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Author(s)	Mabuchi, Toshiki
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ASYMPTOTICS OF POLYBALANCED METRICS UNDER RELATIVE STABILITY CONSTRAINTS

TOSHIKI MABUCHI*

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Abstract

Under the assumption of asymptotic relative Chow-stability for polarized algebraic manifolds (M, L), a series of weighted balanced metrics ω_m, $m \gg 1$, called polybalanced metrics, are obtained from complete linear systems $|L^m|$ on M. Then the asymptotic behavior of the weights as $m \to \infty$ will be studied.

1. Introduction

In this paper, we shall study relative Chow-stability (cf. [5]; see also [11]) for polarized algebraic manifolds (M, L) from the viewpoints of the existence problem of extremal Kähler metrics. As balanced metrics are obtained from Chow-stability on polarized algebraic manifolds, our relative Chow-stability similarly provides us with a special type of weighted balanced metrics called *polybalanced metrics*. As a crucial step in the program of [7], we here study the asymptotic behavior of the weights for such polybalanced metrics.

By a *polarized algebraic manifold* (M, L), we mean a pair of a connected projective algebraic manifold M and a very ample holomorphic line bundle L over M. For a maximal connected linear algebraic subgroup G of the group $\mathrm{Aut}(M)$ of all holomorphic automorphisms of M, let $\mathfrak{g} := \mathrm{Lie}\, G$ denote its Lie algebra. Since the infinitesimal $\mathfrak{g}$-action on M lifts to an infinitesimal bundle $\mathfrak{g}$-action on L, by setting

$$V_m := H^0(M, L^m), \quad m = 1, 2, \dots,$$

we view $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{sl}(V_m)$. We now define a symmetric bilinear form $\langle\ ,\ \rangle_m$ on $\mathfrak{sl}(V_m)$ by

$$\langle X, Y\rangle_m = \mathrm{Tr}(XY)/m^{n+2}, \quad X, Y \in \mathfrak{sl}(V_m),$$

where the asymptotic limit of $\langle\ ,\ \rangle_m$ as $m \to \infty$ often plays an important role in the

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study of K-stability. In fact one can show

$$\langle X, Y\rangle_m = O(1)$$

by using the equivariant Riemann–Roch formula, see [1] and [11]. Let T be an algebraic torus in $\mathrm{SL}(V_m)$ such that the corresponding Lie algebra $\mathfrak{t} := \mathrm{Lie}\, T$ satisfies

$$\mathfrak{t} \subset \mathfrak{g}.$$

Then by the T-action on V_m, we can write the vector space V_m as a direct sum of $\mathfrak{t}$-eigenspaces:

$$V_m = \bigoplus_{k=1}^{\nu_m} V(\chi_k),$$

where $V(\chi_k) := \{v \in V_m;\ g \cdot v = \chi_k(g)v \text{ for all } g \in T\}$ for mutually distinct multiplicative characters $\chi_k \in \mathrm{Hom}(T, \mathbb{C}^*)$, $k = 1, 2, \dots, \nu_m$.

To study V_m, let ω_m be a Kähler metric in the class $c_1(L)_{\mathbb{R}}$, and choose a Hermitian metric h_m for L such that $\omega_m = c_1(L; h_m)_{\mathbb{R}}$. We now endow V_m with the Hermitian L^2 inner product on V_m defined by

$$(u, v)_{L^2} := \int_M (u, v)_{h_m} \omega_m^n, \quad u, v \in V_m, \tag{1.1}$$

where $(u, v)_{h_m}$ denotes the pointwise Hermitian pairing of u, v in terms of h_m. Then by this L^2 inner product, we have $V(\chi_k) \perp V(\chi_{k'})$, $k \neq k'$. Put $N_m := \dim V_m$ and $n_k := \dim_{\mathbb{C}} V(\chi_k)$. For each k, by choosing an orthonormal basis $\{\sigma_{k,i}; i = 1, 2, \dots, n_k\}$ for $V(\chi_k)$, we put

$$B_{m,k}(\omega_m) := \sum_{i=1}^{n_k} |\sigma_{k,i}|^2_{h_m},$$

where $|u|^2_{h_m} := (u, u)_{h_m}$ for each $u \in V_m$. Then ω_m is called a *polybalanced metric*, if there exist real constants $\gamma_{m,k} > 0$ such that

$$B_m^{\circ}(\omega_m) = \sum_{k=1}^{\nu_m} \gamma_{m,k} B_{m,k}(\omega_m) \tag{1.2}$$

is a constant function on M. Here $\gamma_{m,k}$ are called the *weights* of the polybalanced metric ω_m. On the other hand,

$$B_m^{\bullet}(\omega_m) := \sum_{k=1}^{\nu_m} B_{m,k}(\omega_m)$$

is called the *m-th asymptotic Bergman kernel* of ω_m. A smooth real-valued function $f \in C^\infty(M)_{\mathbb{R}}$ on the Kähler manifold (M, ω_m) is said to be *Hamiltonian* if there exists a holomorphic vector field $X \in \mathfrak{g}$ on M such that $i_X\omega_m = \sqrt{-1}\,\bar\partial f$. Put $N'_m := N_m/c_1(L)^n[M]$. In this paper, as the first step in [7], we shall show the following:

