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A CLUSTER OF SETS OF EXCEPTIONAL TIMES OF LINEAR BROWNIAN MOTION

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1. Introduction and the main theorems

Aspandiiarov-Le Gall [1] studied the following random closed sets K^-, K and K': Let $(B_t; t \geq 0)$ be a linear standard Brownian motion starting at 0, and let

$$K^- = \left\{ t \in [0,1]; \int_s^t (B_u - B_t)du \leq 0 \quad \text{for every } s \in [0,t). \right\},$$
$$K = \left\{ t \in K^-; \int_t^s (B_u - B_t)du \leq 0 \quad \text{for every } s \in (t,1]. \right\},$$
$$K' = \left\{ t \in K^-; \int_t^s (B_u - B_t)du \geq 0 \quad \text{for every } s \in (t,1]. \right\}.$$

They computed the Hausdorff dimension of K^-, K and K'.

Theorem ([1]). *It holds* $\dim K^- = 3/4$, $\dim K = 1/2$ *and* $\dim K' \leq 1/2$ *almost surely. The set* K' *is possibly empty or* $\dim K' = 1/2$, *both with positive probability. The same statements hold if the weak inequalities in the definition of* K^-, K *and* K' *are replaced by the strict inequalities.*

In this paper, we consider a cluster of random sets having various dimension.

For $\alpha \geq 0$ and $c > 0$, we define the following functions $V(\alpha, c)$ increasing on $\mathbb{R}$:

$$V(\alpha,c;y) = y^\alpha \quad \text{for } y > 0; V(\alpha,c;0) = 0; V(\alpha,c;y) = -\frac{|y|^\alpha}{c} \quad \text{for } y < 0.$$

Let α, α_+, $\alpha_- \geq 0$, c, c_+, $c_- > 0$ and write V for $V(\alpha, c)$, $V_\pm$ for $V(\alpha_\pm, c_\pm)$. We define the random sets depending on the functions V, V_+ and V_-:

$$K^-(V) = \left\{ t \in [0,1]; \int_s^t V(B_u - B_t)du \leq 0 \quad \text{for every } s \in [0,t). \right\}, \tag{1.1}$$

$$K(V_-;V_+) = \left\{ t \in K^-(V_-); \int_t^s V_+(B_u - B_t)du \leq 0 \quad \text{for every } s \in (t,1]. \right\}, \tag{1.2}$$

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(1.3) $$K'(V_-;V_+) = \left\{ t \in K^-(V_-); \int_t^s V_+(B_u - B_t)du \geq 0 \quad \text{for every } s \in (t,1]. \right\}.$$

These sets consist of exceptional times in the sense that $P[t \in K^-(V)] = 0$ for every $t \in (0,1]$ and $P[t \in K(V_-;V_+)] = P[t \in K'(V_-;V_+)] = 0$ for every $t \in [0,1]$.

Theorem 1. *We define $\nu = 1/(2+\alpha)$, $\nu_- = 1/(2+\alpha_-)$ and $\nu_+ = 1/(2+\alpha_+)$. Let ρ, ρ_-, $\rho_+ \in (0,1)$ be the unique solutions of the equations*

$$c^{\nu} \sin \pi\nu(1-\rho) = \sin \pi\nu\rho,$$
$$c_-^{\nu_-} \sin \pi\nu_-(1-\rho_-) = \sin \pi\nu_-\rho_-,$$
$$c_+^{\nu_+} \sin \pi\nu_+(1-\rho_+) = \sin \pi\nu_+\rho_+$$

respectively.

(a) *For $V = V(\alpha, c)$, we have almost surely* $\dim K^-(V) = 1 - \rho/2$.
For $V_+ = V(\alpha_+, c_+)$ and $V_- = V(\alpha_-, c_-)$ we have (b) *and* (c):

(b) $\dim K(V_-;V_+) \leq 1 - (\rho_- + \rho_+)/2$ *almost surely and*

$$P\left[\dim K(V_-;V_+) \geq 1 - \frac{\rho_- + \rho_+}{2}\right] > 0.$$

(c) $\dim K'(V_-;V_+) \leq (1 - \rho_- + \rho_+)/2$ *almost surely and*

$$P\left[\dim K'(V_-;V_+) \geq \frac{1 - \rho_- + \rho_+}{2}\right] > 0.$$

The behavior of V, V_+ and V_- outside any neighborhood of the origin have no influence on the Hausdorff dimension; We could state the theorem in that fashon. The parameters ρ, ρ_-, $\rho_+ \in (0,1)$ in the statement of Theorem 1 are continuous and increasing in c, c_+, c_- and have the range $(0,1)$ since $\lim_{c\to 0} \rho = 0$ and $\lim_{c\to\infty} \rho = 1$. In fact, they are equal to the probability of some event related to the parameters in the theorem, see the remark 4 in [4].

Note that for fixed α, it holds $\rho = 1/2$ if $c = 1$. Hence the statements in the theorem in [1] for K^- and K' can be included in Theorem 1 since $K^- = K^-(V(1,1))$ and $K' = K'(V(1,1); V(1,1))$. The implication by Theorem 1 on K, however, is weaker than [1], since we have not obtained the almost sure estimate from below.

Let α, $\tilde{\alpha} \geq 0$ and c, $\tilde{c} > 0$. If $V = V(\alpha, c)$ and $\tilde{V} = V(\tilde{\alpha}, \tilde{c})$, then there is no inclusion in general between $K^-(V)$ and $K^-(\tilde{V})$. However it is easy to see, for each α, that $K^-(V(\alpha, c)) \subset K^-(V(\alpha, \tilde{c}))$ if $\tilde{c} < c$. Hence we obtain a family

$$\{K^-(V(\alpha, c)); c \in (0,1)\}$$

of decreasing random sets having strictly decreasing dimension.

