x P^l u\|_m/2t$, we can prove (4.4) for $|\alpha|=2$. □

In the follow, we prove Theorem 1.2 by showing

$$\|r^{|\alpha|}\partial_x^\alpha P^l u\|_m\leq A_0^{|\alpha|+l-1}t^{-|\alpha|}(|\alpha|+l-2)!^\sigma l!^{s-\sigma}\quad \text{for all } l \tag{4.9}$$

for all $|\alpha|\geq 2$. We note that (4.9) for $|\alpha|=2$ hold from (4.4). So we assume (4.9) for $|\beta|<|\alpha|$ and prove (4.9) for a multi-index α satisfying $|\alpha|\geq 3$.

Let γ be a multi-index with $|\gamma|=2$ and we put $\alpha'=\alpha-\gamma$. We estimate the each term of the right hand side of the identity,

$$r^{|\alpha|}\partial_x^\alpha P^l u=\partial_x^\gamma r^{|\alpha|}\partial_x^{\alpha'}P^l u+[r^{|\alpha|},\partial_x^\gamma]\partial_x^{\alpha'}P^l u. \tag{4.10}$$

Lemma 4.2. *Assume that* (4.9) *holds for* $|\beta|<|\alpha|$. *Then we have*

$$\|[r^{|\alpha|},\partial_x^\gamma]\partial_x^{\alpha'}P^l u\|_m\leq C_{25}A_0^{|\alpha'|+l}t^{-|\alpha'|-1}(|\alpha'|+l)!^\sigma l!^{s-\sigma}. \tag{4.11}$$

Proof. Let $\partial_x^\gamma=\partial_j\partial_k$ with $\partial_j=\partial_{x_j}$ and set $r_j=\partial_j r$, $r_k=\partial_k r$. Then, since

$$
\begin{aligned}
[r^{|\alpha|},\partial_x^\gamma]=-\{&|\alpha|r^{|\alpha|-1}r_j\partial_k+|\alpha|r^{|\alpha|-1}r_k\partial_j \\
&+|\alpha|(|\alpha|-1)r^{|\alpha|-2}r_jr_k+|\alpha|r^{|\alpha|-1}(\partial_x^\gamma r)\},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\|[r^{|\alpha|},\partial_x^{\gamma}]\partial_x^{\alpha'}P^l u\|_m \\
&\quad\le |\alpha|\,\|r^{|\alpha|-1}r_j\partial_k\partial_x^{\alpha'}P^l u\|_m+|\alpha|\,\|r^{|\alpha|-1}r_k\partial_j\partial_x^{\alpha'}P^l u\|_m \\
&\qquad+|\alpha|(|\alpha|-1)\|r^{|\alpha|-2}\partial_x^{\alpha'}P^l u\|_m+|\alpha|\,\|(\partial_x^{\gamma}r)\partial_x^{\alpha'}P^l u\|_m.
\end{aligned}
$$

First we treat the case $|\alpha'|\ge 2$. Then from (4.9) for $|\beta|<|\alpha|$, we have

$$
\begin{aligned}
&\|[r^{|\alpha|},\partial_x^{\gamma}]\partial_x^{\alpha'}P^l u\|_m \\
&\quad\le 2C_{26}|\alpha|A_0^{|\alpha'|+l}t^{-|\alpha'|-1}(|\alpha'|+l-1)!^{\sigma}l!^{s-\sigma} \\
&\qquad+|\alpha|(|\alpha|-1)A_0^{|\alpha'|+l-1}t^{-|\alpha'|}(|\alpha'|+l-2)!^{\sigma}l!^{s-\sigma} \\
&\qquad+C_{27}|\alpha|A_0^{|\alpha'|+l-1}t^{-|\alpha'|}(|\alpha'|+l-2)!^{\sigma}l!^{s-\sigma} \\
&\quad\le C_{28}A_0^{|\alpha'|+l}t^{-|\alpha'|-1}(|\alpha'|+l)!^{\sigma}l!^{s-\sigma}.
\end{aligned}
$$