Theorem A. *For a polarized algebraic manifold (M, L) and an algebraic torus T as above, assume that (M, L) is asymptotically Chow-stable relative to T. Then for each $m \gg 1$, there exists a polybalanced metric ω_m in the class $c_1(L)_{\mathbb{R}}$ such that $\gamma_{m,k} = 1 + O(1/m)$, i.e.,*

$$|\gamma_{m,k} - 1| \leq \frac{C_1}{m}, \quad k = 1, 2, \dots, \nu_m\,; m \gg 1, \tag{1.3}$$

for some positive constant C_1 independent of k and m. Moreover, there exist uniformly C^0-bounded functions $f_m \in C^\infty(M)_{\mathbb{R}}$ on M such that

$$B^\bullet_m(\omega_m) = N'_m + f_m m^{n-1} + O(m^{n-2}) \tag{1.4}$$

and that each f_m is a Hamiltonian function on (M, ω_m) satisfying $i_{X_m}\omega_m = \sqrt{-1}\,\bar\partial f_m$ for some holomorphic vector field $X_m \in \mathfrak{t}$ on M.

In view of [8], this theorem and the result of Catlin–Lu–Tian–Yau–Zelditch ([3], [12], [13]) allow us to obtain an approach (cf. [7]) to an extremal Kähler version of Donaldson–Tian–Yau's conjecture. On the other hand, as a corollary to Theorem A, we obtain the following:

Corollary B. *Under the same assumption as in* Theorem A, *suppose further that the classical Futaki character $\mathcal{F}_1 : \mathfrak{g} \to \mathbb{C}$ for M vanishes on $\mathfrak{t}$. Then for each $m \gg 1$, there exists a polybalanced metric ω_m in the class $c_1(L)_{\mathbb{R}}$ such that $\gamma_{m,k} = 1 + O(1/m^2)$. In particular*

$$B^\bullet_m(\omega_m) = N'_m + O(m^{n-2}).$$

2. Asymptotic relative Chow-stability

By the same notation as in the introduction, we consider the algebraic subgroup S_m of $\mathrm{SL}(V_m)$ defined by

$$S_m := \prod_{k=1}^{\nu_m} \mathrm{SL}(V(\chi_k)),$$

where the action of each $\mathrm{SL}(V(\chi_k))$ on V_m fixes $V(\chi_i)$ if $i \neq k$. Then the centralizer H_m of S_m in $\mathrm{SL}(V_m)$ consists of all diagonal matrices in $\mathrm{SL}(V_m)$ acting on each

$V(\chi_k)$ by constant scalar multiplication. Hence the centralizer $Z_m(T)$ of T in $\mathrm{SL}(V_m)$ is $H_m \cdot S_m$ with Lie algebra

$$\mathfrak{z}_m(\mathfrak{t}) = \mathfrak{h}_m + \mathfrak{s}_m,$$

where $\mathfrak{s}_m := \mathrm{Lie}\, S_m$ and $\mathfrak{h}_m := \mathrm{Lie}\, H_m$. For the exponential map defined by $\mathfrak{h}_m \ni X \mapsto \exp(2\pi\sqrt{-1}X) \in H_m$, let $(\mathfrak{h}_m)_{\mathbb{Z}}$ denote its kernel. Regarding $(\mathfrak{h}_m)_{\mathbb{R}} := (\mathfrak{h}_m)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ as a subspace of $\mathfrak{h}_m$, we have a real structure on $\mathfrak{h}_m$, i.e., an involution

$$\mathfrak{h}_m \ni X \mapsto \bar{X} \in \mathfrak{h}_m$$

defined as the associated complex conjugate of $\mathfrak{h}_m$ fixing $(\mathfrak{h}_m)_{\mathbb{R}}$. We then have a Hermitian metric $(\ ,\ )_m$ on $\mathfrak{h}_m$ by setting

$$(X, Y)_m = \langle X, \bar{Y} \rangle_m, \quad X, Y \in \mathfrak{h}_m. \tag{2.1}$$