The estimate in Theorem 1 for $\dim K^-(V)$ is exhaustive in the following sense: Let H be the set of times t when B_t attains its past-maximum:

$$H := \left\{ t \in [0,1]; B_t = \sup_{0 \le s \le t} B_s \right\}.$$

It is well known that $\dim H = 1/2$ a.s. Since $H \subset K^-(V(\alpha, c)) \subset [0,1]$, we have $1/2 \le \dim K^-(V) \le 1$. The range of $1 - \rho/2$ is exactly $(1/2, 1)$ and the trivial case $K^-(V) = H$ or $K^-(V) = [0,1]$ could be included if we allow $c = \infty$ or $c = 0$.

The estimate in Theorem 1 for $\dim K(V_-; V_+)$ is also exhaustive in the following sense: Let τ be the time when the maximum on $[0,1]$ of B is attained: $B_\tau \ge B_t$ for every $t \in [0,1]$. The inclusion $\{\tau\} \subset K(V_-; V_+) \subset [0,1]$ implies $0 \le \dim K(V_-; V_+) \le 1$ and the range of the value $1 - (\rho_- + \rho_+)/2$ is exactly $(0,1)$. The extreme cases could also be included here.

In the same sense as Aspandiiarov and LeGall [1] noted concerning K', $K'(V_-; V_+)$ can be interpreted as a weakened notion of the increasing points of Brownian motion and it is not straightforward to exhibit an element of $K'(V_-; V_+)$.

If both V_- and V_+ are $V(\alpha, c)$ then $(1 - \rho_- + \rho_+)/2 = 1/2$ irrespective of α and c. This motivates the next theorem, which could be a version of settlement of a conjecture at the end of [1]: $\dim K' = 1/2$ a.s. on the event $\{B_1 > 0\}$.

Theorem 2. *Let* $\mathcal{V} = \{V : \mathbb{R} \to \mathbb{R}; V(0) = 0, \ V \text{ is strictly increasing}\}$.

We define $\tilde{K}'(V;V)$ *for* $V \in \mathcal{V}$ *in the same way as* (1.3) *replacing the weak inequalities by strict inequalities in the definition of* $K'(V;V)$:

$$\tilde{K}'(V;V) = \left\{ t \in [0,1]; \int_s^t V(B_u - B_t)du < 0 \quad \text{for every } s \in [0,t), \right.$$
$$\left. \text{and } \int_t^s V(B_u - B_t)du > 0 \quad \text{for every } s \in (t,1]. \right\}.$$

Then we have $P[\dim \tilde{K}'(V;V) = 1/2] > 0$, $P[\tilde{K}'(V;V) \subset \{0,1\}] > 0$ *and*

$$P\left[\dim \tilde{K}'(V;V) = \frac{1}{2} \quad \text{or} \quad \tilde{K}'(V;V) \subset \{0,1\} \right] = 1.$$

REMARK 1. When the set $\tilde{K}'(V;V)$ consists of exceptional times, we have the dichotomy that $\dim \tilde{K}'(V;V) = 1/2$ if it is not empty.

The result of Theorem 2 is stronger than Theorem 1(c) for each strictly increasing functions$V(\alpha, c)$, i.e. $\alpha > 0$, while Theorem 2 says nothing about $V(0, c)$.

Theorem 2 is in fact a corollary of the following Theorem 3 due essentially to Bertoin [3].

Let $V \in \mathcal{V}$, $x \in \mathbb{R}$ and $X = (X(t); t \geq 0)$ be a cadlag path with $\liminf_{t\to\infty} X(t) = +\infty$. We define, inspired by Bertoin [3],

$$K'_\infty(V, x, X) = \left\{ t \in [0,\infty); \int_s^t V(X_u - x)du \leq 0 \quad \text{for every } s \in [0,t), \right.$$
$$\left. \text{and } \int_t^s V(X_u - x)du \geq 0 \quad \text{for every } s \in (t,\infty). \right\},$$
$$K'_1(V, x, X) = \left\{ t \in [0,1]; \int_s^t V(X_u - x)du \leq 0 \quad \text{for every } s \in [0,t), \right.$$
$$\left. \text{and } \int_t^s V(X_u - x)du \geq 0 \quad \text{for every } s \in (t,1]. \right\}.$$

It is then easy to see $\tilde{K}'(V;V)\bigcup\{0,1\} = \cup_{\sharp K'_1(V,x,B)=1} K'_1(V,x,B)$.

In other words, $K'_\infty(V,x,X)$ and $K'_1(V,x,X)$ consist of the locations of the overall minimum of the function $s \mapsto \int_0^s V(X_u - x)du$ on $[0,\infty)$ or $[0,1]$ respectively and $\tilde{K}'(V;V)$ is the collections of such t's that the function $s \mapsto \int_0^s V(B_u - B_t)du$ has the unique minimum at $s = t$.

The following results are proven in Bertoin [3] in the case where $V(y) \equiv y = V(1,1;y)$.

Theorem 3. *Let $V \in \mathcal{V}$ and X be a Lévy process with no positive jump such that $\liminf_{t\to\infty} X(t) = +\infty$ a.s. Let $a(x)$ be the rightmost element of $K'_\infty(V,x,X)$.*
(a) *$\{a(x) - a(0); x \geq 0\}$ and the process $T^X(x) := \inf\{t \geq 0; X_t \geq x\}$ have the same law.*
(b) *For every fixed $x \in \mathbb{R}$, $P^X[\sharp K'_\infty(V,x,X) = 1] = 1$.*
(c) *Let $g(0) = \sup\{t \geq 0; X(t) \leq 0\}$ be the last exit time from $(-\infty, 0]$. If $V \in \mathcal{V}$ satisfies $V(y) = -V(-y)$, then $a(0)$ and $g(0) - a(0)$ are independent and have the same law.*
(d) *If X is a Brownian motion with unit drift, then $\{a(x) - a(0);\ x \geq 0\}$ has the Lévy measure $(2\pi)^{-1/2} y^{-3/2} e^{-y/2} dy$ on $(0,\infty)$. If, moreover, $V \in \mathcal{V}$ satisfies $V(y) = -V(-y)$, then the density of the common law of $a(0)$ and $g(0) - a(0)$ is $2^{-1/4}\Gamma(1/4)^{-1} y^{-3/4} e^{-y/2} dy$ on $(0,\infty)$.*

REMARK 2. The statement (a) and the first sentence in (d) hold for nondecreasing V satisfying $V(0) = 0$. The second sentence in (d) was known to Jean Bertoin(private communication).

