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<thead>
<tr>
<th><strong>Title</strong></th>
<th>A cluster of sets of exceptional times of linear Brownian motion</th>
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</thead>
<tbody>
<tr>
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<td>磯崎, 泰樹</td>
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<td><strong>Citation</strong></td>
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Osaka University
A CLUSTER OF SETS OF EXCEPTIONAL TIMES
OF LINEAR BROWNIAN MOTION

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(Received June 7, 2000)

1. Introduction and the main theorems

Aspandiarov-Le Gall [1] studied the following random closed sets $K^-$, $K$ and $K'$: Let $(B_t; t \geq 0)$ be a linear standard Brownian motion starting at 0, and let

$$K^- = \left\{ t \in [0, 1]; \int_s^t (B_u - B_t)du \leq 0 \text{ for every } s \in [0, t). \right\},$$

$$K = \left\{ t \in K^-; \int_s^t (B_u - B_t)du \leq 0 \text{ for every } s \in (t, 1]. \right\},$$

$$K' = \left\{ t \in K^-; \int_s^t (B_u - B_t)du \geq 0 \text{ for every } s \in (t, 1]. \right\}.$$

They computed the Hausdorff dimension of $K^-$, $K$ and $K'$.

**Theorem** ([1]). It holds $\dim K^- = 3/4$, $\dim K = 1/2$ and $\dim K' \leq 1/2$ almost surely. The set $K'$ is possibly empty or $\dim K' = 1/2$, both with positive probability. The same statements hold if the weak inequalities in the definition of $K^-$, $K$ and $K'$ are replaced by the strict inequalities.

In this paper, we consider a cluster of random sets having various dimension.

For $\alpha \geq 0$ and $c > 0$, we define the following functions $V(\alpha, c)$ increasing on $\mathbb{R}$:

$$V(\alpha, c; y) = y^\alpha \text{ for } y > 0; V(\alpha, c; 0) = 0; V(\alpha, c; y) = -\frac{|y|^\alpha}{c} \text{ for } y < 0.$$  \hspace{1cm} (1.1)

Let $\alpha, \alpha_+, \alpha_- \geq 0$, $c, c_+, c_- > 0$ and write $V$ for $V(\alpha, c)$, $V_+ \text{ for } V(\alpha_+, c_+)$.

We define the random sets depending on the functions $V$, $V_+$ and $V_-:

$$K^-(V) = \left\{ t \in [0, 1]; \int_s^t V(B_u - B_t)du \leq 0 \text{ for every } s \in [0, t). \right\},$$

$$K(V_+; V_+) = \left\{ t \in K^- (V_-); \int_s^t V_+(B_u - B_t)du \leq 0 \text{ for every } s \in (t, 1]. \right\},$$

$$K(V_-; V_+) = \left\{ t \in K^- (V_-); \int_s^t V_+(B_u - B_t)du \leq 0 \text{ for every } s \in (t, 1]. \right\}.$$
These sets consist of exceptional times in the sense that \( P[t \in K^-(V)] = 0 \) for every \( t \in (0, 1] \) and \( P[t \in K^-(V_-; V_+) = 0 \) for every \( t \in [0, 1] \).

**Theorem 1.** We define \( \nu = 1/(2+\alpha), \nu_- = 1/(2+\alpha_-) \) and \( \nu_+ = 1/(2+\alpha_+) \). Let \( \rho, \rho_-, \rho_+ \in (0, 1) \) be the unique solutions of the equations

\[
\begin{align*}
C^\nu \sin \pi \nu(1 - \rho) &= \sin \pi \nu_\rho, \\
C^\nu_- \sin \pi \nu_-(1 - \rho_-) &= \sin \pi \nu_- \rho_-, \\
C^\nu_+ \sin \pi \nu_+(1 - \rho_+) &= \sin \pi \nu_+ \rho_+
\end{align*}
\]

respectively.

(a) For \( V = V(\alpha, c) \), we have almost surely \( \dim K^-(V) = 1 - \rho/2 \).

For \( V_+ = V(\alpha_+, c_+) \) and \( V_- = V(\alpha_-, c_-) \) we have (b) and (c):

(b) \( \dim K(V_-; V_+) \leq 1 - (\rho_+ + \rho_-)/2 \) almost surely and

\[
P \left[ \dim K(V_-; V_+) \geq 1 - \frac{\rho_+ + \rho_-}{2} \right] > 0.
\]

(c) \( \dim K'(V_-; V_+) \leq (1 - \rho_+ + \rho_-)/2 \) almost surely and

\[
P \left[ \dim K'(V_-; V_+) \geq \frac{1 - \rho_+ + \rho_-}{2} \right] > 0.
\]

The behavior of \( V, V_+ \) and \( V_- \) outside any neighborhood of the origin have no influence on the Hausdorff dimension; We could state the theorem in that fashion. The parameters \( \rho, \rho_-, \rho_+ \in (0, 1) \) in the statement of Theorem 1 are continuous and increasing in \( c, c_+, c_- \) and have the range \((0, 1)\) since \( \lim_{c \to 0} \rho = 0 \) and \( \lim_{c \to \infty} \rho = 1 \).

In fact, they are equal to the probability of some event related to the parameters in the theorem, see the remark 4 in [4].

Note that for fixed \( \alpha \), it holds \( \rho = 1/2 \) if \( c = 1 \). Hence the statements in the theorem in [1] for \( K^- \) and \( K' \) can be included in Theorem 1 since \( K^- = K^-(V(1, 1)) \) and \( K' = K'(V(1, 1); V(1, 1)) \). The implication by Theorem 1 on \( K \), however, is weaker than [1], since we have not obtained the almost sure estimate from below.

