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<th>Title</th>
<th>On P-exchange rings</th>
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Osaka University
ON $P$-EXCHANGE RINGS

HIKOJI KAMBARA AND KIYOICHI OSHIRO

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There is a problem concerning the exchange property: which ring $R$ satisfies the condition that every projective right $R$-module satisfies the exchange property. A ring $R$ with the above condition is said to be a right $P$-exchange ring. $P$-exchange rings have been studied in [2], [3], [4], [6], [7], [11], [12], and recently in [9]. Among others, it is shown in [9] that semi-regular rings with right $T$-nilpotent Jacobson radical are right $P$-exchange rings, and the converse holds for commutative rings but not in general. It is still open to determine the structure of $P$-exchange rings. Our main object of this paper is to show that a ring is a right $P$-exchange ring if and only if all Pierce stalks $R_x$ are right $P$-exchange rings.

1. Preliminaries

Throughout this paper, all rings $R$ considered are associative and all $R$-modules are unitary. For an $R$-module $M$, $J(M)$ denotes the Jacobson radical of $M$. For a ring $R$, $B(R)$ represents the Boolean ring consisting of all central idempotents of $R$ and, as usual, $\text{Spec}(B(R))$ denotes the spectrum of all prime (=maximal) ideals of $B(R)$. For a right $R$-module $M$ and an element $a$ in $M$ and $x$ in $\text{Spec}(B(R))$ we put $M_x=M/Mx$ and $a_x=a+Mx(\in M_x)$. $M_x$ is called the Pierce stalk of $M$ for $x$ ([8]). Note that $M_x=M\otimes_R R_x$ and $R_x$ is flat as an $R$-module, hence for a submodule $N$ of $M$, $N_x\subseteq M_x$. For $e$ in $B(R)$, note that $e_x=1_x$ if and only if $e\in B(R)-x$. Let $A$ and $B$ be right $R$-modules and $x$ in $\text{Spec}(B(R))$. Then there exists a canonical homomorphism $\sigma$ from $\text{Hom}_R(A, B)$ to $\text{Hom}_{R_x}(A_x, B_x)$. We denote $f^*=\sigma(f)$ for $f$ in $\text{Hom}_R(A, B)$. We note that if $A$ is projective, then $\sigma$ is an epimorphism.

We will use later the following well known facts [8]:

a) Let $M$ and $N$ be finitely generated right $R$-modules with $M\subseteq N$. If $x\in\text{Spec}(B(R))$ and $M_x=N_x$ then $Me=Ne$ for suitable $e$ in $B(R)-x$.

b) For right $R$-modules $M$ and $N$ with $M\supseteq N$, if $N_x=M_x$ for all $x$ in $\text{Spec}(B(R))$, then $M=N$.

c) A ring $R$ is a commutative regular ring if and only if all stalks $R_x$ are fields, and similarly, a ring $R$ is a strongly regular ring if and only if all $R_x$ are division rings.
For an $R$-module $M$ and a cardinal $\alpha$, $\alpha M$ denotes the direct sum of $\alpha$-copies of $M$.

2. $P$-exchange ring

An $R$-module $M$ is said to satisfy (or have) the exchange property if, for any direct sums

$$X = \bigoplus_i X_i = M \oplus Y$$

of $R$-modules, there exist suitable submodules $X_i \subseteq X_\alpha$ such that

$$X = M \oplus \bigoplus_i X_i.$$

Whenever this property hold for any finite set $I$, $M$ is said to satisfy the finite exchange property. Recently, B. Zimmerman and W. Zimmerman pointed out an important fact that, in the definition above, we can assume that each $X_i$ is isomorphic to $M$. A ring $R$ is said to be an exchange ring (or a suitable ring) if $R$ satisfies the exchange property as a right, or equivalently left, $R$-module.

**Definition** (cf. ([9])). A ring $R$ is a right $P$-exchange ring (resp. $PF$-exchange ring) if every projective right $R$-module satisfies the exchange (resp. finite exchange) property.

For the study of $P$-exchange (and $PF$-exchange) rings, we need the following conditions $(N_1)$ and $(N_2)$ for projective right $R$-modules $P$:

$(N_1)$ For any finite sum $P = \sum_i A_i$, there exist submodules $A_i \subseteq A$ such that $P = \sum_i A_i$.

$(N_2)$ For any sum $P = \sum_{\alpha} a_\alpha R$, there exist suitable submodules $a_\alpha R \subseteq a_\alpha R$ such that $P = \sum a_\alpha R$.