This paper is organized as follows: We prove Theorem 1 in Section 2 using Theorem 4, which contains an asymptotic estimate for some fluctuating additive functionals. Theorems 2 and 3 are proven in Section 3. We prove Theorem 4 in Section 4 using a theorem in [4].

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2. Proof of Theorem 1

The argument here mimics that of Aspandiiarov and Le Gall [1] line by line. We first state Theorem 4, an estimate for the distribution of the first hitting time of $\int_0^t V(B_u)du$, next define suitable approximations of $K^-(V)$, $K(V_-;V_+)$ and $K'(V_-;V_+)$ and obtain some preliminary estimates. From that point on, we only need the straightforward changes.

Theorem 4. *Let* $\alpha \geq 0$, $c > 0$, $V = V(\alpha, c)$, $\nu = 1/(2+\alpha)$ *and* $\rho \in (0,1)$ *be the solution of* $c^\nu \sin \pi\nu(1-\rho) = \sin \pi\nu\rho$. *We denote by* $p(t,x,y;V)$ *the probability* $P[\forall s \in [0,t], x + \int_0^s V(y+B_u)du \leq 0]$.

For any $t > 0$, $x < 0$, $y \in \mathbb{R}$ *and there exist constants* $C_0(t,x,y;V) > 0$, $C_1(\alpha,c) > 0$ *and* $\tilde{C}(x,y) > 0$ *such that it holds*

$$\sup_{\sigma>0} \sigma^{-\rho} p(t, \sigma^{1/\nu}x, \sigma y; V) = C_0(t,x,y;V), \tag{2.4}$$

$$\lim_{\sigma\to 0} \sigma^{-\rho} p(t, \sigma^{1/\nu}x, \sigma y; V) = C_1(\alpha,c) t^{-\rho/2} \tilde{C}(x,y), \tag{2.5}$$

Moreover it holds

$$C_0(t,x,y;V) \leq \mathrm{const}\, t^{-\rho/2}(|x|^{\nu\rho} \vee |y^-|^\rho), \tag{2.6}$$

DEFINITION. Let $\varepsilon \in [0,1/2]$, $a \in [0, 1-\varepsilon]$ and $b \in [\varepsilon, 1]$.
For V, V_+, $V_- \in \cup_{\alpha\geq 0, c>0}\{V(\alpha,c)\}$ we define

$$K^-_{\varepsilon,a}(V) = \left\{ t \in [a+\varepsilon, 1]; \int_s^t V(B_u - B_t)du \leq 0 \quad \text{for every } s \in [a, t-\varepsilon] \right\},$$

$$K^+_{\varepsilon,b}(V) = \left\{ t \in [0, b-\varepsilon]; \int_t^s V(B_u - B_t)du \leq 0 \quad \text{for every } s \in [t+\varepsilon, b] \right\},$$

$$K^*_{\varepsilon,b}(V) = \left\{ t \in [0, b-\varepsilon]; \int_t^s V(B_u - B_t)du \geq 0 \quad \text{for every } s \in [t+\varepsilon, b] \right\},$$

$$K_{\varepsilon,a,b}(V_-;V_+) = K^-_{\varepsilon,a}(V_-) \cap K^+_{\varepsilon,b}(V_+),$$

$$K'_{\varepsilon,a,b}(V_-;V_+) = K^-_{\varepsilon,a}(V_-) \cap K^*_{\varepsilon,b}(V_+).$$

We also define

$$K^-_\varepsilon(V) = K^-_{\varepsilon,0}(V), \qquad K^-(V) = K^-_0(V), \tag{2.7}$$

$$K_\varepsilon(V_-;V_+) = K_{\varepsilon,0,1}(V_-;V_+), \qquad K(V_-;V_+) = K_0(V_-;V_+), \tag{2.8}$$

$$K'_\varepsilon(V_-;V_+) = K'_{\varepsilon,0,1}(V_-;V_+), \qquad K'(V_-;V_+) = K'_0(V_-;V_+). \tag{2.9}$$

Lemma 5. *Let α, α_+, $\alpha_- \geq 0$, c, c_+, $c_- > 0$ and let ρ, ρ_+, ρ_- be defined in the statement of* Theorem 1.

(a) *For any $V = V(\alpha, c)$, $0 < \varepsilon < 1/2$ and $t > a$, it holds*

$$\left(\frac{t-a}{\varepsilon}\right)^{\rho/2} P[t \in K^-_{\varepsilon,a}(V)] < \text{const}.$$

There exists a constant $C_3(V) > 0$ such that it holds

$$P[t \in K^-_{\varepsilon,a}(V)] \sim C_3(V)\left(\frac{\varepsilon}{t-a}\right)^{\rho/2}$$

as $\varepsilon \searrow 0$ for every t.

