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## AN EXAMPLE OF THE COMPLETION OF RANK FUNCTIONS OVER SIMPLE UNIT REGULAR RINGS

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In this note, we are concerned with Von Neumann regular rings having (pseudo-) rank functions. Let  $R$  be a regular ring and  $N$  a pseudo-rank function on  $R$ . Then  $N$  induces a pseudo-metric topology on  $R$ , and  $\bar{R}$ , the completion of  $R$  at this pseudo-metric, is a right and left self-injective regular ring. If  $N$  is an extremal pseudo-rank function,  $\bar{R}$  is simple moreover. It is known that there exist uncountable nonisomorphic simple right and left self-injective regular rings [1, Cor. 2. 9].

From this observation, K.R.Goodearl asked for two different extremal rank functions  $P, Q$  on a given simple unit-regular ring  $R$ , whether the  $P$ -completion of  $R$  is isomorphic to the  $Q$ -completion of  $R$  or not. ([3, Open problem 38]). Now we answer that this problem is negative. Let  $F$  be any field and  $K_i$  ( $i=1, 2$ ) any quadratic extensions of  $F$ . We give an example of a simple regular  $F$ -algebra  $R$  with two extremal rank functions  $P_i$  such that the center of the  $P_i$ -completion of  $R$  is  $K_i$  ( $i=1, 2$ ). In particular, put  $F=\mathbb{Q}$ ,  $K_1=\mathbb{Q}(i)$ , and  $K_2=\mathbb{Q}(\sqrt{2})$ . Then, since  $\mathbb{Q}(i)$  is not isomorphic to  $\mathbb{Q}(\sqrt{2})$  over  $\mathbb{Q}$ , the  $P_1$ -completion of  $R$  is not isomorphic to the  $P_2$ -completion of  $R$ .

We use most of our terminologies and notations from Goodearl's book [3].

### 1. A construction an example

Let  $K_1, K_2$  be quadratic extension fields of a field  $F$  and  $g_i: K_i \rightarrow M_2(F)$  ( $i=1, 2$ ) matrix representations of  $K_i$  over  $F$  with respect to regular representation of  $K_i$ . We shall construct an  $F$ -algebra  $R$ , as a direct limit of a sequence  $R_1 \rightarrow R_2 \rightarrow \dots$  of semisimple  $F$ -algebras. We shall refer to K.R.Goodearl's example [2, Scheme I] and D. Handelman's one [4, p.1144]. Let  $p_1, p_2, \dots$  be integers ( $p_n > 2$ ). Define positive integers  $w(1), w(2), \dots$  by setting  $w_1=1$  and  $w_n=(p_{n-1}+2)(p_{n-2}+2)\dots(p_1+2)$  and put

$$R_n = M_{w(n)}(F) \otimes_F K_1 \oplus M_{w(n)}(F) \otimes_F K_2$$

Next we shall define  $F$ -algebra maps from  $R_n$  to  $R_{n+1}$ . Let  $\{1, v_i\}$  be  $F$ -basis of  $K_i$ . Then any element of  $M_{w(n)}(F) \otimes_F K_i$  is written by the following form;  $x \otimes 1 + y \otimes v_i$ , where  $x, y \in M_{w(n)}(F)$ . We use  $x \boxplus y$  to denote the Kronecker product of matrices  $x, y \in M_{w(n)}(F)$ . Let  $I_n$  be the identity matrix in  $M_n(F)$ .

Define maps  $G_{i_n}: M_{w(n)}(F) \otimes_F K_i \rightarrow M_{2w(n)}(F)$  by the rule;  $z_i = x \otimes 1 + y \otimes v_i \rightarrow x \boxplus I_2 + y \boxplus g_i(v_i)$ . Define maps  $\phi_n: R_n \rightarrow R_{n+1}$  by the rule;

$$\left( \begin{array}{c} [z_1, z_2] \\ \downarrow \phi_n \\ \left( \begin{array}{c} G_{1n}(z_1) \\ \vdots \\ p_n - 2 \\ \vdots \\ z_1 \\ G_{2n}(z_2) \end{array} \right), \left( \begin{array}{c} G_{1n}(z_1) \\ \vdots \\ p_n - 2 \\ \vdots \\ z_2 \\ G_{2n}(z_2) \end{array} \right) \end{array} \right) \in R_{n+1}$$

where  $x, x', y$  and  $y' \in M_{w(n)}(F)$ .