The following is due to Nicholson ([6]).

**Proposition 1.** a) The following are equivalent for a ring $R$:

1) $R$ is right $PF$-exchange.

2) $J(R)$ is right $T$-nilpotent (equivalently, $J(\mathfrak{R}_0 R)$ is small in $\mathfrak{R}_0 R$) and $R/J(R)$ is right $PF$-exchange.

3) $(N_1)$ holds for any projective right $R$-module $P$.

b) If $R$ is right $PF$-exchange, then so is every factor ring of $R$.

Similar results on $P$-exchange ring also hold:

**Proposition 2** (Stock [9]). a) The following are equivalent for a ring $R$:

1) $R$ is right $P$-exchange.

2) $J(R)$ is right $T$-nilpotent and $R/J(R)$ is right $P$-exchange.

3) $(N_2)$ holds for any projective right $R$-module $P$. 
b) If $R$ is right $P$-exchange, then so is every factor ring of $R$.

**Lemma 1.** If $K_aR$ satisfies the condition $(N_2)$ for any countable set $I$, then so does every free (hence every projective) right $R$-module.

Proof. Let $F=\sum a_\lambda R_\lambda$ be a free right $R$-module with $R_\lambda \subseteq R$. Consider a sum $F=\sum a_\lambda R_\lambda$. For subsets $I \subseteq \Lambda$ and $J \subseteq \Gamma$, put $F(I)=\sum a_\lambda R_\lambda$ and $A(J)=\sum a_\lambda R_\lambda$. First we take a finite subset $I_1 \subseteq \Lambda$. Starting from $I_1$, we can proceed to take $J_1 \subseteq \Gamma, J_2 \subseteq \Lambda, J_3 \subseteq \Gamma, \ldots$ such that

1) each $I_i$ and $J_i$ are finite sets,
2) $I_1 \subseteq I_2 \subseteq \cdots, J_1 \subseteq J_2 \subseteq \cdots$,
3) $F(I_1) \subseteq A(J_1) \subseteq F(I_2) \subseteq A(J_2) \subseteq \cdots$.

Putting $\Lambda_1=\bigcup I_i$ and $\Gamma_1=\bigcup J_i$, we see that

4) $|\Lambda_1| \leq \aleph_0$, $|\Gamma_1| \leq \aleph_0$,
5) $F(\Lambda_1)=A(\Gamma_1)$.

Next, we take a finite subset $K_1 \subseteq \Lambda-\Lambda$. And again starting from $K_1$, we take subsets $L_1 \subseteq \Gamma-\Gamma, K_2 \subseteq \Lambda-\Lambda, K_3 \subseteq \Lambda-\Lambda, \cdots$ such that

1) each $K_i$ and $L_i$ are finite sets,
2) $K_1 \subseteq K_2 \subseteq \cdots, L_1 \subseteq L_2 \subseteq \cdots$,
3) $F(\Lambda_1) \oplus F(K_1) \subseteq A(\Gamma_1) + A(L_1) \subseteq F(\Lambda_2) \oplus F(K_2) \subseteq A(\Gamma_2) + A(L_2) \subseteq \cdots$.

Putting $\Lambda_2=\bigcup K_i$ and $\Gamma_2=\bigcup L_i$, we see that

4) $|\Lambda_2| \leq \aleph_0$, $|\Gamma_2| \leq \aleph_0$,
5) $F(\Lambda_2)=A(\Gamma_2)$.

Proceeding this argument transfinite-inductively, we can get a well ordered set $\Omega$ and subfamilies $\{\Lambda_\alpha\}_{\alpha \in \Omega} \subseteq 2^\Lambda$ and $\{\Gamma_\alpha\}_{\alpha \in \Omega} \subseteq 2^\Gamma$ such that

a) for each $\alpha \in \Omega$, $|\Lambda_\alpha| \leq \aleph_0$ and $|\Gamma_\alpha| \leq \aleph_0$,

b) for each $\alpha \in \Omega$, $\sum_{\beta \leq \alpha} F(A_\beta) = \sum_{\beta \leq \alpha} A(\Gamma_\beta)$,

c) $F=\sum_{\alpha \in \Omega} \oplus F(\Lambda_\alpha) = \sum_{\alpha \in \Omega} A(\Gamma_\alpha)$.