(b) *For any $V_+ = V(\alpha_+, c_+)$, $V_- = V(\alpha_-, c_-)$, $0 < \varepsilon < 1/2$ and $t \in (a, b)$,*

$$\left(\frac{t-a}{\varepsilon}\right)^{\rho_-/2}\left(\frac{b-t}{\varepsilon}\right)^{\rho_+/2} P[t \in K_{\varepsilon,a,b}(V_-;V_+)] < \text{const},$$
$$\left(\frac{t-a}{\varepsilon}\right)^{\rho_-/2}\left(\frac{b-t}{\varepsilon}\right)^{(1-\rho_+)/2} P[t \in K'_{\varepsilon,a,b}(V_-;V_+)] < \text{const}.$$

We denote by $V_+(-\,\cdot)$ the function $y \mapsto V_+(-y)$. It holds as $\varepsilon \searrow 0$

$$P[t \in K_{\varepsilon,a,b}(V_-;V_+)] \sim C_3(V_-)C_3(V_+)\left(\frac{\varepsilon}{t-a}\right)^{\rho_-/2}\left(\frac{\varepsilon}{b-t}\right)^{\rho_+/2},$$
$$P[t \in K'_{\varepsilon,a,b}(V_-;V_+)] \sim C_3(V_-)C_3(V_+(-\,\cdot))\left(\frac{\varepsilon}{t-a}\right)^{\rho_-/2}\left(\frac{\varepsilon}{b-t}\right)^{(1-\rho_+)/2}$$

Proof. We only prove (a) since the statement (b) follows by time-reversal $\tilde{B}_s = B_{1-s}$ and by reflection $\tilde{B}_s = -B_s$.

Let $P^V_{(x,y)}$ be the law of the following two-dimensional diffusion $(X(t), Y(t))$:

$$Y(t) = y + B(t), \qquad X(t) = x + \int_0^t V(Y(s))ds.$$

By the strong Markov property,

$$P[t \in K^-_{\varepsilon,a}(V)] = E^V_{(0,0)}[p(t-a-\varepsilon, X(\varepsilon), Y(\varepsilon); V)]$$

Under $P^V_{(0,0)}$, the law of $(X(\varepsilon), Y(\varepsilon))$ is the same as that of $(\varepsilon^{1/2\nu}X(1), \varepsilon^{1/2}Y(1))$. By (2.4) and (2.6), we have for any $\varepsilon > 0$,

$$\begin{aligned}&\varepsilon^{-\rho/2}p(t-a-\varepsilon,\varepsilon^{1/2\nu}X(1),\varepsilon^{1/2}Y(1);V)\\&<\text{const}(t-a-\varepsilon)^{-\rho/2}\left(|X(1)^-|^{\nu\rho}\vee|Y(1)^-|^{\rho}\right)\\&<\text{const}(t-a)^{-\rho/2}\left(|X(1)^-|^{\nu\rho}\vee|Y(1)^-|^{\rho}\right).\end{aligned}$$

The quantity $((t-a)/\varepsilon)^{\rho/2}P[t\in K^-_{\varepsilon,a}(V)]$ is hence bounded. This bound also enables us to prove the second sentence of (a) with $C_3(V)=C_1(\alpha,c)E^V_{(0,0)}[\tilde{C}(X(1),Y(1))]$. □

Lemma 6. *We use the same notations as the previous lemma. It holds for any* $\varepsilon\in(0,1/2)$ *and* $0<s<t<1$,

$$P[\{s,t\}\subset K^-_{\varepsilon,a}(V)]\le\text{const}\,\frac{\varepsilon^{\rho}}{s^{\rho/2}(t-s)^{\rho/2}},\tag{2.10}$$

$$P[\{s,t\}\subset K_{\varepsilon,a,b}(V_-;V_+)]\le\frac{\text{const}\,\varepsilon^{\rho_-+\rho_+}}{s^{\rho_-/2}(t-s)^{(\rho_-+\rho_+)/2}(1-t)^{\rho_+/2}},\tag{2.11}$$

$$P[\{s,t\}\subset K'_{\varepsilon,a,b}(V_-;V_+)]\le\frac{\text{const}\,\varepsilon^{\rho_-+(1-\rho_+)}}{s^{\rho_-/2}(t-s)^{(\rho_-+1-\rho_+)/2}(1-t)^{(1-\rho_+)/2}}.\tag{2.12}$$

The constants here depend on α, α_+, α_- *and* c, c_+, c_-.

Proof. This can be done using Lemma 5. See the proof of Proposition 4 in [1]. □

Lemma 7. *Let* $\mathcal{F}_{a,b}$ *be the* σ*-field* $\sigma(B_u-B_a;u\in[a,b])$ *for* $0\le a<b\le1$.

For any $\alpha\ge0$, $c>0$ *and* $V=V(\alpha,c)$ *there exist* $\mathcal{F}_{a,b}$*-measurable random variables* $U_{a,b,-}$, $U_{a,b,+}$ *and* $U_{a,b,*}$ *such that*

$$P\left[K^-(V)\cap[a,b]\ne\emptyset\mid\mathcal{F}_{a,1}\right]\le(b-a)^{\rho/2}U_{a,b,-},\tag{2.13}$$

$$P\left[K^+(V)\cap[a,b]\ne\emptyset\mid\mathcal{F}_{0,b}\right]\le(b-a)^{\rho/2}U_{a,b,+},\tag{2.14}$$

$$P\left[K^*(V)\cap[a,b]\ne\emptyset\mid\mathcal{F}_{0,b}\right]\le(b-a)^{(1-\rho)/2}U_{a,b,*},\tag{2.15}$$

and $E_0[(U_{a,b,-})^2]\le\text{const}\,a^{-\rho}$, $E_0[(U_{a,b,+})^2]\le\text{const}(1-b)^{-\rho}$, $E_0[(U_{a,b,*})^2]\le\text{const}(1-b)^{-1+\rho}$. *The constants here depend on* α *and* c.

Proof. We prove (2.14) since (2.13), (2.15) and the corresponding moment estimates follow by time-reversal $\tilde{B}_s=B_{1-s}$ and by reflection $\tilde{B}_s=-B_s$.

