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ON *P*-EXCHANGE RINGS

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There is a problem concerning the exchange property: which ring R satisfies the condition that every projective right R-module satisfies the exchange property. A ring R with the above condition is said to be a right P-exchange ring. P-exchange rings have been studied in [2], [3], [4], [6], [7], [11], [12], and recently in [9]. Among others, it is shown in [9] that semi-regular rings with right T-nilpotent Jacobson radical are right P-exchange rings, and the converse holds for commutative rings but not in general. It is still open to determine the structure of P-exchange rings. Our main object of this paper is to show that a ring is a right P-exchange ring if and only if all Pierce stalks R_x are right P-exchange rings.

1. Preliminaries

Throughout this paper, all rings R considered are associative and all R-modules are unitary. For an R-module M, $J(M)$ denotes the Jacobson radical of M. For a ring R, $B(R)$ represents the Boolean ring consisting of all central idempotents of R and, as usual, $\mathrm{Spec}(B(R))$ denotes the spectrum of all prime (=maximal) ideals of $B(R)$. For a right R-module M and an element a in M and x in $\mathrm{Spec}(B(R))$ we put $M_x=M/Mx$ and $a_x=a+Mx(\in M_x)$. M_x is called the Pierce stalk of M for x ([8]). Note that $M_x=M\otimes_R R_x$ and R_x is flat as an R-module, hence for a submodule N of M, $N_x\subseteq M_x$. For e in $B(R)$, note that $e_x=1_x$ if and only if $e\in B(R)-x$. Let A and B be right R-modules and x in $\mathrm{Spec}(B(R))$. Then there exists a canonical homomorphism σ from $\mathrm{Hom}_R(A, B)$ to $\mathrm{Hom}_{R_x}(A_x, B_x)$. We denote $f^x=\sigma(f)$ for f in $\mathrm{Hom}_R(A, B)$. We note that if A is projective, then σ is an epimorphism.

We will use later the following well known facts [8]:

a) Let M and N be finitely generated right R-modules with $M\subseteq N$. If $x\in\mathrm{Spec}(B(R))$ and $M_x=N_x$ then $Me=Ne$ for suitable e in $B(R)-x$.

b) For right R-modules M and N with $M\supseteq N$, if $N_x=M_x$ for all x in $\mathrm{Spec}(B(R))$, then $M=N$.

c) A ring R is a commutative reguler ring if and only if all stalks R_x are fields, and similarly, a ring R is a strongly reguler ring if and only if all R_x are division rings.

For an R-module M and a cardinal α, αM denotes the direct sum of α-copies of M.

2. *P*-exchange ring

An R-module M is said to satisfy (or have) the *exchange* property if, for any direct sums

$$X = \sum_I \oplus X_\alpha = M \oplus Y$$

of R-modules, there exist suitable submodules $X'_\alpha \subseteq X_\alpha$ such that

$$X = M \oplus \sum_I \oplus X'_\alpha .$$

Whenever this property hold for any finite set I, M is said to satisfy the *finite exchange* property. Recently, B. Zimmerman and W. Zimmerman pointed out an important fact that, in the definition above, we can assume that each X_α is isomorphic to M. A ring R is said to be an *exchang* ring (or a *suitable* ring) if R satisfies the exchange property as a right, or equivalently left, R-module.

DEFINITION (cf. ([9]). A ring R is a right *P-exchange* ring (resp. *PF-exchange* ring) if every projective right R-module satisfies the exchange (resp. finite exchange) property.

For the study of P-exchange (and PF-exchange) rings, we need the following conditions (N_1) and (N_2) for projective right R-modules P:

(N_1) *For any finite sum* $P=\sum_{i=1}^{n} A_i$, *there exist submodules* $A_i^* \subseteq A$ *such that* $P=\sum_{i=1}^{n} \oplus A_i^*$.

(N_2) *For any sum* $P=\sum_I a_\alpha R$, *there exist suitable submodules* $a_\alpha^* R \subseteq a_\alpha R$ *such that* $P=\sum_I \oplus a_\alpha^* R$.

The following is due to Nicholson ([6]).