Here, we used $|\alpha|\le 3(|\alpha'|+l)$. Next we treat the case $|\alpha'|=1$. Since $|\alpha|=3$, we get (4.11) by (4.4). This proves (4.11). $\square$

In order to estimate the H^m-norm of the first term of the right hand side of (4.10) we use the estimate (4.1). Then, We have

$$
\begin{aligned}
(4.12)\qquad &\|\partial_x^{\gamma}r^{|\alpha|}\partial_x^{\alpha'}P^l u\|_m\le\|r^{|\alpha|}\partial_x^{\alpha'}P^l u\|_{m+2} \\
&\le C_{29}\Delta r^{|\alpha|}\partial_x^{\alpha'}P^l u\|_m \\
&\le C_{29}\|r^{|\alpha|}\partial_x^{\alpha'}\Delta P^l u\|_m+\|[\Delta,r^{|\alpha|}]\partial_x^{\alpha'}P^l u\|_m\} \\
&\le C_{29}\{\|r^{|\alpha|}\partial_x^{\alpha'}P^{l+1}u\|_m/2t+\|r^{|\alpha|}\partial_x^{\alpha'}x\cdot\nabla_x P^l u\|_m/2t \\
&\qquad+\|r^{|\alpha|}\partial_x^{\alpha'}(P+2)^l[f(t,x,u)]\|_m \\
&\qquad+\|[\Delta,r^{|\alpha|}]\partial_x^{\alpha'}P^l u\|_m\}.
\end{aligned}
$$

Now, we estimate the each term in the right hand side of (4.12).

Lemma 4.3. *Assume that* (4.9) *holds for* $|\beta|<|\alpha|$. *Then, we have*

$$
(4.13)\qquad \|[\Delta,r^{|\alpha|}]\partial_x^{\alpha'}P^l u\|_m\le C_{30}A_0^{|\alpha'|+l}t^{-|\alpha'|-1}(|\alpha'|+l)!^{\sigma}l!^{s-\sigma}.
$$

We can prove this lemma by the same way as in the proof of Lemma 4.2.

Lemma 4.4. *Let σ be a positive number with $\sigma\ge s/2$. Assume that* (4.9) *holds for $|\beta|<|\alpha|$. Then the inequality*

$$
(4.14)\qquad \|r^{|\alpha|}\partial_x^{\alpha'}P^{l+1}u\|_m\le C_{31}A_0^{|\alpha'|+l}t^{-|\alpha'|}(|\alpha'|+l)!^{\sigma}l!^{s-\sigma}
$$

holds with $\alpha'=\alpha-\gamma$ *and* $|\gamma|=2$.

Proof. Since $\sigma\geq s/2$ and $|\alpha'|\geq 1$, we have $(l+1)^{s-\sigma}/(|\alpha'|+l)^{\sigma}\leq 1$. Hence, if $|\alpha'|\geq 2$, we get from (4.9)

$$\begin{aligned}\|r^{|\alpha|}\partial_x^{\alpha'}P^{l+1}u\|_m &\leq \|r^{|\alpha'|}\partial_x^{\alpha'}P^{l+1}u\|_m\\ &\leq A_0^{|\alpha'|+l}t^{-|\alpha'|}(|\alpha'|+l-1)!^{\sigma}(l+1)!^{s-\sigma}\\ &= A_0^{|\alpha'|+l}t^{-|\alpha'|}\{(l+1)^{s-\sigma}/(|\alpha'|+l)\}(|\alpha'|+l)!^{\sigma}l!^{s-\sigma}\\ &\leq A_0^{|\alpha'|+l}t^{-|\alpha'|}(|\alpha'|+l)!^{\sigma}l!^{s-\sigma}\end{aligned}$$

and get (4.14). We also have (4.14) for $|\alpha'|=1$ from (4.4). □

Lemma 4.5. *Assume that* (4.9) *holds for* $|\beta|<|\alpha|$. *Then there exists a constant* C_{32} *such that*

$$\|r^{|\alpha|}\partial_x^{\alpha'}x\cdot\nabla_xP^lu\|_m\leq C_{32}A_0^{|\alpha'|+l}t^{-|\alpha'|-1}(|\alpha'|+l)!^{\sigma}l!^{s-\sigma} \tag{4.15}$$

holds with $\alpha'=\alpha-\gamma$ *and* $|\gamma|=2$.

