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AN EXAMPLE OF THE COMPLETION OF RANK FUNCTIONS OVER SIMPLE UNIT REGULAR RINGS

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In this note, we are concerned with Von Neumann regular rings having (pseudo-) rank functions. Let R be a regular ring and N a pseudo-rank function on R . Then N induces a pseudo-metric topology on R , and \bar{R} , the completion of R at this pseudo-metric, is a right and left self-injective regular ring. If N is an extremal pseudo-rank function, \bar{R} is simple moreover. It is known that there exist uncountable nonisomorphic simple right and left self-injective regular rings [1, Cor. 2. 9].

From this observation, K.R.Goodearl asked for two different extremal rank functions P, Q on a given simple unit-regular ring R , whether the P -completion of R is isomorphic to the Q -completion of R or not. ([3, Open problem 38]). Now we answer that this problem is negative. Let F be any field and K_i ($=1, 2$) any quadratic extensions of F . We give an example of a simple regular F -algebra R with two extremal rank functions P_i such that the center of the P_i -completion of R is K_i ($i=1, 2$). In particular, put $F=\mathbf{Q}$, $K_1=\mathbf{Q}(i)$, and $K_2=\mathbf{Q}(\sqrt{2})$. Then, since $\mathbf{Q}(i)$ is not isomorphic to $\mathbf{Q}(\sqrt{2})$ over \mathbf{Q} , the P_1 -completion of R is not isomorphic to the P_2 -completion of R .

We use most of our terminologies and notations from Goodearl's book [3].

1. A construction an example

Let K_1, K_2 be quadratic extension fields of a field F and $g_i: K_i \rightarrow M_2(F)$ ($i=1, 2$) matrix representations of K_i over F with respect to regular representation of K_i . We shall construct an F -algebra R , as a direct limit of a sequence $R_1 \rightarrow R_2 \rightarrow \dots$ of semisimple F -algebras. We shall refer to K.R.Goodearl's example [2, Scheme I] and D. Handelman's one [4, p.1144]. Let p_1, p_2, \dots be integers ($p_n > 2$). Define positive integers $w(1), w(2), \dots$ by setting $w_1=1$ and $w_n=(p_{n-1}+2)(p_{n-2}+2)\dots(p_1+2)$ and put

$$R_n = M_{w(n)}(F) \otimes_F K_1 \oplus M_{w(n)}(F) \otimes_F K_2$$

Next we shall define F -algebra maps from R_n to R_{n+1} . Let $\{1, v_i\}$ be F -basis of K_i . Then any element of $M_{w(n)}(F) \otimes_F K_i$ is written by the following form; $x \otimes 1 + y \otimes v_i$, where $x, y \in M_{w(n)}(F)$. We use $x \boxplus y$ to denote the Kronecker product of matrices $x, y \in M_{w(n)}(F)$. Let I_n be the identity matrix in $M_n(F)$.

Define maps $G_{i_n}: M_{w(n)}(F) \otimes_F K_i \rightarrow M_{2w(n)}(F)$ by the rule; $z_i = x \otimes 1 + y \otimes v_i \rightarrow x \boxplus I_2 + y \boxplus g_i(v_i)$. Define maps $\phi_n: R_n \rightarrow R_{n+1}$ by the rule;

$$\left(\left(\begin{array}{c} [z_1, z_2] \\ \downarrow \phi_n \\ \left(\begin{array}{c} G_{1n}(z_1) \\ \dots \\ z_1 \dots z_{p_n-2} \\ \dots \\ G_{2n}(z_2) \end{array} \right) \end{array} \right), \left(\begin{array}{c} G_{1n}(z_1) \\ \dots \\ z_2 \dots z_{p_2-2} \\ \dots \\ G_{2n}(z_2) \end{array} \right) \right) \in R_{n+1}$$

where x, x', y and $y' \in M_{w(n)}(F)$.