For each $\alpha \in \Omega$, let $\psi_\alpha: F=\sum_{\alpha} \oplus F(\Lambda_\alpha) \rightarrow F(\Lambda_\alpha)$ be the projection. By b) we see that

$F(\Lambda_\alpha) = \psi_\alpha(A(\Gamma_\alpha))$.

and

$F = \sum_{\alpha} \oplus F(\Lambda_\alpha) = \sum_{\alpha} \psi_\alpha(A(\Gamma_\alpha))$.

Since $F(\Lambda_\alpha) = \sum_{\sigma \in \Lambda_\alpha} \oplus R_{\sigma} = \sum_{\lambda \in \Gamma_\alpha} \psi_\alpha(\alpha R_{\lambda}) = \psi_\alpha(A(\Gamma_\alpha))$, we can take $a_\lambda \in \alpha R$ for all $\lambda \in \Gamma_\alpha$ such that

$\sum_{\lambda \in \Gamma_\alpha} \oplus \psi_\alpha(\alpha R_{\lambda}) = \sum_{\lambda \in \Lambda_\alpha} \oplus R_{\lambda}$.
Since $\psi_\lambda(a^\lambda R)$ is projective, we can take $a^\lambda \in a_\lambda R$ such that the restriction map $\psi_\lambda|_a^\lambda R$ is an isomorphism for each $\lambda \in \Gamma_\alpha$ and $\alpha \in \Omega$. Then we see that

$$F = \sum_{\alpha \in \Omega} \left( \sum_{\lambda \in \Gamma_\alpha} \langle a^\lambda R \rangle \right)$$

as desired.

**Lemma 2.** If $\mathfrak{m}_R$ satisfies the exchange property, then $\mathfrak{m}_R$ satisfies the condition $(N_2)$.

**Proof.** Let $F = \sum_{i=1}^\infty \oplus m_i R$ be a free right $R$-module $R \twoheadrightarrow m_i r$ by $r \mapsto m_i r$. Consider a sum $F = \sum_{i=1}^\infty a_i R$, and let $\psi: \sum_{i=1}^\infty m_i R \rightarrow \sum_{i=1}^\infty a_i R$ be the canonical epimorphism from Lemma 1. Since $F = \sum_{i=1}^\infty a_i R$ is projective, $	ext{Ker} \psi = \sum_{i=1}^\infty m_i R$; say $F = B \oplus \text{Ker} \psi$. Let $\pi: F = B \oplus \text{Ker} \psi \rightarrow B$ be the projection and put $b_i = \pi(m_i)$ for all $i$. Then $\psi(b_i) = a_i$ for all $i$. By assumption, there exist a decomposition $m_i R = n_i R \oplus t_i R$ for each $i$ such that

$$F = (\sum_{i=1}^\infty b_i R) \oplus \text{Ker} \psi = (\sum_{i=1}^\infty a_i R) \oplus \text{Ker} \psi.$$  

Since $\pi(n_i R) \subseteq b_i R$ and $\sum_{i=1}^\infty \pi(n_i R) = \sum_{i=1}^\infty b_i R$, we have that $\sum_{i=1}^\infty \psi \pi(n_i R) = \sum_{i=1}^\infty a_i R$ and $\psi \pi(n_i R) \subseteq a_i R$ for each $i$. Thus $F$ satisfies the condition $(N_2)$.

**Theorem 1.** The following conditions are equivalent for a given ring $R$:

1. $R$ is a right $P$-exchange ring.
2. Every projective right $R$-module satisfies the condition $(N_2)$.
3. $\mathfrak{m}_R$ has the exchange property.
4. $\mathfrak{m}_R$ satisfies the condition $(N_2)$.

**Proof.** The implications 1) $\Rightarrow$ 3) and 2) $\Rightarrow$ 4) are trivial. 1) $\Rightarrow$ 2) is Proposition 2. The implication 4) $\Rightarrow$ 2) is Lemma 1 and 3) $\Rightarrow$ 4) is Lemma 2.

3. **Commutative $P$-exchange ring**

In this section, we study the rings whose Pierce stalks are local right perfect rings. Such rings are right $P$-exchange rings and for commutative rings the converse also holds (Theorem 2 and Corollary 1)

**Lemma 3.** If $R$ is a ring such that all $R_x$ are local right perfect rings, then so is every factor ring of $R$.

**Proof.** Let $I$ be an ideal of $R$, and put $\tilde{R} = R/I$. Let $y$ be in $\text{Spec}(B(\tilde{R}))$ and put $x = \{e \in B(R) | e + I \in y \}$. Then $x \subseteq \text{Spec}(B(R))$ and there is a ring
epimorphism from \( R_x \) to \( R_y \) as a result, \( R_y \) is also a local right perfect ring.