For the orthogonal complement $\mathfrak{t}^{\perp}$ of $\mathfrak{t}$ in $\mathfrak{h}_m$ in terms of this Hermitian metric, let $T^{\perp}$ denote the corresponding algebraic torus in H_m. We now define an algebraic subgroup G_m of $Z_m(T)$ by

$$G_m := T^{\perp} \cdot S_m. \tag{2.2}$$

For the T-equivariant Kodaira embedding $\Phi_m \colon M \hookrightarrow \mathbb{P}^*(V_m)$ associated to the complete linear system $|L^m|$ on M, let $d(m)$ denote the degree of the image $\Phi_m(M)$ in the projective space $\mathbb{P}^*(V_m)$. For the dual space W_m^* of $W_m := S^{d(m)}(V_m)^{\otimes n+1}$, we have the Chow form

$$0 \neq \hat{M}_m \in W_m^*$$

for the irreducible reduced algebraic cycle $\Phi_m(M)$ on $\mathbb{P}^*(V_m)$, so that the corresponding element $[\hat{M}_m]$ in $\mathbb{P}^*(W_m)$ is the Chow point for the cycle $\Phi_m(M)$. Consider the natural action of $\mathrm{SL}(V_m)$ on W_m^* induced by the action of $\mathrm{SL}(V_m)$ on V_m.

DEFINITION 2.3. (1) (M, L^m) is said to be *Chow-stable relative to T* if the orbit $G_m \cdot \hat{M}_m$ is closed in W_m^*.
(2) (M, L) is said to be *asymptotically Chow-stable relative to T* if (M, L^m) is Chow-stable relative to T for each integer $m \gg 1$.

3. Relative Chow-stability for each fixed *m*

In this section, we consider a polarized algebraic manifold (M, L) under the assumption that (M, L^m) is Chow-stable relative to T for a fixed positive integer m. Then we shall show that a polybalanced metric ω_m exists in the class $c_1(L)_{\mathbb{R}}$.

The space $\Lambda_m := \left\{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\nu_m}) \in \mathbb{C}^{\nu_m};\ \sum_{k=1}^{\nu_m} n_k \lambda_k = 0\right\}$ and the Lie algebra $\mathfrak{h}_m$ are identified by an isomorphism

$$\Lambda_m \cong \mathfrak{h}_m, \quad \lambda \leftrightarrow X_\lambda, \tag{3.1}$$

with $(\mathfrak{h}_m)_{\mathbb{R}}$ corresponding to the set $(\Lambda_m)_{\mathbb{R}}$ of the real points in Λ_m, where X_λ is the endomorphism of V_m defined by

$$X_\lambda := \bigoplus_{k=1}^{\nu_m} \lambda_k \mathrm{id}_{V(\chi_k)} \in \bigoplus_{k=1}^{\nu_m} \mathrm{End}(V(\chi_k)) \quad (\subset \mathrm{End}(V_m)).$$

In terms of the identification (3.1), we can write the Hermitian metric $(\ ,\ )_m$ on $\mathfrak{h}_m$ in (2.1) in the form

$$(\lambda, \mu)_m := \sum_{k=1}^{\nu_m} \frac{n_k \lambda_k \bar{\mu}_k}{m^{n+2}},$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\nu_m})$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{\nu_m})$ are in $\mathbb{C}^{\nu_m}$. By the identification (3.1), corresponding to the decomposition $\mathfrak{h}_m = \mathfrak{t} \oplus \mathfrak{t}^\perp$, we have the orthogonal direct sum

$$\Lambda_m = \Lambda(\mathfrak{t}) \oplus \Lambda(\mathfrak{t}^\perp),$$

where $\Lambda(\mathfrak{t})$ and $\Lambda(\mathfrak{t}^\perp)$ are the subspace of Λ_m associated to $\mathfrak{t}$ and $\mathfrak{t}^\perp$, respectively. Take a Hermitian metric ρ_k on $V(\chi_k)$, and for the metric

$$\rho := \bigoplus_{k=1}^{\nu_m} \rho_k$$

on V_m, we see that $V(\chi_k) \perp V(\chi_{k'})$ whenever $k \neq k'$. By choosing an orthonormal basis $\{s_{k,i}\,;\ i = 1, 2, \ldots, n_k\}$ for the Hermitian vector space $(V(\chi_k), \rho_k)$, we now set

$$j(k, i) := i + \sum_{l=1}^{k-1} n_l, \quad i = 1, 2, \ldots, n_k;\ k = 1, 2, \ldots, \nu_m, \tag{3.2}$$

where the right-hand side denotes i in the special case $k = 1$. By writing $s_{k,i}$ as $s_{j(k,i)}$, we have an orthonormal basis