Proposition 1. a) *The following are equivalent for a ring R:*
1) *R is right PF-exchange.*
2) *$J(R)$ is right T-nilpotent (equivalently, $J(\aleph_0 R)$ is small in $\aleph_0 R$) and $R/J(R)$ is right PF-exchange.*
3) *(N_1) holds for any projective right R-module P.*

b) *If R is right PF-exchange, then so is every factor ring of R.*

Similar results on P-exchange ring also hold:

Proposition 2 (Stock [9]). a) *The following are equivalent for a ring R:*
1) *R is right P-exchange.*
2) *$J(R)$ is right T-nilpotent and $R/J(R)$ is right P-exchange.*
3) *(N_2) holds for any projective right R-module P.*

b) *If R is right P-exchange, then so is every factor ring of R.*

Lemma 1. *If $\aleph_0 R$ satisfies the condition (N_2) for any countable set I, then so does every free (hence every projective) right R-module.*

Proof. Let $F=\sum_{\Lambda}\oplus R_\lambda$ be a free right R-module with $R_\lambda \simeq R$. Consider a sum $F=\sum_{\Gamma} a_\alpha R$. For subsets $I\subseteq\Lambda$ and $J\subseteq\Gamma$, put $F(I)=\sum_{I}\oplus R_\alpha$ and $A(J)=\sum_{J} a_\alpha R$. First we take a finite subset $I_1\subseteq\Lambda$. Starting from I_1, we can proceed to take $J_1\subseteq\Gamma$, $I_2\subseteq\Lambda$, $J_2\subseteq\Gamma$, $I_3\subset\Lambda$, $\cdots$ such that

1) each I_i and J_i are finite sets,
2) $I_1\subseteq I_2\subseteq\cdots$, $J_1\subseteq J_2\subseteq\cdots$,
3) $F(I_1)\subseteq A(J_1)\subseteq F(I_2)\subseteq A(J_2)\subseteq\cdots$.

Putting $\Lambda_1=\bigcup_{i=1}^{\infty} I_i$ and $\Gamma_1=\bigcup_{i=1}^{\infty} J_i$, we see that

4) $|\Lambda_1|\leq\aleph_0$, $|\Gamma_1|\leq\aleph_0$,
5) $F(\Lambda_1)=A(\Gamma_1)$.

Next, we take a finite subset $K_1\subseteq\Lambda-\Lambda$. And again starting from K_1, we take subsets $L_1\subseteq\Gamma-\Gamma_1$, $K_2\subseteq\Lambda-\Lambda_1$, $K_3\subseteq\Lambda-\Lambda_1$, $\cdots$ such that

1) each K_i and L_i are finite sets,
2) $K_1\subseteq K_2\subseteq\cdots$, $L_1\subseteq L_2\subseteq\cdots$,
3) $F(\Lambda_1)\oplus F(K_1)\subseteq A(\Gamma_1)+A(L_1)\subseteq F(\Lambda_1)\oplus F(K_2)\subseteq A(\Gamma_1)+A(L_2)\subseteq\cdots$.

Putting $\Lambda_2=\bigcup_{i=1}^{\infty} K_i$ and $\Gamma_2=\bigcup_{i=1}^{\infty} L_i$, we see that

4) $|\Lambda_2|\leq\aleph_0$, $|\Gamma_2|\leq\aleph_0$,
5) $F(\Lambda_1)\oplus F(\Lambda_2)=A(\Gamma_1)+A(\Gamma_2)$.

Proceeding this argument transfinite-inductively, we can get a well ordered set Ω and subfamilies $\{\Lambda_\alpha\}_\Omega\subseteq 2^\Lambda$ and $\{\Gamma_\alpha\}_\Omega\subseteq 2^\Gamma$ such that

a) for each $\alpha\in\Omega$, $|\Lambda_\alpha|\leq\aleph_0$ and $|\Gamma_\alpha|\leq\aleph_0$,
b) for each $\alpha\in\Omega$, $\sum_{\beta\leq\alpha}\oplus F(\Lambda_\beta)=\sum_{\beta\leq\alpha} A(\Gamma_\beta)$,
c) $F=\sum_{\alpha\in\Omega}\oplus F(\Lambda_\alpha)=\sum_{\alpha\in\Omega} A(\Gamma_\alpha)$.

