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AN EXAMPLE OF THE COMPLETION OF RANK FUNCTIONS OVER SIMPLE UNIT REGULAR RINGS

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In this note, we are concerned with Von Neumann regular rings having (pseudo-) rank functions. Let R be a regular ring and N a pseudo-rank function on R. Then N induces a pseudo-metric topology on R, and $\bar{R}$, the completion of R at this pseudo-metric, is a right and left self-injective regular ring. If N is an extremal pseudo-rank function, $\bar{R}$ is simple moreover. It is known that there exist uncountable nonisomorphic simple right and left self-injective regular rings [1, Cor. 2. 9].

From this observation, K.R.Goodearl asked for two different extremal rank functions P, Q on a given simple unit-regular ring R, whether the P-completion of R is isomorphic to the Q-completion of R or not. ([3, Open problem 38]). Now we answer that this problem is negative. Let F be any field and K_i $(=1, 2)$ any qudratic extensions of F. We give an example of a simple regular F-algebra R with two extremal rank functions P_i such that the center of the P_i-completion of R is K_i $(i=1, 2)$. In prarticular, put $F=\boldsymbol{Q}$, $K_1=\boldsymbol{Q}(i)$, and $K_2=\boldsymbol{Q}(\sqrt{2})$. Then, since $\boldsymbol{Q}(i)$ is not isomorphic to $\boldsymbol{Q}(\sqrt{2})$ over $\boldsymbol{Q}$, the P_1-completion of R is not isomorphic to the P_2-completion of R.

We use most of our terminologies and notations from Goodearl's book [3].

1. A construction an example

Let K_1, K_2 be quadratic extension fields of a field F and g_i: $K_i \to M_2(F)$ $(i=1, 2)$ matrix representations of K_i over F with respect to regular representation of K_i. We shall constract an F-algebra R, as a direct limit of a sequence $R_1 \to R_2 \to \cdots$ of semisimple F-algebras. We shall refer to K.R. Goodearl's example [2, Scheme I] and D. Handelman's one [4, p.1144]. Let p_1, p_2, $\cdots$ be integers $(p_n > 2)$. Define positive integers $w(1)$, $w(2)$, $\cdots$ by setting $w_1 = 1$ and $w_n = (p_{n-1}+2)(p_{n-2}+2)\cdots(p_1+2)$ and put

$$R_n = M_{w(n)}(F) \otimes_F K_1 \oplus M_{w(n)}(F) \otimes_F K_2$$

Next we shall define F-algebra maps from R_n to R_{n+1}. Let $\{1, v_i\}$ be F-basis of K_i. Then any elemant of $M_{w(n)}(F) \otimes_F K_i$ is written by the following form; $x \otimes 1 + y \otimes v_i$, where $x, y \in M_{w(n)}(F)$. We use $x \boxplus y$ to denote the Kronecker product of matrices $x, y \in M_{w(n)}(F)$. Let I_n be the identity matrix in $M_n(F)$.

Define maps G_{in}: $M_{w(n)}(F)\otimes_F K_i \to M_{2w(n)}(F)$ by the rule; $z_i = x\otimes 1 + y\otimes v_i \to x\boxplus I_2 + y\boxplus g_i(v_i)$. Define maps ϕ_n: $R_n \to R_{n+1}$ by the rule;

$$
\begin{array}{ccc}
[z_1, z_2] & & \in R_n \\
\downarrow \phi_n & & \\
\left(\begin{pmatrix} G_{1n}(z_1) & & & & \\ & \overbrace{z_1 & & & }^{p_n-2} \\ & & \ddots & & \\ & & & z_1 & \\ & & & & G_{2n}(z_2) \end{pmatrix}, \begin{pmatrix} G_{1n}(z_1) & & & & \\ & \overbrace{z_2 & & & }^{p_2-2} \\ & & \ddots & & \\ & & & z_2 & \\ & & & & G_{2n}(z_2) \end{pmatrix}\right) & & \in R_{n+1}
\end{array}
$$

where x, x', y and $y'\in M_{w(n)}(F)$.

