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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 48(3) P.845-P.856</td>
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<tr>
<td>Issue Date</td>
<td>2011-09</td>
</tr>
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<td>Text Version</td>
<td>publisher</td>
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<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/10932">https://doi.org/10.18910/10932</a></td>
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ASYMPTOTICS OF POLYBALANCED METRICS UNDER RELATIVE STABILITY CONSTRAINTS

TOSHIKI MABUCHI

(Received May 14, 2010, revised October 15, 2010)

Abstract

Under the assumption of asymptotic relative Chow-stability for polarized algebraic manifolds \((M, L)\), a series of weighted balanced metrics \(\omega_m, m \gg 1\), called polybalanced metrics, are obtained from complete linear systems \([L^m]\) on \(M\). Then the asymptotic behavior of the weights as \(m \to \infty\) will be studied.

1. Introduction

In this paper, we shall study relative Chow-stability (cf. [5]; see also [11]) for polarized algebraic manifolds \((M, L)\) from the viewpoints of the existence problem of extremal Kähler metrics. As balanced metrics are obtained from Chow-stability on polarized algebraic manifolds, our relative Chow-stability similarly provides us with a special type of weighted balanced metrics called polybalanced metrics. As a crucial step in the program of [7], we here study the asymptotic behavior of the weights for such polybalanced metrics.

By a polarized algebraic manifold \((M, L)\), we mean a pair of a connected projective algebraic manifold \(M\) and a very ample holomorphic line bundle \(L\) over \(M\). For a maximal connected linear algebraic subgroup \(G\) of the group \(\text{Aut}(M)\) of all holomorphic automorphisms of \(M\), let \(\mathfrak{g} := \text{Lie } G\) denote its Lie algebra. Since the infinitesimal \(\mathfrak{g}\)-action on \(M\) lifts to an infinitesimal bundle \(\mathfrak{g}\)-action on \(L\), by setting

\[ V_m := H^0(M, L^m), \quad m = 1, 2, \ldots, \]

we view \(\mathfrak{g}\) as a Lie subalgebra of \(\mathfrak{sl}(V_m)\). We now define a symmetric bilinear form \(\langle \cdot, \cdot \rangle_m\) on \(\mathfrak{sl}(V_m)\) by

\[ \langle X, Y \rangle_m = \text{Tr}(XY)/m^{n+2}, \quad X, Y \in \mathfrak{sl}(V_m), \]

where the asymptotic limit of \(\langle \cdot, \cdot \rangle_m\) as \(m \to \infty\) often plays an important role in the

2010 Mathematics Subject Classification. Primary 14L24; Secondary 53C55.

*Supported by JSPS Grant-in-Aid for Scientific Research (A) No. 20244005.
study of K-stability. In fact one can show
\[ \langle X, Y \rangle_m = O(1) \]
by using the equivariant Riemann–Roch formula, see [1] and [11]. Let \( T \) be an algebraic torus in \( \text{SL}(V_m) \) such that the corresponding Lie algebra \( t := \text{Lie } T \) satisfies
\[ t \subset g. \]
Then by the \( T \)-action on \( V_m \), we can write the vector space \( V_m \) as a direct sum of \( t \)-eigenspaces:
\[ V_m = \bigoplus_{k=1}^{v_m} V(\chi_k), \]
where \( V(\chi_k) := \{ v \in V_m; g \cdot v = \chi_k(g)v \text{ for all } g \in T \} \) for mutually distinct multiplicative characters \( \chi_k \in \text{Hom}(T, \mathbb{C}^\times), k = 1, 2, \ldots, v_m \).