Let \( \alpha, \bar{\alpha} \geq 0 \) and \( \bar{c} > 0 \). If \( V = V(\alpha, c) \) and \( \bar{V} = V(\bar{\alpha}, \bar{c}) \), then there is no inclusion in general between \( K^-(V) \) and \( K^-(\bar{V}) \). However it is easy to see, for each \( \alpha \), that \( K^-(V(\alpha, c)) \subset K^-(V(\alpha, \bar{c})) \) if \( \bar{c} < c \). Hence we obtain a family

\[
\{K^-(V(\alpha, c)); c \in (0, 1)\}
\]

of decreasing random sets having strictly decreasing dimension.
The estimate in Theorem 1 for $\dim K^-(V)$ is exhaustive in the following sense: Let $H$ be the set of times $t$ when $B_t$ attains its past-maximum:

$$H := \left\{ t \in [0, 1]; B_t = \sup_{0 \leq s \leq t} B_s \right\}.$$

It is well known that $\dim H = 1/2$ a.s. Since $H \subset K^-(V(\alpha, c)) \subset [0, 1]$, we have $1/2 \leq \dim K^-(V) \leq 1$. The range of $1 - \rho/2$ is exactly $(1/2, 1)$ and the trivial case $K^-(V) = H$ or $K^-(V) = [0, 1]$ could be included if we allow $c = \infty$ or $c = 0$.

The estimate in Theorem 1 for $\dim K(V_-; V_+)$ is also exhaustive in the following sense: Let $\tau$ be the time when the maximum on $[0, 1]$ of $B$ is attained: $B_\tau \geq B_t$ for every $t \in [0, 1]$. The inclusion $\{\tau\} \subset K(V_-; V_+) \subset [0, 1]$ implies $0 \leq \dim K(V_-; V_+) \leq 1$ and the range of the value $1 - (\rho_- + \rho_+)/2$ is exactly $(0, 1)$. The extreme cases could also be included here.

In the same sense as Aspandiiarov and LeGall [1] noted concerning $K', K'(V_-; V_+)$ can be interpreted as a weakened notion of the increasing points of Brownian motion and it is not straightforward to exhibit an element of $K'(V_-; V_+)$. If both $V_-$ and $V_+$ are $V(\alpha, c)$ then $(1 - \rho_- + \rho_+)/2 = 1/2$ irrespective of $\alpha$ and $c$. This motivates the next theorem, which could be a version of settlement of a conjecture at the end of [1]: $\dim K' = 1/2$ a.s. on the event $\{B_1 > 0\}$.

**Theorem 2.** Let $\mathcal{V} = \{V : \mathbb{R} \rightarrow \mathbb{R}; V(0) = 0, \text{ } V \text{ is strictly increasing}\}$.

We define $\tilde{K}'(V; V)$ for $V \in \mathcal{V}$ in the same way as (1.3) replacing the weak inequalities by strict inequalities in the definition of $K'(V; V)$:

$$\tilde{K}'(V; V) = \left\{ t \in [0, 1]; \int_s^t V(B_u - B_s) du < 0 \quad \text{for every } s \in [0, t), \right.$$  
$$\text{and} \quad \int_t^s V(B_u - B_t) du > 0 \quad \text{for every } s \in (t, 1]. \right\}.$$

Then we have $P[\dim \tilde{K}'(V; V) = 1/2] > 0$, $P[\tilde{K}'(V; V) \subset \{0, 1\}] > 0$ and

$$P \left[ \dim \tilde{K}'(V; V) = \frac{1}{2} \text{ or } \tilde{K}'(V; V) \subset \{0, 1\} \right] = 1.$$

**Remark 1.** When the set $\tilde{K}'(V; V)$ consists of exceptional times, we have the dichotomy that $\dim \tilde{K}'(V; V) = 1/2$ if it is not empty.

The result of Theorem 2 is stronger than Theorem 1(c) for each strictly increasing functions $V(\alpha, c)$, i.e. $\alpha > 0$, while Theorem 2 says nothing about $V(0, c)$.

Theorem 2 is in fact a corollary of the following Theorem 3 due essentially to Bertoin [3].
Let $V \in \mathcal{V}$, $x \in \mathbb{R}$ and $X = (X(t); t \geq 0)$ be a cadlag path with $\liminf_{t \to -\infty} X(t) = +\infty$. We define, inspired by Bertoin [3],

$$K'_\infty(V, x, X) = \left\{ t \in [0, \infty); \int_s^t V(X_u - x)du \leq 0 \quad \text{for every } s \in [0, t), \right.$$  

$$\text{and } \int_s^t V(X_u - x)du \geq 0 \quad \text{for every } s \in (t, \infty). \right\},$$

$$K'_1(V, x, X) = \left\{ t \in [0, 1]; \int_s^t V(X_u - x)du \leq 0 \quad \text{for every } s \in [0, t), \right.$$  

$$\text{and } \int_s^t V(X_u - x)du \geq 0 \quad \text{for every } s \in (t, 1]. \right\}.$$  

It is then easy to see $\tilde{K}'(V; V) \cup \{0, 1\} = \cup_{B \in \mathcal{B}} K'_1(V, x, B)$.

In other words, $K'_\infty(V, x, X)$ and $K'_1(V, x, X)$ consist of the locations of the overall minimum of the function $s \mapsto \int_0^s V(X_u - x)du$ on $[0, \infty)$ or $[0, 1]$ respectively and $\tilde{K}'(V; V)$ is the collection of such $t$'s that the function $s \mapsto \int_0^s V(B_u - B_t)du$ has the unique minimum at $s = t$.

The following results are proven in Bertoin [3] in the case where $V(y) \equiv y = V(1, 1; y)$.

**Theorem 3.** Let $V \in \mathcal{V}$ and $X$ be a Lévy process with no positive jump such that $\liminf_{t \to -\infty} X(t) = +\infty$ a.s. Let $a(x)$ be the rightmost element of $K'_\infty(V, x, X)$.

(a) $\{a(x) - a(0); x \geq 0\}$ and the process $T^X_x := \inf\{t \geq 0; X_t \geq x\}$ have the same law.

(b) For every fixed $x \in \mathbb{R}$, $P^X_x[\#K'_\infty(V, x, X) = 1] = 1$.

(c) Let $g(0) = \sup\{t \geq 0; X(t) \leq 0\}$ be the last exit time from $(-\infty, 0]$. If $V \in \mathcal{V}$ satisfies $V(y) = -V(-y)$, then $a(0)$ and $g(0) - a(0)$ are independent and have the same law.