Let $\eta_{a,b}$ be the amplitude of B_s on $[a,b]$. Note that V is increasing. By modifying the argument in the proof of Lemma 7 in [1], we can take

$$U_{a,b,+}=(b-a)^{-\rho/2}p(1-b,(b-a)V(-\eta_{a,b}),-\eta_{a,b};V).$$

The bound of the moment follows by (2.6) and by the fact that $\eta_{a,b}$ has the same law as $(b-a)^{1/2}\eta_{0,1}$. □

Proof of Theorem 1. The upper estimates for the Hausdorff dimension is obtained by the argument in the proof Proposition 6 in [1].

To obtain the lower estimates, we define the normalized Lebesgue measures: For any Borel subset F of $[0,1]$, let

$$\begin{aligned}\mu_\varepsilon^-(F) &= \varepsilon^{-\rho/2}|F\cap K_\varepsilon^-(V)|,\\ \mu_\varepsilon(F) &= \varepsilon^{-(\rho_-+\rho_+)/2}|F\cap K_\varepsilon(V_-;V_+)|,\\ \mu'_\varepsilon(F) &= \varepsilon^{-(\rho_-+1-\rho_+)/2}|F\cap K'_\varepsilon(V_-;V_+)|.\end{aligned}$$

We denote by $\mathcal{M}_f$ the Polish space of all finite measures on $[0,1]$ equipped with the topology of weak convergence, and by $C([0,1])$ the Banach space of all continuous map from $[0,1]$ to $\mathbb{R}$.

Let (ε_n) be a sequence strictly decreasing to 0. We define the random variables ζ^n taking values in $\mathcal{M}_f\times C([0,1])$ by $\zeta^n=(\mu_{\varepsilon_n},(B_t;0\le t\le 1))$. We define $\zeta^{-,n}$ and $\zeta^{\prime,n}$ in the same way using $\mu^-_{\varepsilon_n}$ and μ'_{ε_n}. The argument in [1] ensures that we may assume the sequence (ζ^n) is weakly convergent by extracting a subsequence. Skorohod's representation theorem says that there is a probalility space carrying a sequence of random variables $\overline{\zeta^n}=(\mu^n,(B^n_t;0\le t\le 1))$ and a random variable $\overline{\zeta^\infty}=(\mu^\infty,(B^\infty_t;0\le t\le 1))$ such that $\overline{\zeta^n}$ and ζ^n have the same law and $\overline{\zeta^n}$ converges to $\overline{\zeta^\infty}$ almost surely.

Let $K(V_-;V_+;B^\infty)$ be defined in the same way as $K(V_-;V_+)$ replacing B by B^∞. To prove that μ^∞ is a.s. supported on $K(V_-;V_+;B^\infty)$, we change the definition of $G(\eta,\gamma)$ appearing in the proof of Lemma 9 in [1].

$$G(\eta,\gamma)=\left\{t<1-\eta;\ \sup_{t+\eta<s\le 1}\int_t^s V_+(B^\infty_u-B^\infty_t)du>\gamma\right\}.$$

Since V_+ has no discontinuities of the second kind, it is locally bounded and hence we can deduce, from the occupation time formula, that $G(\eta,\gamma)$ is an open set.

On the other hand, μ^n is a.s. supported on

$$\left\{t\le 1-\varepsilon_n;\ \sup_{t+\varepsilon_n<s\le 1}\int_t^s V_+(B^n_u-B^n_t)du\le 0\right\}.$$

To deduce that $\mu^\infty(G(\eta,\gamma))=0$ and μ^∞ is a.s. supported on $K(V_-;V_+;B^\infty)$ by the argument in the proof of Lemma 9 in [1], we need only to prove the following:

(2.16) For fixed s and t, $\displaystyle\int_t^s V_+(B^n_u-B^n_t)du\to\int_t^s V_+(B^\infty_u-B^\infty_t)du$ as $n\to\infty$.

To prove (2.16), let ε, ε' be arbitrary positive numbers and let

$$R^\infty(\varepsilon',s):=\{x\in\mathbb{R};\exists u<s,|x-B^\infty_u|<2\varepsilon'\}.$$

Since V_+ has discontinuity only at the origin (when $\alpha = 0$), there exists $0 < \delta < \varepsilon'$ such that for any $x, y \in R^\infty(\varepsilon', s)$ satisfying $|x - y| < \delta$ and $|x| > \varepsilon'$, it holds $|V_+(x) - V_+(y)| < \varepsilon$.

We can make $\int_t^s 1_{\{|B_u^\infty - B_t^\infty| \le 3\varepsilon'\}} du$ arbitrarily small by taking ε' small, and hence $\int_t^s V_+(B_u^n - B_t^n) 1_{\{|B_u^n - B_t^n| \le \varepsilon'\}} du$ is also small if $\|B^n - B^\infty\| < \varepsilon'$, since V_+ is bounded on $R^\infty(\varepsilon', s)$.

For $u \in [t, s]$ satisfying $|B_u^\infty - B_t^\infty| > \varepsilon'$, we have $|V_+(B_u^n - B_t^n) - V_+(B_u^\infty - B_t^\infty)| < \varepsilon$ if $\|B^n - B^\infty\| < \delta/2$, which is satisfied for all large n.

We have thus proven (2.16).

Using Lemma 5 and the weak convergence we have

$$\begin{aligned} E[\mu^{-,\infty}([0,1])] &= \int_0^1 dt\, t^{-\rho/2} C_3(V) > 0, \\ E[\mu^{\infty}([0,1])] &= \int_0^1 dt\, t^{-\rho_-/2} C_3(V_-)(1-t)^{-\rho_+/2} C_3(V_+) > 0, \\ E[\mu^{\prime,\infty}([0,1])] &= \int_0^1 dt\, t^{-\rho_-/2} C_3(V_-)(1-t)^{-(1-\rho_+)/2} C_3(V_+(-\,\cdot)) > 0. \end{aligned}$$

The positivity of these values is, through Frostman's lemma, related to the positivity of $P[\dim K^-(V) \le 1 - \rho/2]$ and its companions; The a.s. estimate from below follows by the scaling property of Brownian motion as in [1]. □

3. Proof of Theorems 3 and 2

In this section, V is an strictly increasing function with $V(0) = 0$ and $a(x)$ is the rightmost element in $K'_\infty(V, x, X)$.