For each $\alpha\in\Omega$, let ψ_α: $F=\sum_{\Omega}\oplus F(\Lambda_\alpha)\to F(\Lambda_\alpha)$ be the projection. By b) we see that

$$F(\Lambda_\alpha)=\psi_\alpha(A(\Gamma_\alpha))\,.$$

and

$$F=\sum_{\Omega}\oplus F(\Lambda_\alpha)=\sum_{\Omega}\psi_\alpha(A(\Gamma_\alpha))\,.$$

Since $F(\Lambda_\alpha)=\sum_{\sigma\in\Lambda_\alpha}\oplus R_\sigma=\sum_{\lambda\in\Gamma_\alpha}\psi_\alpha(a_\lambda R)=\psi_\alpha(A(\Gamma_\alpha))$, we can take $a_\lambda^\alpha\in a_\lambda R$ for all $\lambda\in\Gamma_\alpha$ such that

$$\sum_{\lambda\in\Gamma_\alpha}\oplus\psi_\alpha(a^\alpha R)=\sum_{\sigma\in\Lambda_\alpha}\oplus R_\sigma$$

Since $\psi_\alpha(a_\lambda^\alpha R)$ is projective, we can take $a_\lambda^\alpha \in a_\lambda R$ such that the restriction map $\psi_\alpha | a_\lambda^\alpha R$ is an isomorphism for each $\lambda \in \Gamma_\alpha$ and $\alpha \in \Omega$. Then we see that

$$F = \sum_{\alpha\in\Omega} \oplus (\sum_{\lambda\in\Gamma_\alpha} \oplus a_\lambda^\alpha R))$$

as desired.

Lemma 2. *If $\aleph_0 R$ satisfies the exchange property, then $\aleph_0 R$ satisfies the condition (N_2).*

Proof. Let $F=\sum_{i=1}^\infty \oplus m_i R$ be a free right R-module $R \simeq m_i R$ by $r \leftrightarrow m_i r$. Consider a sum $F=\sum_{i=1}^\infty a_i R$, and let $\psi\colon \sum_{i=1}^\infty \oplus m_i R \to \sum_{i=1}^\infty a_i R$ be the canonical epimorphism from Lemma 1. Since $F=\sum_{i=1}^\infty a_i R$ is projective, Ker $\psi \lessdot \oplus F$; say $F=B\oplus \mathrm{Ker}\,\psi$. Let $\pi\colon F=B\oplus\mathrm{Ker}\,\psi \to B$ be the projection and put $b_i=\pi(m_i)$ for all i. Then $\psi(b_i)=a_i$ for all i. By assumption, there exist a decomposition $m_i R = n_i R \oplus t_i R$ for each i such that

$$\begin{aligned} F &= (\sum_{i=1}^\infty b_i R)\oplus \mathrm{Ker}\,\psi \\ &= (\sum_{i=1}^\infty \oplus n_i R)\oplus \mathrm{Ker}\,\psi\,. \end{aligned}$$

Since $\pi(n_i R)\subseteq b_i R$ and $\sum_{i=1}^\infty \oplus\ \pi(n_i R)=\sum_{i=1}^\infty b_i R$, we have that $\sum_{i=1}^\infty \oplus \psi\pi(n_i R)=\sum_{i=1}^\infty a_i R$ and $\psi\pi(n_i R)\subseteq a_i R$ for each i. Thus F satisfies the condition (N_2).

Theorem 1. *The following conditions are equivalent for a given ring R:*

1) *R is a right P-exchange ring.*
2) *Every projective right R-module satisfies the condition (N_2).*
3) *$\aleph_0 R$ has the exchange property.*
4) *$\aleph_0 R$ satisfies the condition (N_2).*

Proof. The implications 1)$\Rightarrow$3) and 2)$\Rightarrow$4) are trivial. 1)$\Leftrightarrow$2) is Proposition 2. The implication 4)$\Rightarrow$2) is Lemma 1 and 3)$\Rightarrow$4) is Lemma 2.