Now define R to be the limit of $\{R_n, \phi_n\}$ and let $\theta_n: R_n \rightarrow R$ natural embeddings. Obviously R is a simple unit-regular F -algebra with the center F .

Next we shall determine all (extremal) rank functions on R . We use $P(R)$ to denote the set of all rank functions on R . Put $R'_n = M_{w(n)}(F) \oplus M_{w(n)}(F)$ for each n . We consider R'_n as a sub- F -algebra of R_n by the embedding $[x, y] \rightarrow [x \otimes 1, y \otimes 1]$ where $x, y \in M_{w(n)}(F)$. Put $\phi'_n = \phi_n|_{R'_n}$, then ϕ'_n is as follows:

$$\left(\left(\begin{array}{c} [x, y] \\ \downarrow \phi'_n \\ \left(\begin{array}{c} x \\ \dots \\ x \\ \dots \\ y \\ \dots \\ y \end{array} \right) \end{array} \right), \left(\begin{array}{c} x \\ \dots \\ y \\ \dots \\ y \end{array} \right) \right) \in R'_{n+1}$$

Define R' to be the limit of $\{R', \phi'_n\}$.

Lemma 1. $P(R)$ is affinely homeomorphic to $P(R')$ by the restriction map.

Proof. For any $N \in P(R_n)$ (resp. $P(R'_n)$),

$$N([A, B]) = \frac{\alpha \text{rank}(A) + \beta \text{rank}(B)}{w(n)}$$

, where $A \in M_{w(n)}(F) \otimes K_1, B \in M_{w(n)}(F) \otimes K_2$ (resp. $A, B \in M_{w(n)}(F)$), $\alpha = N([I_n, O])$, and $\beta = N([O, I_n])$ by [3, Cor. 16.6]. Then $P(R_n)$ is affinely homeomorphic to $P(R'_n)$ by the restriction map for all n . Since $P(R)$ (resp. $P(R')$) is the inverse limit of $\{P(R_n), \phi_n^*\}$ (resp. $\{P(R'_n), \phi_n^*\}$), by [3, Prop. 16.21], $P(R)$ is affinely homeomorphic to $P(R')$.

The structure of $P(R')$ has been determined by K.R. Goodearl [2, pp. 277–280]. For the sake of completeness, we shall again explain it.

Put $u(1)=1$ and $u(n+1)=(p_n-2) \cdots (p_1-2)$ for all $n \geq 1$. For $N \in P(R')$, there exist positive real numbers $\alpha_n(i)$ ($n=1, 2, \dots; i=1, 2$) such that

$$(1) \quad \alpha_n(1) + \alpha_n(2) = 1 \text{ for all } n$$

$$(2) \quad \alpha_{n+1}(i) = \frac{(p_n+2)\alpha_n(i)-2}{p_n-2} \quad \text{for all } n, i$$

$$(3) \quad N([A, B]) = \frac{\alpha_n(1) \text{rank}(x) + \alpha_n(2) \text{rank}(y)}{w(n)} \quad \text{for all } [A, B] \in R',$$

where $[A, B] = \theta_n([x, y])$ for some n and $[x, y] \in R'_n$. Conversely, if $\{\alpha_n(i)\}$ are any positive real numbers satisfying (1) and (2), then (3) defines a rank function N on R' .

Now we assume that $\lim_{n \rightarrow \infty} \frac{u(n)}{w(n)} > 0$. Put $\lambda = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{u(n)}{w(n)}$

