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ASYMPTOTICS OF POLYBALANCED METRICS UNDER RELATIVE STABILITY CONSTRAINTS

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Abstract

Under the assumption of asymptotic relative Chow-stability for polarized algebraic manifolds (M, L), a series of weighted balanced metrics ω_m , $m \gg 1$, called polybalanced metrics, are obtained from complete linear systems $|L^m|$ on M. Then the asymptotic behavior of the weights as $m \to \infty$ will be studied.

1. Introduction

In this paper, we shall study relative Chow-stability (cf. [5]; see also [11]) for polarized algebraic manifolds (M, L) from the viewpoints of the existence problem of extremal Kähler metrics. As balanced metrics are obtained from Chow-stability on polarized algebraic manifolds, our relative Chow-stability similarly provides us with a special type of weighted balanced metrics called *polybalanced metrics*. As a crucial step in the program of [7], we here study the asymptotic behavior of the weights for such polybalanced metrics.

By a *polarized algebraic manifold* (M, L), we mean a pair of a connected projective algebraic manifold M and a very ample holomorphic line bundle L over M. For a maximal connected linear algebraic subgroup G of the group Aut(M) of all holomorphic automorphisms of M, let g := Lie G denote its Lie algebra. Since the infinitesimal g-action on M lifts to an infinitesimal bundle g-action on L, by setting

$$V_m := H^0(M, L^m), \quad m = 1, 2, \dots,$$

we view \mathfrak{g} as a Lie subalgebra of $\mathfrak{sl}(V_m)$. We now define a symmetric bilinear form \langle , \rangle_m on $\mathfrak{sl}(V_m)$ by

$$\langle X, Y \rangle_m = \operatorname{Tr}(XY)/m^{n+2}, \quad X, Y \in \mathfrak{sl}(V_m),$$

where the asymptotic limit of \langle , \rangle_m as $m \to \infty$ often plays an important role in the

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study of K-stability. In fact one can show

$$\langle X, Y \rangle_m = O(1)$$

by using the equivariant Riemann–Roch formula, see [1] and [11]. Let T be an algebraic torus in $SL(V_m)$ such that the corresponding Lie algebra $\mathfrak{t} := \text{Lie } T$ satisfies

 $\mathfrak{t} \subset \mathfrak{g}.$

Then by the *T*-action on V_m , we can write the vector space V_m as a direct sum of t-eigenspaces:

$$V_m = \bigoplus_{k=1}^{\nu_m} V(\chi_k),$$

where $V(\chi_k) := \{v \in V_m; g \cdot v = \chi_k(g)v \text{ for all } g \in T\}$ for mutually distinct multiplicative characters $\chi_k \in \text{Hom}(T, \mathbb{C}^*), k = 1, 2, ..., v_m$.

To study V_m , let ω_m be a Kähler metric in the class $c_1(L)_{\mathbb{R}}$, and choose a Hermitian metric h_m for L such that $\omega_m = c_1(L;h_m)_{\mathbb{R}}$. We now endow V_m with the Hermitian L^2 inner product on V_m defined by

(1.1)
$$(u, v)_{L^2} := \int_M (u, v)_{h_m} \omega_m^n, \quad u, v \in V_m,$$

where $(u, v)_{h_m}$ denotes the pointwise Hermitian pairing of u, v in terms of h_m . Then by this L^2 inner product, we have $V(\chi_k) \perp V(\chi_{k'})$, $k \neq k'$. Put $N_m := \dim V_m$ and $n_k := \dim_{\mathbb{C}} V(\chi_k)$. For each k, by choosing an orthonormal basis $\{\sigma_{k,i}; i = 1, 2, ..., n_k\}$ for $V(\chi_k)$, we put

$$B_{m,k}(\omega_m) := \sum_{i=1}^{n_k} |\sigma_{k,i}|^2_{h_m},$$

where $|u|_{h_m}^2 := (u, u)_{h_m}$ for each $u \in V_m$. Then ω_m is called a *polybalanced metric*, if there exist real constants $\gamma_{m,k} > 0$ such that

(1.2)
$$B_m^{\circ}(\omega_m) = \sum_{k=1}^{\nu_m} \gamma_{m,k} B_{m,k}(\omega_m)$$

is a constant function on M. Here $\gamma_{m,k}$ are called the *weights* of the polybalanced metric ω_m . On the other hand,

$$B_m^{\bullet}(\omega_m) := \sum_{k=1}^{\nu_m} B_{m,k}(\omega_m)$$

is called the *m*-th asymptotic Bergman kernel of ω_m . A smooth real-valued function $f \in C^{\infty}(M)_{\mathbb{R}}$ on the Kähler manifold (M, ω_m) is said to be Hamiltonian if there exists a holomorphic vector field $X \in \mathfrak{g}$ on M such that $i_X \omega_m = \sqrt{-1} \,\overline{\partial} f$. Put $N'_m := N_m/c_1(L)^n[M]$. In this paper, as the first step in [7], we shall show the following:

