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ON A LATTICE ORDERED GROUPOID

Dedicated to Professor Keizo Asano for his 60th birthday

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In most cases a multiplicative partially ordered system satisfies the ditributive law: $a(b\cup c)=ab\cup ac$ (e.g. a lo-semigroup of the ideals in a ring, lo-semigroups of the normal subgroups of a group, etc.). But there are more general examples of multiplicative systems in each of which a weak distributive law: $a(b\cup c)=ab\cup ac\cup(ab)c$ is satisfied. The purpose of the present paper is to develop the theory of normal chain and regular union of a partially ordered groupoid satisfying the weak distributive law.

In §1 we define a lattice ordered groupoid with some conditions and define normal elements and a normal closure in this system and give their properties. In §2 we treat a classification of our system M and show that the classified system also satisfies the same conditions for M. In §3 we define a normal chain in our system and give some results of the chain. In §4 we consider the modularity of our system and give an extension of direct union, called a regular union, and study some results of the union. In §5 we show that the results of the preceding sections are applicable to the family of subgroups of a group and that of the ideals in commutative ring, and list the applied results.

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1. Definitions and elementary properties

Let M be a non-void set with the following five conditions (M1~M5).

M1. M is a commutative groupoid,

M2. M is a complete (upper and lower) lattice,

M3. $ab\leq a\cup b$ for all $a, b\in M$,

M4. $a(b\cup c)=ab\cup ac\cup(ab)c$, if $bc\leq b$ or $bc\leq c$.

An element b of M is said to be *normal* with respect to a, or shortly *a-normal*, if $ba\leq b$. For the greatest element e of M, an *e-normal* element of M is simply said to be normal. We shall denote by N and N_a the set of all normal elements of M and that of all a-normal elements of M respectively.

M5. $(ab)c\leq(bc)a\cup(ca)b$ holds for normal elements a, b and c.

EXAMPLES. (1) Let $\mathfrak{G}$ be a set consisting of subgroups of a group G. Then $\mathfrak{G}$ satisfies the above conditions M1, $\cdots$, M5 under the commutator-product and the set-inclusion[1]. In this case normal subgroups of G are normal elements of $\mathfrak{G}$.

(2) The set $\mathfrak{R}$ consisting of the subrings of a commutative ring R satisfies the above five conditions under the module-product and the set-inclusion. In this case the multiplication is associative, and every ideal is evidently normal.

We shall list some elementary properties of M.

Proposition 1. (1) *$a\leq b$ implies $ac\leq bc$ for all $c\in M$.*
(2) *$(ab)b\leq ab$ for all a, $b\in M$.*
(3) *$N\subseteq N_a$ for every a of M.*
(4) *$a(b\cup c)=ab\cup ac$, if a is normal and b is c-normal.*
(5) *$ab\leq a\cap b$ holds for a, $b\in N$.*
(6) *N is closed under the join, meet and multiplication.*

Proof. For (1), since $ab\leq a\cup b=b$ (by M3), by using M4 we have $bc=(a\cup b)c=ac\cup bc\cup(ac)b\geq ac$. For (2), since $bb\leq b\cup b=b$, by using M4 we have $ab=a(b\cup b)=ab\cup ab\cup(ab)b=ab\cup(ab)b$, and hence $ab\geq(ab)b$. For (3), let b be any element of N, then $b\geq be\geq ba$ (by (1)), hence we have $b\in N_a$. For (4), since a is normal, we have $ab\leq a$ (by (3)). Hence $(ab)c\leq ac$. Therefore we obtain $a(b\cup c)=ab\cup ac\cup(ab)c=ab\cup ac$. (5) is obvious. For (6), let a and b be any two elements of N. Then we have $e(a\cup b)=ea\cup eb\leq a\cup b$. Hence $a\cup b\in N$. Since $e(a\cap b)\leq ea\leq a$ and similarly $e(a\cap b)\leq b$, we have $e(a\cap b)\leq a\cap b$. Hence $a\cap b\in N$. By using 5 we have $e(ab)\leq(ea)b\cup(eb)a\leq ab\cup ab=ab$. Hence $ab\in N$.

DEFINITION 1. The greatest lower bound of the set $\{x\,|\,x\geq a,\ xe\leq x\}$ is called a *normal closure* of a, and is denoted by $\bar{a}$.

The normal closure has the following properties.