Now define R to be the limit of $\{R_n, \phi_n\}$ and let θ_n: $R_n \to R$ natural embeddings. Obviously R is a simple unit-regular F-algebra with the center F.

Next we shall determine all (exremal) rank functions on R. We use $P(R)$ to denote the set of all rank functions on R. Put $R'_n = M_{w(n)}(F)\oplus M_{w(n)}(F)$ for each n. We consider R'_n as a sub-F-algebra of R_n by the embedding $[x, y]\to[x\otimes 1, y\otimes 1]$ where $x, y\in M_{w(n)}(F)$. Put $\phi'_n = \phi_n | R'_n$, then ϕ'_n is as follows:

$$
\begin{array}{ccc}
[x, y] & & \in R'_n \\
\downarrow & & \\
\left(\begin{pmatrix} \overbrace{x & & }^{p_n} & & \\ & \ddots & & & \\ & & x & & \\ & & & y & \\ & & & & y \end{pmatrix}, \begin{pmatrix} x & & & & \\ & x & & & \\ & & \overbrace{y & & }^{p_n} & \\ & & & \ddots & \\ & & & & y \end{pmatrix}\right) & & \in R'_{n+1}
\end{array}
$$

Define R' to be the limit of $\{R', \phi'_n\}$.

Lemma 1. *$P(R)$ is affinely homeomorphic to $P(R')$ by the restriction map.*

Proof. For any $N\in P(R_n)$ (resp. $P(R'_n)$),

$$N([A, B]) = \frac{\alpha\, rank\,(A) + \beta\, rank\,(B)}{w(n)}$$

, where $A\in M_{w(n)}(F)\otimes K_1$, $B\in M_{w(n)}(F)\otimes K_2$ (resp. A, $B\in M_{w(n)}(F)$), $\alpha = N([I_n, O])$, and $\beta = N([O, I_n])$ by [3, Cor. 16.6]. Then $P(R_n)$ is affinely homeomorphic to $P(R'_n)$ by the restriction map for all n. Since $P(R)$ (resp. $P(R')$) is the inverse limit of $\{P(R_n), \phi_n^*\}$ (resp. $\{P(R'_n), \phi_n^*\}$), by [3, Prop. 16.21], $P(R)$ is affinely homeomorphic to $P(R')$.

The structure of $P(R')$ has been determined by K.R. Goodearl [2, pp. 277–280]. For the sake of copmleteness, we shall again explain it.

Put $u(1)=1$ and $u(n+1)=(p_n-2)\cdots(p_1-2)$ for all $n\geqq 1$. For $N\in P(R')$, there exist positive real numbers $\alpha_n(i)$ ($n=1, 2, \cdots$; $i=1, 2$) such that

(1) $\alpha_n(1)+\alpha_n(2)=1$ for all n

(2) $\alpha_{n+1}(i)=\dfrac{(p_n+2)\alpha_n(i)-2}{p_n-2}$ for all n, i

(3) $N([A, B])=\dfrac{\alpha_n(1)\, rank\,(x)+\alpha_n(2)\, rank\,(y)}{w(n)}$ for all $[A, B]\in R'$,

where $[A, B]=\theta_n([x, y])$ for some n and $[x, y]\in R_n'$. Convercely, if $\{\alpha_n(i)\}$ are any positive real numbers satisfing (1) and (2), then (3) defines a rank function N on R'.

Now we assume that $\underset{\longrightarrow}{\lim}\dfrac{u(n)}{w(n)}>0$. Put $\lambda=\dfrac{1}{2}\underset{\longrightarrow}{\lim}\dfrac{u(n)}{w(n)}$