Theorem A. For a polarized algebraic manifold (M, L) and an algebraic torus T as above, assume that (M, L) is asymptotically Chow-stable relative to T. Then for each $m \gg 1$, there exists a polybalanced metric ω_m in the class $c_1(L)_{\mathbb{R}}$ such that $\gamma_{m,k} = 1 + O(1/m)$, i.e.,

(1.3)
$$|\gamma_{m,k}-1| \leq \frac{C_1}{m}, \quad k = 1, 2, \ldots, \nu_m; m \gg 1,$$

for some positive constant C_1 independent of k and m. Moreover, there exist uniformly C^0 -bounded functions $f_m \in C^{\infty}(M)_{\mathbb{R}}$ on M such that

(1.4)
$$B_m^{\bullet}(\omega_m) = N_m' + f_m m^{n-1} + O(m^{n-2})$$

and that each f_m is a Hamiltonian function on (M, ω_m) satisfying $i_{X_m}\omega_m = \sqrt{-1} \bar{\partial} f_m$ for some holomorphic vector field $X_m \in \mathfrak{t}$ on M.

In view of [8], this theorem and the result of Catlin–Lu–Tian–Yau–Zelditch ([3], [12], [13]) allow us to obtain an approach (cf. [7]) to an extremal Kähler version of Donaldson–Tian–Yau's conjecture. On the other hand, as a corollary to Theorem A, we obtain the following:

Corollary B. Under the same assumption as in Theorem A, suppose further that the classical Futaki character $\mathcal{F}_1 : \mathfrak{g} \to \mathbb{C}$ for M vanishes on \mathfrak{t} . Then for each $m \gg 1$, there exists a polybalanced metric ω_m in the class $c_1(L)_{\mathbb{R}}$ such that $\gamma_{m,k} = 1 + O(1/m^2)$. In particular

$$B_m^{\bullet}(\omega_m) = N'_m + O(m^{n-2}).$$

2. Asymptotic relative Chow-stability

By the same notation as in the introduction, we consider the algebraic subgroup S_m of $SL(V_m)$ defined by

$$S_m := \prod_{k=1}^{\nu_m} \mathrm{SL}(V(\chi_k)),$$

where the action of each $SL(V(\chi_k))$ on V_m fixes $V(\chi_i)$ if $i \neq k$. Then the centralizer H_m of S_m in $SL(V_m)$ consists of all diagonal matrices in $SL(V_m)$ acting on each $V(\chi_k)$ by constant scalar multiplication. Hence the centralizer $Z_m(T)$ of T in $SL(V_m)$ is $H_m \cdot S_m$ with Lie algebra

$$\mathfrak{z}_m(\mathfrak{t}) = \mathfrak{h}_m + \mathfrak{s}_m,$$

where $\mathfrak{s}_m := \text{Lie } S_m$ and $\mathfrak{h}_m := \text{Lie } H_m$. For the exponential map defined by $\mathfrak{h}_m \ni X \mapsto \exp(2\pi\sqrt{-1}X) \in H_m$, let $(\mathfrak{h}_m)_{\mathbb{Z}}$ denote its kernel. Regarding $(\mathfrak{h}_m)_{\mathbb{R}} := (\mathfrak{h}_m)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ as a subspace of \mathfrak{h}_m , we have a real structure on \mathfrak{h}_m , i.e., an involution

$$\mathfrak{h}_m \ni X \mapsto \overline{X} \in \mathfrak{h}_m$$

defined as the associated complex conjugate of \mathfrak{h}_m fixing $(\mathfrak{h}_m)_{\mathbb{R}}$. We then have a Hermitian metric $(,)_m$ on \mathfrak{h}_m by setting