Lemma 8. (a) *If $x_0 < x_1$ and there exists a triple (t_0, t_1, t_2) such that*

$$\begin{aligned} t_0 &\in K'_\infty(V, x_0, X) \setminus K'_\infty(V, x_1, X), \\ t_1 &\in K'_\infty(V, x_0, X) \cap K'_\infty(V, x_1, X), \\ t_2 &\in K'_\infty(V, x_1, X) \setminus K'_\infty(V, x_0, X), \end{aligned}$$

then it holds $t_0 < t_1 < t_2$.
(b) *The cardinality of $K'_\infty(V, x_0, X) \cap K'_\infty(V, x_1, X)$ are 0 or 1 for all $x_0 < x_1$. For all but countable x's, the cardinality of $K'_\infty(V, x, X)$'s are 1.*
(c) *If $\int_0^t V(X_u - x) du$ is continuous in t and x, then $a(x)$ is right contnuous.*

Proof. We first note that for $s < t$, $\int_s^t V(X_u - x) du$ is strictly decreasing in x.

(a) Assume $t_1 < t_0$. We then have $\int_{t_1}^{t^0} V(X_u - x_0) du = 0$ and $\int_{t_1}^{t^0} V(X_u - x_1) du > 0$, which is a contradiction. We can prove $t_1 < t_2$ by the same argument and time-reversal.

(b) If both t_0 and t_1 with $t_0 < t_1$ belong to $K'_\infty(V, x_0, X) \cap K'_\infty(V, x_1, X)$ then we have $\int_{t_0}^{t_1} V(X_u - x_0)du = 0 = \int_{t_0}^{t_1} V(X_u - x_1)du$, which provides a contradiction.

By (a) and the first part of (b), we have for any $x_0 < x_1$, $t_0 \in K'_\infty(V, x_0, X)$ and $t_1 \in K'_\infty(V, x_1, X)$, $t_1 - t_0 \geq \sum_{x \in (x_0, x_1)} \operatorname{diam} K'_\infty(V, x, X)$. Hence at most countably many x's admit $\operatorname{diam} K'_\infty(V, x, X) > 0$.

(c) For any sequence $t_n \to t_\infty$ and $x_n \to x_\infty$ such that $t_n \in K'_\infty(V, x_n, X)$, we prove $t_\infty \in K'_\infty(V, x_\infty, X)$. If s is greater than t_∞, then eventually $s > t_n$. By the definition of $t_n \in K'_\infty(V, x_n, X)$,

$$0 \leq \int_{t_n}^{s} V(X_r - x_n)dr \to \int_{t_\infty}^{s} V(X_r - x_\infty)dr.$$

If $s < t_\infty$, $\int_s^{t_\infty} V(X_r - x_\infty)dr \leq 0$ by the same argument and this establishes $t_\infty \in K'_\infty(V, x_\infty, X)$.

We have thus proven that $a(x+) \equiv \lim_{\delta \searrow 0} a(x + \delta)$ is in $K'_\infty(V, x, X)$. It follows from (a) that $a(x+)$ dominates every element in $K'_\infty(V, x, X)$ and hence $a(x+) = a(x)$. $\square$

Lemma 9. *If X is a Lévy process with no positive jumps which satisfies $\lim_{t\to\infty} X_t = \infty$, then for any $x \geq 0$, the two processes $(X_t - x; 0 \leq t \leq a(x))$ and $(X - x) \circ \theta_{a(x)} \equiv (X_{a(x)+t} - x; t \geq 0)$ are independent. Moreover, the law of the latter process does not depend on x.*

Proof. It can be proved by the same argument in Bertoin [3].

We define $I_s^x = \int_0^s V(X_u - x)du$ and $m_s^x = \inf_{0 \leq t \leq s} I_t^x$. Then $a(x)$ is the last exit time for the process $(X_t - x, I_t^x - m_t^x)$ from the point $(0, 0)$, which is finite almost surely. It can also be proved $X_{a(x)} = x$. This enables us to apply the result by Getoor on the last exit decomposition as in Bertoin [3]. $\square$

Proof of Theorem 3(a). To use Lemma 8(c), we first show that $f(x, t) = \int_0^t V(X_u - x)du$ is jointly continuous in t and x. Fix an $\tau > 0$ and $\xi > 0$. The set

$$R(\tau, \xi) = \{X_t - x; 0 \leq t \leq \tau, |x| < \xi\}$$

is bounded and so is its image by $V(\cdot)$. This implies $f(x, t)$ is uniformly continuous in t on the rectangle $\{0 \leq t \leq \tau, |x| < \xi\}$.

Single point sets are not essentially polar for a Lévy process with no positive jump diverging to $+\infty$. There exist local times $L_t(\,\cdot\,)$, the sojourn time density, so that

$$f(x, t) = \int_{R(\tau,\xi)} V(y) L_t(y + x)dy$$

for $0 \leq t \leq \tau$ and $|x| < \xi$. See e.g. Bertoin [2]. Let a and x' be two points such that $|x| < \xi$, $|x'| < \xi$. By making x' arbitraily close to x, the $\mathcal{L}^1$-norm of $L_t(y + x') -$

$L_t(y+x)$ with respect to dy can be made arbitrarily small since $L_t(\cdot)$ is integrable. The boundedness of V on $R(\tau, \xi)$ enables us to conclude that $f(x,t)$ is continuous in x. Local uniform continuity in t combined with this implies continuity in two variables.