We define

$$\alpha_n(1) = \frac{1}{2} + \lambda \frac{w(n)}{u(n)} \quad \beta_n(1) = \frac{1}{2} - \lambda \frac{w(n)}{u(n)}$$

$$\alpha_n(2) = \frac{1}{2} - \lambda \frac{w(n)}{u(n)} \quad \beta_n(2) = \frac{1}{2} + \lambda \frac{w(n)}{u(n)}$$

for all $n \geq 1$. Then $\{\alpha_n(i)\}$ and $\{\beta_n(i)\}$ satisfy the above conditions (1) and (2). Let N_1 (resp. N_2) be the rank function determined by $\{\alpha_n(i)\}$ (resp. $\{\beta_n(i)\}$). by [2, Lemma 27], N_1 and N_2 are all extremal rank functions on R' . Therefore, by Lemma 1, N_1 and N_2 can be extended to extremal rank functions on R . N_i ($i=1, 2$) induce metrics on R given by the rule; $d_i(x, y) = N_i(x - y)$ for $x, y \in R$, which we call the N_i -metric [3, § 19]. Let T_i be the completion of R with respect to N_i -metric ($i=1, 2$). Then T_i are simple regular, right and left self-injective F -algebras by [3, Th. 19. 14].

2. Calculation of the centers of T_i

In this note, we shall calculate the center $Z(T_i)$ of T_i . Let I_k be the identity matrix for $M_k(F)$ and θ_n the natural embedding: $R_n \rightarrow R$.

Lemma 2. *If $\sum_{i=1}^{\infty} 1/(P_n+2) < \infty$, then $\{\theta_n([I_{w(n)} \otimes \alpha, 0])\}$ (resp. $\theta_n([0, I_{w(n)} \otimes \beta])$) is a Cauchy sequence with respect to N_1 -metric (resp. N_2 -metric) for each $\alpha \in K_1$ (resp. $\beta \in K_2$).*

Proof. Put $K = K_1$ and $N = N_1$. For $\alpha \in K$ and each n , we see that

$$\phi_n([I_{w(n)} \otimes \alpha, 0]) = \left(\begin{array}{c} \left(\begin{array}{c} g_1(\alpha) \quad w(n) \\ \vdots \\ g_1(\alpha) \end{array} \right) \\ I_{w(n)} \otimes \alpha \quad p_n - 2 \\ \vdots \\ I_{w(n)} \otimes \alpha \\ 0 \\ \vdots \\ 0 \end{array} \right), \left(\begin{array}{c} \left(\begin{array}{c} g_1(\alpha) \quad w(n) \\ \vdots \\ g_1(\alpha) \end{array} \right) \\ 0 \\ \vdots \\ 0 \end{array} \right) \end{array} \right).$$

Therefore, we have

$$= \left(\begin{array}{c} [I_{w(n+1)} \otimes \alpha, 0] - \phi_n([I_{w(n)} \otimes \alpha, 0]) \\ \left(\begin{array}{c} \alpha I_2 - g_1(\alpha) \quad \swarrow w(n) \\ \vdots \\ \alpha I_2 - g_1(\alpha) \\ 0 \\ \vdots \\ 0 \\ \alpha \quad \swarrow 2w(n) \\ \vdots \\ \alpha \end{array} \right) \\ \left(\begin{array}{c} -g_1(\alpha) \quad \swarrow w(n) \\ \vdots \\ -g_1(\alpha) \\ 0 \\ \vdots \\ 0 \end{array} \right) \end{array} \right)$$

We can calculate that

$$\begin{aligned} & N(\theta_{n+1}([I_{w(n+1)} \otimes \alpha, 0]) - \theta_n([I_{w(n)} \otimes \alpha, 0])) \\ &= N([I_{w(n+1)} \otimes \alpha, 0] - \phi_n([I_{w(n)} \otimes \alpha, 0])) \\ &= \frac{1}{w(n+1)} \cdot \{ \alpha_{n+1}(1)(w(n) \text{ rank } (\alpha I_2 - g_1(\alpha)) + 2w(n)) + \alpha_{n+1}(2)w(n) \text{ rank } (g_1(\alpha)) \} \\ &< \frac{1}{w(n+1)} \cdot \left\{ \left(\frac{1}{2} + \lambda \frac{w(n+1)}{u(n+1)} \right) 4w(n) + \left(\frac{1}{2} - \lambda \frac{w(n+1)}{u(n+1)} \right) 4w(n) \right\} \\ &< 4/(p_n + 2) \end{aligned}$$

Then $\{\theta_n([I_{w(n)} \otimes \alpha, 0])\}$ is a Cauchy sequence.