To study \( V_m \), let \( \omega_m \) be a Kähler metric in the class \( c_1(L) \mathbb{R} \), and choose a Hermitian metric \( h_m \) for \( L \) such that \( \omega_m = c_1(L; h_m) \mathbb{R} \). We now endow \( V_m \) with the Hermitian \( L^2 \) inner product on \( V_m \) defined by
\[ (u, v)_{L^2} := \int_M (u, v)_{h_m} \omega_m^n, \quad u, v \in V_m, \]
where \( (u, v)_{h_m} \) denotes the pointwise Hermitian pairing of \( u, v \) in terms of \( h_m \). Then by this \( L^2 \) inner product, we have \( V(\chi_k) \perp V(\chi_{k'}), k \neq k' \). Put \( N_m := \dim V_m \) and \( n_k := \dim_{\mathbb{C}} V(\chi_k) \). For each \( k \), by choosing an orthonormal basis \( \{ \sigma_{k,i}; i = 1, 2, \ldots, n_k \} \) for \( V(\chi_k) \), we put
\[ B_{m,k}(\omega_m) := \sum_{i=1}^{n_k} |\sigma_{k,i}|_{h_m}^2, \]
where \( |u|_{h_m}^2 := (u, u)_{h_m} \) for each \( u \in V_m \). Then \( \omega_m \) is called a polybalanced metric, if there exist real constants \( \gamma_{m,k} > 0 \) such that
\[ (1.2) \quad B_{m}^\gamma(\omega_m) = \sum_{k=1}^{v_m} \gamma_{m,k} B_{m,k}(\omega_m) \]
is a constant function on \( M \). Here \( \gamma_{m,k} \) are called the weights of the polybalanced metric \( \omega_m \). On the other hand,
\[ B_{m}^\ast(\omega_m) := \sum_{k=1}^{v_m} B_{m,k}(\omega_m) \]
is called the \(m\)-th asymptotic Bergman kernel of \(\omega_m\). A smooth real-valued function \(f \in C^\infty(M,\mathbb{R})\) on the Kähler manifold \((M,\omega_m)\) is said to be Hamiltonian if there exists a holomorphic vector field \(X \in \mathfrak{g}\) on \(M\) such that \(i_X\omega_m = \sqrt{-1} \bar{\partial} f\). Put \(N_m := N_m/c_1(L)^n[M]\). In this paper, as the first step in [7], we shall show the following:

**Theorem A.** For a polarized algebraic manifold \((M, L)\) and an algebraic torus \(T\) as above, assume that \((M, L)\) is asymptotically Chow-stable relative to \(T\). Then for each \(m \gg 1\), there exists a polybalanced metric \(\omega_m\) in the class \(c_1(L)\) such that \(\gamma_{m,k} = 1 + O(1/m)\), i.e.,

\[
|\gamma_{m,k} - 1| \leq \frac{C_1}{m}, \quad k = 1, 2, \ldots, v_m; \quad m \gg 1,
\]

for some positive constant \(C_1\) independent of \(k\) and \(m\). Moreover, there exist uniformly \(C^0\)-bounded functions \(f_m \in C^\infty(M,\mathbb{R})\) on \(M\) such that

\[
B_m^*(\omega_m) = N_m^\prime + f_m m^{n-1} + O(m^{n-2})
\]

and that each \(f_m\) is a Hamiltonian function on \((M, \omega_m)\) satisfying \(i_{X_m}\omega_m = \sqrt{-1} \bar{\partial} f_m\) for some holomorphic vector field \(X_m \in \mathfrak{t}\) on \(M\).

In view of [8], this theorem and the result of Catlin–Lu–Tian–Yau–Zelditch ([3], [12], [13]) allow us to obtain an approach (cf. [7]) to an extremal Kähler version of Donaldson–Tian–Yau’s conjecture. On the other hand, as a corollary to Theorem A, we obtain the following:

**Corollary B.** Under the same assumption as in Theorem A, suppose further that the classical Futaki character \(F_1: \mathfrak{g} \rightarrow \mathbb{C}\) for \(M\) vanishes on \(\mathfrak{t}\). Then for each \(m \gg 1\), there exists a polybalanced metric \(\omega_m\) in the class \(c_1(L)\) such that \(\gamma_{m,k} = 1 + O(1/m^2)\).

In particular

\[
B_m^*(\omega_m) = N_m^\prime + O(m^{n-2}).
\]

2. Asymptotic relative Chow-stability

By the same notation as in the introduction, we consider the algebraic subgroup \(S_m\) of \(SL(V_m)\) defined by

\[
S_m := \prod_{k=1}^{v_m} SL(V(\chi_k)),
\]

where the action of each \(SL(V(\chi_k))\) on \(V_m\) fixes \(V(\chi_i)\) if \(i \neq k\). Then the centralizer \(H_m\) of \(S_m\) in \(SL(V_m)\) consists of all diagonal matrices in \(SL(V_m)\) acting on each
V(\chi_k) by constant scalar multiplication. Hence the centralizer \( Z_m(T) \) of \( T \) in \( \text{SL}(V_m) \) is \( H_m \cdot S_m \) with Lie algebra

\[ J_m(t) = h_m + s_m, \]