Proposition 2. (1) *$\bar{a}$ is normal,* (2) *$a\leq\bar{a}$,* (3) *$a\leq b$ implies $\bar{a}\leq\bar{b}$,* (4) *$\bar{a}=\bar{\bar{a}}$,* (5) *$\overline{a\cup b}=\bar{a}\cup\bar{b}$,* (6) *$\overline{ab}\leq\bar{a}\bar{b}$.*

Proof. For (1), $\bar{a}e=(\inf\{x\,|\,x\geq a,\ xe\leq x\})e\leq\inf\{xe\,|\,x\geq a,\ xe\leq x\}\leq\inf\{x\,|\,x\geq a,\ xe\leq x\}=\bar{a}$. (2), (3) and (4) are obvious. For (5), since $\bar{a}\cup\bar{b}$ is normal (by Proposition 1 (6)), we have $\overline{\bar{a}\cup\bar{b}}=\bar{a}\cup\bar{b}$ and hence $\overline{a\cup b}\leq\overline{\bar{a}\cup\bar{b}}=\bar{a}\cup\bar{b}$. On the other hand, since $\overline{a\cup b}\geq\bar{a}$ and $\overline{a\cup b}\geq\bar{b}$, we have $\overline{a\cup b}\geq\bar{a}\cup\bar{b}$. Hence we obtain $\overline{a\cup b}=\bar{a}\cup\bar{b}$. For (6), Since $\bar{a}\bar{b}$ is normal (by Proposition 1 (6)), we have $\overline{ab}\leq\overline{\bar{a}\bar{b}}=\bar{a}\bar{b}$.

Lemma 1. *$a\cup ab$ is b-normal for all a, $b\in M$.*

1) See § 5 of this paper.

Proof. By Proposition 1 (2) $a(ab)\leq ab$, and hence we have $b(a\cup ab)=ba\cup b(ab)\cup(ab)(ab)=ab\leq a\cup ab$.

Theorem 1. *If $a\cup b=e$, then ab is normal.*

Proof. By Lemma 1 $b(a\cup ab)\leq a\cup ab$, and hence we have $(ab)e=(ab)(a\cup b)=(ab)(a\cup ab\cup b)=(ab)((a\cup ab)\cup b)=(ab)(a\cup ab)\cup(ab)b\cup((ab)(a\cup ab))b$. Since $a(ab)\leq ab$, by using M4 we have $(ab)\,(a\cup ab)=a(ab)\cup(ab)(ab)\cup((ab)a)(ab)\leq ab\cup ab\cup(ab)(ab)=ab$, and hence $((ab)(a\cup ab))b\leq(ab)b\leq ab$. Therefore we obtain $(ab)e\leq ab$.

Theorem 2. *$\bar{a}=a\cup ae$ for any $a\in M$.*

Proof. Since $a\leq\bar{a}$ and $ae\leq\bar{a}e\leq\bar{a}$, we have $\bar{a}\geq a\cup ae$. On the other hand, by Lemma 1 $a\cup ae$ is normal. By the definition of the normal closure we have $\bar{a}\leq a\cup ae$. Therefore we obtain $\bar{a}=a\cup ae$.

Corollary 3. *If $a\cup b=e$, then $\bar{a}=a\cup ab$.*

Proof. By Theorem 2 and M4 we have $\bar{a}=a\cup ae=a\cup a(a\cup b)=a\cup a((a\cup ab)\cup b)=a\cup a(a\cup ab)\cup ab\cup(a(a\cup ab))b$. Since $a(ab)\leq ab$, we have $a(a\cup ab)=aa\cup a(ab)\cup(aa)(ab)\leq a\cup ab$. Since $a\cup ab$ is b-normal (by Lemma 1), we have $(a(a\cup ab))b\leq(a\cup ab)b\leq a\cup ab$. Hence $\bar{a}\leq a\cup ab$. On the other hand, since $\bar{a}=a\cup ae$ we have $\bar{a}\geq a\cup ab$. Therefore we obtain $\bar{a}=a\cup ab$.

Corollary 4. *If $a\cup b=e$, $a\geq n$ and $an\leq n$, then $n\cup ab$ is normal.*

Proof. Since e and ab are normal, we have

$$\begin{aligned}
e(n\cup ab) &= en\cup e(ab) \qquad \text{(by Proposition 1 (3))}\\
&\leq(a\cup b)n\cup ab=(a\cup(ab\cup b))n\cup ab\\
&= an\cup(ab\cup b)n\cup(an)(ab\cup b)\cup ab \qquad \text{(by M4)}\\
&\leq n\cup(ab\cup b)n\cup n(ab\cup b)\cup ab \qquad \text{(because } an\leq n)\\
&= n\cup(ab\cup b)n\cup ab=n\cup(ab)n\cup bn\cup((ab)n)b\cup ab \qquad \text{(by M4)}\\
&= n\cup ab \text{ (because } ((ab)n)b\leq(ab)b\leq ab \text{ and } nb\leq ab)\,.
\end{aligned}$$

Hence $n\cup ab$ is normal.