(2.1)
$$(X, Y)_m = \langle X, \bar{Y} \rangle_m, \quad X, Y \in \mathfrak{h}_m$$

For the orthogonal complement \mathfrak{t}^{\perp} of \mathfrak{t} in \mathfrak{h}_m in terms of this Hermitian metric, let T^{\perp} denote the corresponding algebraic torus in H_m . We now define an algebraic subgroup G_m of $Z_m(T)$ by

$$(2.2) G_m := T^{\perp} \cdot S_m$$

For the *T*-equivariant Kodaira embedding $\Phi_m \colon M \hookrightarrow \mathbb{P}^*(V_m)$ associated to the complete linear system $|L^m|$ on *M*, let d(m) denote the degree of the image $\Phi_m(M)$ in the projective space $\mathbb{P}^*(V_m)$. For the dual space W_m^* of $W_m := S^{d(m)}(V_m)^{\otimes n+1}$, we have the Chow form

$$0 \neq \hat{M}_m \in W_m^*$$

for the irreducible reduced algebraic cycle $\Phi_m(M)$ on $\mathbb{P}^*(V_m)$, so that the corresponding element $[\hat{M}_m]$ in $\mathbb{P}^*(W_m)$ is the Chow point for the cycle $\Phi_m(M)$. Consider the natural action of $SL(V_m)$ on W_m^* induced by the action of $SL(V_m)$ on V_m .

DEFINITION 2.3. (1) (M, L^m) is said to be *Chow-stable relative to T* if the orbit $G_m \cdot \hat{M}_m$ is closed in W_m^* .

(2) (M, L) is said to be asymptotically Chow-stable relative to T if (M, L^m) is Chow-stable relative to T for each integer $m \gg 1$.

3. Relative Chow-stability for each fixed m

In this section, we consider a polarized algebraic manifold (M, L) under the assumption that (M, L^m) is Chow-stable relative to T for a fixed positive integer m. Then we shall show that a polybalanced metric ω_m exists in the class $c_1(L)_{\mathbb{R}}$.

The space $\Lambda_m := \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\nu_m}) \in \mathbb{C}^{\nu_m}; \sum_{k=1}^{\nu_m} n_k \lambda_k = 0\}$ and the Lie algebra \mathfrak{h}_m are identified by an isomorphism

(3.1)
$$\Lambda_m \cong \mathfrak{h}_m, \quad \lambda \leftrightarrow X_\lambda,$$

with $(\mathfrak{h}_m)_{\mathbb{R}}$ corresponding to the set $(\Lambda_m)_{\mathbb{R}}$ of the real points in Λ_m , where X_{λ} is the endomorphism of V_m defined by

$$X_{\lambda} := \bigoplus_{k=1}^{\nu_m} \lambda_k \operatorname{id}_{V(\chi_k)} \in \bigoplus_{k=1}^{\nu_m} \operatorname{End}(V(\chi_k)) \quad (\subset \operatorname{End}(V_m)).$$

In terms of the identification (3.1), we can write the Hermitian metric (,)_m on \mathfrak{h}_m in (2.1) in the form

$$(\lambda, \mu)_m := \sum_{k=1}^{\nu_m} \frac{n_k \lambda_k \bar{\mu}_k}{m^{n+2}},$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\nu_m})$ and $\mu = (\mu_1, \mu_2, \dots, \mu_{\nu_m})$ are in \mathbb{C}^{ν_m} . By the identification (3.1), corresponding to the decomposition $\mathfrak{h}_m = \mathfrak{t} \oplus \mathfrak{t}^{\perp}$, we have the orthogonal direct sum

$$\Lambda_m = \Lambda(\mathfrak{t}) \oplus \Lambda(\mathfrak{t}^{\perp}),$$

where $\Lambda(\mathfrak{t})$ and $\Lambda(\mathfrak{t}^{\perp})$ are the subspace of Λ_m associated to \mathfrak{t} and \mathfrak{t}^{\perp} , respectively. Take a Hermitian metric ρ_k on $V(\chi_k)$, and for the metric

$$\rho := \bigoplus_{k=1}^{\nu_m} \rho_k$$

on V_m , we see that $V(\chi_k) \perp V(\chi_{k'})$ whenever $k \neq k'$. By choosing an orthonormal basis $\{s_{k,i}; i = 1, 2, ..., n_k\}$ for the Hermitian vector space $(V(\chi_k), \rho_k)$, we now set

(3.2)
$$j(k,i) := i + \sum_{l=1}^{k-1} n_l, \quad i = 1, 2, \dots, n_k; k = 1, 2, \dots, \nu_m,$$

where the right-hand side denotes *i* in the special case k = 1. By writing $s_{k,i}$ as $s_{j(k,i)}$, we have an orthonormal basis

$$\mathcal{S} := \{s_1, s_2, \ldots, s_{N_m}\}$$

for (V_m, ρ) . By this basis, the vector space V_m and the algebraic group $SL(V_m)$ are identified with $\mathbb{C}^{N_m} = \{(z_1, \ldots, z_{N_m})\}$ and $SL(N_m, \mathbb{C})$, respectively. In terms of S, the Kodaira embedding Φ_m is given by