where \( s_m := \text{Lie } S_m \) and \( h_m := \text{Lie } H_m \). For the exponential map defined by \( h_m \ni X \mapsto \exp(2\pi \sqrt{-1}X) \in H_m \), let \( (h_m)_\mathbb{Z} \) denote its kernel. Regarding \( (h_m)_\mathbb{R} := (h_m)_\mathbb{Z} \otimes \mathbb{Z} \mathbb{R} \) as a subspace of \( h_m \), we have a real structure on \( h_m \), i.e., an involution

\[ h_m \ni X \mapsto \bar{X} \in h_m \]

defined as the associated complex conjugate of \( h_m \) fixing \( (h_m)_\mathbb{R} \). We then have a Hermitian metric \((\quad,\quad)_m\) on \( h_m \) by setting

\[
(X, Y)_m = \langle X, \bar{Y} \rangle_m, \quad X, Y \in h_m. 
\]

For the orthogonal complement \( t^\perp \) of \( t \) in \( h_m \) in terms of this Hermitian metric, let \( T^\perp \) denote the corresponding algebraic torus in \( H_m \). We now define an algebraic subgroup \( G_m \) of \( Z_m(T) \) by

\[
G_m := T^\perp \cdot S_m. 
\]

For the \( T \)-equivariant Kodaira embedding \( \Phi_m : M \hookrightarrow \mathbb{P}^s(V_m) \) associated to the complete linear system \([L^m]\) on \( M \), let \( d(m) \) denote the degree of the image \( \Phi_m(M) \) in the projective space \( \mathbb{P}^s(V_m) \). For the dual space \( W^*_m \) of \( W_m := S^{d(m)}(V_m)^{\otimes n+1} \), we have the Chow form

\[ 0 \neq \tilde{M}_m \in W^*_m \]

for the irreducible reduced algebraic cycle \( \Phi_m(M) \) on \( \mathbb{P}^s(V_m) \), so that the corresponding element \([\tilde{M}_m]\) in \( \mathbb{P}^s(W_m) \) is the Chow point for the cycle \( \Phi_m(M) \). Consider the natural action of \( \text{SL}(V_m) \) on \( W^*_m \) induced by the action of \( \text{SL}(V_m) \) on \( V_m \).

**Definition 2.3.** (1) \((M, L^m)\) is said to be **Chow-stable relative to** \( T \) if the orbit \( G_m \cdot \tilde{M}_m \) is closed in \( W^*_m \).

(2) \((M, L)\) is said to be **asymptotically Chow-stable relative to** \( T \) if \((M, L^m)\) is Chow-stable relative to \( T \) for each integer \( m \gg 1 \).

### 3. Relative Chow-stability for each fixed \( m \)

In this section, we consider a polarized algebraic manifold \((M, L)\) under the assumption that \((M, L^m)\) is Chow-stable relative to \( T \) for a fixed positive integer \( m \). Then we shall show that a polybalanced metric \( c_{om} \) exists in the class \( c_1(L)_\mathbb{R} \).
The space \( \Lambda_m := \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_v) \in \mathbb{C}^v; \sum_{k=1}^{v} n_k \lambda_k = 0 \} \) and the Lie algebra \( \mathfrak{h}_m \) are identified by an isomorphism

\[
\Lambda_m \cong \mathfrak{h}_m, \quad \lambda \leftrightarrow X_{\lambda},
\]

with \( (\mathfrak{h}_m)_{\mathbb{R}} \) corresponding to the set \( (\Lambda_m)_{\mathbb{R}} \) of the real points in \( \Lambda_m \), where \( X_{\lambda} \) is the endomorphism of \( V_m \) defined by

\[
X_{\lambda} := \bigoplus_{k=1}^{v} \lambda_k \text{id}_{V(\chi_k)} \in \bigoplus_{k=1}^{v} \text{End}(V(\chi_k)) \quad (\subset \text{End}(V_m)).
\]

In terms of the identification (3.1), we can write the Hermitian metric \( \langle \, , \rangle_m \) on \( \mathfrak{h}_m \) in (2.1) in the form

\[
\langle \lambda, \mu \rangle_m := \sum_{k=1}^{v} \frac{n_k \lambda_k \mu_k}{m^{n+2}},
\]

where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_v) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_v) \) are in \( \mathbb{C}^v \). By the identification (3.1), corresponding to the decomposition \( \mathfrak{h}_m = t \oplus t^\perp \), we have the orthogonal direct sum

\[
\Lambda_m = \Lambda(t) \oplus \Lambda(t^\perp),
\]

where \( \Lambda(t) \) and \( \Lambda(t^\perp) \) are the subspace of \( \Lambda_m \) associated to \( t \) and \( t^\perp \), respectively. Take a Hermitian metric \( \rho_k \) on \( V(\chi_k) \), and for the metric