**Proposition 3.** Let \( R \) be a ring whose Pierce stalks are local right perfect rings. Then

1) \( J(R) \) is right \( T \)-nilpotent,
2) \( J(E) \) coincides with the set of all nilpotent elements of \( R \).

Proof. 1) Let \( \{a_i | i = 1, 2, \ldots \} \) be a subset of \( J(R) \) and let \( x \in \text{Spec}(\beta(B(R))) \). Since \( J(R)_x \subseteq J(R_x) \), \( \{(a_i)_x | i = 1, 2, \ldots \} \subseteq J(R_x) \). Hence there exists \( n \) such that \((a_n)_x(a_{n-1})_x \cdots (a_1)_x = 0 \). So there exists a neighborhood \( N(x) \) of \( x \) such that \((a_n a_{n-1} \cdots a_1)_x = 0 \) for all \( z \) in \( N(x) \). Hence by the partition property of \( \text{Spec}(\beta(B(R))) \), we can have neighborhoods \( N_1, \ldots, N_k \) and \( n_1, \ldots, n_k \) such that \( \text{Spec}(\beta(B(R))) = N_1 \cup \cdots \cup N_k \) and \((a_n a_{n-1} \cdots a_1)_x = O_x \) for all \( x \) in \( N_i \) for \( i = 1, \ldots, k \). Hence if we put \( m = \max \{n_i \} \), then \((a_m a_{m-1} \cdots a_1)_x = O_x \) for all \( x \) in \( \text{Spec}(\beta(B(R))) \), hence \( a_m a_{m-1} \cdots a_1 = 0 \).

2) By 1) \( J(R) \) is nil. For \( x \) in \( \text{Spec}(\beta(B(R))) \), we denote by \( M(x) \) the unique maximal (right) ideal of \( R \) containing \( R_x \). Then we see that \( \{M(x) | x \in \text{Spec}(\beta(B(R)))\} \) is just the family of all maximal right ideals of \( R \). For, if \( M \) is a maximal right ideal of \( R \), then \( \{e \in B(R) | e \in M\} \subseteq \text{Spec}(\beta(B(R))) \). As a result, we have \( J(R) = \cap \{M(x) | x \in \text{Spec}(\beta(B(R)))\} \). Now, let \( a \) be a nilpotent element of \( R \). Since \( M(x) | R_x = J(R_x) \), we see that \( a \in M(x) \). (Note that \( R_x \) is local). Hence \( a \in \cap \{M(x) | x \in \text{Spec}(\beta(B(R)))\} = J(R) \). Accordingly \( J(R) \) coincides with the set of all nilpotent elements of \( R \).

**Lemma 4.** Let \( R \) be a ring such that \( J(R) = 0 \) and all stalks \( R_x \) are local right perfect rings. Then \( R \) is a strongly regular ring.

Proof. We may show that all stalks are division rings. Let \( x \in \text{Spec}(\beta(B(R))) \). Let \( a \) be in \( R \) such that \( a_x \in J(R_x) \). Then there exists \( n \) such that \((a_n)^x = (a_n^x) = 0 \), so \( a^n e = O \) for a suitable \( e \) in \( B(R) - x \). Since \((ae)^n = a^n e = O \), Proposition 4 shows that \( ae \in J(R) = 0 \), so \( a_x = O_x \). Thus \( J(R_x) = O \). Since \( R_x \) is a right perfect ring, it follows that \( R_x \) is a division ring.

**Notation.** For a ring \( R \), we denote by \( I(R) \) the set of all idempotents of \( R \). Of course \( B(R) \subseteq I(R) \).

**Lemma 5.** For a ring \( R \), the following are equivalent:

1) \( I(R) = B(R) \).
2) \( I(R_x) = \{1_x, O_x\} \) for all \( x \) in \( \text{Spec}(B(R)) \).