By Lemma 2, we define $\tau_1(\alpha) = \lim \theta_n([I_{w(n)} \otimes \alpha, 0])$ (resp. $\tau_2(\beta) = \theta_n([0, I_{w(n)} \otimes \beta])$) for each $\alpha \in K_1$ (resp. $\beta \in K_2$). Then $\tau_i: K_i \rightarrow T_i$ is a map as F -algebra for $i=1, 2$.

Lemma 3.

- (1) $\tau_i(K_i) \subseteq Z(T_i)$ for $i=1, 2$.
- (2) $\tau_i(a) = a$ for all $a \in F$.

Proof. (1) For any $r \in R$ and $\alpha \in K_1$, we shall show that $\tau_1(\alpha)r = r\tau_1(\alpha)$. Let $r = \theta_n([x, y])$ for some n and $[x, y] \in R_n$. Since $[I_{w(k)} \otimes \alpha, 0][x, y] = [x, y][I_{w(k)} \otimes \alpha, 0]$ for all $k > n$, we have, that $\tau_1(\alpha)r = r\tau_1(\alpha)$. Since T_1 is the completion of R with respect to N_1 -metric, we have that $\tau_1(\alpha)x = x\tau_1(\alpha)$ for all $x \in T_1$.

(2) Since $a = \theta_n([I_{w(n)} \otimes a, I_{w(n)} \otimes a])$ for all $a \in F$ and all n , we see that

$$\begin{aligned} & N_1(a - \theta_n([I_{w(n)} \otimes a, 0])) \\ &= N_1([0, I_{w(n)} \otimes a]) \\ &= \alpha_n(2) \end{aligned}$$

Therefore we have that $a = \lim_{\rightarrow} \theta_n([I_{w(n)} \otimes a, 0])$, because $\lim_{\rightarrow} \alpha_n(2) = 0$.

Lemma 4. *Let p_1, p_2, \dots be integers such that $\lim_{n \rightarrow \infty} \frac{u(n)}{w(n)} > 0$ and $\sum_{n=1}^{\infty} 4/(p_n+2) < \infty$. Then $\tau_i: K_i \rightarrow Z(T_i)$ is an isomorphism over F .*

Proof. Put $T=T_1$ and $N=N_1$. We shall show that $\tau_1(K_1)=Z(T)$. First for any $x \in Z(T)$ and any real number $\varepsilon > 0$, there exists $r \in R$ such that $N(x-r) < \varepsilon/4$. And there exists $r_{k(1)} \in R_{k(1)}$ such that $r = \theta_{k(1)}(r_{k(1)})$. We note that $r = \theta_m(r_m)$ for all $m \geq k(1)$, where some $r_m \in R_m$. For any r_m , there exist $z_m \in Z(R_m)$ and $y_m \in R_m$ such that $N(r_m - z_m) \leq N(r_m y_m - y_m r_m)$ by [1, Cor. 2.4].

$$\begin{aligned} \text{Since } N(r_m - z_m) &\leq N(r_m y_m - y_m r_m) \\ &\leq N((r-x)\theta_m(y_m)) + N(\theta_m(y_m)(x-r)) \\ &\leq 2N(x-r) \\ &< \varepsilon/2, \end{aligned}$$

we see that for any $m \geq k(1)$,

$$\begin{aligned} (*) \quad N(x - \theta_m(z_m)) &\leq N(x-r) + N(r - \theta_m(z_m)) \\ &\leq N(x-r) + N(r_m - z_m) \\ &< \varepsilon \cdot 3/4. \end{aligned}$$

Put $z_m = [I_{w(n)} \otimes \alpha_m, I_{w(n)} \otimes \beta_m]$ for some $\alpha_m \in K_1$ and $\beta_m \in K_2$. Since $\lim_{n \rightarrow \infty} \alpha_n(2) = 0$, there exists $k(2)$ such that $\alpha_m(2) < \varepsilon/4$ for all $m \geq k(2)$.