Now define  $R$  to be the limit of  $\{R_n, \phi_n\}$  and let  $\theta_n: R_n \rightarrow R$  natural embeddings. Obviously  $R$  is a simple unit-regular  $F$ -algebra with the center  $F$ .

Next we shall determine all (extremal) rank functions on  $R$ . We use  $P(R)$  to denote the set of all rank functions on  $R$ . Put  $R'_n = M_{w(n)}(F) \oplus M_{w(n)}(F)$  for each  $n$ . We consider  $R'_n$  as a sub- $F$ -algebra of  $R_n$  by the embedding  $[x, y] \rightarrow [x \otimes 1, y \otimes 1]$  where  $x, y \in M_{w(n)}(F)$ . Put  $\phi'_n = \phi_n|_{R'_n}$ , then  $\phi'_n$  is as follows:

$$\left( \begin{array}{c} [x, y] \\ \downarrow \phi'_n \\ \left( \begin{array}{c} x \\ \vdots \\ p_n \\ \vdots \\ x \\ y \\ y \end{array} \right), \left( \begin{array}{c} x \\ x \\ y \\ \vdots \\ p_n \\ y \end{array} \right) \end{array} \right) \in R'_{n+1}$$

Define  $R'$  to be the limit of  $\{R', \phi'_n\}$ .

**Lemma 1.**  $P(R)$  is affinely homeomorphic to  $P(R')$  by the restriction map.

Proof. For any  $N \in P(R_n)$  (resp.  $P(R'_n)$ ),

$$N([A, B]) = \frac{\alpha \text{rank}(A) + \beta \text{rank}(B)}{w(n)}$$

, where  $A \in M_{w(n)}(F) \otimes K_1$ ,  $B \in M_{w(n)}(F) \otimes K_2$  (resp.  $A, B \in M_{w(n)}(F)$ ),  $\alpha = N([I_n, O])$ , and  $\beta = N([O, I_n])$  by [3, Cor. 16.6]. Then  $P(R_n)$  is affinely homeomorphic to  $P(R'_n)$  by the restriction map for all  $n$ . Since  $P(R)$  (resp.  $P(R')$ ) is the inverse limit of  $\{P(R_n), \phi_n^*\}$  (resp.  $\{P(R'_n), \phi_n^*\}$ ), by [3, Prop. 16.21],  $P(R)$  is affinely homeomorphic to  $P(R')$ .

The structure of  $P(R')$  has been determined by K.R. Goodearl [2, pp. 277–280]. For the sake of completeness, we shall again explain it.

Put  $u(1)=1$  and  $u(n+1)=(p_n-2)\cdots(p_1-2)$  for all  $n \geq 1$ . For  $N \in P(R')$ , there exist positive real numbers  $\alpha_n(i)$  ( $n=1, 2, \dots; i=1, 2$ ) such that

$$(1) \quad \alpha_n(1) + \alpha_n(2) = 1 \text{ for all } n$$

- (2)  $\alpha_{n+1}(i) = \frac{(p_n+2)\alpha_n(i)-2}{p_n-2}$  for all  $n, i$
- (3)  $N([A, B]) = \frac{\alpha_n(1) \text{rank}(x) + \alpha_n(2) \text{rank}(y)}{w(n)}$  for all  $[A, B] \in R'$ ,

where  $[A, B] = \theta_n([x, y])$  for some  $n$  and  $[x, y] \in R'_n$ . Conversely, if  $\{\alpha_n(i)\}$  are any positive real numbers satisfying (1) and (2), then (3) defines a rank function  $N$  on  $R'$ .