Proof. 1) \( \Rightarrow \) 2): Let \( a \in R \) such that \( a_x \in I(R_x) \) (where \( x \in \text{Spec}(\beta(B(R))) \). Since \((a_x)^x = a_x, a x e = ae \) for some \( e \) in \( B(R) - x \). Then \( ae \in I(R) = B(R) \), we see that \( a_x = (ae)_x \) is either \( 1_x \) or \( O_x \). 2) \( \Rightarrow \) 1): Let \( a \in I(R) \) and \( x \in \text{Spec}(B(R)) \). Then \( a_x = 1_x \) or \( a_x = 0_x \) since \( a_x \in I(R_x) \). Here using the partition property of
Spec(B(R)), we can take a suitable $e$ in $B(R)$ such that $ae=e$ and $a(1-e)=0$, whence $a=e \in B(R)$. Thus $I(R)=B(R)$.

We are now ready to show the following.

**Theorem 2.** The following conditions are equivalent for a given ring $R$:
1) $R$ is a right $P$-exchange ring and $I(R)=B(R)$.
2) $R/J(R)$ is a strongly regular ring, $J(R)$ is right $T$-nilpotent and $I(R)=B(R)$.
3) All stalks are local right perfect rings.

Proof. 1)$\Rightarrow$3): By Proposition 2 (b) and Lemmas 3 and 5, each $R_x$ is a $P$-exchange ring with $I(R_x)=B(R_x)$, whence $R_x$ is a right perfect ring by [11, Theorem 8]. The implication 2)$\Rightarrow$1) follows from Proposition 2. The implication 3)$\Rightarrow$2) follows from Proposition 3 and Lemmas 3 and 4.

**Corollary 1.** The following conditions are equivalent for a commutative ring $R$.
1) $R$ is $P$-exchange ring.
2$^*$) $R/J(R)$ is a regular ring and $J(R)$ is $T$-nilpotent.
3) All stalks are local perfect rings.

REMARK 3. The equivalence of 1) and 2) in Theorem 2 above is shown in [9]. It should be noted that an exchange ring with $T$-nilpotent Jacobson radical need not be a $P$-exchange ring, because there exist a non-regular commutative exchange ring $R$ with $J(R)=0$ ([5]).

4. Main Theorem

As we see later, or by [9] the equivalence of 1) and 2) in Corollary 1 does not hold in general. However we show that 1) and 3) are equivalent, that is, the following holds:

**Theorem 3.** A ring $R$ is a right $P$-exchange ring if and only if all Pierce stalks $R_x$ are $P$-exchange rings.

**Lemma 6.** Let $P$ be a projective right $R$-module and let $x \in \text{Spec}(B(R))$.
1) If $A$ is a finitely generated submodule with $A_x \lhd \bigoplus P_x$, then $Ae \lhd \bigoplus P$ for a suitable $e$ in $B(R)\setminus x$. 2) If $P$ is finitely generated and $A_1$ and $A_2$ are finitely generated submodules of $P$ with $P_x=(A_1)_x \oplus (A_2)_x$, then $Pe=A_1e \oplus A_2e$ for a suitable $e$ in $B(R)\setminus x$.

Proof. As 1) follows from 2), we may only show 2). Let $\tau_i$ be the

$^*$ Prof. Y. Kurata informed the authors that commutative rings $R$ which satisfy the condition 2) in Corollary 2 are studied in [1].
inclusion mapping: $A_i \to P$ for $i = 1, 2$. Since $P$ is projective, there exist $\pi_1: P \to A_1$ and $\pi_2: P \to A_2$ such that $(\pi_i)^*\ (i = 1, 2)$. Noting that $P$, $A_1$ and $A_2$ are finitely generated, we can take a suitable $e$ in $B(R) - x$ such that
\[
(1 - (\tau_1\pi_1 + \tau_2\pi_2))(Pe) = 0, \quad ((\pi_1 - (\pi_2)^*)(Pe) = 0, \quad (\tau_1\pi_1, \tau_2\pi_2)(Pe) = 0 \quad \text{for } i \neq j.
\]
Then it follows that $Pe = A_1e \oplus A_2e$.

**Lemma 7.** Let $P$ be a projective right $R$-module with a sum $P = \sum_{i=1}^n a_i R$, and let $x \in \text{Spec}(B(R))$. If $P_x = \sum_{i=1}^n (a_i R)_x$, then there exists $\{e_i\}_{i=1}^n \subseteq B(R) - x$ such that $\sum_{i=1}^n a_i e_i R = \sum_{i=1}^n a_i e_i R \oplus P$ for all $n$.