$$\mathcal{S} := \{s_1, s_2, \ldots, s_{N_m}\}$$

for (V_m, ρ). By this basis, the vector space V_m and the algebraic group $\mathrm{SL}(V_m)$ are identified with $\mathbb{C}^{N_m} = \{(z_1, \ldots, z_{N_m})\}$ and $\mathrm{SL}(N_m, \mathbb{C})$, respectively. In terms of $\mathcal{S}$, the Kodaira embedding Φ_m is given by

$$\Phi_m(x) := (s_1(x) : \cdots : s_{N_m}(x)), \quad x \in M.$$

Consider the associated *Chow norm* $W_m^* \ni \xi \mapsto \|\xi\|_{\mathrm{CH}(\rho)} \in \mathbb{R}_{\geq 0}$ as in Zhang [14] (see also [4]). Then by the closedness of $G_m \cdot \hat{M}_m$ in W_m^* (cf. (2.2) and Definition 2.3), the Chow norm on the orbit $G_m \cdot \hat{M}_m$ takes its minimum at $g_m \cdot \hat{M}_m$ for some $g_m \in G_m$. Note that, by complexifying

$$K_m := \prod_{k=1}^{\nu_m} \mathrm{SU}(V(\chi_k)),$$

we obtain the reductive algebraic group S_m. For each $\kappa \in K_m$ and each diagonal matrix Δ in $\mathfrak{sl}(N_m, \mathbb{C})$, we put

$$e(\kappa, \Delta) := \exp\{\mathrm{Ad}(\kappa)\Delta\}.$$

Then g_m is written as $\kappa_1 \cdot e(\kappa_0, D)$ for some $\kappa_0, \kappa_1 \in K_m$ and a diagonal matrix $D = (d_j)_{1\leq j\leq N_m}$ in $\mathfrak{sl}(N_m, \mathbb{C})$ with the j-th diagonal element d_j. Put $g'_m := e(\kappa_0, D)$. In view of $\|g'_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)} = \|g_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)}$ (cf. [10], Proposition 4.1), we obtain

$$\|g'_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)} \leq \|e(\kappa_0, t(X_\lambda + A)) \cdot g'_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)}, \quad t \in \mathbb{C}, \tag{3.3}$$

for all $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\nu_m}) \in \Lambda(\mathfrak{t}^\perp)$ and all diagonal matrices $A = (a_j)_{1\leq j\leq N_m}$ in $\mathfrak{s}_m$, where each a_j denotes the j-th diagonal element of A. We now write $a_{j(k,i)}$ as $a_{k,i}$ for simplicity. Put

$$s'_j := \kappa_0^{-1} \cdot s_j, \quad b_{k,i} := \lambda_k + a_{k,i}, \quad c_{k,i} := \exp d_{j(k,i)}.$$

Then we shall now identify V_m with $\mathbb{C}^{N_m} = \{(z'_1, \dots, z'_{N_m})\}$ by the orthonormal basis $\mathcal{S}' := \{s'_1, s'_2, \dots, s'_{N_m}\}$ for V_m. In view (3.2), we rewrite s'_j, z'_j as $s'_{k,i}$, $z'_{k,i}$, respectively by

$$s'_{k,i} := s'_{j(k,i)}, \quad z'_{k,i} := z'_{j(k,i)},$$

where $k = 1, 2, \dots, \nu_m$ and $i = 1, 2, \dots, n_k$. By writing $b_{k,i}$, $c_{k,i}$ also as $b_{j(k,i)}$, $c_{j(k,i)}$, respectively, we consider the diagonal matrices B and C of order N_m with the j-th diagonal elements b_j and c_j, respectively. Note that the right-hand side of (3.3) is

$$\|(\exp tB) \cdot C \cdot \kappa_0^{-1} \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)},$$

and its derivative at $t = 0$ vanishes by virtue of the inequality (3.3). Hence, by setting $\Theta := (\sqrt{-1}/2\pi)\,\partial\bar{\partial} \log\bigl(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} z'_{k,i}|^2\bigr)$, we obtain the equality (see for instance (4.4) in [4])

$$\int_M \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} b_{k,i} |c_{k,i} s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2} \Phi'^*_m(\Theta^n) = 0 \tag{3.4}$$

for all $\lambda \in \Lambda(\mathfrak{t}^{\perp})$ and all diagonal matrices A in the Lie algebra $\mathfrak{s}_m$, where $\Phi'_m \colon M \hookrightarrow \mathbb{P}^*(V_m)$ is the Kodaira embedding of M by $\mathcal{S}'$ which sends each $x \in M$ to $\big(s'_1(x) : s'_2(x) : \cdots : s'_{N_m}(x)\big) \in \mathbb{P}^*(V_m)$. Here we regard