$$\Phi_m(x) := (s_1(x) : \cdots : s_{N_m}(x)), \quad x \in M.$$

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Consider the associated *Chow norm* $W_m^* \ni \xi \mapsto ||\xi||_{CH(\rho)} \in \mathbb{R}_{\geq 0}$ as in Zhang [14] (see also [4]). Then by the closedness of $G_m \cdot \hat{M}_m$ in W_m^* (cf. (2.2) and Definition 2.3), the Chow norm on the orbit $G_m \cdot \hat{M}_m$ takes its minimum at $g_m \cdot \hat{M}_m$ for some $g_m \in G_m$. Note that, by complexifying

$$K_m := \prod_{k=1}^{\nu_m} \mathrm{SU}(V(\chi_k)),$$

we obtain the reductive algebraic group S_m . For each $\kappa \in K_m$ and each diagonal matrix Δ in $\mathfrak{sl}(N_m, \mathbb{C})$, we put

$$e(\kappa, \Delta) := \exp\{\operatorname{Ad}(\kappa)\Delta\}.$$

Then g_m is written as $\kappa_1 \cdot e(\kappa_0, D)$ for some $\kappa_0, \kappa_1 \in K_m$ and a diagonal matrix $D = (d_j)_{1 \le j \le N_m}$ in $\mathfrak{sl}(N_m, \mathbb{C})$ with the *j*-th diagonal element d_j . Put $g'_m := e(\kappa_0, D)$. In view of $\|g'_m \cdot \hat{M}_m\|_{\operatorname{CH}(\rho)} = \|g_m \cdot \hat{M}_m\|_{\operatorname{CH}(\rho)}$ (cf. [10], Proposition 4.1), we obtain

(3.3)
$$\|g'_m \cdot \hat{M}_m\|_{\operatorname{CH}(\rho)} \leq \|e(\kappa_0, t(X_\lambda + A)) \cdot g'_m \cdot \hat{M}_m\|_{\operatorname{CH}(\rho)}, \quad t \in \mathbb{C},$$

for all $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\nu_m}) \in \Lambda(\mathfrak{t}^{\perp})$ and all diagonal matrices $A = (a_j)_{1 \leq j \leq N_m}$ in \mathfrak{s}_m , where each a_j denotes the *j*-th diagonal element of *A*. We now write $a_{j(k,i)}$ as $a_{k,i}$ for simplicity. Put

$$s'_{j} := \kappa_{0}^{-1} \cdot s_{j}, \quad b_{k,i} := \lambda_{k} + a_{k,i}, \quad c_{k,i} := \exp d_{j(k,i)}.$$

Then we shall now identify V_m with $\mathbb{C}^{N_m} = \{(z'_1, \ldots, z'_{N_m})\}$ by the orthonormal basis $S' := \{s'_1, s'_2, \ldots, s'_{N_m}\}$ for V_m . In view (3.2), we rewrite s'_i, z'_i as $s'_{k,i}, z'_{k,i}$, respectively by

$$s'_{k,i} := s'_{j(k,i)}, \quad z'_{k,i} := z'_{j(k,i)},$$

where $k = 1, 2, ..., v_m$ and $i = 1, 2, ..., n_k$. By writing $b_{k,i}$, $c_{k,i}$ also as $b_{j(k,i)}$, $c_{j(k,i)}$, respectively, we consider the diagonal matrices *B* and *C* of order N_m with the *j*-th diagonal elements b_j and c_j , respectively. Note that the right-hand side of (3.3) is

$$\|(\exp t B) \cdot C \cdot \kappa_0^{-1} \cdot \hat{M}_m\|_{CH(\rho)},$$

and its derivative at t = 0 vanishes by virtue of the inequality (3.3). Hence, by setting $\Theta := (\sqrt{-1}/2\pi) \,\partial\bar{\partial} \log \left(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} z'_{k,i}|^2 \right)$, we obtain the equality (see for instance (4.4) in [4])

(3.4)
$$\int_{M} \frac{\sum_{k=1}^{\nu_{m}} \sum_{i=1}^{n_{k}} b_{k,i} |c_{k,i}s'_{k,i}|^{2}}{\sum_{k=1}^{\nu_{m}} \sum_{i=1}^{n_{k}} |c_{k,i}s'_{k,i}|^{2}} \Phi_{m}^{\prime*}(\Theta^{n}) = 0$$

for all $\lambda \in \Lambda(\mathfrak{t}^{\perp})$ and all diagonal matrices A in the Lie algebra \mathfrak{s}_m , where $\Phi'_m \colon M \hookrightarrow \mathbb{P}^*(V_m)$ is the Kodaira embedding of M by S' which sends each $x \in M$ to $(s'_1(x) : s'_2(x) : \cdots : s'_{N_m}(x)) \in \mathbb{P}^*(V_m)$. Here we regard