**Proof.** Since $(a_i R)_x \subseteq P_x$, there exists $e_i \in B(R) - x$ such that $a_i e_i R \subseteq P_x$ by Lemma 6. Since $(a_i R \oplus a_i R)_x \subseteq P_x$, there exists $e_i \in B(R) - x$ such that $a_i e_i R \oplus a_i e_i R \subseteq P_x$. Put $e_2 = e_1 e_2$. Then we see that
\[
a_i e_i R + a_2 e_2 R = a_i e_i R \oplus a_2 e_2 R \subseteq P_x.
\]
By similar argument, we can take $\{e_i\}_{i=1}^n \subseteq B(R) - x$ such that $e_n e_{n+1} = e_{n+1}$ for $n = 1, 2, \ldots$ and
\[
a_i e_i R + \cdots + a_n e_n R = a_i e_i R \oplus \cdots \oplus a_n e_n R \subseteq P
\]
for $n = 1, 2, \ldots$.

**Lemma 8.** Let $P$ be a finitely generated projective right $R$-module such that all stalks $P_x$ have the exchange property. Then $P$ has the exchange property.

**Proof.** Since $P$ is finitely generated, we may show that $P$ satisfies the condition $(N_1)$ (Proposition 1). So, let $P = A + B$, where $A$ and $B$ are finitely generated submodules. Let $x \in \text{Spec}(B(R))$. Since $P_x$ satisfies $(N_1)$, we can take finitely generated submodules $A_x \subseteq A$ and $B_x \subseteq B$ such that $P_x = (A_x)_x \oplus (B_x)_x$. Then, by Lemma 6, $Pe = A_x e \oplus B_x e$ for a suitable $e$ in $B(R) - x$. Using the partition property of $\text{Spec}(B(R))$, we can take orthogonal idempotents $e_1, \ldots, e_n$ in $B(R)$ and finitely generated submodules $A^*_1, \ldots, A^*_n$ of $A$ and $B^*_1, \ldots, B^*_n$ of $B$ such that
\[
P = A^*_1 e_1 \oplus \cdots \oplus A^*_n e_n \oplus B^*_1 e_1 \oplus \cdots \oplus B^*_n e_n.
\]
Hence putting $A^* = A^*_1 e_1 \oplus \cdots \oplus A^*_n e_n$ and $B^* = B^*_1 e_1 \oplus \cdots \oplus B^*_n e_n$, we have that $P = A^* \oplus B^*$. 

**Proof of Theorem 3.** If $R$ is a right $P$-exchange ring, then all $R_x$ are right $P$-exchange rings by Proposition 2. Conversely, assume that all $R_x$ are right $P$-exchange rings. We may show that $R_x R$ satisfies the condition $(N_2)$. Let
\( F = \sum_{i=1}^{\infty} R_i \oplus \oplus R \) be a free right \( R \)-module with \( R_i \cong R \) for all \( i \), so \( F = \oplus_{i=1}^{\infty} R_i \). We put \( F(s) = R_i \oplus \oplus \oplus R \) for \( s=1, 2, \ldots \). Now, consider a sum \( F = \sum_{i=1}^{\infty} a_i R \). For any \( x \) in Spec(\( B(R) \)), as \( F_\xi \) satisfies \( (N_2) \), we can take by Lemma 7 \( \{ b_1^\xi \in a_i R \mid i=1, 2, \ldots \} \) such that \( F_\xi = \sum_{i=1}^{\infty} (b_i^\xi R)_\xi \) and

\[
F \oplus b_1^\xi R + \cdots + b_n^\xi R = b_1^\xi R \oplus \cdots \oplus b_n^\xi R
\]

for all \( n \).

Let \( x \in \text{Spec}(B(R)) \) and take any \( s_1 \geq 1 \). Then there exists \( n(x) \) such that

\[
F(s_1)_x \subseteq \sum_{i=1}^{n(x)} (b_i^\xi R)_x
\]

and so there exists \( e(x) \) in \( B(R) - x \) such that

\[
F(s_1) e(x) \subseteq \sum_{i=1}^{n(x)} b_i^\xi e(x) R \subseteq F
\]

Using the partition property, we have \( x_1, \ldots, x_m \) in Spec(\( B(R) \)), orthogonal idempotents \( \{ e(x_1), \ldots, e(x_m) \} \subseteq B(R) \) and \( m_1 \) such that \( 1 = \sum_{i=1}^{m_1} e(x_i) \) and

\[
F(s_1) \subseteq \sum_{i=1}^{m_1} b_i^\xi e(x_i) R \oplus \cdots \oplus \sum_{i=1}^{m_1} b_i^\xi e(x_i) R \subseteq F
\]