Now we assume that  $\lim_{n \rightarrow \infty} \frac{u(n)}{w(n)} > 0$ . Put  $\lambda = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{u(n)}{w(n)}$

We define

$$\begin{aligned} \alpha_n(1) &= \frac{1}{2} + \lambda \frac{w(n)}{u(n)} & \beta_n(1) &= \frac{1}{2} - \lambda \frac{w(n)}{u(n)} \\ \alpha_n(2) &= \frac{1}{2} - \lambda \frac{w(n)}{u(n)} & \beta_n(2) &= \frac{1}{2} + \lambda \frac{w(n)}{u(n)} \end{aligned}$$

for all  $n \geq 1$ . Then  $\{\alpha_n(i)\}$  and  $\{\beta_n(i)\}$  satisfy the above conditions (1) and (2). Let  $N_1$  (resp.  $N_2$ ) be the rank function determined by  $\{\alpha_n(i)\}$  (resp.  $\{\beta_n(i)\}$ ). by [2, Lemma 27],  $N_1$  and  $N_2$  are all extremal rank functions on  $R'$ . Therefore, by Lemma 1,  $N_1$  and  $N_2$  can be extended to extremal rank functions on  $R$ .  $N_i$  ( $i=1, 2$ ) induce metrics on  $R$  given by the rule;  $d_i(x, y) = N_i(x - y)$  for  $x, y \in R$ , which we call the  $N_i$ -metric [3, § 19]. Let  $T_i$  be the completion of  $R$  with respect to  $N_i$ -metric ( $i=1, 2$ ). Then  $T_i$  are simple regular, right and left self-injective  $F$ -algebras by [3, Th. 19. 14].

## 2. Calculation of the centers of $T_i$

In this note, we shall calculate the center  $Z(T_i)$  of  $T_i$ . Let  $I_k$  be the identity matrix for  $M_k(F)$  and  $\theta_n$  the natural embedding:  $R_n \rightarrow R$ .

**Lemma 2.** *If  $\sum_{i=1}^{\infty} 1/(p_n+2) < \infty$ , then  $\{\theta_n([I_{w(n)} \otimes \alpha, 0])\}$  (resp.  $\theta_n([0, I_{w(n)} \otimes \beta])$ ) is a Cauchy sequence with respect to  $N_1$ -metric (resp.  $N_2$ -metric) for each  $\alpha \in K_1$  (resp.  $\beta \in K_2$ ).*

**Proof.** Put  $K = K_1$  and  $N = N_1$ . For  $\alpha \in K$  and each  $n$ , we see that

$$\begin{aligned} & \phi_n([I_{w(n)} \otimes \alpha, 0]) \\ &= \left( \begin{array}{c} \left( \begin{array}{c} g_1(\alpha) \quad w(n) \\ \vdots \quad \vdots \\ g_1(\alpha) \end{array} \right) \\ I_{w(n)} \otimes \alpha \quad p_n - 2 \\ \vdots \quad \vdots \\ I_{w(n)} \otimes \alpha \\ 0 \quad \vdots \quad 0 \end{array} \right), \left( \begin{array}{c} \left( \begin{array}{c} g_1(\alpha) \quad w(n) \\ \vdots \quad \vdots \\ g_1(\alpha) \end{array} \right) \\ 0 \\ \vdots \quad \vdots \quad \vdots \\ 0 \end{array} \right) \end{array} \right). \end{aligned}$$

Therefore, we have

$$= \left[ \begin{pmatrix} \alpha I_2 - g_1(\alpha) & & & & & \\ & \ddots & & & & \\ & & \alpha I_2 - g_1(\alpha) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ & & & & & & \alpha & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & \alpha & & \\ & & & & & & & & & 2w(n) & \\ & & & & & & & & & & \alpha \end{pmatrix}, \begin{pmatrix} -g_1(\alpha) & & & & & \\ & \ddots & & & & \\ & & -g_1(\alpha) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \right].$$

We can calculate that

$$\begin{aligned} & N(\theta_{n+1}([I_{w(n+1)} \otimes \alpha, 0]) - \theta_n([I_{w(n)} \otimes \alpha, 0])) \\ &= N([I_{w(n+1)} \otimes \alpha, 0] - \phi_n([I_{w(n)} \otimes \alpha, 0])) \\ &= \frac{1}{w(n+1)} \cdot \{ \alpha_{n+1}(1)(w(n) \operatorname{rank}(\alpha I_2 - g_1(\alpha)) + 2w(n)) + \alpha_{n+1}(2)w(n) \operatorname{rank}(g_1(\alpha)) \} \\ &< \frac{1}{w(n+1)} \cdot \left\{ \left( \frac{1}{2} + \lambda \frac{w(n+1)}{u(n+1)} \right) 4w(n) + \left( \frac{1}{2} - \lambda \frac{w(n+1)}{u(n+1)} \right) 4w(n) \right\} \\ &< 4/(p_n + 2) \end{aligned}$$

Then  $\{\theta_n([I_{w(n)} \otimes \alpha, 0])\}$  is a Cauchy sequence.