(d) If $X$ is a Brownian motion with unit drift, then $\{a(x) - a(0); x \geq 0\}$ has the Lévy measure $(2\pi)^{-1/2}y^{-3/2}e^{-y^2/2}dy$ on $(0, \infty)$. If, moreover, $V \in \mathcal{V}$ satisfies $V(y) = -V(-y)$, then the density of the common law of $a(0)$ and $g(0) - a(0)$ is $2^{-1/4}\Gamma(1/4)^{-1}y^{-3/4}e^{-y^2/2}dy$ on $(0, \infty)$.

**Remark 2.** The statement (a) and the first sentence in (d) hold for nondecreasing $V$ satisfying $V(0) = 0$. The second sentence in (d) was known to Jean Bertoin (private communication).

This paper is organized as follows: We prove Theorem 1 in Section 2 using Theorem 4, which contains an asymptotic estimate for some fluctuating additive functionals. Theorems 2 and 3 are proven in Section 3. We prove Theorem 4 in Section 4 using a theorem in [4].
ACKNOWLEDGEMENT. The author would express his gratitude to Professor Shin’ichi Kotani for his careful reading of a draft of this paper and for his helpful comment. Thanks also goes to the anonymous referee for advice on improving the presentation.

2. Proof of Theorem 1

The argument here mimics that of Aspandiiarov and Le Gall [1] line by line. We first state Theorem 4, an estimate for the distribution of the first hitting time of \( \int_0^t V(B_u)du \), next define suitable approximations of \( K^{-}(V) \), \( K(V_-; V_+) \) and \( K'(V_-; V_+) \) and obtain some preliminary estimates. From that point on, we only need the straightforward changes.

**Theorem 4.** Let \( \alpha \geq 0, \ c > 0, \ V = V(\alpha, c) \), \( \nu = 1/(2 + \alpha) \) and \( \rho \in (0, 1) \) be the solution of \( c' \sin \pi \nu (1 - \rho) = \sin \pi \nu \rho \). We denote by \( p(t, x, y; V) \) the probability \( P[\forall s \in [0, t], x + \int_0^s V(y + B_u)du \leq 0] \).

For any \( t > 0, \ x < 0, \ y \in \mathbb{R} \) and there exist constants \( C_0(t, x, y; V) > 0, \ C_1(\alpha, c) > 0 \) and \( \tilde{C}(x, y) > 0 \) such that it holds

\[
\sup_{\sigma > 0} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) = C_0(t, x, y; V), \tag{2.4}
\]

\[
\lim_{\sigma \to 0} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) = C_1(\alpha, c)t^{-\rho/2}\tilde{C}(x, y), \tag{2.5}
\]

Moreover it holds

\[
C_0(t, x, y; V) \leq \text{const} t^{-\rho/2}(|x|^{\nu \rho} \lor |y|^{-\rho}), \tag{2.6}
\]

**Definition.** Let \( \varepsilon \in [0, 1/2], \ a \in [0, 1 - \varepsilon] \) and \( b \in [\varepsilon, 1] \).

For \( V, V_+, V_- \in \cup_{\alpha \geq 0, c > 0} \{V(\alpha, c)\} \) we define

\[
K^{-}_{\varepsilon, a}(V) = \left\{ t \in [a + \varepsilon, 1]; \int_s^t V(B_u - B_t)du \leq 0 \right. \text{ for every } s \in [a, t - \varepsilon] \right\},
\]

\[
K^{+}_{\varepsilon, b}(V) = \left\{ t \in [0, b - \varepsilon]; \int_0^s V(B_u - B_t)du \leq 0 \right. \text{ for every } s \in [t + \varepsilon, b] \right\},
\]

\[
K^{\pm}_{\varepsilon, b}(V) = \left\{ t \in [0, b - \varepsilon]; \int_t^s V(B_u - B_t)du \geq 0 \right. \text{ for every } s \in [t + \varepsilon, b] \right\},
\]

\[
K^{-}_{\varepsilon, a}(V_-; V_+) = K^{-}_{\varepsilon, a}(V_-) \cap K^{+}_{\varepsilon, b}(V_+),
\]

\[
K^{+}_{\varepsilon, a}(V_-; V_+) = K^{-}_{\varepsilon, a}(V_-) \cap K^{+}_{\varepsilon, b}(V_+).
\]

We also define

\[
K^{-}(V) = K^{-}_{\varepsilon, 0}(V), \quad K^{+}(V) = K^{+}_{\varepsilon, 0}(V), \tag{2.7}
\]

\[
K(V_-; V_+) = K_{\varepsilon, 0, 1}(V_-; V_+), \quad K(V_-; V_+) = K_{0}(V_-; V_+), \tag{2.8}
\]
Lemma 5. Let $\alpha, \alpha_+, \alpha_- \geq 0$, $c, c_+, c_- > 0$ and let $\rho, \rho_+, \rho_-$ be defined in the statement of Theorem 1.

(a) For any $V = V(\alpha, c)$, $0 < \varepsilon < 1/2$ and $t > a$, it holds
\[
\left( \frac{t-a}{\varepsilon} \right)^{\rho/2} P[t \in K^-_{\varepsilon a}(V)] < \text{const}.
\]
There exists a constant $C_3(V) > 0$ such that it holds
\[
P[t \in K^-_{\varepsilon a}(V)] \sim C_3(V) \left( \frac{\varepsilon}{t-a} \right)^{\rho/2}
\]
as $\varepsilon \searrow 0$ for every $t$.