2. A classification of M

Let a be an arbitrary fixed element of N. We now define an equivalence relation of M by putting $u\sim v(a)$, if $u\cup a=v\cup a$, where u, $v\in M$. It is easily verified that this relation is stable for the join and the multiplication. That is, $\sim(a)$ is a congruence relation with respect to the join and the multiplication, which is called an *a-congruence* relation of M. The a-congruence class containing an element u is denoted by $K_a(u)$. The join and the multiplication of the classes are defined by $K_a(u) \cup K_a(v) = K_a(u \cup v)$ and $K_a(u)K_a(v) = K_a(uv)$ respectively. Then the set M/a of the classes forms a partially ordered groupoid with the following properties. (1) $K_a(u) = K_a(a)$ if and only if $u \leq a$. (2) $K_a(u) \leq K_a(v)$ if and only if $u \leq v \cup a$. In particular, $u \leq v$ implies $K_a(u) \leq K_a(v)$. (3) $K_a(e)$ and $K_a(a)$ are the greatest element and least element of M/a, respectively.

Lemma 2. (1) $\sup_\alpha \{K_a(x_\alpha)\} = K_a(\sup_\alpha \{x_\alpha\})$.

(2) $\inf_\alpha \{K_a(x_\alpha)\} = K_a(\inf_\alpha \{x_\alpha \cup a\})$.

Proof. (1) is obvious. For (2), put $b = \inf_\alpha \{x_\alpha \cup a\}$. Then, since $b \leq x_\alpha \cup a$ for all α, we have $K_a(b) \leq K_a(x_\alpha)$ (by (2) of the properties of M/a). Suppose that $K_a(c)$ is any lower bound of the set $\{K_a(x_\alpha)\}$. Then, we have $K_a(c) \leq K_a(x_\alpha)$ for all α, hence $c \leq x_\alpha \cup a$ (again by the property (2) of M/a). From this, we have $c \leq \inf_\alpha \{x_\alpha \cup a\} = b$. Thus $K_a(c) \leq K_a(b)$. That is, $K_a(b)$ is the greatest lower bound of the set $\{K_a(x_\alpha)\}$.

Theorem 5. *M/a satisfies the conditions $M1 \sim M5$.*[2)]

Proof. It is evident that M/a satisfies M1, M2, M3 and M4. For M5, we begin by showing that, if $K_a(u)$ is normal in M/a then $u \cup a$ is normal in M. Let $K_a(u)$ be normal, then we have $K_a((u \cup a)e) = K_a(u \cup a)K_a(e) = K_a(u)K_a(e) \leq K_a(u)$. Hence we obtain $(u \cup a)e \leq u \cup a$. Let $K_a(u)$, $K_a(v)$ and $K_a(w)$ be normal, we have

$$\begin{aligned}
&(K_a(u)K_a(v))K_a(w) = (K_a(u \cup a)K_a(v \cup a))K_a(w \cup a) \\
&= K_a(((u \cup a)(v \cup a))(w \cup a)) \\
&\leq K_a(((u \cup a)(w \cup a))(v \cup a) \cup ((v \cup a)(w \cup a))(u \cup a)) \qquad \text{(by M5)} \\
&= (K_a(u)K_a(w))K_a(v) \cup (K_a(v)K_a(w))K_a(u) .
\end{aligned}$$

Lemma 3. $\overline{K_a(b)} = K_a(\bar{b})$ *for all $b \in M$.*

Proof. Since $K_a(\bar{b})$ is normal, we have $\overline{K_a(b)} \leq \overline{K_a(\bar{b})} = K_a(\bar{b})$. On the other hand, put $\overline{K_a(b)} = K_a(c)$ then $K_a(b) \leq K_a(c)$, and hence $b \leq c \cup a$. Since $K_a(c)$ is normal in M/a, $c \cup a$ is normal in M. Hence we have $\bar{b} \leq c \cup a$. Therefore we obtain $K_a(\bar{b}) \leq K_a(c \cup a) = K_a(c) = \overline{K_a(b)}$.

REMARK. It can be proved that if M is a modular lattice then so is M/a.

Theorem 6. *If $a \cup b = e$, then $\bar{a}\bar{b} = \overline{ab}$.*

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2) The normality and the normal closure of elements of M/a are similarly defined as M.