Proof. Using the boundedness of $\operatorname{supp} r(x)$, we have from (4.9) for $|\beta|=|\alpha|-1$ and $\beta=\alpha'$

$$\begin{aligned}&\|r^{|\alpha|}\partial_x^{\alpha'}x\cdot\nabla_xP^lu\|_m\\ &\quad\leq\sum_{j=1}^n\|r^{|\alpha|}x_j\partial_k\partial_x^{\alpha'}\partial_jP^lu\|_m+\sum_{j=1}^n\alpha'_j\|r^{|\alpha|}\partial_x^{\alpha'}P^lu\|_m\\ &\quad\leq C_{32}A_0^{|\alpha'|+l}t^{-|\alpha'|-1}(|\alpha'|+l)!^{\sigma}l!^{s-\sigma}\end{aligned}$$

This proves (4.14). □

Lemma 4.6. *Let* $f(t,x,u)$ *be a function satisfying Assumption* 1.3. *Assume that* (4.9) *holds for* $|\beta|<|\alpha|$. *Then, we have*

$$\|r^{|\alpha|}\partial_x^{\alpha'}(P+2)^l[f(t,x,u(t,x))]\|_m\leq C_{33}A_0^{|\alpha'|+l}t^{-|\alpha'|}(|\alpha'|+l)!^{\sigma}l!^{s-\sigma}. \tag{4.16}$$

Proof. We note that we have

$$\begin{aligned}(|\beta|+l-2)! &= (|\beta|+l)!/\{(|\beta|+l-1)(|\beta|+l)\}\\ &\leq C_{34}(|\beta|+l)!/(|\beta|+l+1)^2,\end{aligned}$$

for $|\beta|+l\geq 2$, which and (4.9) for $|\beta|<|\alpha|$ yield

$$\|r^{|\beta|}\partial_x^\beta P^l u\|_m \le C_{34} A_0^{|\beta|+l-1} t^{-|\beta|}(|\beta|+l)(|\beta|+l-1)!^{\sigma-1} \times l!^{s-\sigma}/(|\beta|+l+1)^2 \tag{4.17}$$

for $2\le|\beta|<|\alpha|$. We note that, from (4.4), the above (4.17) holds also for $|\beta|+l\ge 1$ with $\beta=0$ or $|\beta|=1$. Moreover, from the Assumption 1.3 we have

$$\begin{aligned} &\|(r^{|\alpha|}\partial_x^{\alpha'}(P+2)^l f)(t,\cdot,u(t,\cdot))\|_m \le C_{35} A_8^{|\alpha'|+l}\alpha'!^\sigma l!^s, \\ &|(\partial_x^\gamma \partial_u^j \partial_{\bar u}^{j'} r^{|\alpha|}\partial_x^\alpha (P+2)^l \partial^k u \partial_{\bar u}^{k'} f)(t,x,u(t,x))| \\ &\qquad \le C_{36} A_9^{|\alpha|+l+k+k'}\alpha!^\sigma l!^s k! k'! (k+k')!^{\sigma-1} \\ &\qquad\qquad \text{for } j+j'+|\gamma|\le m \end{aligned} \tag{4.18}$$

Using these estimates we prove (4.16). Since (4.16) is trivial when $|\alpha'|+l=0$, we may assume $|\alpha'|+l\ge 1$. Then, from the differentiation of composite function and (4.17)–(4.18) we have