Hence right continuity of $a(x)$ follows from Lemma 8(c). Let $\tilde{a}(y)$ be the rightmost location of the overall minimum of $\int_0^t V(X_{a(x)+s} - x - y)ds$. By Lemma 8(a), we have $a(x+y) = a(x) + \tilde{a}(y)$ for $x \geq 0$ and $y > 0$. The rest can be done just like the proof of Theorem 1 in Bertoin [3]. □

Proof of Theorem 3(b). For any $0 \leq x < x_1$, the event $\{\sharp K'_\infty(V, x_1, X) \geq 2\}$ is independent of $(X_t - x; 0 \leq t \leq a(x))$ because it is the event that $\int_0^s V(X_{a(x)+t} - x_1)dt$ attains its overall minimum at least twice. Hence $P^X[\sharp K'_\infty(V, x, X) \geq 2]$ is the same value for all $x \geq 0$. If it is positive, then with a positive probability, $\{x \in [0,\infty); \sharp K'_\infty(V, x, X) \geq 2\}$ has positive mass with respect to the Lebesgue measure. This contradicts Lemma 8(b).

In the case where $x < 0$, we just condition on the event that I_t^x hits 0. We resort to the strong Markov property at the first time $X_t = 0$ after $I_t^x = 0$ and finally use $P^X[\sharp K'_\infty(V, 0, X) = 1] = 1$. □

Proof of Theorem 3(c). We follow the argument by Bertoin [3]. Independence is proven in Lemma 9. By (b), $a(0)$ is the unique location of the overall minimum of $\int_0^t V(X_s)ds$. We define a new process $\hat{X}$ by $\hat{X}_t = -X_{g(0)-t-0}$ for $0 \leq t \leq g(0)$, $\hat{X}_t = X_t$ for $t > g(0)$. It is known that $\hat{X}$ and X have the same law. Since V is an odd function,

$$\hat{I}_t = \int_0^t V(\hat{X}_u)du = \int_{g(0)-t}^{g(0)} V(-X_u)du = I_{g(0)-t} - \int_0^{g(0)} V(X_u)du.$$

The unique location of the minimum of $\hat{I}_t$ is $g(0) - a(0)$ and has the same law as that of $a(0)$. □

Proof of Theorem 3(d). This is proven in the same way as the final part of Theorem 1 in [3]. □

Now we restate Theorem 2 as the following Lemma. Note that $K'(V, B_1/2, B) \subset (0,1)$ if $B_1 > 0$ and the following lemma implies $\dim \tilde{K}'(V;V) = 1/2$ a.s. on the event $\{B_1 > 0\}$.

Lemma 10. *Let $a_1(x)$ be the rightmost element in $K'_1(V, x, B)$. It holds $\dim \tilde{K}'(V;V) = 1/2$ a.s. on $\{\exists x, K'_1(V, x, B) \subset (0,1)\} = \{\exists x, 0 < a_1(x) < 1\}$, and $\tilde{K}'(V;V) \subset \{0,1\}$ a.s. on $\{\forall x, K'_1(V, x, B) = \{0\}$ or $1 \in K'_1(V, x, B)\} = \{\forall x, a_1(x) = 0$ or $1\}$.*

Proof. We first note that, by the continuity of $B(t)$, $B(a_1(x)) = x$ if $0 < a_1(x) < 1$ and hence $\tilde{K}'(V;V) \cup \{0,1\} = \{a_1(x); \sharp K_1'(V,x,B) = 1\}$. The symmetric difference of $\tilde{K}'(V;V)$ and $\{a_1(x); x \in \mathbb{R}\}$ is at most a countable set, which has no effect on the Hausdorff dimension.

We define the event for $\xi \in \mathbb{R}$, $\eta > 0$, $x \in R$ and $\varepsilon > 0$,

$$E(\xi,\eta,x,\varepsilon) = \{K_1'(V,x,B) \subset (0,1), B_1 - x - \varepsilon \geq \xi, I_1^{x+\varepsilon} - m_1^{x+\varepsilon} \geq \eta\}.$$

Let $\tilde{B}$ be the process conditioned on $E(\xi,\eta,x,\varepsilon)$. Since $P[E(\xi,\eta,x,\varepsilon)] > 0$, the law of $\tilde{B}$ is absolutely continuous with respect to the law of a standard Brownian motion on $[0,1]$, and hence to the law of a Brownian motion on $[0,1]$ with unit drift.

If X is a Brownian motion on $[0,\infty)$ with unit drift independent of B, then

$$P\left[\forall t \geq 0, \eta + \int_0^t V(X_u + \xi)du > 0\right] > 0.$$

Let $\tilde{X}$ be the conditioned process on this event.

Now we define Z by $Z_t = \tilde{B}_t$ for $t \in [0,1]$ and $Z_t = \tilde{B}_1 + \tilde{X}_{t-1}$ for $t > 1$. The law of Z is absolutely continuous with respect to the law of a Brownian motion on $[0,\infty)$ with unit drift. For all $x' < x + \varepsilon$, it follows from definition $K_1'(V,x',\tilde{B}) \equiv K_\infty'(V,x',Z) \subset (0,1)$ and hence $0 < a_1(x') = a(x') < 1$.