Proof. By Corollary 3, we have $K_{\overline{ab}}(\bar{a}\bar{b})=K_{\overline{ab}}(\bar{a})K_{\overline{ab}}(\bar{b})=K_{\overline{ab}}(a\cup ab)$ $K_{\overline{ab}}(b\cup ab)=K_{\overline{ab}}(a)K_{\overline{ab}}(b)=K_{\overline{ab}}(ab)$, and hence $\bar{a}\bar{b}\leq\overline{ab}$ (because $K_{\overline{ab}}(\bar{a}\bar{b})=K_{\overline{ab}}(ab)$ is the least element in $M/\overline{ab}$). On the other hand, by Proposition 2 (6) we have $\overline{ab}\leq\bar{a}\bar{b}$.

Theorem 7. *If* $\bigcup_{i=1}^{n} a_i=e$, *then* $\overline{\bigcup_{r\neq s} a_r a_s}=\bigcup_{r\neq s}\bar{a}_r\bar{a}_s$ $(r=1, 2, \cdots, n;\ s=1, 2, \cdots, n)$.

Proof. First we show that $\bar{a}_1(\bar{a}_2\cup\cdots\cup\bar{a}_n)\leq\bigcup_{r\neq s}\overline{a_r a_s}$. By Theorem 6, we have $\bar{a}_1(\bar{a}_2\cup\cdots\cup\bar{a}_n)=\bar{a}_1\overline{(a_2\cup\cdots\cup a_n)}=\overline{a_1(a_2\cup\cdots\cup a_n)}$. Put $a_3\cup\cdots\cup a_n=b_2$, then we have

$$\begin{aligned} a_1(a_2\cup b_2) &= a_1(a_2\cup(a_2b_2\cup b_2))\\ &=a_1a_2\cup a_1(a_2b_2\cup b_2)\cup(a_1a_2)(a_2b_2\cup b_2) \qquad \text{(by M4)}\\ &\leq\overline{a_1a_2}\cup a_1(\overline{a_2b_2}\cup b_2)\cup(\overline{a_1a_2})(a_2b_2\cup b_2)\\ &=\overline{a_1a_2}\cup a_1(\overline{a_2b_2})\cup a_1b_2\cup(a_1(\overline{a_2b_2}))b_2\leq\overline{a_1a_2}\cup\overline{a_2b_2}\cup\overline{a_1b_2}\,. \end{aligned}$$

Hence we have $\overline{a_1(a_2\cup b_2)}\leq\overline{\overline{a_1a_2}\cup\overline{a_2b_2}\cup\overline{a_1b_2}}=\overline{a_1a_2}\cup\overline{a_2b_2}\cup\overline{a_1b_2}$. Let us assume that $\overline{a_1(a_2\cup\cdots\cup a_n)}=\bigcup_{\substack{r\neq s\\ r=1, s=1}}^{k}\overline{a_r a_s}\cup(\bigcup_{i=1}^{k}\overline{a_i b_k})$, where $b_k=a_{k+1}\cup\cdots\cup a_n$. Since $\bigcup_{i=1}^{k}\overline{a_i b_k}=\bigcup_{i=1}^{k}\overline{a_i(a_{k+1}\cup b_{k+1})}\leq\bigcup_{i=1}^{k}(\overline{a_i a_{k+1}}\cup\overline{a_i b_{k+1}}\cup\overline{a_{k+1}b_{k+1}})=\bigcup_{i=1}^{k}\overline{a_i a_{k+1}}\cup\bigcup_{i=1}^{k+1}\overline{a_i b_{k+1}}$, we have $\overline{a_1(a_2\cup\cdots\cup a_n)}=\bigcup_{\substack{r\neq s\\ r=1, s=1}}^{k+1}\overline{a_r a_s}\cup(\bigcup_{i=1}^{k+1}\overline{a_i b_{k+1}})$. Putting $k=n-1$ we have $\overline{a_1(a_2\cup\cdots\cup a_n)}\leq\bigcup_{\substack{r\neq s\\ r=1, s=1}}^{n}\overline{a_r a_s}$. Similarly we obtain $\bar{a}_i(\bar{a}_1\cup\cdots\cup\bar{a}_{i-1}\cup\bar{a}_{i+1}\cup\cdots\cup\bar{a}_n)$ $\leq\bigcup_{\substack{r\neq s\\ r=1, s=1}}^{n}\overline{a_r a_s}$. Since $\bigcup_{r\neq s}\bar{a}_r\bar{a}_s=\bar{a}_1(\bar{a}_2\cup\cdots\cup\bar{a}_n)\cup\cdots\cup\bar{a}_i(\bar{a}_2\cup\cdots\cup\bar{a}_{i-1}\cup\bar{a}_{i+1}\cup\cdots\cup\bar{a}_n)\cup\cdots\cup\bar{a}_n(\bar{a}_1\cup\cdots\cup\bar{a}_{n-1})$, we obtain $\bigcup_{r\neq s}\bar{a}_r\bar{a}_s\leq\bigcup_{r\neq s}\overline{a_r a_s}$. On the other hand, by using Proposition 2(6) we have $\bigcup_{r\neq s}\bar{a}_r\bar{a}_s\geq\overline{\bigcup_{r\neq s} a_r a_s}$.