$$s_{k,i}' = \Phi'^* z_{k,i}'.$$

Let $k_0 \in \{1, 2, ..., v_m\}$ and let $i_1, i_2 \in \{1, 2, ..., n_{k_0}\}$ with $i_1 \neq i_2$. Using Kronecker's delta, we first specify the real diagonal matrix *B* by

$$\lambda_k = 0$$
 and $a_{k,i} = \delta_{kk_0}(\delta_{ii_1} - \delta_{ii_2}),$

where $k = 1, 2, ..., v_m$; $i = 1, 2, ..., n_k$. By (3.4) applied to this *B*, and let (i_1, i_2) run through the set of all pairs of two distinct integers in $\{1, 2, ..., n_{k_0}\}$, where positive integer k_0 varies from 1 to v_m . Then there exists a positive constant $\beta_k > 0$ independent of the choice of *i* in $\{1, 2, ..., n_k\}$ such that, for all *i*,

(3.5)
$$\int_{M} \frac{|c_{k,i}s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i}s'_{k,i}|^2} \Phi_m^{\prime*}(\Theta^n) = \beta_k, \quad k = 1, 2, \dots, \nu_m.$$

Let k_0, i_1, i_2 be as above, and let κ_2 be the element in K_m such that

$$\kappa_2 z'_{k_0,i_1} = \frac{1}{\sqrt{2}} (z'_{k_0,i_1} - z'_{k_0,i_2}), \quad \kappa_2 z'_{k_0,i_2} = \frac{1}{\sqrt{2}} (z'_{k_0,i_1} + z'_{k_0,i_2})$$

and that κ_2 fixes all other $z_{k,i}$'s. Let κ_3 be the element in K_m such that

$$\kappa_3 z'_{k_0,i_1} = \frac{1}{\sqrt{2}} (z'_{k_0,i_1} + \sqrt{-1} z'_{k_0,i_2}), \quad \kappa_3 z'_{k_0,i_2} = \frac{1}{\sqrt{2}} (\sqrt{-1} z'_{k_0,i_1} + z'_{k_0,i_2})$$

and that κ_3 fixes all other $z'_{k,i}$'s. Now

$$\|\kappa_{2}g'_{m}\cdot\hat{M}_{m}\|_{\mathrm{CH}(\rho)} = \|\kappa_{3}g'_{m}\cdot\hat{M}_{m}\|_{\mathrm{CH}(\rho)} = \|g'_{m}\cdot\hat{M}_{m}\|_{\mathrm{CH}(\rho)},$$

and note that

$$2z'_{k_0,i_1}\overline{z}'_{k_0,i_2} = (|\kappa_2 z'_{k_0,i_2}|^2 - |\kappa_2 z'_{k_0,i_1}|^2) - \sqrt{-1}(|\kappa_3 z'_{k_0,i_2}|^2 - |\kappa_3 z'_{k_0,i_1}|^2).$$

Hence replacing g'_m by $\kappa_{\alpha}g'_m$, $\alpha = 2, 3$, in (3.3), we obtain the case k' = k'' of the following by an argument as in deriving (3.5) from (3.3):

(3.6)
$$\int_{M} \frac{s'_{k',i'}\bar{s}'_{k'',i''}}{\sum_{k=1}^{\nu_{m}} \sum_{i=1}^{n_{k}} |c_{k,i}s'_{k,i}|^{2}} \Phi_{m}^{\prime*}(\Theta^{n}) = 0, \quad \text{if} \quad (k',i') \neq (k'',i'')$$

Here (3.6) holds easily for $k' \neq k''$, since for every element g of the maximal compact subgroup of T, we have:

L.H.S. of (3.6) =
$$\int_{M} g^{*} \left\{ \frac{s'_{k',i'} \overline{s}'_{k',i''}}{\sum_{k=1}^{\nu_{m}} \sum_{i=1}^{n_{k}} |c_{k,i}s'_{k,i}|^{2}} \Phi_{m}^{'*}(\Theta^{n}) \right\}$$
$$= \frac{\chi_{k''}(g)}{\chi_{k'}(g)} \int_{M} \frac{s'_{k',i'} \overline{s}'_{k',i''}}{\sum_{k=1}^{\nu_{m}} \sum_{i=1}^{n_{k}} |c_{k,i}s'_{k,i}|^{2}} \Phi_{m}^{'*}(\Theta^{n}).$$