\[
\rho := \bigoplus_{k=1}^{v} \rho_k
\]

on \( V_m \), we see that \( V(\chi_k) \perp V(\chi_{k'}) \) whenever \( k \neq k' \). By choosing an orthonormal basis \( \{ s_{k,i}; i = 1, 2, \ldots, n_k \} \) for the Hermitian vector space \( (V(\chi_k), \rho_k) \), we now set

\[
j(k, i) := i + \sum_{l=1}^{k-1} n_l, \quad i = 1, 2, \ldots, n_k; \quad k = 1, 2, \ldots, v_m,
\]

where the right-hand side denotes \( i \) in the special case \( k = 1 \). By writing \( s_{k,i} \) as \( s_{j(k,i)} \), we have an orthonormal basis

\[
\mathcal{S} := \{ s_1, s_2, \ldots, s_{N_m} \}
\]

for \( (V_m, \rho) \). By this basis, the vector space \( V_m \) and the algebraic group \( \text{SL}(V_m) \) are identified with \( \mathbb{C}^{N_m} = \{ (z_1, \ldots, z_{N_m}) \} \) and \( \text{SL}(N_m, \mathbb{C}) \), respectively. In terms of \( \mathcal{S} \), the Kodaira embedding \( \Phi_m \) is given by

\[
\Phi_m(x) := (s_1(x) : \cdots : s_{N_m}(x)), \quad x \in M.
\]
Consider the associated Chow norm \( W_m \ni \xi \mapsto \|\xi\|_{CH(p)} \in \mathbb{R}_{\geq 0} \) as in Zhang [14] (see also [4]). Then by the closedness of \( G_m \cdot \hat{M}_m \) in \( W_m \) (cf. (2.2) and Definition 2.3), the Chow norm on the orbit \( G_m \cdot \hat{M}_m \) takes its minimum at \( g_m \cdot \hat{M}_m \) for some \( g_m \in G_m \).

Note that, by complexifying respectively, we consider the diagonal matrices \( \Sigma \). Then we shall now identify \( \Sigma \) with the reductive algebraic group \( S_m \). For each \( \kappa \in K_m \) and each diagonal matrix \( \Delta \) in \( \mathfrak{sl}(N_m, \mathbb{C}) \), we put

\[
e(\kappa, \Delta) := \exp\{\text{Ad}(\kappa)\Delta\}.
\]

Then \( g_m \) is written as \( \kappa_1 \cdot e(\kappa_0, D) \) for some \( \kappa_0, \kappa_1 \in K_m \) and a diagonal matrix \( D = (d_j)_{1 \leq j \leq N} \) in \( \mathfrak{sl}(N_m, \mathbb{C}) \) with the \( j \)-th diagonal element \( d_j \). Put \( g_m' := e(\kappa_0, D). \) In view of \( \|g_m' \cdot \hat{M}_m\|_{CH(p)} = \|g_m \cdot \hat{M}_m\|_{CH(p)} \) (cf. [10], Proposition 4.1), we obtain

\[
(3.3) \quad \|g_m' \cdot \hat{M}_m\|_{CH(p)} \leq \|e(\kappa_0, t(X_\lambda + A)) \cdot g_m' \cdot \hat{M}_m\|_{CH(p)}, \quad t \in \mathbb{C},
\]

for all \( \lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_{n_m}\} \subset \Lambda(t^+) \) and all diagonal matrices \( A = (a_j)_{1 \leq j \leq N} \) in \( \mathfrak{sl}_m \), where each \( a_j \) denotes the \( j \)-th diagonal element of \( A \). We now write \( a_{j(k,i)} \) as \( a_{k,i} \) for simplicity. Put

\[
s'_{j} := \kappa_0^{-1} \cdot s_j, \quad b_{k,i} := \lambda_k + a_{k,i}, \quad c_{k,i} := \exp d_{j(k,i)}.
\]