(b) For any $V_+ = V(\alpha_+, c_+)$, $V_- = V(\alpha_-, c_-)$, $0 < \varepsilon < 1/2$ and $t \in (a, b)$,
\[
\left( \frac{t-a}{\varepsilon} \right)^{\rho_-/2} \left( \frac{b-t}{\varepsilon} \right)^{\rho_+/2} P[t \in K_{\varepsilon a b}(V_-; V_+)] < \text{const},
\]
\[
\left( \frac{t-a}{\varepsilon} \right)^{\rho_-/2} \left( \frac{b-t}{\varepsilon} \right)^{(1-\rho_+)/2} P[t \in K'_{\varepsilon a b}(V_-; V_+)] < \text{const}.
\]

We denote by $V_+(- \cdot)$ the function $y \mapsto V_+(-y)$. It holds as $\varepsilon \searrow 0$
\[
P[t \in K_{\varepsilon a b}(V_-; V_+)] \sim C_3(V_-) C_3(V_+) \left( \frac{\varepsilon}{t-a} \right)^{\rho_-/2} \left( \frac{\varepsilon}{b-t} \right)^{\rho_+/2},
\]
\[
P[t \in K'_{\varepsilon a b}(V_-; V_+)] \sim C_3(V_-) C_3(V_+(- \cdot)) \left( \frac{\varepsilon}{t-a} \right)^{\rho_-/2} \left( \frac{\varepsilon}{b-t} \right)^{(1-\rho_+)/2}
\]

Proof. We only prove (a) since the statement (b) follows by time-reversal $\tilde{B}_s = B_{1-s}$ and by reflection $\tilde{B}_s = -B_s$.

Let $P^V_{\alpha c y}$ be the law of the following two-dimensional diffusion $(X(t), Y(t))$: $Y(t) = y + B(t), \quad X(t) = x + \int_0^t V(Y(s))ds$.

By the strong Markov property,
\[
P[t \in K^-_{\varepsilon a}(V)] = E^V_{0,0}[P(t-a - \varepsilon, X(\varepsilon), Y(\varepsilon); V)]
\]
Under $P^V_{0,0}$, the law of $(X(\varepsilon), Y(\varepsilon))$ is the same as that of $(\varepsilon^{1/2} X(1), \varepsilon^{1/2} Y(1))$. By (2.4) and (2.6), we have for any $\varepsilon > 0$,
\[ \varepsilon^{-\rho/2} p(t - a - \varepsilon^1, \varepsilon^{1/2} x_1, \varepsilon^{1/2} y_1; V) \leq \text{const}(t - a - \varepsilon)^{1/2} \left( |x_1|^{1/2} + |y_1|^{1/2} \right) \]

The quantity \( (t - a)\varepsilon^{1/2} p(t \in K_{x,a}^-(V)) \) is hence bounded. This bound also enables us to prove the second sentence of (a) with \( C_3(V) = C_1(\alpha, c) E(0, 0), \tilde{C}(x_1, y_1) \).

**Lemma 6.** We use the same notations as the previous lemma. It holds for any \( \varepsilon \in (0, 1/2) \) and \( 0 < s < t < 1 \).

\[
\begin{align*}
(2.10) & \quad P[\{s,t\} \in K_{x,a}^-(V)] \leq \text{const} \frac{\varepsilon^\rho}{s^{\rho/2}(t - s)^{\rho/2}}, \\
(2.11) & \quad P[\{s,t\} \in K_{x,a,b}^-(V_-; V_+)] \leq \text{const} \frac{\varepsilon^\rho - \eta}{s^{\rho/2}(t - s)^{(\rho^2 + \rho_1)/2(1 - 1)^{\rho_2}/2}}, \\
(2.12) & \quad P[\{s,t\} \in K_{x,a,b}^-(V_-; V_+)] \leq \text{const} \frac{\varepsilon^\rho - \eta}{s^{\rho/2}(t - s)^{(\rho^2 + \rho_1)/2(1 - 1)^{\rho_2}/2}}.
\end{align*}
\]

The constants here depend on \( \alpha, \alpha_+, \alpha_- \) and \( c_+, c_- \).

**Proof.** This can be done using Lemma 5. See the proof of Proposition 4 in [1].

**Lemma 7.** Let \( F_{a,b} \) be the \( \sigma \)-field \( \sigma(B_t - B_a; u \in [a,b]) \) for \( 0 \leq a < b \leq 1 \).

For any \( \alpha \geq 0, c > 0 \) and \( V = V(\alpha, c) \) there exist \( F_{a,b} \)-measurable random variables \( U_{a,b,-}, U_{a,b,+} \) and \( U_{a,b,\ast} \) such that

\[
\begin{align*}
(2.13) & \quad P\left[ K^-(V) \cap [a,b] \neq \emptyset \mid F_{a,b} \right] \leq (b - a)^{-\rho/2} U_{a,b,-}, \\
(2.14) & \quad P\left[ K^+(V) \cap [a,b] \neq \emptyset \mid F_{b,b} \right] \leq (b - a)^{-\rho/2} U_{a,b,+}, \\
(2.15) & \quad P\left[ K^\ast(V) \cap [a,b] \neq \emptyset \mid F_{b,b} \right] \leq (b - a)^{1-\rho/2} U_{a,b,\ast},
\end{align*}
\]

and \( E_0(U_{a,b,-}^2) \leq \text{const} a^{-\rho}, E_0(U_{a,b,\ast}^2) \leq \text{const}(1 - b)^{1-\rho}, E_0(U_{a,b,\ast}^2) \leq \text{const}(1 - b)^{-1+\rho}. \) The constants here depend on \( \alpha \) and \( c \).

**Proof.** We prove (2.14) since (2.13), (2.15) and the corresponding moment estimates follow by time-reversal \( \tilde{B}_s = B_{1-s} \) and by reflection \( \tilde{B}_s = -B_s \).

Let \( \eta_{a,b} \) be the amplitude of \( B_s \) on \([a,b] \). Note that \( V \) is increasing. By modifying the argument in the proof of Lemma 7 in [1], we can take

\[
U_{a,b,+} = (b - a)^{-\rho/2} p(1 - b, (b - a) V(-\eta_{a,b}, -\eta_{a,b}; V)).
\]

The bound of the moment follows by (2.6) and by the fact that \( \eta_{a,b} \) has the same law as \( (b - a)^{1/2} \eta_{0,1} \).
Proof of Theorem 1. The upper estimates for the Hausdorff dimension is obtained by the argument in the proof Proposition 6 in [1].