Put $\beta := (\beta_1, \beta_2, \dots, \beta_{\nu_m}) \in \mathbb{R}^{\nu_m}$ and $\beta_0 := (\sum_{k=1}^{\nu_m} n_k \beta_k)/N_m$, where β_k is given in (3.5). In view of $N_m = \sum_{k=1}^{\nu_m} n_k$, by setting $\underline{\beta}_k := \beta_k - \beta_0$, we have

$$\underline{\beta} := \left(\underline{\beta}_1, \underline{\beta}_2, \ldots, \underline{\beta}_{\nu_m}\right) \in \Lambda_m.$$

Next for each $\lambda \in \Lambda(\mathfrak{t}^{\perp})$, by setting $a_{k,i} = 0$ for all (k, i), the equality (3.4) above implies $0 = \sum_{k=1}^{\nu_m} (n_k \lambda_k) \beta_k$. From this together with the equality $\sum_{k=1}^{\nu_m} n_k \lambda_k = 0$, we obtain $(\lambda, \beta)_m = 0$, i.e.,

$$(3.7) \qquad \qquad \beta \in \Lambda(\mathfrak{t}).$$

We now define a Hermitian metric h_{FS} (cf. [14]) for L^m as follows. Let u be a local section for L^m . Then¹

(3.8)
$$|u|_{h_{\rm FS}}^2 := \frac{|u|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2}.$$

For the Hermitian metric $h_m := (h_{\rm FS})^{1/m}$ for *L*, we consider the associated Kähler metric $\omega_m := c_1(L; h_m)_{\mathbb{R}}$ on *M*. In view of (3.5),

(3.9)
$$\beta_0 = \frac{\sum_{k=1}^{\nu_m} n_k \beta_k}{N_m} = N_m^{-1} \int_M \Phi_m^*(\Theta^n)$$
$$= N_m^{-1} m^n c_1(L)^n [M] = n! \left\{ 1 + O\left(\frac{1}{m}\right) \right\}$$

Then for $\gamma_{m,k} := \beta_k / \beta_0$ and $\sigma_{k,i} := c_{k,i} s'_{k,i} (m^n \beta_k^{-1})^{1/2}$, we have

(3.10)
$$\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \gamma_{m,k} |\sigma_{k,i}|_{h_m}^2 = \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \gamma_{m,k} |\sigma_{k,i}|_{h_{\rm FS}}^2 = \frac{m^n}{\beta_0} \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i}s'_{k,i}|_{h_{\rm FS}}^2 = \frac{m^n}{\beta_0}.$$

¹In view of (3.8), there is some error in [4]. Actually, for the numerator of (5.7) in the paper [4], please read $(N_m + 1)|s|^2$.

By operating $(\sqrt{-1}/2\pi) \,\overline{\partial} \partial \log$ on (3.8), we obtain

(3.11)
$$\Phi_m^{\prime*}(\Theta) = c_1(L^m; h_{\rm FS})_{\mathbb{R}} = mc_1(L; h_m)_{\mathbb{R}} = m\omega_m.$$

Then in terms of the Hermitian L^2 inner product (1.1), we see from (3.5), (3.6), (3.8) and (3.11) that $\{\sigma_{k,i}; k = 1, ..., v_m, i = 1, ..., n_k\}$ is an orthonormal basis for V_m . Moreover, (3.10) is rewritten as

$$B_m^{\circ}(\omega_m) = \frac{m^n}{\beta_0},$$

where $B_m^{\circ}(\omega_m)$ is as in (1.2). Hence ω_m is a polybalanced metric, and the proof of Theorem A is reduced to showing (1.3) and (1.4). By summing up, we obtain

Theorem C. If (M, L^m) is Chow-stable relative to T for a positive integer m, then the Kähler class $c_1(L)_{\mathbb{R}}$ admits a polybalanced metric ω_m with the weights $\gamma_{m,k}$ as above.

4. The asymptotic behavior of the weights $\gamma_{m,k}$

The purpose of this section is to prove (1.3). If $\underline{\beta} = 0$, then we are done. Hence, we may assume that $\beta \neq 0$. Consider the sphere

$$\sum := \{ X \in \mathfrak{t}_{\mathbb{R}} \, ; \, \langle X, X \rangle_0 = 1 \}$$

in $\mathfrak{t}_{\mathbb{R}} := \mathfrak{t} \cap (\mathfrak{h}_m)_{\mathbb{R}}$, where \langle , \rangle_0 denotes the positive definite symmetric bilinear form on \mathfrak{g} as in [2]. Since all components $\underline{\beta}_k$ of $\underline{\beta}$ are real, we see from (3.7) that, in view of (3.1), $\lambda := r_m \beta$ satisfies