By Lemma 2, we define  $\tau_1(\alpha) = \lim \theta_n([I_{w(n)} \otimes \alpha, 0])$  (resp.  $\tau_2(\beta) = \theta_n([0, I_{w(n)} \otimes \beta])$ ) for each  $\alpha \in K_1$  (resp.  $\beta \in K_2$ ). Then  $\tau_i: K_i \rightarrow T_i$  is a map as  $F$ -algebra for  $i=1, 2$ .

**Lemma 3.**

- (1)  $\tau_i(K_i) \subseteq Z(T_i)$  for  $i=1, 2$ .
- (2)  $\tau_i(a) = a$  for all  $a \in F$ .

Proof. (1) For any  $r \in R$  and  $\alpha \in K_1$ , we shall show that  $\tau_1(\alpha)r = r\tau_1(\alpha)$ . Let  $r = \theta_n([x, y])$  for some  $n$  and  $[x, y] \in R_n$ . Since  $[I_{w(k)} \otimes \alpha, 0][x, y] = [x, y][I_{w(k)} \otimes \alpha, 0]$  for all  $k > n$ , we have, that  $\tau_1(\alpha)r = r\tau_1(\alpha)$ . Since  $T_1$  is the completion of  $R$  with respect to  $N_1$ -metric, we have that  $\tau_1(\alpha)x = x\tau_1(\alpha)$  for all  $x \in T_1$ .

- (2) Since  $a = \theta_n([I_{w(n)} \otimes a, I_{w(n)} \otimes a])$  for all  $a \in F$  and all  $n$ , we see that

$$\begin{aligned} & N_1(a - \theta_n([I_{w(n)} \otimes a, 0])) \\ &= N_1([0, I_{w(n)} \otimes a]) \\ &= \alpha_n(2) \end{aligned}$$

Therefore we have that  $a = \varinjlim \theta_n([I_{w(n)} \otimes a, 0])$ , because  $\varinjlim \alpha_n(2) = 0$ .

**Lemma 4.** *Let  $p_1, p_2, \dots$  be integers such that  $\lim_{n \rightarrow \infty} \frac{u(n)}{w(n)} > 0$  and  $\sum_{n=1}^{\infty} 4/(p_n+2) < \infty$ . Then  $\tau_i: K_i \rightarrow Z(T_i)$  is an isomorphism over  $F$ .*

*Proof.* Put  $T=T_1$  and  $N=N_1$ . We shall show that  $\tau_1(K_1)=Z(T)$ . First for any  $x \in Z(T)$  and any real number  $\varepsilon > 0$ , there exists  $r \in R$  such that  $N(x-r) < \varepsilon/4$ . And there exists  $r_{k(1)} \in R_{k(1)}$  such that  $r = \theta_{k(1)}(r_{k(1)})$ . We note that  $r = \theta_m(r_m)$  for all  $m \geq k(1)$ , where some  $r_m \in R_m$ . For any  $r_m$ , there exist  $z_m \in Z(R_m)$  and  $y_m \in R_m$  such that  $N(r_m - z_m) \leq N(r_m y_m - y_m r_m)$  by [1, Cor. 2.4].

$$\begin{aligned} \text{Since } N(r_m - z_m) &\leq N(r_m y_m - y_m r_m) \\ &\leq N((r-x)\theta_m(y_m)) + N(\theta_m(y_m)(x-r)) \\ &\leq 2N(x-r) \\ &< \varepsilon/2, \end{aligned}$$

we see that for any  $m \geq k(1)$ ,

$$\begin{aligned} (*) \quad N(x - \theta_m(z_m)) &\leq N(x-r) + N(r - \theta_m(z_m)) \\ &\leq N(x-r) + N(r_m - z_m) \\ &< \varepsilon \cdot 3/4. \end{aligned}$$

Put  $z_m = [I_{w(n)} \otimes \alpha_m, I_{w(n)} \otimes \beta_m]$  for some  $\alpha_m \in K_1$  and  $\beta_m \in K_2$ . Since  $\lim_{n \rightarrow \infty} \alpha_n(2) = 0$ , there exists  $k(2)$  such that  $\alpha_m(2) < \varepsilon/4$  for all  $m \geq k(2)$ .