3. Normal chain

In this and the next sections, we shall assume that $ao=o$ for any element a of M and the least element o of M and that $(\sup X)n=\sup(Xn)$ for any subset X of N and any element n of N.

DEFINITION 2. The chain $\{a^{(0)}, a^{(1)}, \cdots, a^{(n-1)}, a^{(n)}, \cdots\}$ with $a^{(0)}=\bar{a}$ and $a^{(n)}=a^{(n-1)}e$ is called a *minimal normal chain* of a determind by e (shortly *a-e-chain*).The chain $\{a^{[0]}, a^{[1]}, \cdots, a^{[n-1]}, a^{[n]},\cdots\}$ with $a^{[0]}=a$ and $a^{[n]}=a^{[n-1]}a$ is called an *a-a-chain*.

The following properties are immediate.

(1) $a^{(n)}$ is normal and $a^{(n)}\geq a^{(n+1)}$ for every whole number n.

(2) $a^{[n]}$ is a-normal and $a^{[n]} \geq a^{[n+1]}$ for every whole number n.

Theorem 8. *$(\bigcup_{i=1}^{n} a_i)^{(p)} = \bigcup_{i=1}^{n} a_i^{(p)}$ for any $a \in M$.*

Proof. By Proposition 2 (5) $(\bigcup_{i=1}^{n} a_i)^{(0)} = \overline{\bigcup_{i=1}^{n} a_i} = \bigcup_{i=3}^{n} \bar{a}_i = \bigcup_{i=1}^{n} a_i^{(0)}$. Hence the theorem holds for $p=0$. Let us assume that the theorem holds for $p=k-1$. Then we have $(\bigcup_{i=1}^{n} a_i)^{(k)} = (\bigcup_{i=1}^{n} a_i)^{(k-1)} e = (\bigcup_{i=1}^{n} a_i^{(k-1)}) e = \bigcup_{i=1}^{n} (a_i^{(k-1)} e) = \bigcup_{i=1}^{n} a_i^{(k)}$. This completes the proof.

Theorem 9. *$e^{[p-1]} a \leq a^{(p)}$ for any $a \in M$.*

Proof. If $p=1$, this is trivial. Let us now assume that this holds for $p=k-1$. Then we have

$$\begin{aligned} e^{[k-1]} a &\leq (e^{[k-2]} e) \bar{a} \\ &\leq (e\bar{a}) e^{[k-2]} \cup (\bar{a} e^{[k-2]}) e && \text{(by M5)} \\ &\leq a^{(1)} e^{[k-2]} \cup \bar{a}^{(k-1)} e && \text{(by the assumption)} \\ &\leq (a^{(1)})^{(k-1)} \cup a^{(k)} = a^{(k)} \cup a^{(k)} = a^{(k)} . \end{aligned}$$

This completes the proof.

Theorem 10. *If $\bigcup_{i=1}^{n} a_i = e$, then $(\bigcup_{r \neq s} a_r a_s)^{(p)} = (\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(p)}$.*

Proof. This is easily verified by the induction on p.

DEFINITION 3. The least upper bound of the set $\{x \mid xe=o, x \in N\}$ is called an *annihilator* of e. A chain $o=c_0 \leq c_1 \leq c_2 \leq \cdots \leq c_n \leq \cdots$ is called an *upper normal chain*, if c_n is normal and $K_{c_n}(c_{n+1})$ is an annihilator of $K_{c_n}(e)$ in M/c_n for every whole number n.

Lemma 4. *Let a be an annihilator of e. Then the equality $ae=o$ holds.*

Proof. Since a is normal (by the definition of the annihilator), $ae = (\sup \{x \mid xe=o, x \in N\}) e = \sup \{xe\} = o$ (by the assumption of this section).

Theorem 11. *Let $o=c_0 \leq c_1 \leq c_2 \leq \cdots \leq c_n \leq \cdots$ be an upper normal chain. Then $(c_n)^{(n)}=o$, and if $a^{(n)}=o$ for some $a \in M$ then $a \leq c_n$.*

Proof. We show that $c_n^{(k)} \leq c_{n-k}$. Since $K_{c_{n-1}}(c_n)$ is an annihilator of $K_{c_{n-1}}(e)$, we have $K_{c_{n-1}}(c_n^{(1)}) = K_{c_{n-1}}(c_n e) = K_{c_{n-1}}(c_n) K_{c_{n-1}}(e) = K_{c_{n-1}}(c_{n-1})$. Hence we obtain $c_n^{(1)} \leq c_{n-1}$. Let us assume that $c_n^{(k-1)} \leq c_{n-k+1}$. Then we have $c_n^{(k)} = c_n^{(k-1)} e \leq c_{n-k+1}^{(1)} \leq c_{n-k}$. Therefore we obtain $c_n^{(n)} \leq c_0 = o$ if $k=n$.