for some positive real number r_m . Hence by writing $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\nu_m})$, we obtain positive constants C_2 , C_3 independent of k and m such that (see for instance [5], Lemma 2.6)

$$(4.2) -C_2m \le \lambda_k \le C_3m.$$

Put $g(t) := \exp(tX_{\lambda})$ and $\gamma(t) := \log \|g(t) \cdot C \cdot \kappa_0^{-1} \cdot \hat{M}_m\|_{CH(\rho)}$, $t \in \mathbb{R}$, by using the notation in Section 3. Since g(t) commutes with $C \cdot \kappa_0^{-1}$, we see that g(t) defines a holomorphic automorphism of $\Phi'_m(M)$. In view of Theorem 4.5 in [4], it follows from Remark 4.6 in [4] that (cf. [14])

(4.3)
$$\dot{\gamma}(t) = \dot{\gamma}(0) = \sum_{k=1}^{\nu_m} n_k \lambda_k \beta_k, \quad -\infty < t < +\infty.$$

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Consider the classical Futaki invariant $\mathcal{F}_1(X_{\lambda})$ associated to the holomorphic vector field X_{λ} on (M, L). Since M is smooth, this coincides with the corresponding Donaldson–Futaki's invariant for test configurations. Then by applying Lemma 4.8 in [6] to the product configuration of (M, L) associated to the one-parameter group generated by X_{λ} on the central fiber, we obtain (see also [1], [11])

(4.4)
$$\lim_{t \to -\infty} \dot{\gamma}(t) = C_4 \left\{ \mathcal{F}_1(X_\lambda) + O\left(\frac{1}{m}\right) \right\} m^n,$$

where $C_4 := (n+1)! c_1(L)^n[M] > 0$. Hence by $\sum_{k=1}^{\nu_m} n_k \lambda_k = 0$ and $\lambda = r_m \underline{\beta}$, it now follows from (4.3) and (4.4) that

(4.5)
$$C_4 \left\{ \mathcal{F}_1(X_\lambda) + O\left(\frac{1}{m}\right) \right\} m^n = r_m^{-1} \sum_{k=1}^{\nu_m} n_k \lambda_k^2$$
$$= r_m^{-1} m^{n+2} \langle X_\lambda, X_\lambda \rangle_m,$$

where by (4.1) above, (7) in [1] (see also [11]) implies $\langle X_{\lambda}, X_{\lambda} \rangle_m \geq C_5$ for some positive real constant C_5 independent of m. Furthermore $\mathcal{F}_1(X_{\lambda}) = O(1)$ again by (4.1). Hence from (4.5), we obtain

(4.6)
$$r_m^{-1} = O\left(\frac{1}{m^2}\right).$$

In view of (3.9), since $\underline{\beta}_k = r_m^{-1}\lambda_k$, (4.2) and (4.6) imply the required estimate (1.3) as follows:

$$\gamma_{m,k} - 1 = \frac{\beta_k}{\beta_0} - 1 = \frac{\underline{\beta}_k}{\beta_0} = O\left(\frac{1}{m}\right).$$

5. Proof of (1.4) and Corollary B

In this section, keeping the same notation as in the preceding sections, we shall prove (1.4) and Corollary B.

Proof of (1.4). For X_{λ} in (4.1), the associated Hamiltonian function $f_{\lambda} \in C^{\infty}(M)_{\mathbb{R}}$ on the Kähler manifold (M, ω_m) is

$$f_{\lambda} = \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \lambda_k \gamma_{m,k} |\sigma_{k,i}|_{h_m}^2}{m \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \gamma_{m,k} |\sigma_{k,i}|_{h_m}^2} = \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \lambda_k |c_{k,i}s'_{k,i}|^2}{m \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i}s'_{k,i}|^2},$$

where by (4.2), when *m* runs through the set of all sufficiently large integers, the function f_{λ} is uniformly C^0 -bounded. Now by (3.10),

(5.1)
$$\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \lambda_k \gamma_{m,k} |\sigma_{k,i}|_{h_m}^2 = \frac{m^{n+1} f_{\lambda}}{\beta_0}$$

We now define a function I_m on M by

(5.2)
$$I_m := \frac{r_m^{-1}}{\beta_0} \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \{1 - (\gamma_{m,k})^{-1}\} \lambda_k \gamma_{m,k} |\sigma_{k,i}|_{h_m}^2.$$

Then by (1.3), (3.9), (3.10), (4.2) and (4.6), we easily see that

(5.3)
$$I_m = O\left(m^{-2} \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \gamma_{m,k} |\sigma_{k,i}|_{h_m}^2\right) = O(m^{n-2}).$$