We define
$$\alpha_n(1)=\frac{1}{2}+\lambda\frac{w(n)}{u(n)}\qquad \beta_n(1)=\frac{1}{2}-\lambda\frac{w(n)}{u(n)}$$
$$\alpha_n(2)=\frac{1}{2}-\lambda\frac{w(n)}{u(n)}\qquad \beta_n(2)=\frac{1}{2}+\lambda\frac{w(n)}{u(n)}$$

for all $n\geqq 1$. Then $\{\alpha_n(i)\}$ and $\{\beta_n(i)\}$ satisfy the above conditions (1) and (2). Let N_1 (resp. N_2) be the rank function determined by $\{\alpha_n(i)\}$ (resp. $\{\beta_n(i)\}$). by [2, Lemma 27], N_1 and N_2 are all extremal rank functions on R'. Therefore, by Lemma 1, N_1 and N_2 can be extended to extremal rank functions on R. N_i $(i=1, 2)$ induce metrics on R given by the rule; $d_i(x, y)=N_i(x-y)$ for $x, y\in R$, which we call the N_i-metric [3, § 19]. Let T_i be the completion of R with respect to N_i-metric $(i=1, 2)$. Then T_i are simple regular, right and left self-injective F-algebras by [3, Th. 19. 14].

2. Caluculation of the centers of T_i

In this note, we shall calculate the center $Z(T_i)$ of T_i. Let I_k be the identity matrix for $M_k(F)$ and θ_n the natural embedding: $R_n\to R$.

Lemma 2. *If* $\sum_1^\infty 1/(P_n+2)<\infty$, *then* $\{\theta_n([I_{w(n)}\otimes\alpha, 0])\}$ (*resp.* $\theta_n([0, I_{w(n)}\otimes\beta])$) *is a Cauchy sequence with respect to* N_1-*metric* (*resp.* N_2-*metric*) *for each* $\alpha\in K_1$ (*resp.* $\beta\in K_2$).

Proof. Put $K=K_1$ and $N=N_1$. For $\alpha\in K$ and each n, we see that

$$\phi_n([I_{w(n)}\otimes\alpha, 0])$$
$$=\left[\begin{pmatrix} \overbrace{g_1(\alpha)\ \ddots\ g_1(\alpha)}^{w(n)} & & & & \\ & \overbrace{I_{w(n)}\otimes\alpha\ \ddots\ I_{w(n)}\otimes\alpha}^{p_n-2} & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} \overbrace{g_1(\alpha)\ \ddots\ g_1(\alpha)}^{w(n)} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}\right].$$

Therefore, we have

$$
\begin{aligned}
&[I_{w(n+1)}\otimes\alpha,\ 0]-\phi_n([I_{w(n)}\otimes\alpha,\ 0])\\
&=\left[\begin{pmatrix}
\overbrace{\alpha I_2-g_1(\alpha) \quad \ddots \quad \alpha I_2-g_1(\alpha)}^{w(n)} & & \\
& 0 \ \ddots \ 0 & \\
& & \overbrace{\alpha \ \ddots \ \alpha}^{2w(n)}
\end{pmatrix},\
\begin{pmatrix}
\overbrace{-g_1(\alpha) \quad \ddots \quad -g_1(\alpha)}^{w(n)} & \\
& 0 \ \ddots \ \ddots \ \ddots \ 0
\end{pmatrix}\right].
\end{aligned}
$$

We can calculate that

$$
\begin{aligned}
&N(\theta_{n+1}([I_{w(n+1)}\otimes\alpha,\ 0])-\theta_n([I_{w(n)}\otimes\alpha,\ 0]))\\
&=N([I_{w(n+1)}\otimes\alpha,\ 0]-\phi_n[I_{w(n)}\otimes\alpha,\ 0])\\
&=\frac{1}{w(n+1)}\cdot\{\alpha_{n+1}(1)(w(n)\operatorname{rank}(\alpha I_2-g_1(\alpha))+2w(n))+\alpha_{n+1}(2)w(n)\operatorname{rank}(g_1(\alpha))\}\\
&<\frac{1}{w(n+1)}\cdot\left\{\left(\frac{1}{2}+\lambda\frac{w(n+1)}{u(n+1)}\right)4w(n)+\left(\frac{1}{2}-\lambda\frac{w(n+1)}{u(n+1)}\right)4w(n)\right\}\\
&<4/(p_n+2)
\end{aligned}
$$

Then $\{\theta_n([I_{w(n)}\otimes\alpha,\ 0])\}$ is a Cauchy sequence.