Then we shall now identify \( V_m \) with \( \mathbb{C}^{N_m} = \{(z_1', \ldots, z_{N_m}')\} \) by the orthonormal basis \( \{s_1', s_2', \ldots, s_{N_m}'\} \) for \( V_m \). In view (3.2), we rewrite \( s'_j \) as \( s'_{k,i} \) and \( \zeta_j' \) as \( \zeta'_{k,i} \), respectively by

\[
s'_{k,i} := s'_{j(k,i)}, \quad \zeta'_{k,i} := \zeta'_{j(k,i)},
\]

where \( k = 1, 2, \ldots, N \) and \( i = 1, 2, \ldots, n_k \). By writing \( b_{k,i}, c_{k,i} \) also as \( b_{j(k,i)}, c_{j(k,i)} \), respectively, we consider the diagonal matrices \( B \) and \( C \) of order \( N_m \) with the \( j \)-th diagonal elements \( b_j \) and \( c_j \), respectively. Note that the right-hand side of (3.3) is

\[
\|e(tB) \cdot C \cdot \kappa_0^{-1} \cdot \hat{M}_m\|_{CH(p)},
\]

and its derivative at \( t = 0 \) vanishes by virtue of the inequality (3.3). Hence, by setting \( \Theta := (\sqrt{-1}/2\pi) \partial \bar{\partial} \log (\sum_{k=1}^{N} \sum_{i=1}^{n_k} |c_{k,i}z_{k,i}'|^2) \), we obtain the equality (see for instance (4.4) in [4])

\[
(3.4) \quad \int_M \frac{\sum_{k=1}^{N} \sum_{i=1}^{n_k} b_{k,i} |c_{k,i}z_{k,i}'|^2}{\sum_{k=1}^{N} \sum_{i=1}^{n_k} |c_{k,i}z_{k,i}'|^2} \Phi_m^\sigma(\Theta^\sigma) = 0
\]
for all \( \lambda \in \Lambda(t^k) \) and all diagonal matrices \( A \) in the Lie algebra \( s_m \), where \( \Phi'_m: M \hookrightarrow \mathbb{P}^*(V_m) \) is the Kodaira embedding of \( M \) by \( S' \) which sends each \( x \in M \) to 
\[ (s'_1(x): s'_2(x): \cdots : s'_{N_m}(x)) \in \mathbb{P}^*(V_m). \]
Here we regard 
\[ s'_k, i = \Phi'^* z'_{k, i}. \]
Let \( k_0 \in \{1, 2, \ldots, v_m\} \) and let \( i_1, i_2 \in \{1, 2, \ldots, n_k\} \) with \( i_1 \neq i_2 \). Using Kronecker’s delta, we first specify the real diagonal matrix \( B \) by 
\[ \lambda_k = 0 \quad \text{and} \quad a_{k, i} = \delta_{k_0 i} (\delta_{i_1 i} - \delta_{ii_2}), \]
where \( k = 1, 2, \ldots, v_m; \ i = 1, 2, \ldots, n_k \). By (3.4) applied to this \( B \), and let \( (i_1, i_2) \) run through the set of all pairs of two distinct integers in \( \{1, 2, \ldots, n_k\} \), where positive integer \( k_0 \) varies from \( 1 \) to \( v_m \). Then there exists a positive constant \( \beta_k > 0 \) independent of the choice of \( i \) in \( \{1, 2, \ldots, n_k\} \) such that, for all \( i \),
\[ \int_M \frac{|s'_{k, i} s_{k, i} |^2}{\sum_{k=1}^{n_k} \sum_{j=1}^{n_k} |s_{k, i} s_{k, j} |^2} \Phi'^* (\Theta^k) = \beta_k, \quad k = 1, 2, \ldots, v_m. \]
Let \( k_0, i_1, i_2 \) be as above, and let \( \kappa_2 \) be the element in \( K_m \) such that
\[ \kappa_2 z'_{k_0, i_1} = \frac{1}{\sqrt{2}} (z'_1 - z'_{k_0, i_2}), \quad \kappa_2 z'_{k_0, i_2} = \frac{1}{\sqrt{2}} (z'_1 + z'_{k_0, i_2}) \]
and that \( \kappa_2 \) fixes all other \( z_{k_0, i} \)'s. Let \( \kappa_3 \) be the element in \( K_m \) such that
\[ \kappa_3 z'_{k_0, i_1} = \frac{1}{\sqrt{2}} (z'_1 + \sqrt{-1} z'_{k_0, i_2}), \quad \kappa_3 z'_{k_0, i_2} = \frac{1}{\sqrt{2}} (\sqrt{-1} z'_1 + z'_{k_0, i_2}) \]
and that \( \kappa_3 \) fixes all other \( z'_{k_0, i} \)'s. Now
\[ \| \kappa_2 g'_m \cdot \tilde{M}_m \|_{CH(\rho)} = \| \kappa_3 g'_m \cdot \tilde{M}_m \|_{CH(\rho)} = \| g'_m \cdot \tilde{M}_m \|_{CH(\rho)}, \]
and note that
\[ 2z'_{k_0, i_1} z'_{k_0, i_2} = (|\kappa_2 z'_{k_0, i_1}|^2 - |\kappa_2 z'_{k_0, i_2}|^2) - \sqrt{-1} (|\kappa_3 z'_{k_0, i_1}|^2 - |\kappa_3 z'_{k_0, i_2}|^2). \]
Hence replacing \( g'_m \) by \( \kappa_\alpha g'_m, \ \alpha = 2, 3 \), in (3.3), we obtain the case \( k' = k'' \) of the following by an argument as in deriving (3.5) from (3.3):
\[ \int_M \frac{s'_{k, i} s'_{k', i}^*}{\sum_{k=1}^{n_k} \sum_{i=1}^{n_k} |s_{k, i} s_{k, i} |^2} \Phi'^* (\Theta^k) = 0, \quad \text{if} \quad (k', i') \neq (k'', i''). \]
Here (3.6) holds easily for \(k' \neq k''\), since for every element \(g\) of the maximal compact subgroup of \(T\), we have:

\[
\text{L.H.S. of (3.6)} = \int_M g^n \left\{ \sum_{k=1}^{n_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2 \Phi^m_n(\Theta^n) \right\} = \frac{\chi_{k'}(g)}{\chi_{k''}(g)} \int_M \sum_{k=1}^{n_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2 \Phi^m_n(\Theta^n).
\]

Put \(\beta := (\beta_1, \beta_2, \ldots, \beta_{n_m}) \in \mathbb{R}^{n_m}\) and \(\beta_0 := \left(\sum_{k=1}^{n_m} n_k \beta_k\right)/N_m\), where \(\beta_k\) is given in (3.5). In view of \(N_m = \sum_{k=1}^{n_m} n_k\), by setting \(\beta_{k-} := \beta_k - \beta_0\), we have

\[
\beta := (\beta_1, \beta_2, \ldots, \beta_{n_m}) \in \Lambda_m.
\]

Next for each \(\lambda \in \Lambda(t^1)\), by setting \(a_{k,i} = 0\) for all \((k, i)\), the equality (3.4) above implies \(0 = \sum_{k=1}^{n_m} (n_k \lambda_k) \beta_k\). From this together with the equality \(\sum_{k=1}^{n_m} n_k \lambda_k = 0\), we obtain \((\lambda, \beta)_m = 0\), i.e.,

\[
(3.7) \quad \beta \in \Lambda(t).
\]

We now define a Hermitian metric \(h_{FS}\) (cf. [14]) for \(L^m\) as follows. Let \(u\) be a local section for \(L^m\). Then\(^1\)

\[
(3.8) \quad |u|_{h_{FS}}^2 := \frac{|u|^2}{\sum_{k=1}^{n_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2}.
\]

For the Hermitian metric \(h_m := (h_{FS})^{1/m}\) for \(L\), we consider the associated Kähler metric \(\omega_m := c_1(L; \Phi^n_m)\) on \(M\). In view of (3.5),

\[
(3.9) \quad \beta_0 = \frac{\sum_{k=1}^{n_m} n_k \beta_k}{N_m} = N_m^{-1} \int_M \Phi^m_n(\Theta^n)
\]

\[
= N_m^{-1} m^n c_1(L)^n[M] = n! \left\{ 1 + O \left( \frac{1}{m} \right) \right\}.
\]

Then for \(\gamma_{m,k} := \beta_k/\beta_0\) and \(\sigma_{k,i} := c_{k,i} s'_{k,i}(m^n \beta_k)^{-1/2}\), we have

\[
(3.10) \quad \sum_{k=1}^{n_m} \sum_{i=1}^{n_k} \gamma_{m,k} |\sigma_{k,i}|^2_{h_m} = \sum_{k=1}^{n_m} \sum_{i=1}^{n_k} \gamma_{m,k} |\sigma_{k,i}|^2_{h_{FS}} = \frac{m^n}{\beta_0} \sum_{k=1}^{n_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2_{h_{FS}} = \frac{m^n}{\beta_0}.
\]

\(^1\)In view of (3.8), there is some error in [4]. Actually, for the numerator of (5.7) in the paper [4], please read \((N_m + 1)|\lambda|^2\).
By operating \((\sqrt{-1}/2\pi) \bar{\partial} \partial \log\) on (3.8), we obtain

\[
\Phi_{m}^{\mu}(\Theta) = c_{1}(L^{m}; h_{\text{FS}})_{\mathbb{R}} = mc_{1}(L; h_{m})_{\mathbb{R}} = m\omega_{m}.
\]