By (3.10) together with (5.1) and (5.2), it now follows that

$$\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |\sigma_{k,i}|_{h_m}^2 = \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \gamma_{m,k} |\sigma_{k,i}|_{h_m}^2 - \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} (\gamma_{m,k} - 1) |\sigma_{k,i}|_{h_m}^2$$
$$= \frac{m^n}{\beta_0} - \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \frac{\beta_k}{\beta_0} |\sigma_{k,i}|_{h_m}^2$$
$$= \frac{m^n}{\beta_0} + I_m - \frac{r_m^{-1}}{\beta_0} \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \lambda_k \gamma_{m,k} |\sigma_{k,i}|_{h_m}^2$$
$$= \frac{m^n}{\beta_0} + I_m - \frac{r_m^{-1}m^2}{\beta_0^2} f_{\lambda} m^{n-1}.$$

By (3.9), $m^n/\beta_0 = N'_m$. Moreover, by (4.6), $r_m^{-1}m^2/\beta_0^2 = O(1)$. Since

$$f_m := -\frac{r_m^{-1}m^2}{\beta_0^2} f_\lambda, \quad m \gg 1,$$

are uniformly C^0 -bounded Hamiltonian functions on (M, ω_m) associated to holomorphic vector fields in t, in view of (5.3), we obtain

$$B_m^{\bullet}(\omega_m) = \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |\sigma_{k,i}|_{h_m}^2 = N_m' + f_m m^{n-1} + O(m^{n-2}),$$

as required.

Proof of Corollary B. Since the classical Futaki character \mathcal{F}_1 vanishes on t, we have $\mathcal{F}_1(X_{\lambda}) = 0$ in (4.5), so that

$$r_m^{-1} = O\left(\frac{1}{m^3}\right).$$

Then from (3.9) and $\underline{\beta}_k = r_m^{-1}\lambda_k$, by looking at (4.2), we obtain the following required estimate:

$$\gamma_{m,k}-1=\frac{\underline{\beta}_k}{\overline{\beta}_0}=O\bigg(\frac{1}{m^2}\bigg).$$

Hence $B_m^{\bullet}(\omega_m) = \{1 + O(1/m^2)\}B_m^{\circ}(\omega_m)$. Integrating this over *M* by the volume form ω_m^n , in view of (3.12), we see that

$$N'_m = \left\{ 1 + O\left(\frac{1}{m^2}\right) \right\} \frac{m^n}{\beta_0}.$$

Therefore, from (3.12) and $B_m^{\bullet}(\omega_m) = \{1 + O(1/m^2)\}B_m^{\circ}(\omega_m)$, we now conclude that $B_m^{\bullet}(\omega_m) = N'_m + O(m^{n-2})$, as required.

References

- [1] S.K. Donaldson: Lower bounds on the Calabi functional, J. Differential Geom. **70** (2005), 453–472.
- [2] A. Futaki and T. Mabuchi: *Bilinear forms and extremal Kähler vector fields associated with Kähler classes*, Math. Ann. **301** (1995), 199–210.
- [3] Z. Lu: On the lower order terms of the asymptotic expansion of Tian–Yau–Zelditch, Amer. J. Math. **122** (2000), 235–273.
- [4] T. Mabuchi: Stability of extremal Kähler manifolds, Osaka J. Math. 41 (2004), 563-582.
- [5] T. Mabuchi: An energy-theoretic approach to the Hitchin–Kobayashi correspondence for manifolds, II, Osaka J. Math. 46 (2009), 115–139.
- [6] T. Mabuchi: K-stability of constant scalar curvature polarization, arXiv:math.DG/0812.4093.
- [7] T. Mabuchi: *Relative stability and extremal metrics*, in preparation.
- [8] T. Mabuchi and Y. Nitta: K-stability and asymptotic Chow stability, in preparation.
- [9] D. Mumford, J. Fogarty and F. Kirwan: Geometric Invariant Theory, third edition, Ergebnisse der Mathematik und ihrer Grenzgebiete (2) 34, Springer, Berlin, 1994.
- [10] Y. Sano: On stability criterion of complete intersections, J. Geom. Anal. 14 (2004), 533-544.
- [11] G. Székelyhidi: Extremal metrics and K-stability, Bull. Lond. Math. Soc. 39 (2007), 76–84.
- [12] G. Tian: On a set of polarized Kähler metrics on algebraic manifolds, J. Differential Geom. 32 (1990), 99–130.
- [13] S. Zelditch: Szegő kernels and a theorem of Tian, Internat. Math. Res. Notices (1998), 317–331.
- [14] S. Zhang: Heights and reductions of semi-stable varieties, Compositio Math. 104 (1996), 77–105.

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