We see that for all $m \geq \max(k(1), k(2))$,

$$\begin{aligned} &N(x - \theta_m([I_{w(n)} \otimes \alpha_m, 0])) \\ &\leq N(x - \theta_m(z_m)) + N([0, I_{w(n)} \otimes \beta_m]) \\ &\leq N(x - \theta_m(z_m)) + \alpha_m(2) \\ &< \varepsilon. \end{aligned} \tag{by (*)}$$

Since $\sum_{n=1}^{\infty} 4/(p_n+2) < \infty$, for ε , there exists a natural number $k(3)$ such that

$$(**) \quad \sum_{i=n}^l 4/(p_i+2) < \varepsilon \quad \text{for all } l > n \geq k(3).$$

Select some $k \geq \max(k(i) \ i=1, 2, 3)$. Then we have already seen that $\alpha_k \in K_1$ and $N(x - \theta_k([I_{w(k)} \otimes \alpha_k, 0])) < \varepsilon$. Put $\gamma = \alpha_k$. We shall show that $N(x - \tau_1(\gamma)) < \varepsilon$. There exists a positive integer $k(4) > k$ such that for any $m \geq k(4)$,

$$(***) \quad N(\theta_m([I_{w(m)} \otimes \gamma, 0]) - \tau_1(\gamma)) < \varepsilon.$$

We see that for some $m \geq \max\{k(i); i=1, 2, 3, 4\}$,

$$\begin{aligned}
 N(x - \tau_1(\gamma)) &\leq N(x - \theta_k([I_{w(k)} \otimes \gamma, 0])) \\
 &\quad + N(\theta_k([I_{w(k)} \otimes \gamma, 0]) - \theta_{k+1}([I_{w(k+1)} \otimes \gamma, 0])) \\
 &\quad \dots\dots\dots \\
 &\quad + N(\theta_{m-1}([I_{w(m-1)} \otimes \gamma, 0]) - \theta_m([I_{w(m)} \otimes \gamma, 0])) \\
 &\quad + N(\theta_m([I_{w(m)} \otimes \gamma, 0]) - \tau_1(\gamma))
 \end{aligned}$$

, using the inequality in the proof of Lemma 2, (**) and (***)

$$\begin{aligned}
 &< \varepsilon + \sum_{i=k}^{m-1} 4/(p_i + 2) + \varepsilon \\
 &< 6\varepsilon .
 \end{aligned}$$

Since T is a simple ring, $Z(T)$ is a field, so if ε is less than $1/6$, $N(x - \tau_1(\gamma)) = 0$. Therefore x belongs to $\tau_1(K_1)$.

Now we shall give a negative answer for the Goodearl's problem No. 38 [3, p. 348].

EXAMPLE There exists a simple unit-regular ring R such that

- (1) R has two extremal rank functions N_1, N_2 .
- (2) The N_1 -completion of R is not isomorphic to the N_2 -completion of R .

Proof. Set $F = \mathbf{Q}, K_1 = \mathbf{Q}(i)$ and $K_2 = \mathbf{Q}(\sqrt{2})$. Put $p_n = n^2 + 4n + 2$ for all n , and construct R according to the previous method. Since $\frac{w(n)}{u(n)} = \frac{2(n+2)(n+3)}{9n(n+1)}$, we have $\lim_{n \rightarrow \infty} \frac{u(n)}{w(n)} = \frac{2}{9}$. And we see that $\sum_{n=1}^{\infty} \frac{1}{p_n + 2} < \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} = \frac{5}{12}$. By Lemma 4, the N_1 -completion T_1 of R is not isomorphic to the N_2 -completion T_2 of R , because $Z(T_1) = \mathbf{Q}(i)$ is not isomorphic to $Z(T_2) = \mathbf{Q}(\sqrt{2})$ over \mathbf{Q} .

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