To obtain the lower estimates, we define the normalized Lebesgue measures: For any Borel subset \( F \) of \([0, 1] \), let

\[
\mu_{\varepsilon}^{-}(F) = \varepsilon^{-\rho/2}|F \cap K_{\varepsilon}^{-}(V)|, \\
\mu_{\varepsilon}(F) = \varepsilon^{-(\rho-\rho')/2}|F \cap K_{\varepsilon}(V_{-}; V_{+})|, \\
\mu_{\varepsilon}'(F) = \varepsilon^{-(\rho-1-\rho')/2}|F \cap K_{\varepsilon}'(V_{-}; V_{+})|.
\]

We denote by \( \mathcal{M}_{f} \) the Polish space of all finite measures on \([0, 1] \) equipped with the topology of weak convergence, and by \( \mathcal{C}([0, 1]) \) the Banach space of all continuous map from \([0, 1] \) to \( \mathbb{R} \).

Let \( (\varepsilon_{n}) \) be a sequence strictly decreasing to 0. We define the random variables \( \zeta_{n} \) taking values in \( \mathcal{M}_{f} \times \mathcal{C}([0, 1]) \) by \( \zeta_{n} = (\mu_{\varepsilon_{n}}(B_{t}; 0 \leq t \leq 1)) \). We define \( \zeta_{-1} \) and \( \zeta_{-1}' \) in the same way using \( \mu_{\varepsilon_{n}} \) and \( \mu_{\varepsilon_{n}}' \). The argument in [1] ensures that we may assume the sequence \( \zeta \) is weakly convergent by extracting a subsequence. Skorohod’s representation theorem says that there is a probability space carrying a sequence of random variables \( \zeta_{n} = (\mu_{\varepsilon_{n}}(B_{t}; 0 \leq t \leq 1)) \) and a random variable \( \zeta_{\infty} = (\mu_{\infty}(B_{t}; 0 \leq t \leq 1)) \) such that \( \zeta_{n} \) and \( \zeta \) have the same law and \( \zeta_{n} \) converges to \( \zeta_{\infty} \) almost surely.

Let \( K(V_{-}; V_{+}; B_{-}^{\infty}) \) be defined in the same way as \( K(V_{-}; V_{+}; B_{-}) \) replacing \( B_{-} \) by \( B_{-}^{\infty} \). To prove that \( \mu_{\infty} \) is a.s. supported on \( K(V_{-}; V_{+}; B_{-}^{\infty}) \), we change the definition of \( G(\eta, \gamma) \) appearing in the proof of Lemma 9 in [1].

\[
G(\eta, \gamma) = \left\{ t < 1 - \eta; \sup_{t + \eta < s \leq 1} \int_{t}^{s} V_{s}(B_{u}^{\infty} - B_{t}^{\infty})du > \gamma \right\}.
\]

Since \( V_{+} \) has no discontinuities of the second kind, it is locally bounded and hence we can deduce, from the occupation time formula, that \( G(\eta, \gamma) \) is an open set.

On the other hand, \( \mu_{n} \) is a.s. supported on

\[
\left\{ t \leq 1 - \varepsilon_{n}; \sup_{t + \varepsilon_{n} < s \leq 1} \int_{t}^{s} V_{s}(B_{u}^{n} - B_{t}^{n})du \leq 0 \right\}.
\]

To deduce that \( \mu_{\infty}(G(\eta, \gamma)) = 0 \) and \( \mu_{\infty} \) is a.s. supported on \( K(V_{-}; V_{+}; B_{-}^{\infty}) \) by the argument in the proof of Lemma 9 in [1], we need only to prove the following:

\[
(2.16) \quad \text{For fixed } s \text{ and } t, \quad \int_{t}^{s} V_{s}(B_{u}^{n} - B_{t}^{n})du \to \int_{t}^{s} V_{s}(B_{u}^{\infty} - B_{t}^{\infty})du \quad \text{as } n \to \infty.
\]

To prove (2.16), let \( \varepsilon, \varepsilon' \) be arbitrary positive numbers and let

\[
R_{\infty}(\varepsilon', s) := \{ x \in \mathbb{R}; \exists u < s, |x - B_{u}^{\infty}| < 2\varepsilon' \}.
\]
Since $V_\epsilon$ has discontinuity only at the origin (when $\alpha = 0$), there exists $0 < \delta < \epsilon'$ such that for any $x, y \in \mathbb{R}^\infty(\epsilon', s)$ satisfying $|x - y| < \delta$ and $|x| > \epsilon'$, it holds $|V_\epsilon(x) - V_\epsilon(y)| < \epsilon'$.

We can make $\int_t^s 1(|B^\infty_u - B^\infty_t| \leq \epsilon')du$ arbitrarily small by taking $\epsilon'$ small, and hence $\int_t^s V_\epsilon(B^\infty_u - B^\infty_t)1(|B^\infty_u - B^\infty_t| \leq \epsilon')du$ is also small if $\|B^\infty - B^\infty\| < \epsilon'$, since $V_\epsilon$ is bounded on $\mathbb{R}^\infty(\epsilon', s)$.

For $u \in [t, s]$ satisfying $|B^\infty_u - B^\infty_t| > \epsilon'$, we have $|V_\epsilon(B^\infty_u - B^\infty_t) - V_\epsilon(B^\infty_u - B^\infty_t)| < \epsilon$ if $\|B^\infty - B^\infty\| < \delta/2$, which is satisfied for all large $\eta$.

We have thus proven (2.16).

Using Lemma 5 and the weak convergence we have

$$E[\mu^{-\infty}([0, 1])] = \int_0^1 dt t^{-\rho/2}C_3(V) > 0,$$

$$E[\mu^{\infty}([0, 1])] = \int_0^1 dt t^{-\rho/2}C_3(V)(1 - t)^{-\rho/2}C_3(V_\epsilon) > 0,$$

$$E[\mu^{\rho^{\infty}}([0, 1])] = \int_0^1 dt t^{-\rho/2}C_3(V)(1 - t)^{-(1 - \rho)/2}C_3(V_\epsilon(- \cdot)) > 0.$$

The positivity of these values is, through Frostman’s lemma, related to the positivity of $P(\dim K^-(V) \leq 1 - \rho/2)$ and its companions; The a.s. estimate from below follows by the scaling property of Brownian motion as in [1].