For the second part of the theorem, we show that $c_k \geq a^{(n-k)}$. By the assumption $a^{(n)} = a^{(n-1)} e = o$, we have $K_{c_0}(a^{(n-1)}) K_{c_0}(e) = K_{c_0}(o)$. Since $K_{c_0}(c_1)$

is an annihilator of $K_{c_0}(e)$, by the definition of the annihilator we have $K_{c_0}(c_1) \geq K_{c_0}(a^{(n-1)})$. This shows that $c_1 \geq c_0 \cup a^{(n-1)}$, and hence $c_1 \geq a^{(n-1)}$. Let us assume that $c_{k-1} \geq a^{(n-k+1)}$. Then, since $a^{(n-k+1)} = a^{(n-k)}e$ we have $K_{c_{k-1}}(c_{k-1}) \geq K_{c_{k-1}}(a^{(n-k)})K_{c_{k-1}}(e)$. Since $K_{c_{k-1}}(c_k)$ is an annihilator of $K_{c_{k-1}}(e)$, we have $K_{c_{k-1}}(c_k) \geq K_{c_{k-1}}(a^{(n-k)})$. Hence $c_k = c_k \cup c_{k-1} \geq a^{(n-k)}$. Putting $k=n$, we obtain $c_n \geq a^{(0)} = \bar{a} \geq a$, as desired.

DEFINITION 4. An element a is said to be *nilpotent* if $a^{[n]} = o$ for some positive integer n. An element a is said to be *semi-nilpotent* if there exists a finite chain $a = a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n = o$ with $a_{i-1}a_{i-1} \geq a_i$ $(i=1, 2, \cdots, n)$.

Proposition 3. (1) *If a is nilpotent, then a is semi-nilpotent.*

(2) *If e is nilpotent, then a is nilpotent for all $a \in M$ and $K_b(e)$ is nilpotent in M/b for all $b \in N$.*

Proof. (1) If a is nilpotent, then $a = a^{[0]} \geq a^{[1]} \geq \cdots \geq a^{[n]} = o$ and $a^{[i-1]}a^{[i-1]} \leq aa^{[i-1]} = a^{[i]}$. Therefore a is semi-nilpotent.

(2) Since $a^{[i]} \leq e^{[i]}$ and $(K_b(e))^{[i]} = K_b(e^{[i]})$, this is obvious.

Theorem 12. *If $e(\neq o)$ is nilpotent, then the annihilator of e is not o.*

Proof. Suppose that $e^{[n]} = o$ for some positive integer n. Then $e^{[i]} = o$ and $e^{[i-1]} \neq o$ for some i $(1 \leq i \leq n)$. Since $e^{[i-1]}e = e^{[i]} = o$, $e^{[i-1]}$ precedes the annihilator of e.

4. Regular unions

In this section we shall assume the following condition.

M6. If $b \leq a \cup c$ and $ac \leq a$ or $ac \leq c$, then $b \leq (a \cap (b \cup c)) \cup (c \cap (a \cup b))$ for $a, b, c \in M$.

Lemma 5. *If $a \cup c = b \cup c$, $a \cap c = b \cap c$, $a \leq b$ and $ac \leq a$, then $a = b$.*

Proof. Since $b \leq a \cup c$ and $ac \leq a$, by M6 we have $b \leq (a \cap (b \cup c)) \cup (c \cap (a \cup b)) = (a \cap (a \cup c)) \cup (c \cap b) = a \cup (a \cap c) = a$. Hence we obtain $a = b$.

Lemma 6. *If a and c are b-normal and $a \leq c$, then $a \cup (b \cap c) = (a \cup b) \cap c$.*

Proof. Put $a' = a \cup (b \cap c)$, $b' = (a \cup b) \cap c$ and $c' = b$. Then, since $a(b \cap c) \leq ab \leq a$, by M4 we have $a'c' = (a \cup (b \cap c))b = ab \cup b(b \cap c) \cup (ab)(b \cap c)$. Since a and $b \cap c$ are b-normal, $ab \leq a$ and $b(b \cap c) \leq b \cap c$, and we have $a(b \cap c) \leq ab \cap ac \leq a \cap c$. Therefore $a'c' \leq a \cup (b \cap c) \cup (a \cap c) = a \cup (b \cap c) = a'$. And we have $a' \cup c' = b' \cup c'$, $a' \cap c' = b' \cap c'$ and $a' \leq b'$. Hence by using Lemma 5, we obtain $a' = b'$.