3. Commutative P-exchange ring

In this section, we study the rings whose Pierce stalks are local right perfect rings. Such rings are right P-exchange rings and for commutative rings the converse also holds (Theorem 2 and Corollary 1)

Lemma 3. *If R is a ring such that all R_x are local right perfect rings, then so is every factor ring of R.*

Proof. Let I be an ideal of R, and put $\bar{R}=R/I$. Let y be in $\mathrm{Spec}(B(\bar{R}))$ and put $x=\{e\in B(R) \mid e+I\in y\}$. Then $x\in\mathrm{Spec}(B(R))$ and there is a ring epimorphism from R_x to R_y, as a result, R_y is also a local right perfect ring.

Propositton 3. *Let R be a ring whose Pierce stalks are local right perfect rings. Then*

1) *$J(R)$ is right T-nilpotent,*
2) *$J(E)$ coincides with the set of all nilpotent elements of R.*

Proof. 1) Let $\{a_i \mid i=1, 2, \cdots\}$ be a subset of $J(R)$ and let $x \in \mathrm{Spec}(B(R))$. Since $J(R)_x \subseteq J(R_x)$, $\{(a_i)_x \mid i=1, 2, \cdots\} \subseteq J(R_x)$. Hence there exists n such that $(a_n)_x(a_{n-1})_x \cdots (a_1)_x=0$. So there exists a neighborhood $N(x)$ of x such that $(a_n a_{n-1} \cdots a_1)_z = 0_z$ for all z in $N(x)$. Hence by the partition property of $\mathrm{Spec}(B(R))$, we can have neighborhoods $N_1, \cdots, N_k$ and $n_1, \cdots, n_k$ such that $\mathrm{Spec}(B(R))=N_1 \cup \cdots \cup N_k$ and $(a_{n_i} a_{n_{i-1}} \cdots a_1)_x=O_x$ for all x in N_i for $i=1, \cdots, k$. Hence if we put $m=\max\{n_i\}$, then $(a_m a_{m-1} \cdots a_1)_x=O_x$ for all x in $\mathrm{Spec}(B(R))$, hence $a_m a_{m-1} \cdots a_1=O$.

2) By 1) $J(R)$ is nil. For x in $\mathrm{Spec}(B(R))$, we denote by $M(x)$ the unique maximal (right) ideal of R containing Rx. Then we see that $\{M(x) \mid x \in \mathrm{Spec}(B(R))\}$ is just the family of all maximal right ideals of R. For, if M is a maximal right ideal of R, then $\{e \in B(R) \mid e \in M\} \in \mathrm{Spec}(B(R))$. As a result, we have $J(R)= \cap \{M(x) \mid x \in \mathrm{Spec}(B(R))\}$. Now, let a be a nilpotent element of R. Since $M(x)/Rx=J(R_x)$, we see that $a \in M(x)$. (Note that R_x is local). Hence $a \in \cap \{M(x) \mid x \in \mathrm{Spec}(B(R))\} =J(R)$. Accordingly $J(R)$ coincides with the set of all nilpotent elements of R.

Lemma 4. *Let R be a ring such that $J(R)=O$ and all stalks R_x are local right perfect rings. Then R is a strongly regular ring.*

Proof. We may show that all stalks are division rings. Let $x \in \mathrm{Spec}(B(R))$. Let a be in R such that $a_x \in J(R_x)$. Then there exists n such that $(a_x)^n=(a_x^n)$ $=O$, so $a^n e=O$ for a suitable e in $B(R)-x$. Since $(ae)^n=a^n e=O$, Proposition 4 shows that $ae \in J(R)=O$, so $a_x=O_x$. Thus $J(R_x)=O$. Since R_x is a right perfect ring, it follows that R_x is a division ring.

NOTATION. For a ring R, we denote by $I(R)$ the set of all idempotents of R. Of course $B(R) \subseteq I(R)$.

Lemma 5. *For a ring R, the following are equivalent:*

1) *$I(R)=B(R)$.*
2) *$I(R_x)=\{1_x, O_x\}$ for all x in $\mathrm{Spec}(B(R))$.*

Proof. 1)$\Rightarrow$2): Let $a \in R$ such that $a_x \in I(R_x)$ (where $x \in \mathrm{Spec}(B(R))$. Since $(a^2)_x=a_x$, $a^2 e=ae$ for some e in $B(R)-x$. Then $ae \in I(R)=B(R)$, we see that $a_x(=(ae)_x)$ is either 1_x or O_x. 2)$\Rightarrow$1): Let $a \in I(R)$ and $x \in \mathrm{Spec}(B(R))$. Then $a_x=1_x$ or $a_x=0_x$ since $a_x \in I(R_x)$. Here using the partition property of

Spec$(B(R))$, we can take a suitable e in $B(R)$ such that $ae=e$ and $a(1-e)=0$, whence $a=e\in B(R)$. Thus $I(R)=B(R)$.