3. Proof of Theorems 3 and 2

In this section, $V$ is an strictly increasing function with $V(0) = 0$ and $a(x)$ is the rightmost element in $K'_\infty(V, x, X)$.

**Lemma 8.** (a) If $x_0 < x_1$ and there exists a triple $(t_0, t_1, t_2)$ such that

$t_0 \in K'_\infty(V, x_0, X) \setminus K'_\infty(V, x_1, X)$,

$t_1 \in K'_\infty(V, x_0, X) \cap K'_\infty(V, x_1, X)$,

$t_2 \in K'_\infty(V, x_1, X) \setminus K'_\infty(V, x_0, X)$,

then it holds $t_0 < t_1 < t_2$.

(b) The cardinality of $K'_\infty(V, x_0, X) \cap K'_\infty(V, x_1, X)$ are 0 or 1 for all $x_0 < x_1$. For all but countable $x$’s, the cardinality of $K'_\infty(V, x, X)$’s are 1.

(c) If $\int_t^s V(X_u - x)du$ is continuous in $t$ and $x$, then $a(x)$ is right continuous.

**Proof.** We first note that for $s < t$, $\int_t^s V(X_u - x)du$ is strictly decreasing in $x$.

(a) Assume $t_1 < t_0$. We then have $\int_t^{t_1} V(X_u - x_0)du = 0$ and $\int_{t_1}^t V(X_u - x_1)du > 0$, which is a contradiction. We can prove $t_1 < t_2$ by the same argument and time-reversal.
(b) If both \( t_0 \) and \( t_1 \) with \( t_0 < t_1 \) belong to \( K'_\infty(V, x_0, X) \cap K'_\infty(V, x_1, X) \) then we have
\[
\int_{t_0}^{t_1} V(X_u - x_0) \, du = 0 = \int_{t_0}^{t_1} V(X_u - x_1) \, du,
\]
which provides a contradiction.

By (a) and the first part of (b), we have for any \( x_0 < x_1 \), \( t_0 \in K'_\infty(V, x_0, X) \) and \( t_1 \in K'_\infty(V, x_1, X) \),
\[
t_1 - t_0 \geq \sum_{x \in \{x_0, x_1\}} \text{diam} K'_\infty(V, x, X).
\]
Hence at most countably many \( x \)'s admit diam \( K'_\infty(V, x, X) > 0 \).

(c) For any sequence \( t_n \to t_\infty \) and \( x_n \to x_\infty \) such that \( t_n \in K'_\infty(V, x_n, X) \),
we prove \( t_\infty \in K'_\infty(V, x_\infty, X) \).
If \( s \) is greater than \( t_\infty \), then eventually \( s > t_n \).
By the definition of \( t_n \in K'_\infty(V, x_n, X) \),
\[
0 \leq \int_{t_n}^{s} V(X_r - x_n) \, dr \to \int_{t_\infty}^{s} V(X_r - x_\infty) \, dr.
\]
If \( s < t_\infty \), \( \int_{t_n}^{t_\infty} V(X_r - x_\infty) \, dr = 0 \) by the same argument and this establishes \( t_\infty \in K'_\infty(V, x_\infty, X) \).

We have thus proven that \( a(x+) \equiv \lim_{\delta \downarrow 0} a(x + \delta) \) is in \( K'_\infty(V, x, X) \).
It follows from (a) that \( a(x+) \) dominates every element in \( K'_\infty(V, x, X) \) and hence \( a(x+) = a(x) \). \( \square \)

**Lemma 9.** If \( X \) is a Lévy process with no positive jumps which satisfies \( \lim_{t \to \infty} X_t = \infty \),
then for any \( x \geq 0 \), the two processes \( (X_t - x; 0 \leq t \leq a(x)) \) and \( (X - x) \circ \theta_{a(x)} \equiv (X_{a(x)+t} - x; t \geq 0) \) are independent.
Moreover, the law of the latter process does not depend on \( x \).

Proof. It can be proved by the same argument in Bertoin [3].
We define \( I^x_s = \int_0^s V(X_u - x) \, du \) and \( m^x_s = \inf_{0 \leq t \leq s} I^x_t \).
Then \( a(x) \) is the last exit time for the process \( (X_t - x, I^x_t - m^x_t) \) from the point \((0, 0)\),
which is finite almost surely. It can also be proved \( X_{a(x)} = x \). This enables us to apply the result by Getoor
on the last exit decomposition as in Bertoin [3]. \( \square \)

Proof of Theorem 3(a). To use Lemma 8(c), we first show that
\[
f(x, t) = \int_0^t V(X_u - x) \, du
\]
is jointly continuous in \( t \) and \( x \). Fix an \( \tau > 0 \) and \( \xi > 0 \).
The set
\[
R(\tau, \xi) = \{ X_t - x; 0 \leq t \leq \tau, |x| < \xi \}
\]
is bounded and so is its image by \( V(\cdot) \). This implies \( f(x, t) \) is uniformly continuous in \( t \)
on the rectangle \( \{0 \leq t \leq \tau, |x| < \xi\} \).

Single point sets are not essentially polar for a Lévy process with no positive jump diverging to \( +\infty \).
There exist local times \( L_t(\cdot) \), the sojourn time density, so that
\[
f(x, t) = \int_{R(\tau, \xi)} V(y)L_t(y+x) \, dy
\]
for \( 0 \leq t \leq \tau \) and \( |x| < \xi \). See e.g. Bertoin [2]. Let \( a \) and \( x' \) be two points such that \( |x| < \xi, |x'| < \xi \).
By making \( x' \) arbitrarily close to \( x \), the \( L^1 \)-norm of \( L_t(y+x') -
$L_t(y+x)$ with respect to $dy$ can be made arbitrarily small since $L_t(\cdot)$ is integrable. The boundedness of $V$ on $\mathbb{R}(\tau, \xi)$ enables us to conclude that $f(x, t)$ is continuous in $x$. Local uniform continuity in $t$ combined with this implies continuity in two variables. Hence right continuity of $\tilde{a}(x)$ follows from Lemma 8(c). Let $\tilde{a}(y)$ be the rightmost location of the overall minimum of $\int_0^t V(X_{\tilde{a}(x)+t}-x-y)ds$. By Lemma 8(a), we have $a(x+y) = a(x) + \tilde{a}(y)$ for $x \geq 0$ and $y > 0$. The rest can be done just like the proof of Theorem 1 in Bertoin [3].