By Lemma 2, we define $\tau_1(\alpha)=\lim\theta_n([I_{w(n)}\otimes\alpha,\ 0])$ (resp. $\tau_2(\beta)=\theta_n([0,\ I_{w(n)}\otimes\beta])$ for each $\alpha\in K_1$ (resp. $\beta\in K_2$). Then $\tau_i\colon K_i\to T_i$ is a map as F-algebra for $i=1, 2$.

Lemma 3.
(1) $\tau_i(K_i)\subseteq Z(T_i)$ *for* $i=1, 2$.
(2) $\tau_i(a)=a$ *for all* $a\in F$.

Proof. (1) For any $r\in R$ and $\alpha\in K_1$, we shall show that $\tau_1(\alpha)r=r\tau_1(\alpha)$. Let $r=\theta_n([x, y])$ for some n and $[x, y]\in R_n$. Since $[I_{w(k)}\otimes\alpha,\ 0][x, y]=[x, y][I_{w(k)}\otimes\alpha,\ 0]$ for all $k>n$, we have, that $\tau_1(\alpha)r=r\tau_1(\alpha)$. Since T_1 is the completion of R with respect to N_1-metric, we have that $\tau_1(\alpha)x=x\tau_1(\alpha)$ for all $x\in T_1$.

(2) Since $a=\theta_n([I_{w(n)}\otimes a,\ I_{w(n)}\otimes a])$ for all $a\in F$ and all n, we see that

$$
\begin{aligned}
&N_1(a-\theta_n([I_{w(n)}\otimes a,\ 0]))\\
&=N_1([0,\ I_{w(n)}\otimes a])\\
&=\alpha_n(2)
\end{aligned}
$$

Therefore we have that $a=\varinjlim\theta_n([I_{w(n)}\otimes a,\ 0]$, because $\varinjlim\alpha_n(2)=0$.

Lemma 4. *Let $p_1, p_2, \cdots$ be integers such that* $\underset{\longrightarrow}{\lim}\frac{u(n)}{w(n)}>0$ *and* $\sum_{n=1}^{\infty} 4/(p_n+2)<\infty$. *Then τ_i: $K_i \to Z(T_i)$ is an isomorphism over F.*

Proof. Put $T=T_1$ and $N=N_1$. We shall show that $\tau_1(K_1)=Z(T)$. First for any $x\in Z(T)$ and any real number $\varepsilon>0$, there exists $r\in R$ such that $N(x-r)<\varepsilon/4$. And there exists $r_{k(1)}\in R_{k(1)}$ such that $r=\theta_{k(1)}(r_{k(1)})$. We note that $r=\theta_m(r_m)$ for all $m\geqq k(1)$, where some $r_m\in R_m$. For any r_m, there exist $z_m\in Z(R_m)$ and $y_m\in R_m$ such that $N(r_m-z_m)\leqq N(r_m y_m-y_m r_m)$ by [1, Cor. 2.4].

Since
$$\begin{aligned} N(r_m-z_m) &\leqq N(r_m y_m-y_m r_m)\\ &\leqq N((r-x)\theta_m(y_m))+N(\theta_m(y_m)(x-r))\\ &\leqq 2N(x-r)\\ &<\varepsilon/2\,, \end{aligned}$$

we see that for any $m\geqq k(1)$,

$$\begin{aligned} (*)\quad N(x-\theta_m(z_m)) &\leqq N(x-r)+N(r-\theta_m(z_m))\\ &\leqq N(x-r)+N(r_m-z_m)\\ &<\varepsilon\cdot 3/4\,. \end{aligned}$$

Put $z_m=[I_{w(n)}\otimes\alpha_m,\ I_{w(n)}\otimes\beta_m]$ for some $\alpha_m\in K_1$ and $\beta_m\in K_2$. Since $\underset{\longrightarrow}{\lim}\,\alpha_n(2)=0$, there exists $k(2)$ such that $\alpha_m(2)<\varepsilon/4$ for all $m\geqq k(2)$.