Then in terms of the Hermitian \(L^{2}\) inner product (1.1), we see from (3.5), (3.6), (3.8) and (3.11) that \(\{\sigma_{k,i}; k = 1, \ldots, v_{m}, i = 1, \ldots, n_{k}\}\) is an orthonormal basis for \(V_{m}\). Moreover, (3.10) is rewritten as

\[
B_{m}^{*}(\omega_{m}) = \frac{m^{n}}{\beta_{0}},
\]

where \(B_{m}^{*}(\omega_{m})\) is as in (1.2). Hence \(\omega_{m}\) is a polybalanced metric, and the proof of Theorem A is reduced to showing (1.3) and (1.4). By summing up, we obtain

**Theorem C.** If \((M, L^{m})\) is Chow-stable relative to \(T\) for a positive integer \(m\), then the Kähler class \(c_{1}(L)_{\mathbb{R}}\) admits a polybalanced metric \(\omega_{m}\) with the weights \(\gamma_{m,k}\) as above.

## 4. The asymptotic behavior of the weights \(\gamma_{m,k}\)

The purpose of this section is to prove (1.3). If \(\beta = 0\), then we are done. Hence, we may assume that \(\beta \neq 0\). Consider the sphere

\[
\sum := \{X \in t_{\mathbb{R}}; \langle X, X \rangle_{0} = 1\}
\]

in \(t_{\mathbb{R}} := t \cap (h_{m})_{\mathbb{R}}\), where \(\langle , \rangle_{0}\) denotes the positive definite symmetric bilinear form on \(g\) as in [2]. Since all components \(\beta_{k}\) of \(\beta\) are real, we see from (3.7) that, in view of (3.1), \(\lambda := r_{m}\beta\) satisfies

\[
X_{k} \in \sum
\]

for some positive real number \(r_{m}\). Hence by writing \(\lambda = (\lambda_{1}, \lambda_{2}, \ldots, \lambda_{v_{m}})\), we obtain positive constants \(C_{2}, C_{3}\) independent of \(k\) and \(m\) such that (see for instance [5], Lemma 2.6)

\[
-C_{2}m \leq \lambda_{k} \leq C_{3}m.
\]

Put \(g(t) := \exp(tX_{k})\) and \(\gamma(t) := \log\|g(t) \cdot C \cdot \kappa_{0}^{-1} \cdot \tilde{M}_{m}\|_{H_{0}}\), \(t \in \mathbb{R}\), by using the notation in Section 3. Since \(g(t)\) commutes with \(C \cdot \kappa_{0}^{-1}\), we see that \(g(t)\) defines a holomorphic automorphism of \(\Phi_{m}^{\mu}(M)\). In view of Theorem 4.5 in [4], it follows from Remark 4.6 in [4] that (cf. [14])

\[
\dot{\gamma}(t) = \dot{\gamma}(0) = \sum_{k=1}^{v_{m}} n_{k}\lambda_{k}\beta_{k}, \quad -\infty < t < +\infty.
\]
Consider the classical Futaki invariant $F_1(X_\lambda)$ associated to the holomorphic vector field $X_\lambda$ on $(M, L)$. Since $M$ is smooth, this coincides with the corresponding Donaldson–Futaki’s invariant for test configurations. Then by applying Lemma 4.8 in [6] to the product configuration of $(M, L)$ associated to the one-parameter group generated by $X_\lambda$ on the central fiber, we obtain (see also [1], [11])

\[
\lim_{t \to -\infty} \dot{\gamma}(t) = C_4 \left( F_1(X_\lambda) + O \left( \frac{1}{m} \right) \right) m^n,
\]

where $C_4 := (n + 1)! c_1(L)^n |M| > 0$. Hence by $\sum_{k=1}^{n_m} n_k \lambda_k = 0$ and $\lambda = r_m \beta$, it now follows from (4.3) and (4.4) that

\[
C_4 \left( F_1(X_\lambda) + O \left( \frac{1}{m} \right) \right) m^n = r_m^{-1} \sum_{k=1}^{n_m} n_k \lambda_k^2 = r_m^{-1} m^{n+2} \langle X_\lambda, X_\lambda \rangle_m,
\]

where by (4.1) above, (7) in [1] (see also [11]) implies $\langle X_\lambda, X_\lambda \rangle_m \geq C_3$ for some positive real constant $C_3$ independent of $m$. Furthermore $F_1(X_\lambda) = O(1)$ again by (4.1). Hence from (4.5), we obtain

\[
r_m^{-1} = O \left( \frac{1}{m^2} \right).
\]