We are now ready to show the following.

Theorem 2. *The following conditions are equivalent for a given ring R:*

1) *R is a right P-exchange ring and $I(R)=B(R)$.*

2) *$R/J(R)$ is a strongly regular ring, $J(R)$ is right T-nilpotent and $I(R)=B(R)$.*

3) *All stalks are local right perfect rings.*

Proof. 1)$\Rightarrow$3): By Proposition 2 (b) and Lemmas 3 and 5, each R_x is a P-exchange ring with $I(R_x)=B(R_x)$, whence R_x is a right perfect ring by [11, Theorem 8]. The implication 2)$\Rightarrow$1) follows from Proposition 2. The implication 3)$\Rightarrow$2) follows from Proposition 3 and Lemmas 3 and 4.

Corollary 1. *The following conditions are equivalent for a commutative ring R.*

1) *R is P-exchange ring.*

2*)) *$R/J(R)$ is a regular ring and $J(R)$ is T-nilpotent.*

3) *All stalks are lodal perfect rings.*

REMARK 3. The equivalence of 1) and 2) in Theorem 2 above is shown in [9]. It should be noted that an exchange ring with T-nilpotent Jacobson radical need not be a P-exchange ring, becouse there exist a non-regular commutative exchange ring R with $J(R)=0$ ([5]).

4. Main Theorem

As we see later, or by [9] the equivalence of 1) and 2) in Corollary 1 does not hold in general. However we show that 1) and 3) are equivalent, that is, the following holds:

Theorem 3. *A ring R is a right P-exchange ring if and only if all Pierce stalks R_x are P-exchange rings.*

Lemma 6. *Let P be a projective right R-module and let $x\in Spec(B(R))$. 1) If A is a finitely generated submodule with $A_x\langle\oplus P_x$, then $Ae\langle\oplus P$ for a suitable e in $B(R)-x$. 2) If P is finitely generated and A_1 and A_2 are finitely generated submodules of P with $P_x=(A_1)_x\oplus(A_2)_x$, then $Pe=A_1e\oplus A_2e$ for a suitable e in $B(R)-x$.*

Proof. As 1) follows from 2), we may only show 2). Let τ_i be the

*) Prof. Y. Kurata imformed the authers that commutative rings R which satisfy the condition 2) in Corollary 2 are studied in [1].

inclusion mapping: $A_i \to P$ for $i=1, 2$. Since P is projective, there exist $\pi_1: P \to A_1$ and $\pi_2: P \to A_2$ such that $(\pi_i)^x$ is the projection: $P_x=(A_1)_x \oplus (A_2)_x$ for $i=1, 2$. Noting that P, A_1 and A_2 are finitely generated, we can take a suitable e in $B(R)-x$ such that

$$(1-(\tau_1\pi_1+\tau_2\pi_2))(Pe)=0\,,\quad ((\tau_i\pi_i-(\tau_i\pi_i)^2)(Pe)=0\,,$$
$$(\tau_i\pi_i\tau_j\pi_j)(Pe)=0 \qquad \text{for} \quad i \neq j\,.$$

Then it follows that $Pe=A_1e \oplus A_2e$.

Lemma 7. *Let P be a projective right R-module with a sum $P=\sum_{i=1}^{\infty} a_i R$, and let $x \in Spec(B(R))$. If $P_x=\sum_{i=1}^{\infty} \oplus (a_iR)_x$, then there exists $\{e_i\}_{i=i}^{\infty} \subseteq B(R)-x$ such that $\sum_{i=1}^{n} a_ie_iR=\sum_{i=1}^{n} \oplus a_ie_iR \langle \oplus P$ for all n.*