By a general theory for subordinators, for every $\varepsilon > 0$, $\dim\{a(x'); x < x' < x + \varepsilon\} = 1/2$ a.s. on the event $\{0 < a(x) < a(x+\varepsilon) < 1\}$. See e.g. Bertoin [2] Theorem III.15. Now we have a.s. on $E(\xi,\eta,x,\varepsilon)$,

$$\frac{1}{2} = \dim\{a(x'); x < x' < x + \varepsilon\} = \dim\{a_1(x'); x < x' < x + \varepsilon\}.$$

Let

$$F(\xi,\eta,x,\varepsilon) := \{a_1(x'); x < x' < x + \varepsilon, E(\xi,\eta,x,\varepsilon) \text{ occurs}\},$$

a random subset which is nonempty only on the event $E(\xi,\eta,x,\varepsilon)$. Since $\tilde{K}'(V;V)\backslash\{0,1\}$ is the same as a countable union of the random subsets of the form $F(\xi,\eta,x,\varepsilon)$, the dichotomy that $\dim(\tilde{K}'(V;V)\backslash\{0,1\}) = 1/2$ or $\tilde{K}'(V;V)\backslash\{0,1\} = \emptyset$ holds.

Finally, if $K_1'(V,x,B) \neq \{0\}$ and $1 \notin K_1'(V,x,B)$ for some x, then there exists an x' such that $K_1'(V,x',B) \subset (0,1)$ by the continuity of $\int_0^t V(B_s - x')ds$ in x'. □

4. Proof of Theorem 4

We quote a theorem in [4] and the proof of Theorem 4 is based on it. We fix $\alpha \geq 0$ $c > 0$ and write $V(y)$ for $V(\alpha,c;y)$. Throughout this section we set $\nu = 1/(\alpha+2)$ and let $\rho \in (0,1)$ be the unique solution of $c^\nu \sin \pi\nu(1-\rho) = \sin \pi\nu\rho$ and $\tilde{C}(x,y)$ be defined for $x \le 0,\ y \in \mathbb{R}$ by

$$(4.17)\ \tilde{C}(x,y) = \Gamma(\nu)^{-1}|x|^{1-\nu+\nu\rho} \exp\left\{\frac{-2\nu^2(y^+)^{1/\nu}}{|x|}\right\} \times \int_0^\infty dt e^{-t}\left(|x|t + 2\nu^2 c^{-1}|y^-|^{1/\nu}\right)^{\nu\rho}\left(|x|t + 2\nu^2(y^+)^{1/\nu}\right)^{-1+\nu-\nu\rho}.$$

Now we have

Theorem 11 ([4]). *For $\mu \ge 0$, $V = V(\alpha, c)$ there exists a constant $C_4(\alpha, c) > 0$ such that it holds*

$$\lim_{\sigma\to 0}\int_0^\infty dt\mu e^{-\mu t}\sigma^{-\rho} p(t, \sigma^{1/\nu}x, \sigma y; V) = C_4(\alpha, c)\mu^{\rho/2}\tilde{C}(x,y). \tag{4.18}$$

Proof of Theorem 4. Since the integrand above, $\sigma^{-\rho}p(t, \sigma^{1/\nu}x, \sigma y; V)$, is decreasing in t,

$$\limsup_{\sigma\to 0}\sigma^{-\rho}p(t, \sigma^{1/\nu}x, \sigma y; V)$$

must be finite for every $t > 0$ and it is trivial that $\sigma^{-\rho}p(t, \sigma^{1/\nu}x, \sigma y; V) < \sigma^{-\rho} < 1$ for large σ. Hence we know the overall supremum is finite, verifying (2.4), and we denote it by $C_0(t, x, y; V)$, which is clearly monotone decreasing in t and inherites the scaling property from $p(t, x, y, V)$:

$$C_0(t,x,y;V) = \sigma^{-\rho}C_0(t, \sigma^{1/\nu}x, \sigma y; V) = \sigma^{-\rho}C_0(\sigma^{-2}t, x, y; V).$$

It is sufficient to prove (2.6) when $x < 0$ and $y < 0$. We deduce from the scaling property and the monotonicity that

$$\begin{aligned} C_0(t,x,y;V) &= |x|^{\nu\rho}C_0\left(t, -1, \frac{y}{|x|^\nu}; V\right) \\ &\le |x|^{\nu\rho}C_0\left(t, (-1)\wedge\frac{-|y|^{-1/\nu}}{|x|}, (-1)\wedge\frac{y}{|x|^\nu}; V\right) \\ &= C_0\left(t, x\wedge(-|y|^{-1/\nu}), (-|x|^\nu)\wedge y; V\right) \\ &= \left(|x|^{\nu\rho}\vee|y|^\rho\right)C_0(t,-1,-1;V). \end{aligned}$$

Combining this with $C_0(t,x,y;V) = t^{-\rho/2}C_0(1,x,y;V)$, we obtain (2.6).

To prove (2.5), we note that the family $\left\{\sigma^{-\rho}p(t, \sigma^{1/\nu}x, \sigma y; V); \sigma > 0\right\}$ of decreasing functions has an upper bound $C_0(t,x,y;V)$, which satisfies

$$\int_0^\infty dt\mu e^{-\mu t}C_0(t,x,y;V) < \text{const}\int_0^\infty dt\mu e^{-\mu t}t^{-\rho/2} < \infty.$$

Given any sequence $\sigma_n \searrow 0$, we can choose a subsequence σ'_n such that the functions $(\sigma'_n)^{-\rho} p(t, (\sigma'_n)^{1/\nu} x, \sigma'_n y; V)$ converge pointwise on a dense subset of $\{t > 0\}$ and that

$$\begin{aligned}&\int_0^\infty dt \mu e^{-\mu t} (\sigma'_n)^{-\rho} p(t, (\sigma'_n)^{1/\nu} x, \sigma'_n y; V)\\ &\quad \to \int_0^\infty dt \mu e^{-\mu t} \lim_{n\to\infty} (\sigma'_n)^{-\rho} p(t, (\sigma'_n)^{1/\nu} x, \sigma'_n y; V).\end{aligned}$$

By uniqueness of the Laplace transform, we deduce, for any $t > 0$,

$$\lim_{\sigma\to 0} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) = \frac{C_4(\alpha, c)\tilde{C}(x, y) t^{-\rho/2}}{\Gamma(1-\rho/2)}. \qquad \square$$

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