Proof of Theorem 3(b). For any $0 \leq x < x_1$, the event $\{\hat{z}K'_{\infty}(V, x_1, X) \geq 2\}$ is independent of $(X_t-x; 0 \leq t \leq a(x))$ because it is the event that $\int_0^t V(X_{\tilde{a}(x)+t}-x_1)dt$ attains its overall minimum at least twice. Hence $P^X[\hat{z}K'_{\infty}(V, x, X) \geq 2]$ is the same value for all $x \geq 0$. If it is positive, then with a positive probability, $\{x \in [0, \infty); \hat{z}K'_{\infty}(V, x, X) \geq 2\}$ has positive mass with respect to the Lebesgue measure. This contradicts Lemma 8(b).

In the case where $x < 0$, we just condition on the event that $I_t^x$ hits 0. We resort to the strong Markov property at the first time $X_t = 0$ after $I_t^x = 0$ and finally use $P^X[\hat{z}K'_{\infty}(V, 0, X) = 1] = 1$.

Proof of Theorem 3(c). We follow the argument by Bertoin [3]. Independence is proven in Lemma 9. By (b), $a(0)$ is the unique location of the overall minimum of $\int_0^t V(X_u)du$. We define a new process $\hat{X}$ by $\hat{X}_t = -X_{g(0)-t-0}^{g(0)}$ for $0 \leq t \leq g(0)$, $\hat{X}_t = X_t$ for $t > g(0)$. It is known that $\hat{X}$ and $X$ have the same law. Since $V$ is an odd function,

$$\hat{L}_t = \int_0^t V(\hat{X}_u)du = \int_0^{g(0)} V(-X_u)du = I_{g(0)-t} - \int_0^{g(0)} V(X_u)du.$$

The unique location of the minimum of $\hat{L}_t$ is $g(0) - a(0)$ and has the same law as that of $a(0)$.

Proof of Theorem 3(d). This is proven in the same way as the final part of Theorem 1 in [3].

Now we restate Theorem 2 as the following Lemma. Note that $K'(V, B_2/2, B) \subset (0, 1)$ if $B_2 > 0$ and the following lemma implies $\dim \tilde{K}'(V; V) = 1/2$ a.s. on the event $\{B_1 > 0\}$.

**Lemma 10.** Let $a_1(x)$ be the rightmost element in $K'_1(V, x, B)$. It holds

$$\dim \tilde{K}'(V; V) = 1/2 \text{ a.s. on } \{\exists x, K'_1(V, x, B) \subset (0, 1)\} = \{\exists x, 0 < a_1(x) < 1\}, \text{ and } \tilde{K}'(V; V) \subset \{0, 1\} \text{ a.s. on } \{\forall x, K'_1(V, x, B) = \{0\} \text{ or } 1 \in K'_1(V, x, B)\} = \{\forall x, a_1(x) = 0 \text{ or } 1\}.$$
Proof. We first note that, by the continuity of \( B(t), B(a_t(x)) = x \) if \( 0 < a_t(x) < 1 \) and hence \( K'(V; V) \cup \{0, 1\} = \{a_t(x); \exists K'_t(V, x, B) = 1\} \). The symmetric difference of \( K'(V; V) \) and \( \{a_t(x); x \in \mathbb{R}\} \) is at most a countable set, which has no effect on the Hausdorff dimension.

We define the event for \( \xi \in \mathbb{R}, \eta > 0, x \in R \) and \( \varepsilon > 0, \)

\[
E(\xi, \eta, x, \varepsilon) = \{K'_t(V, x, B) \subset (0, 1), B_1 - x - \varepsilon \geq \xi, I^{1+\varepsilon}_1 - m^{1+\varepsilon}_1 \geq \eta\}.
\]

Let \( \tilde{B} \) be the process conditioned on \( E(\xi, \eta, x, \varepsilon) \). Since \( P[E(\xi, \eta, x, \varepsilon)] > 0 \), the law of \( \tilde{B} \) is absolutely continuous with respect to the law of a Brownian motion on \([0, 1]\), and hence to the law of a Brownian motion on \([0, 1]\) with unit drift.

If \( X \) is a Brownian motion on \([0, \infty)\) with unit drift independent of \( B \), then

\[
P\left[\forall t \geq 0, \eta + \int_0^t V(X_u + \xi)du > 0\right] > 0.
\]

Let \( \tilde{X} \) be the conditioned process on this event.

Now we define \( Z \) by \( Z_t = \tilde{B}_t \) for \( t \in [0, 1] \) and \( Z_t = \tilde{B}_t + \tilde{X}_{t-} \) for \( t > 1 \). The law of \( Z \) is absolutely continuous with respect to the law of a Brownian motion on \([0, \infty)\) with unit drift. For all \( x' < x + \varepsilon \), it follows from definition \( K'_t(V, x', B) \equiv K'_{\infty}(V, x', Z) \subset (0, 1) \) and hence \( 0 < a_t(x') = a(x') < 1 \).