Proof. Since $(a_1R)_x \langle \oplus P_x$, there exists $e_1 \in B(R)-x$ such that $a_1e_1R \langle \oplus P$ by Lemma 6. Since $(a_1R \oplus a_2R)_x \langle \oplus P_x$, there exists $e_2' \in B(R)-x$ such that $a_1e_2'R \oplus a_2e_2'R \langle \oplus P$. Put $e_2=e_1e_2'$. Then we see that

$$a_1e_1R+a_2e_2R=a_1e_1R \oplus a_2e_2R \langle \oplus P\,.$$

By similar argument, we can take $\{e_i\}_{i=1}^{\infty} \subseteq B(R)-x$ such that $e_n e_{n+1}=e_{n+1}$ for $n=1, 2, \cdots$ and

$$a_1e_1R+\cdots+a_ne_nR=a_1e_1R \oplus \cdots \oplus a_ne_nR \langle \oplus P$$

for $n=1, 2, \cdots$.

Lemma 8. *Let P be a finitely generated projective right R-module such that all stalks P_x have the exchange property. Then P has the exchange property.*

Proof. Since P is finitely generated, we may show that P satisfies the condition (N_1) (Proposition 1). So, let $P=A+B$, where A and B are finitely generated submodules. Let $x \in \mathrm{Spec}(B(R))$. Since P_x satisfies (N_1), we can take finitely generated submodules $A^x \subseteq A$ and $B^x \subseteq B$ such that $P_x=(A^x)_x \oplus (B^x)_x$. Then, by Lemma 6, $Pe=A^xe \oplus B^xe$ for a suitable e in $B(R)-x$. Using the partition property of $\mathrm{Spec}(B(R))$, we can take orthogonal idempotents $e_1, \cdots, e_n$ in $B(R)$ and finitely generated submodules $A^{x_1}, \cdots, A^{x_n}$ of A and $B^{x_1}, \cdots, B^{x_n}$ of B such that

$$P=A^{x_1}e_1 \oplus B^{x_1}e_1 \oplus \cdots \oplus A^{x_n}e_n \oplus B^{x_n}e_n\,.$$

Hence putting $A^*=A^{x_1}e_1 \oplus \cdots \oplus A^{x_n}e_n$ and $B^*=B^{x_1}e_1 \oplus \cdots \oplus B^{x_n}e_n$, we have that $P=A^* \oplus B^*$..

Proof of Theorem 3. If R is a right P-exchange ring, then all R_x are right P-exchange rings by Proposition 2. Conversely, assume that all R_x are right P-exchange rings. We may show that $\aleph_0 R$ satisfies the condition (N_2). Let

$F=\sum_{i=1}^{\infty}\oplus R_i$ be a free right R-module with $R_i\simeq R$ for all i, so $F=\aleph_0 R$. We put $F(s)=R_1\oplus\cdots\oplus R_s$ for $s=1, 2, \cdots$. Now, consider a sum $F=\sum_{i=1}^{\infty} a_iR$. For any x in $\mathrm{Spec}(B(R))$, as F_x satisfies (N_2), we can take by Lemma 7 $\{b_i^x\in a_iR \mid i=1, 2, \cdots\}$ such that $F_x=\sum_{i=1}^{\infty}\oplus(b_i^xR)_x$ and

$$F\oplus> b_1^xR+\cdots+b_n^xR=b_1^xR\oplus\cdots\oplus b_n^xR$$

for all n.

Let $x\in\mathrm{Spec}(B(R))$ and take any $s_1\geq 1$. Then there exists $n(x)$ such that

$$F(s_1)_x\subseteq\sum_{i=1}^{n(x)}\oplus(b_i^xR)_x$$

and so there exists $e(x)$ in $B(R)-x$ such that

$$F(s_1)e(x)\subseteq\sum_{i=1}^{n(x)}\oplus b_i^x e(x)R<\oplus F$$

Using the partition property, we have $x_1, \cdots, x_n$ in $\mathrm{Spec}(B(R))$, orthogonal idempotents $\{e(x_1), \cdots, e(x_n)\}\subseteq B(R)$ and m_1 such that $1=\sum_{i=1}^{n} e(x_i)$ and

$$F(s_1)\subseteq\sum_{i=1}^{m_1}\oplus b_i^{x_1}e(x_1)R\oplus\cdots\oplus\sum_{i=1}^{m_1}\oplus b_i^{x_n}e(x_n)R<\oplus F$$