$$s'_{k,i} = \Phi'^* z'_{k,i}.$$

Let $k_0 \in \{1, 2, \ldots, \nu_m\}$ and let $i_1, i_2 \in \{1, 2, \ldots, n_{k_0}\}$ with $i_1 \neq i_2$. Using Kronecker's delta, we first specify the real diagonal matrix B by

$$\lambda_k = 0 \quad \text{and} \quad a_{k,i} = \delta_{kk_0}(\delta_{ii_1} - \delta_{ii_2}),$$

where $k = 1, 2, \ldots, \nu_m$; $i = 1, 2, \ldots, n_k$. By (3.4) applied to this B, and let (i_1, i_2) run through the set of all pairs of two distinct integers in $\{1, 2, \ldots, n_{k_0}\}$, where positive integer k_0 varies from 1 to ν_m. Then there exists a positive constant $\beta_k > 0$ independent of the choice of i in $\{1, 2, \ldots, n_k\}$ such that, for all i,

$$\int_M \frac{|c_{k,i} s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2} \Phi'^*_m(\Theta^n) = \beta_k, \quad k = 1, 2, \ldots, \nu_m. \tag{3.5}$$

Let k_0, i_1, i_2 be as above, and let κ_2 be the element in K_m such that

$$\kappa_2 z'_{k_0,i_1} = \frac{1}{\sqrt{2}}(z'_{k_0,i_1} - z'_{k_0,i_2}), \quad \kappa_2 z'_{k_0,i_2} = \frac{1}{\sqrt{2}}(z'_{k_0,i_1} + z'_{k_0,i_2})$$

and that κ_2 fixes all other $z_{k,i}$'s. Let κ_3 be the element in K_m such that

$$\kappa_3 z'_{k_0,i_1} = \frac{1}{\sqrt{2}}\big(z'_{k_0,i_1} + \sqrt{-1} z'_{k_0,i_2}\big), \quad \kappa_3 z'_{k_0,i_2} = \frac{1}{\sqrt{2}}\big(\sqrt{-1} z'_{k_0,i_1} + z'_{k_0,i_2}\big)$$

and that κ_3 fixes all other $z'_{k,i}$'s. Now

$$\|\kappa_2 g'_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)} = \|\kappa_3 g'_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)} = \|g'_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)},$$

and note that

$$2 z'_{k_0,i_1} \bar{z}'_{k_0,i_2} = (|\kappa_2 z'_{k_0,i_2}|^2 - |\kappa_2 z'_{k_0,i_1}|^2) - \sqrt{-1}(|\kappa_3 z'_{k_0,i_2}|^2 - |\kappa_3 z'_{k_0,i_1}|^2).$$

Hence replacing g'_m by $\kappa_\alpha g'_m$, $\alpha = 2, 3$, in (3.3), we obtain the case $k' = k''$ of the following by an argument as in deriving (3.5) from (3.3):

$$\int_M \frac{s'_{k',i'} \bar{s}'_{k'',i''}}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2} \Phi'^*_m(\Theta^n) = 0, \quad \text{if} \quad (k', i') \neq (k'', i''). \tag{3.6}$$

Here (3.6) holds easily for $k' \neq k''$, since for every element g of the maximal compact subgroup of T, we have:

$$\begin{aligned}\text{L.H.S. of (3.6)} &= \int_M g^* \left\{ \frac{s'_{k',i'}\bar{s}'_{k'',i''}}{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|c_{k,i}s'_{k,i}|^2}\Phi'^*_m(\Theta^n)\right\}\\ &= \frac{\chi_{k''}(g)}{\chi_{k'}(g)}\int_M \frac{s'_{k',i'}\bar{s}'_{k'',i''}}{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|c_{k,i}s'_{k,i}|^2}\Phi'^*_m(\Theta^n).\end{aligned}$$

Put $\beta := (\beta_1, \beta_2, \dots, \beta_{\nu_m}) \in \mathbb{R}^{\nu_m}$ and $\beta_0 := \big(\sum_{k=1}^{\nu_m} n_k\beta_k\big)/N_m$, where β_k is given in (3.5). In view of $N_m = \sum_{k=1}^{\nu_m} n_k$, by setting $\underline{\beta}_k := \beta_k - \beta_0$, we have

$$\underline{\beta} := \big(\underline{\beta}_1, \underline{\beta}_2, \dots, \underline{\beta}_{\nu_m}\big) \in \Lambda_m.$$