In view of (3.9), since $\beta_k = r_m^{-1} \lambda_k$, (4.2) and (4.6) imply the required estimate (1.3) as follows:

\[
\gamma_{m,k} - 1 = \frac{\beta_k}{\beta_0} - 1 = \frac{\beta_k}{\beta_0} = O \left( \frac{1}{m} \right).
\]

5. Proof of (1.4) and Corollary B

In this section, keeping the same notation as in the preceding sections, we shall prove (1.4) and Corollary B.

Proof of (1.4). For $X_\lambda$ in (4.1), the associated Hamiltonian function $f_\lambda \in C^\infty(M)_{\mathbb{R}}$ on the Kähler manifold $(M, \omega_m)$ is

\[
f_\lambda = \sum_{k=1}^{n_m} \sum_{i=1}^{n_{\lambda_k}} \lambda_k \gamma_{m,k} |\sigma_{k,i}|^2 \frac{1}{h_m} = \sum_{k=1}^{n_m} \sum_{i=1}^{n_{\lambda_k}} \lambda_k |c_{k,i} s'_{k,i}|^2 \frac{1}{h_m},
\]

where by (4.2), when $m$ runs through the set of all sufficiently large integers, the function $f_\lambda$ is uniformly $C^0$-bounded. Now by (3.10),

\[
\sum_{k=1}^{n_m} \sum_{i=1}^{n_{\lambda_k}} \lambda_k \gamma_{m,k} |\sigma_{k,i} s'_{k,i}|^2 \frac{1}{h_m} = \frac{m^{n+1} f_\lambda}{\beta_0}.
\]
We now define a function $I_m$ on $\mathcal{M}$ by

$$I_m := \frac{r_m^{-1}}{\beta_0} \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \{1 - (\gamma_{m,k})^{-1}\} \lambda_k \gamma_{m,k} |\sigma_{k,i}|^2_h.$$  

Then by (1.3), (3.9), (3.10), (4.2) and (4.6), we easily see that

$$I_m = O \left( m^{-2} \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \gamma_{m,k} |\sigma_{k,i}|^2_h \right) = O(m^{-2}).$$

By (3.10) together with (5.1) and (5.2), it now follows that

$$\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |\sigma_{k,i}|^2_h = \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \gamma_{m,k} |\sigma_{k,i}|^2_h - \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} (\gamma_{m,k} - 1) |\sigma_{k,i}|^2_h = \frac{m^n}{\beta_0} - \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \frac{\lambda_k}{\beta_0} |\sigma_{k,i}|^2_h.$$ 

By (3.9), $m^n/\beta_0 = N'_m$. Moreover, by (4.6), $r_m^{-1}m^2/\beta_0 = O(1)$. Since

$$f_m := -\frac{r_m^{-1}m^2}{\beta_0^2} f_\lambda,$$ 

are uniformly $C^0$-bounded Hamiltonian functions on $(M,\omega_m)$ associated to holomorphic vector fields in $t$, in view of (5.3), we obtain

$$B^*_{\omega_m} = \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |\sigma_{k,i}|^2_h = N'_m + f_m m^{n-1} + O(m^{-2}),$$

as required. $\square$

Proof of Corollary B. Since the classical Futaki character $\mathcal{F}_1$ vanishes on $t$, we have $\mathcal{F}_1(X_\lambda) = 0$ in (4.5), so that

$$r_m^{-1} = O\left( \frac{1}{m^3} \right).$$
Then from (3.9) and $\beta_k = r_m^{-1} \lambda_k$, by looking at (4.2), we obtain the following required estimate:

$$\gamma_{m,k} - 1 = \frac{\beta_k}{\beta_0} = O\left(\frac{1}{m^2}\right).$$

Hence $B_m^*(\omega_m) = \{1 + O(1/m^2)\} B_m^*(\omega_m)$. Integrating this over $M$ by the volume form $\omega_m^n$, in view of (3.12), we see that

$$N'_m = \left\{1 + O\left(\frac{1}{m^2}\right)\right\} \frac{m^n}{\beta_0}.$$

Therefore, from (3.12) and $B_m^*(\omega_m) = \{1 + O(1/m^2)\} B_m^*(\omega_m)$, we now conclude that $B_m^*(\omega_m) = N'_m + O(m^{n-2})$, as required. \hfill \square

References