DEFINITION 5. A finite number of elements $a_1, a_2, \cdots, a_n$ of M is said to be *normally independent*, if $a_i \cap \overline{(a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_n)} = o$ for $i=1, 2, \cdots, n$.

DEFINITION 6. An element b is called a *regular union* of $a_1, a_2, \cdots, a_n$, and is denoted by $b=a_1 \cup^{(R)} a_2 \cup^{(R)} \cdots \cup^{(R)} a_n$, if $b=a_1 \cup a_2 \cup \cdots \cup a_n$ and if $a_1, a_2, \cdots, a_n$ are normally independent.

An element b is called a *k-th nilpotent union* of $a_1, a_2, \cdots, a_n$, and is denoted by $b=a_1 \cup^{(k)} a_2 \cup^{(k)} \cdots \cup^{(k)} a_n$, if $b=a_1 \cup^{(R)} a_2 \cup^{(R)} \cdots \cup^{(R)} a_n$ and if $(\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(k)} = o$ but $(\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(k-1)} \neq o$ $(r, s=1, 2, \cdots, n)$. In particular, 0–th nilpotent union is called a *direct union*.

An element b is called a *free union* of $a_1, a_2, \cdots, a_n$, and is denoted by $b=a_1 \cup^{(F)} a_2 \cup^{(F)} \cdots \cup^{(F)} a_n$, if $b=a_1 \cup^{(R)} a_2 \cup^{(R)} \cdots \cup^{(R)} a_n$ and if $(\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(m)} \neq o$ and $(\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(m)} \lneqq (\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(m-1)}$ for every whole number m.

Lemma 7. $\bigcup_{r \neq s} \bar{a}_r \bar{a}_s \leq \overline{a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_n}$ $(r, s=1, 2, \cdots, n)$ *for each* i $(1 \leq i \leq n)$.

Proof. Since $\bar{a}_r \bar{a}_i \leq \bar{a}_r$ and $\bar{a}_r \bar{a}_s \leq \bar{a}_r \cup \bar{a}_s$, we have $\bigcup_{r \neq s} \bar{a}_r \bar{a}_s = \bar{a}_1 \bar{a}_i \cup \cdots \cup \bar{a}_{i-1} \bar{a}_i \cup \bar{a}_{i+1} \bar{a}_i \cup \cdots \cup \bar{a}_n \bar{a}_i \cup (\bigcup_{\substack{r \neq s \\ r \neq i, s \neq i}} \bar{a}_r \bar{a}_s) \leq \bar{a}_1 \cup \cdots \cup \bar{a}_{i-1} \cup \bar{a}_{i+1} \cup \cdots \cup \bar{a}_n = \overline{a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_n}$.

Lemma 8. *If the elements* $a_1, a_2, \cdots, a_n$ *are normally independent and* $a_i \geq c_i$ $(i=1, 2, \cdots, n)$, *then* $c_1, c_2, \cdots, c_n$ *are normally independent.*

Proof. $c_i \cap \overline{(c_1 \cup \cdots \cup c_{i-1} \cup c_{i+1} \cup \cdots \cup c_n)} \leq a_i \cap \overline{(a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_n)} = o$.

Lemma 9. *If the elements* $a_1, a_2, \cdots, a_n$ *are normally independent and* $c \leq \bigcup_{r \neq s} \bar{a}_r \bar{a}_s$, *then* $K_c(a_1), K_c(a_2), \cdots, K_c(a_n)$ *are normally independent, where* $c \in N$.

Proof. We have

$$\begin{aligned}
& K_c(a_i) \cap \overline{(K_c(a_1) \cup \cdots \cup K_c(a_{i-1}) \cup K_c(a_{i+1}) \cup \cdots \cup K_c(a_n))} \\
&= K_c(a_i) \cap (K_c(\bar{a}_1) \cup \cdots \cup K_c(\bar{a}_{i-1}) \cup K_c(\bar{a}_{i+1}) \cup \cdots \cup K_c(\bar{a}_n)) \\
&\qquad\qquad \text{(by Proposition 2 (5) and Lemma 3)} \\
&= K_c((a_i \cup c) \cap (\bar{a}_1 \cup \cdots \cup \bar{a}_{i-1} \cup \bar{a}_{i+1} \cup \cdots \cup \bar{a}_n \cup c)) \qquad \text{(by Lemma 2)} \\
&= K_c((a_i \cup c) \cap \overline{(a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_n)}) \qquad \text{(by Lemma 7)} \\
&= K_c(c \cup (a_i \cap \overline{(a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_n)})) \qquad \text{(by Lemma 6)} \\
&= K_c(c \cup o) = K_c(c).
\end{aligned}$$

Theorem 13. *If* $\bigcup_{i=1}^{n}{}^{(R)} a_i = b$, $c_i \leq a_i$ $(i=1, 2, \cdots, n)$ *and* $\bigcup_{i=1}^{n} c_i = d$, *then* $\bigcup_{i=1}^{n}{}^{(R)} c_i = d$.