Next for each $\lambda \in \Lambda(\mathfrak{t}^\perp)$, by setting $a_{k,i} = 0$ for all (k,i), the equality (3.4) above implies $0 = \sum_{k=1}^{\nu_m}(n_k\lambda_k)\beta_k$. From this together with the equality $\sum_{k=1}^{\nu_m} n_k\lambda_k = 0$, we obtain $(\lambda, \underline{\beta})_m = 0$, i.e.,

$$\underline{\beta} \in \Lambda(\mathfrak{t}). \tag{3.7}$$

We now define a Hermitian metric h_{FS} (cf. [14]) for L^m as follows. Let u be a local section for L^m. Then[1]

$$|u|^2_{h_{\mathrm{FS}}} := \frac{|u|^2}{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|c_{k,i}s'_{k,i}|^2}. \tag{3.8}$$

For the Hermitian metric $h_m := (h_{\mathrm{FS}})^{1/m}$ for L, we consider the associated Kähler metric $\omega_m := c_1(L; h_m)_{\mathbb{R}}$ on M. In view of (3.5),

$$\begin{aligned}\beta_0 &= \frac{\sum_{k=1}^{\nu_m} n_k\beta_k}{N_m} = N_m^{-1}\int_M \Phi^*_m(\Theta^n)\\ &= N_m^{-1}m^n c_1(L)^n[M] = n!\left\{1 + O\left(\frac{1}{m}\right)\right\}.\end{aligned} \tag{3.9}$$

Then for $\gamma_{m,k} := \beta_k/\beta_0$ and $\sigma_{k,i} := c_{k,i}s'_{k,i}(m^n\beta_k^{-1})^{1/2}$, we have

$$\begin{aligned}\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\gamma_{m,k}|\sigma_{k,i}|^2_{h_m} &= \sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\gamma_{m,k}|\sigma_{k,i}|^2_{h_{\mathrm{FS}}}\\ &= \frac{m^n}{\beta_0}\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|c_{k,i}s'_{k,i}|^2_{h_{\mathrm{FS}}} = \frac{m^n}{\beta_0}.\end{aligned} \tag{3.10}$$

[1]In view of (3.8), there is some error in [4]. Actually, for the numerator of (5.7) in the paper [4], please read $(N_m + 1)|s|^2$.

By operating $(\sqrt{-1}/2\pi)\,\bar\partial\partial\log$ on (3.8), we obtain

$$\Phi_m'^{*}(\Theta) = c_1(L^m; h_{\mathrm{FS}})_{\mathbb{R}} = mc_1(L; h_m)_{\mathbb{R}} = m\omega_m. \tag{3.11}$$

Then in terms of the Hermitian L^2 inner product (1.1), we see from (3.5), (3.6), (3.8) and (3.11) that $\{\sigma_{k,i}\,;\, k = 1, \dots, \nu_m,\ i = 1, \dots, n_k\}$ is an orthonormal basis for V_m. Moreover, (3.10) is rewritten as

$$B_m^{\circ}(\omega_m) = \frac{m^n}{\beta_0}, \tag{3.12}$$

where $B_m^{\circ}(\omega_m)$ is as in (1.2). Hence ω_m is a polybalanced metric, and the proof of Theorem A is reduced to showing (1.3) and (1.4). By summing up, we obtain

Theorem C. *If (M, L^m) is Chow-stable relative to T for a positive integer m, then the Kähler class $c_1(L)_{\mathbb{R}}$ admits a polybalanced metric ω_m with the weights $\gamma_{m,k}$ as above.*

4. The asymptotic behavior of the weights $\gamma_{m,k}$

The purpose of this section is to prove (1.3). If $\underline{\beta} = 0$, then we are done. Hence, we may assume that $\underline{\beta} \neq 0$. Consider the sphere

$$\sum := \{X \in \mathfrak{t}_{\mathbb{R}}\,;\, \langle X, X\rangle_0 = 1\}$$

in $\mathfrak{t}_{\mathbb{R}} := \mathfrak{t} \cap (\mathfrak{h}_m)_{\mathbb{R}}$, where $\langle\ ,\ \rangle_0$ denotes the positive definite symmetric bilinear form on $\mathfrak{g}$ as in [2]. Since all components $\underline{\beta}_k$ of $\underline{\beta}$ are real, we see from (3.7) that, in view of (3.1), $\lambda := r_m\underline{\beta}$ satisfies

$$X_\lambda \in \sum \tag{4.1}$$

for some positive real number r_m. Hence by writing $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\nu_m})$, we obtain positive constants C_2, C_3 independent of k and m such that (see for instance [5], Lemma 2.6)