Put \( b_i^\xi = \sum_{j=1}^{m_1} b_i^\xi e(x_j) \) for \( i = 1, \ldots, m_1 \). Then \( b_i^\xi \in a_i R \) and

\[
F(s_1) \subseteq \sum_{i=1}^{m_1} b_i^\xi R \subseteq F
\]

Put \( G_1 = \sum_{i=1}^{m_1} b_i^\xi R \). Then \( G_1 \subseteq F(s_2) \) for a suitable \( s_2 > s_1 \). By the same argument as above, we can take \( m_2 \) and \( b_i^\xi \in a_i R \) for \( i=1, \ldots, m_2 \) such that

\[
F(s_2) \subseteq \sum_{i=1}^{m_2} b_i^\xi R
\]

Put \( G_2 = \sum_{i=1}^{m_2} b_i^\xi R \). Then \( G_2 \subseteq F(s_3) \) for some \( s_3 > s_2 > s_1 \). Continuing this argument, we can take \( s_1 < s_2 < s_3 < \cdots \) and \( G_1 = \sum_{i=1}^{m_1} b_i^\xi R \), \( G_2 = \sum_{i=1}^{m_2} b_i^\xi R \), \( \cdots \) such that \( b_i^\xi \in a_i R \) for all \( i, k \), each \( G_i \) is a direct summand of \( F \) and

\[
F(s_1) \subseteq G_1 \subseteq F(s_2) \subseteq G_2 \subseteq F(s_3) \subseteq \cdots
\]

Since \( \bigcup_{i=1}^{\infty} G_i \subseteq \bigcup_{i=1}^{\infty} F(s_i) \), we see \( F = \sum_{i=1}^{\infty} G_i \). Since \( G_{n+1} \) has the exchange property by Lemma 8, there exists \( \{ c_i^\xi \in b_i^\xi R \mid i=1, \ldots, m_n \} \) such that

\[
G_n = G_{n-1} \oplus \sum_{i=1}^{m_n} c_i^\xi R.
\]

In particular, put \( c_i^\xi = b_i^\xi \) for \( i=1, \ldots, m_1 \). Then we see that

\[
F = \sum_{i=1}^{m_1} c_i^\xi R \oplus \sum_{i=1}^{m_2} c_i^\xi R \oplus \cdots \fection 1000.0x729.0
Then \( A_i \subseteq a_i R \) for all \( i \) and \( F = \sum_{i=1}^{\infty} A_i \). This completes the proof.

**Corollary 2.** If \( R \) is a ring such that all Pierce stalks are right perfect rings, then \( R \) is a right \( P \)-exchange ring.

By making use of the corollary, we shall give a right \( P \)-exchange ring.

**Example.** Let \( P \) be an indecomposable right perfect ring and \( Q \) an indecomposable right perfect subring of \( P \) with the same identity. Consider the rings \( W = \prod_{\alpha} P_\alpha \) and \( V = \prod_{\alpha} Q_\alpha \), where \( Q_\alpha \simeq Q \) and \( P_\alpha \simeq P \) for all \( \alpha \in I \). Then the ring \( W \) is an extension ring of \( V \) and becomes a right \( V \)-module. Put \( R = \sum_{\alpha} P_\alpha + 1Q \), where 1 is the identity of \( W \). Then \( R \) is a ring such that \( B(R) = \sum_{\alpha} B(P_\alpha)^* + B(Q) \). We can easily see that \( \text{Spec}(B(R)) = \{ x_0 \} \cup \{ x_\alpha | \alpha \in I \} \), where \( x_0 = \sum_{\alpha} B(P_\alpha) \) and \( x_\alpha = \sum_{\alpha} B(P_\alpha) + 1B(Q) \). Further we see that \( R_{x_0} \simeq Q \) and \( R_{x_\alpha} \simeq P \) for all \( \alpha \in I \). Hence Corollary 3 says that \( R \) is a right \( P \)-exchange ring. In particular, if we take \( \begin{pmatrix} F & F \\ F & F \end{pmatrix} \) and \( \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix} \) as \( P \) and \( Q \), respectively, where \( F \) is a division ring, then \( R \) is a non-singular, right \( P \)-exchange ring with \( J(R) = 0 \).

**References**


\(^{*)}\) Note that \( B(P) = B(Q) = GF(2) \)


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