We see that for all  $m \geq \max(k(1), k(2))$ ,

$$\begin{aligned} &N(x - \theta_m([I_{w(n)} \otimes \alpha_m, 0])) \\ &\leq N(x - \theta_m(z_m)) + N([0, I_{w(n)} \otimes \beta_m]) \\ &\leq N(x - \theta_m(z_m)) + \alpha_m(2) \\ &< \varepsilon. \end{aligned} \quad (\text{by } *)$$

Since  $\sum_{n=1}^{\infty} 4/(p_n+2) < \infty$ , for  $\varepsilon$ , there exists a natural number  $k(3)$  such that

$$(**) \quad \sum_{i=n}^l 4/(p_i+2) < \varepsilon \quad \text{for all } l > n \geq k(3).$$

Select some  $k \geq \max(k(i) \mid i=1, 2, 3)$ . Then we have already seen that  $\alpha_k \in K_1$  and  $N(x - \theta_k([I_{w(k)} \otimes \alpha_k, 0])) < \varepsilon$ . Put  $\gamma = \alpha_k$ . We shall show that  $N(x - \tau_1(\gamma)) < \varepsilon$ . There exists a positive integer  $k(4) > k$  such that for any  $m \geq k(4)$ ,

$$(***) \quad N(\theta_m([I_{w(m)} \otimes \gamma, 0] - \tau_1(\gamma))) < \varepsilon.$$

We see that for some  $m \geq \max\{k(i) \mid i=1, 2, 3, 4\}$ ,

$$\begin{aligned}
N(x - \tau_1(\gamma)) &\leq N(x - \theta_k([I_{w(k)} \otimes \gamma, 0])) \\
&\quad + N(\theta_k([I_{w(k)} \otimes \gamma, 0]) - \theta_{k+1}([I_{w(k+1)} \otimes \gamma, 0])) \\
&\quad \dots\dots\dots \\
&\quad + N(\theta_{m-1}([I_{w(m-1)} \otimes \gamma, 0]) - \theta_m([I_{w(m)} \otimes \gamma, 0])) \\
&\quad + N(\theta_m([I_{w(m)} \otimes \gamma, 0]) - \tau_1(\gamma))
\end{aligned}$$

, using the inequality in the proof of Lemma 2, (\*\*) and (\*\*\*)

$$\begin{aligned}
&< \varepsilon + \sum_{i=k}^{m-1} 4/(p_i + 2) + \varepsilon \\
&< 6\varepsilon.
\end{aligned}$$

Since  $T$  is a simple ring,  $Z(T)$  is a field, so if  $\varepsilon$  is less than  $1/6$ ,  $N(x - \tau_1(\gamma)) = 0$ . Therefore  $x$  belongs to  $\tau_1(K_1)$ .

Now we shall give a negative answer for the Goodearl's problem No. 38 [3, p. 348].

EXAMPLE There exists a simple unit-regular ring  $R$  such that

- (1)  $R$  has two extremal rank functions  $N_1, N_2$ .
- (2) The  $N_1$ -completion of  $R$  is not isomorphic to the  $N_2$ -completion of  $R$ .

Proof. Set  $F = \mathbf{Q}$ ,  $K_1 = \mathbf{Q}(i)$  and  $K_2 = \mathbf{Q}(\sqrt{2})$ . Put  $p_n = n^2 + 4n + 2$  for all  $n$ , and construct  $R$  according to the previous method. Since  $\frac{w(n)}{u(n)} = \frac{2(n+2)(n+3)}{9n(n+1)}$ , we have  $\lim_{n \rightarrow \infty} \frac{u(n)}{w(n)} = \frac{2}{9}$ . And we see that  $\sum_{n=1}^{\infty} \frac{1}{p_n + 2} < \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} = \frac{5}{12}$ . By Lemma 4, the  $N_1$ -completion  $T_1$  of  $R$  is not isomorphic to the  $N_2$ -completion  $T_2$  of  $R$ , because  $Z(T_1) = \mathbf{Q}(i)$  is not isomorphic to  $Z(T_2) = \mathbf{Q}(\sqrt{2})$  over  $\mathbf{Q}$ .

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