$$-C_2 m \le \lambda_k \le C_3 m. \tag{4.2}$$

Put $g(t) := \exp(tX_\lambda)$ and $\gamma(t) := \log\|g(t)\cdot C\cdot\kappa_0^{-1}\cdot\hat{M}_m\|_{\mathrm{CH}(\rho)}$, $t \in \mathbb{R}$, by using the notation in Section 3. Since $g(t)$ commutes with $C\cdot\kappa_0^{-1}$, we see that $g(t)$ defines a holomorphic automorphism of $\Phi_m'(M)$. In view of Theorem 4.5 in [4], it follows from Remark 4.6 in [4] that (cf. [14])

$$\dot\gamma(t) = \dot\gamma(0) = \sum_{k=1}^{\nu_m} n_k\lambda_k\beta_k, \quad -\infty < t < +\infty. \tag{4.3}$$

Consider the classical Futaki invariant $\mathcal{F}_1(X_\lambda)$ associated to the holomorphic vector field X_λ on (M, L). Since M is smooth, this coincides with the corresponding Donaldson–Futaki's invariant for test configurations. Then by applying Lemma 4.8 in [6] to the product configuration of (M, L) associated to the one-parameter group generated by X_λ on the central fiber, we obtain (see also [1], [11])

$$\lim_{t\to-\infty} \dot{\gamma}(t) = C_4\left\{\mathcal{F}_1(X_\lambda) + O\left(\frac{1}{m}\right)\right\}m^n, \tag{4.4}$$

where $C_4 := (n+1)!\, c_1(L)^n[M] > 0$. Hence by $\sum_{k=1}^{\nu_m} n_k\lambda_k = 0$ and $\lambda = r_m\underline{\beta}$, it now follows from (4.3) and (4.4) that

$$\begin{aligned} C_4\left\{\mathcal{F}_1(X_\lambda) + O\left(\frac{1}{m}\right)\right\}m^n &= r_m^{-1}\sum_{k=1}^{\nu_m} n_k\lambda_k^2 \\ &= r_m^{-1}m^{n+2}\langle X_\lambda, X_\lambda\rangle_m, \end{aligned} \tag{4.5}$$

where by (4.1) above, (7) in [1] (see also [11]) implies $\langle X_\lambda, X_\lambda\rangle_m \geq C_5$ for some positive real constant C_5 independent of m. Furthermore $\mathcal{F}_1(X_\lambda) = O(1)$ again by (4.1). Hence from (4.5), we obtain

$$r_m^{-1} = O\left(\frac{1}{m^2}\right). \tag{4.6}$$

In view of (3.9), since $\underline{\beta}_k = r_m^{-1}\lambda_k$, (4.2) and (4.6) imply the required estimate (1.3) as follows:

$$\gamma_{m,k} - 1 = \frac{\beta_k}{\beta_0} - 1 = \frac{\underline{\beta}_k}{\beta_0} = O\left(\frac{1}{m}\right).$$

5. Proof of (1.4) and Corollary B

In this section, keeping the same notation as in the preceding sections, we shall prove (1.4) and Corollary B.

Proof of (1.4). For X_λ in (4.1), the associated Hamiltonian function $f_\lambda \in C^\infty(M)_{\mathbb{R}}$ on the Kähler manifold (M, ω_m) is

$$f_\lambda = \frac{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\lambda_k\gamma_{m,k}|\sigma_{k,i}|^2_{h_m}}{m\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\gamma_{m,k}|\sigma_{k,i}|^2_{h_m}} = \frac{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\lambda_k|c_{k,i}s'_{k,i}|^2}{m\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|c_{k,i}s'_{k,i}|^2},$$

where by (4.2), when m runs through the set of all sufficiently large integers, the function f_λ is uniformly C^0-bounded. Now by (3.10),

$$\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\lambda_k\gamma_{m,k}|\sigma_{k,i}|^2_{h_m} = \frac{m^{n+1}f_\lambda}{\beta_0}. \tag{5.1}$$

We now define a function I_m on M by

$$I_m := \frac{r_m^{-1}}{\beta_0}\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\{1-(\gamma_{m,k})^{-1}\}\lambda_k\gamma_{m,k}|\sigma_{k,i}|^2_{h_m}. \tag{5.2}$$

Then by (1.3), (3.9), (3.10), (4.2) and (4.6), we easily see that

$$I_m = O\left(m^{-2}\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\gamma_{m,k}|\sigma_{k,i}|^2_{h_m}\right) = O(m^{n-2}). \tag{5.3}$$