By a general theory for subordinators, for every \( \varepsilon > 0 \), \( \dim\{a(x'); x < x' < x + \varepsilon\} = 1/2 \) a.s. on the event \( \{0 < a(x) < a(x + \varepsilon) < 1\} \). See e.g. Bertoin [2] Theorem III.15. Now we have a.s. on \( E(\xi, \eta, x, \varepsilon), \)

\[
\frac{1}{2} = \dim\{a(x') \geq x < x' < x + \varepsilon\} = \dim\{a_t(x') \geq x < x' < x + \varepsilon\}.
\]

Let

\[
F(\xi, \eta, x, \varepsilon) := \{a_t(x'); x < x' < x + \varepsilon, E(\xi, \eta, x, \varepsilon) \text{ occurs}\},
\]

a random subset which is nonempty only on the event \( E(\xi, \eta, x, \varepsilon) \). Since \( K'(V; V) \setminus \{0, 1\} \) is the same as a countable union of the random subsets of the form \( F(\xi, \eta, x, \varepsilon) \), the dichotomy that \( \dim(K'(V; V) \setminus \{0, 1\}) = 1/2 \) or \( \dim(K'(V; V) \setminus \{0, 1\}) = 0 \) holds.

Finally, if \( K'_t(V, x, B) \neq \{0\} \) and \( 1 \notin K'_t(V, x, B) \) for some \( x \), then there exists an \( x' \) such that \( K'_t(V, x', B) \subset (0, 1) \) by the continuity of \( \int_0^t V(B_s - x')ds \) in \( x' \).

4. Proof of Theorem 4

We quote a theorem in [4] and the proof of Theorem 4 is based on it. We fix \( \alpha = 0 \) \( c > 0 \) and write \( V(y) \) for \( V(\alpha, c; y) \). Throughout this section we set \( \nu = 1/(\alpha + 2) \) and let \( \rho \in (0, 1) \) be the unique solution of \( e^{\nu \pi \sin \pi (1 - \rho)} = \pi \nu \rho \) and \( \tilde{C}(x, y) \) be
defined for \( x \leq 0, y \in \mathbb{R} \) by

\[
(4.17) \quad \tilde{C}(x, y) = \Gamma(\nu)^{-1}|x|^{1-\nu+\nu \rho} \exp \left\{ \frac{-2\nu^2(y^+)^{1/\nu}}{|x|} \right\} \times \int_0^\infty dt e^{-t} \left( |x|t + 2\nu^2c^{-1}|y^+|^{1/\nu} \right)^{-\nu \rho} \left( |x|t + 2\nu^2(y^+)^{1/\nu} \right)^{-1+\nu-\nu \rho}.
\]

Now we have

**Theorem 11** ([4]). For \( \mu \geq 0 \), \( V = V(\alpha, c) \) there exists a constant \( C_d(\alpha, c) > 0 \) such that it holds

\[
(4.18) \quad \lim_{\sigma \to 0} \int_0^\infty dt \mu e^{-\mu t} \sigma^{-\mu} p(t, \sigma^{1/\nu} x, \sigma y; V) = C_d(\alpha, c)\mu^{\nu/2} \tilde{C}(x, y).
\]

Proof of Theorem 4. Since the integrand above, \( \sigma^{-\mu} p(t, \sigma^{1/\nu} x, \sigma y; V) \), is decreasing in \( t \),

\[
\lim_{\sigma \to 0} \sup_{x, y} \sigma^{-\mu} p(t, \sigma^{1/\nu} x, \sigma y; V)
\]

must be finite for every \( t > 0 \) and it is trivial that \( \sigma^{-\mu} p(t, \sigma^{1/\nu} x, \sigma y; V) < \sigma^{-\mu} < 1 \) for large \( \sigma \). Hence we know the overall supremum is finite, verifying (2.4), and we denote it by \( C_0(t, x, y; V) \), which is clearly monotone decreasing in \( t \) and inherits the scaling property from \( p(t, x, y; V) \):

\[
C_0(t, x, y; V) = \sigma^{-\mu} C_0(t, \sigma^{1/\nu} x, \sigma y; V) = \sigma^{-\mu} C_0(\sigma^{-2} t, x, y; V).
\]

It is sufficient to prove (2.6) when \( x < 0 \) and \( y < 0 \). We deduce from the scaling property and the monotonicity that

\[
C_0(t, x, y; V) = |x|^{\nu \rho} C_0 \left( t, -1, \frac{y}{|x|^\nu}; V \right)
\]

\[
\leq |x|^{\nu \rho} C_0 \left( t, (1) \wedge \frac{-|y|^{-1/\nu}}{|x|}, (1) \wedge \frac{y}{|x|^\nu}; V \right)
\]

\[
= C_0 \left( t, x \wedge (-|y|^{-1/\nu}), (-|x|^{\nu}) \wedge y; V \right)
\]

\[
= \left( |x|^{\nu \rho} \vee |y|^\rho \right) C_0(t, -1, -1; V).
\]

Combining this with \( C_0(t, x, y; V) = t^{-\rho/2} C_0(1, x, y; V) \), we obtain (2.6).

To prove (2.5), we note that the family \( \{ \sigma^{-\mu} p(t, \sigma^{1/\nu} x, \sigma y; V); \sigma > 0 \} \) of decreasing functions has an upper bound \( C_0(t, x, y; V) \), which satisfies

\[
\int_0^\infty dt \mu e^{-\mu t} C_0(t, x, y; V) < \text{const} \int_0^\infty dt \mu e^{-\mu t} t^{-\rho/2} < \infty.
\]
Given any sequence $\sigma_n \downarrow 0$, we can choose a subsequence $\sigma'_n$ such that the functions 
$(\sigma'_n)^{-\rho} p(t, (\sigma'_n)^{1/\nu} x, \sigma'_n y; V)$ converge pointwise on a dense subset of \{\(t > 0\)\} and that

$$
\int_0^\infty dt \mu e^{-\mu t} (\sigma'_n)^{-\rho} p(t, (\sigma'_n)^{1/\nu} x, \sigma'_n y; V) 
\to \int_0^\infty dt \mu e^{-\mu t} \lim_{n \to \infty} (\sigma'_n)^{-\rho} p(t, (\sigma'_n)^{1/\nu} x, \sigma'_n y; V).
$$

By uniqueness of the Laplace transform, we deduce, for any $t > 0$,

$$
\lim_{\sigma \to 0} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) = \frac{C_4(\alpha, c) \bar{C}(x, y) t^{-\rho/2}}{\Gamma(1 - \rho/2)}.
$$

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References


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