$$\begin{aligned} &\|r^{|\alpha|}\partial_x^{\alpha'}(P+2)^l[f(t,x,u(t,x))]\|_m \\ &\le \|(r^{|\alpha|}\partial_x^{\alpha'}(P+2)^l f)(t,x,u(t,x))\|_m \\ &\quad + C_{37}\sum_{\substack{\beta'+\beta''=\alpha' \\ l'+l''=l \\ |\beta'|+l'\ne 0}}\ \sum_{1\le k+k'\le|\beta'|+l'} \frac{\alpha'! l!}{\beta''! l''! k! k'!} \\ &\quad \times \sup_{|x|\le R+1,\, j+j'+|\gamma|\le m} |\partial_x^\gamma \partial_u^j \partial_{\bar u}^{j'}(r^{|\beta''|}\partial_x^{\beta''}(P+2)^{l''}\partial_u^k \partial_{\bar u}^{k'} f)| \\ &\quad \times C_4^{k+k'-1}\sum_{\substack{\beta_1+\cdots+\beta_{k+k'}=\beta' \\ l_1+\cdots+l_{k+k'}=l' \\ |\beta_j|+l_j\ne 0}}\ \prod_{j=1}^{k}\frac{1}{\beta_j! l_j!}\|r^{|\beta_j|}\partial_x^{\beta_j}P^{l_j}u\|_m \\ &\qquad\qquad \times \prod_{j=k+1}^{k+k'}\frac{1}{\beta_j! l_j!}\|r^{|\beta_j|}\partial_x^{\beta_j}P^{l_j}\bar u\|_m \\ &\le C_{35}A_8^{|\alpha'|+l}\alpha'!^\sigma l!^s \\ &\quad + C_{37}\sum_{\substack{v'+v''=|\alpha'|+l \\ v'\ne 0}}\ \sum_{\substack{\beta'+\beta''=\alpha' \\ l'+l''=l \\ |\beta'|+l'\ne v'}}\ \sum_{1\le k+k'\le|\beta'|+l'} \binom{\alpha'}{\beta'}\binom{l}{l'} \\ &\quad \times C_{36}A_9^{|\beta''|+l''+k+k'}\beta''!^\sigma l''!^s (k+k')!^{\sigma-1} \\ &\quad \times C_4^{k+k'-1}K_{\beta',l',k+k'}, \end{aligned}$$

where

$$K_{\beta',l',k+k'}=\beta'!l'! \sum_{\substack{\beta_1+\cdots+\beta_{k+k'}=\beta' \\ l_1+\cdots+l_{k+k'}=l' \\ |\beta_j|+l_j\neq 0}} \prod_{j=1}^{k+k'} C_{34}A_0^{|\beta_j|+l_j-1}$$

$$\times t^{-|\beta_j|}\frac{(|\beta_j|+l_j)!}{\beta_j!l_j!}(|\beta_j|+l_j-1)!^{\sigma-1}l_j!^{s-\sigma}/(|\beta_j|+l_j+1)^2.$$

Now, we use Proposition 2.6 and Proposition 2.2. Then, we have

$$K_{\beta',l',k+k'}\leq C_{34}^{k+k'}A_0^{|\beta'|+l'-k-k'}t^{-|\beta'|}(|\beta'|+l'-k-k')!^{\sigma-1}l'!^{s-\sigma}$$

$$\times \sum_{\nu_1+\cdots+\nu_{k+k'}=|\beta'|+l'} \beta'!l'!$$

$$\times \sum_{\substack{\beta_1+\cdots+\beta_{k+k'}=\beta' \\ l_1+\cdots+l_{k+k'}=l' \\ |\beta_j|+l_j=\nu_j}} \prod_{j=1}^{k+k'} \frac{(|\beta_j|+l_j)!}{\beta_j!l_j!}\frac{1}{(|\beta_j|+l_j+1)^2}$$

$$\leq C_{34}^{k+k'}A_0^{|\beta'|+l'-k-k'}t^{-|\beta'|}(|\beta'|+l')!(|\beta'|+l'-k-k')!^{\sigma-1}l'!^{s-\sigma}$$

$$\times \sum_{\nu_1+\cdots+\nu_{k+k'}=|\beta'|+l'} \prod_{j=1}^{k+k'} \frac{1}{(\nu_j+1)^2}$$

$$\leq C_3^{k+k'-1}C_{34}^{k+k'}A_0^{|\beta'|+l'-k-k'}t^{-|\beta'|}(|\beta'|+l')!(|\beta'|+l'-k-k')!^{\sigma-1}$$