Put $b_i^1=\sum_{j=1}^{n} b_i^{x_j}e(x_j)$ for $i=1, \cdots, m_1$. Then $b_i^1\in a_iR$ and

$$F(s_1)\subseteq\sum_{i=1}^{m_1}\oplus b_i^1R<\oplus F\,.$$

Put $G_1=\sum_{i=1}^{m_1}\oplus b_i^1R$. Then $G_1\subseteq F(s_2)$ for a suitable $s_2>s_1$. By the same argument as above, we can take m_2 and $b_i^x\in a_iR$ for $i=1, \cdots, m_2$ such that

$$F(s_2)\subseteq\sum_{i=1}^{m_2}\oplus b_i^2R\,.$$

Put $G_2=\sum_{i=1}^{m_2}\oplus b_i^2R$. Then $G_2\subseteq F(s_3)$ for some $s_3>s_2>s_1$. Continuing this argument, we can take $s_1<s_2<s_3<\cdots$ and $G_1=\sum_{i=1}^{m_1}\oplus b_i^1R$, $G_2=\sum_{i=1}^{m_2}\oplus b_i^2R$, $\cdots$ such that $b_i^k\in a_iR$ for all i, k, each G_i is a direct summand of F and

$$F(s_1)\subseteq G_1\subseteq F(s_2)\subseteq G_2\subseteq F(s_3)\subseteq\cdots\,.$$

Since $\cup_{i=1}^{\infty} G_i\subseteq\cup_{i=1}^{\infty} F(s_i)$, we see $F=\sum_{i=1}^{\infty} G_i$. Since G_{n-1} has the exchange property by Lemma 8, there exists $\{c_i^n\in b_i^nR \mid i=1, \cdots, m_n\}$ such that

$$G_n=G_{n-1}\oplus\sum_{i=1}^{m_n}\oplus c_i^nR\,.$$

In particular, put $c_i^1=b_i^1$ for $i=1, \cdots, m_1$. Then we see that

$$F=\sum_{i=1}^{m_1}\oplus c_i^1R\oplus\sum_{i=1}^{m_2}\oplus c_i^2R\oplus\sum_{i=1}^{m_2}\oplus c_i^3R\oplus\cdots\,.$$

We put

$$A_k = \sum_{i=1}^{\infty} \oplus c_k^i R \qquad \text{for} \quad k = 1, 2, \cdots .$$

Then $A_i \subseteq a_i R$ for all i and $F = \sum_{i=1}^{\infty} \oplus A_i$. This completes the proof.

Corollary 2. *If R is a ring such that all Pierce stalks are right perfect rings, then R is a right P-exchange ring.*

By making use of the corollary, we shall give a right P-exchange ring.

EXAMPLE. Let P be an indecomposable right perfect ring and Q an indecomposable right perfect subring of P with the same identity. Consider the rings $W = \prod_I P_\alpha$ and $V = \prod_I Q_\alpha$, where $Q_\alpha \simeq Q$ and $P_\alpha \simeq P$ for all $\alpha \in I$. Then the ring W is an extension ring of V and becomes a right V-module. Put $R = \sum_I \oplus P_\alpha + 1Q$, where 1 is the identity of W. Then R is a ring such that $B(R) = \sum_I \oplus B(P_\alpha)^{*)} + 1B(Q)$. We can easily see that $\mathrm{Spec}(B(R)) = \{x_0\} \cup \{x_\alpha \mid \alpha \in I\}$, where $x_0 = \sum_I \oplus B(P_\alpha)$ and $x_\alpha = \sum_{I-\{\alpha\}} \oplus B(P_\beta) + 1B(Q)$. Further we see that $R_{x_0} \simeq Q$ and $R_{x_\alpha} \simeq P$ for all $\alpha \in I$. Hence Corollary 3 says that R is a right P-exchange ring. In paticular, if we take $\begin{pmatrix} F & F \\ F & F \end{pmatrix}$ and $\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ as P and Q, respectively, where F is a division ring, then R is a non-singular, right P-exchange ring with $J(R) = 0$.

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*) Note that $B(P) = B(Q) = GF(2)$

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