By (3.10) together with (5.1) and (5.2), it now follows that

$$\begin{aligned}\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|\sigma_{k,i}|^2_{h_m} &= \sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\gamma_{m,k}|\sigma_{k,i}|^2_{h_m} - \sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}(\gamma_{m,k}-1)|\sigma_{k,i}|^2_{h_m}\\ &= \frac{m^n}{\beta_0} - \sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\frac{\underline{\beta}_k}{\beta_0}|\sigma_{k,i}|^2_{h_m}\\ &= \frac{m^n}{\beta_0} + I_m - \frac{r_m^{-1}}{\beta_0}\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\lambda_k\gamma_{m,k}|\sigma_{k,i}|^2_{h_m}\\ &= \frac{m^n}{\beta_0} + I_m - \frac{r_m^{-1}m^2}{\beta_0^2}f_\lambda m^{n-1}.\end{aligned}$$

By (3.9), $m^n/\beta_0 = N'_m$. Moreover, by (4.6), $r_m^{-1}m^2/\beta_0^2 = O(1)$. Since

$$f_m := -\frac{r_m^{-1}m^2}{\beta_0^2}f_\lambda, \quad m \gg 1,$$

are uniformly C^0-bounded Hamiltonian functions on (M,ω_m) associated to holomorphic vector fields in $\mathfrak{t}$, in view of (5.3), we obtain

$$B_m^\bullet(\omega_m) = \sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|\sigma_{k,i}|^2_{h_m} = N'_m + f_m m^{n-1} + O(m^{n-2}),$$

as required. □

Proof of Corollary B. Since the classical Futaki character $\mathcal{F}_1$ vanishes on $\mathfrak{t}$, we have $\mathcal{F}_1(X_\lambda) = 0$ in (4.5), so that

$$r_m^{-1} = O\left(\frac{1}{m^3}\right).$$

Then from (3.9) and $\underline{\beta}_k = r_m^{-1}\lambda_k$, by looking at (4.2), we obtain the following required estimate:

$$\gamma_{m,k} - 1 = \frac{\underline{\beta}_k}{\beta_0} = O\left(\frac{1}{m^2}\right).$$

Hence $B_m^{\bullet}(\omega_m) = \{1 + O(1/m^2)\}B_m^{\circ}(\omega_m)$. Integrating this over M by the volume form ω_m^n, in view of (3.12), we see that

$$N_m' = \left\{1 + O\left(\frac{1}{m^2}\right)\right\}\frac{m^n}{\beta_0}.$$

Therefore, from (3.12) and $B_m^{\bullet}(\omega_m) = \{1 + O(1/m^2)\}B_m^{\circ}(\omega_m)$, we now conclude that $B_m^{\bullet}(\omega_m) = N_m' + O(m^{n-2})$, as required. □

References

[1] S.K. Donaldson: *Lower bounds on the Calabi functional*, J. Differential Geom. **70** (2005), 453–472.

[2] A. Futaki and T. Mabuchi: *Bilinear forms and extremal Kähler vector fields associated with Kähler classes*, Math. Ann. **301** (1995), 199–210.

[3] Z. Lu: *On the lower order terms of the asymptotic expansion of Tian–Yau–Zelditch*, Amer. J. Math. **122** (2000), 235–273.

[4] T. Mabuchi: *Stability of extremal Kähler manifolds*, Osaka J. Math. **41** (2004), 563–582.

[5] T. Mabuchi: *An energy-theoretic approach to the Hitchin–Kobayashi correspondence for manifolds*, II, Osaka J. Math. **46** (2009), 115–139.

[6] T. Mabuchi: *K-stability of constant scalar curvature polarization*, arXiv:`math.DG/0812.4093`.

[7] T. Mabuchi: *Relative stability and extremal metrics*, in preparation.

[8] T. Mabuchi and Y. Nitta: *K-stability and asymptotic Chow stability*, in preparation.

[9] D. Mumford, J. Fogarty and F. Kirwan: Geometric Invariant Theory, third edition, Ergebnisse der Mathematik und ihrer Grenzgebiete (2) **34**, Springer, Berlin, 1994.

[10] Y. Sano: *On stability criterion of complete intersections*, J. Geom. Anal. **14** (2004), 533–544.

[11] G. Székelyhidi: *Extremal metrics and K-stability*, Bull. Lond. Math. Soc. **39** (2007), 76–84.

[12] G. Tian: *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), 99–130.

[13] S. Zelditch: *Szegő kernels and a theorem of Tian*, Internat. Math. Res. Notices (1998), 317–331.

[14] S. Zhang: *Heights and reductions of semi-stable varieties*, Compositio Math. **104** (1996), 77–105.

Department of Mathematics
Osaka University
Toyonaka, Osaka, 560-0043
Japan