We see that for all $m\geqq\max(k(1), k(2))$,

$$\begin{aligned} &N(x-\theta_m([I_{w(n)}\otimes\alpha_m,\ 0]))\\ &\leqq N(x-\theta_m(z_m))+N([0,\ I_{w(n)}\otimes\beta_m])\\ &\leqq N(x-\theta_m(z_m))+\alpha_m(2)\\ &<\varepsilon\,. \qquad\qquad \text{(by (*))} \end{aligned}$$

Since $\sum_{n=1}^{\infty} 4/(p_n+2)<\infty$, for ε, there exists a natural number $k(3)$ such that

$$(**)\quad \textstyle\sum_{i=n}^{l} 4/(p_i+2)<\varepsilon \qquad \text{for all}\quad l>n\geqq k(3)\,.$$

Select some $k\geqq\max(k(i)\ i=1, 2, 3)$. Then we have already seen that $\alpha_k\in K_1$ and $N(x-\theta_k([I_{w(k)}\otimes\alpha_k,\ 0]))<\varepsilon$. Put $\gamma=\alpha_k$. We shall show that $N(x-\tau_1(\gamma))<\varepsilon$. There exists a positive integer $k(4)>k$ such that for any $m\geqq k(4)$,

$$(***)\quad N(\theta_m([I_{w(m)}\otimes\gamma,\ 0]-\tau_1(\gamma)))<\varepsilon\,.$$

We see that for some $m\geqq\max\{k(i);\ i=1, 2, 3, 4\}$,

$$
\begin{aligned}
N(x-\tau_1(\gamma)) \leqq\; & N(x-\theta_k([I_{w(k)}\otimes\gamma,\ 0])) \\
& +N(\theta_k([I_{w(k)}\otimes\gamma,\ 0])-\theta_{k+1}([I_{w(k+1)}\otimes\gamma,\ 0])) \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
& +N(\theta_{m-1}([I_{w(m-1)}\otimes\gamma,\ 0])-\theta_m([I_{w(m)}\otimes\gamma,\ 0])) \\
& +N(\theta_m([I_{w(m)}\otimes\gamma,\ 0])-\tau_1(\gamma)))
\end{aligned}
$$

, using the inequality in the proof of Lemma 2, (**) and (***)

$$
\begin{aligned}
& <\varepsilon+\textstyle\sum_{i=k}^{m-1} 4/(p_i+2)+\varepsilon \\
& <6\varepsilon\,.
\end{aligned}
$$

Since T is a simple ring, $Z(T)$ is a field, so if ε is less than $1/6$, $N(x-\tau_1(\gamma))=0$. Therefore x belongs to $\tau_1(K_1)$.

Now we shall give a negative answer for the Goodearl's problem No. 38 [3, p. 348].

EXAMPLE There exists a simple unit-regular ring R such that
(1) R has two extremal rank functions N_1, N_2.
(2) The N_1-completion of R is not isomorphic to the N_2-completion of R.

Proof. Set $F=\boldsymbol{Q}$, $K_1=\boldsymbol{Q}(i)$ and $K_2=\boldsymbol{Q}(\sqrt{2})$. Put $p_n=n^2+4n+2$ for all n, and construct R accoding to the previous method. Since $\dfrac{w(n)}{u(n)}=\dfrac{2(n+2)(n+3)}{9n(n+1)}$, we have $\lim\limits_{\longrightarrow}\dfrac{u(n)}{w(n)}=\dfrac{2}{9}$. And we see that $\sum_{n=1}^{\infty}\dfrac{1}{p_n+2}<\sum_{n=1}^{\infty}\dfrac{1}{(n+1)(n+3)}=\dfrac{5}{12}$. By Lemma 4, the N_1-completion T_1 of R is not isomorphic to the N_2-completion T_2 of R, because $Z(T_1)=\boldsymbol{Q}(i)$ is not isomorphic to $Z(T_2)=\boldsymbol{Q}(\sqrt{2})$ over $\boldsymbol{Q}$.

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