$$\times l'!^{s-\sigma}/(|\beta'|+l'+1)^2.$$

Hence, using Proposition 2.5 now, we have from $|\beta'|+l'=\nu'$ and $|\beta''|+l''=\nu''$

$$\|r^{|\alpha|}\partial_x^{\alpha'}(P+2)^l[f(t,x,u(t,x))]\|_m$$

$$\leq C_{35}A_8^{|\alpha'|+l}\alpha'!^\sigma l!^s$$

$$+C_{37}\sum_{\substack{\nu'+\nu''=|\alpha'|+l \\ \nu'\neq 0}} \sum_{\substack{\beta'+\beta''=\beta \\ l'+l''=l \\ |\beta'|+l'=\nu'}} \sum_{1\leq k+k'\leq \nu'} \binom{\alpha'}{\beta'}\binom{l}{l'}$$

$$\times C_{36}A_9^{\nu''+k+k'}\beta''!^\sigma l''!^s(k+k')!^{\sigma-1}$$

$$\times C_4^{k+k'-1}C_3^{k+k'-1}C_{34}^{k+k'}A_0^{\nu'-k-k'}t^{-|\beta'|}\nu'!$$

$$\times(\nu'-k-k')!^{\sigma-1}l'!^{s-\sigma}/(\nu'+1)^2$$

$$\leq C_{35}A_8^{|\alpha'|+l}\alpha'!^\sigma l!^s$$

$$+(C_{37}C_{36}/C_4C_3)A_0^{|\alpha'|+l}t^{-|\alpha'|}(|\alpha'|+l)!^{\sigma-1}l!^{s-\sigma}$$

$$\times \sum_{\nu'+\nu''=|\alpha'|+l}(A_9/A_0)^{\nu''}/(\nu'+1)^2$$

$$\times\{\nu'!\nu''!\sum_{|\beta'|+l'=\nu'}\binom{\alpha'}{\beta'}\binom{l}{l'}\}$$

$$\times \{ \sum_{1 \le k+k' \le v'} (C_3 C_4 C_{34} A_9 / A_0)^{k+k'} \}.$$

Here, we used $\beta''!^\sigma l''!^\sigma \le v''!^\sigma$ and $\{v''!(k+k')!(v'-k-k')!\}^{\sigma-1} \le (|\alpha'|+l)!^{\sigma-1}$. Hence, assumming $A_0 \ge 2A_9$ and $A_0 \ge 2C_3C_4C_{34}A_9$, we get

$$\begin{aligned}
&\|r^{|\alpha|}\partial_x^{\alpha'}(P+2)^l[f(t,x,u(t,x))]\|_m \\
&\quad \le C_{35} A_0^{|\alpha'|+l}(|\alpha'|+l)!^\sigma l!^{s-\sigma} \\
&\qquad + 16(C_{37}C_{36}/C_4C_3)A_0^{|\alpha'|+l} t^{-|\alpha'|}(|\alpha'|+l)!^\sigma l!^{s-\sigma}.
\end{aligned}$$

This proves (4.16). □

Now, we are prepared to prove Theorem 1.2.

Proof of Theorem 1.2. For any fixed positive constant R we take a C^∞-function $r(x)$ satisfying (4). In order to prove (1.2), we have only to prove (4.9) for any α with $|\alpha| \ge 2$. Note that (4.9) for $|\alpha|=2$ holds from (4.4). So it suffices to show (4.9) for $|\alpha|=N\ge 3$ under the assumption that (4.9) holds for $|\alpha|<N$. Let γ be a multi-index with $|\gamma|=2$ and let $\alpha'=\alpha-\gamma$. From Lemmas 4.2–4.6, we have

$$\|r^\alpha \partial_x^\alpha P^l u\|_m \le (C_{25}+C_{29}(C_{30}+C_{31}+C_{32}+C_{33}))A_0^{|\alpha'|+l} t^{-|\alpha'|-1}(|\alpha'|+l)!^\sigma l!^{s-\sigma}.$$

Retaking the constant A_0 so large that $A_0 \ge C_{25}+C_{29}(C_{30}+C_{31}+C_{32}+C_{33})$ we have the inequality (4.9). □

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