Proof. This is obvious by Lemma 8.

Theorem 14. *If* $\bigcup_{i=1}^{n}{}^{(R)} a_i = b$ *and* $\bar{c} \le \bigcup_{r \neq s} \bar{a}_r \bar{a}_s$, *then* $\bigcup_{i=1}^{n}{}^{(R)} K_{\bar{c}}(a_i) = K_{\bar{c}}(b)$.

Proof. This is obvious by Lemma 9.

Theorem 15. *If* $a \cup^{(R)} b = e$ *and* a *is normal, then* $a \cup^{(0)} b = e$ *and* b *is normal.*

Proof. Since $\bar{a}\bar{b} \le \bar{a} \cap \bar{b} = a \cap \bar{b} = o$, we have $a \cup^{(0)} b = e$. And we have $be \le b(a \cup b) = ab \cup bb \cup (ab)b \le b$ (because $ab = o$). Hence b is normal.

Theorem 16. *If* $\bigcup_{i=1}^{n}{}^{(F)} a_i = b$, *then* $\bigcup_{i=1}^{n}{}^{(k)} (K_{(\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(k)}}(a_i)) = K_{(\bigcup_{r \neq s} a_r a_s)^{(k)}}(b)$.

Proof. Put $c = (\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(k)}$. Then, by Theorem 14 $\bigcup_{i=1}^{n}{}^{(R)} (K_c(a_i)) = K_c(b)$. And we have $(\bigcup_{r \neq s} \overline{K_c(a_r)}\, \overline{K_c(a_s)})^{(k)} = ((\cdots (\bigcup_{r \neq s} \overline{K_c(a_r)}\, \overline{K_c(a_s)}) K_c(e)) \cdots) K_c(e)) = K_c(((\cdots (\bigcup_{r \neq s} \bar{a}_r \bar{a}_s) e) \cdots) e)) = K_c((\bigcup_{r \neq s} \bar{a}_r \bar{a})_s{}^{(k)}) = K_c(c)$, but we have $(\bigcup_{r \neq s} \overline{K_c(a_r)}\, \overline{K_c(a_s)})^{(k-1)} = K_c((\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(k-1)}) \neq K(c)$. This completes the proof.

Corollary 17. *If* $\bigcup_{i=1}^{n}{}^{(R)} a_i = b$, *then* $\bigcup_{i=1}^{n}{}^{(0)} (K_{\bigcup_{r \neq s} \bar{a}_r \bar{a}_s}(a_i)) = K_{\bigcup_{r \neq s} \bar{a}_r \bar{a}_s}(b)$.

Proof. This is obvious by Theorem 16.

Theorem 18. *If* $a \cup^{(k)} b = e$ *and* $a = \bigcup_{i=1}^{n} a_i$, *then* $(\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(p)} = (\bigcup_{r \neq s} \bar{a}_r^a \bar{a}_s^a)^{(p)}$ *for* $p \ge k-1$, *where* $\bar{a}_i^a = a_i \cup a_i a$ $(i = 1, 2, \cdots, n)$.

Proof. By Corollary 4 $\bar{a}_i^a \cup ab$ is normal and $a_i \le \bar{a}_i^a \cup ab$, and hence we have $\bar{a}_i \le \bar{a}_i^a \cup ab$. By using Proposition 1 (4) and M4, we have $\bigcup_{r \neq s} \bar{a}_r \bar{a}_s \le \bigcup_{r \neq s} ((\bar{a}_r^a \cup ab)(\bar{a}_s^a \cup ab)) = \bigcup_{r \neq s} ((\bar{a}_r^a \cup ab) \bar{a}_s^a \cup (\bar{a}_r^a \cup ab)(ab)) = \bigcup_{r \neq s} (\bar{a}_r^a \bar{a}_s^a \cup \bar{a}_s^a (ab) \cup (\bar{a}_r^a \bar{a}_s^a)(ab) \cup (\bar{a}_r^a \cup ab)(ab)) \le \bigcup_{r \neq s} (\bar{a}_r^a \bar{a}_s^a \cup (ab) e) = (\bigcup_{r \neq s} \bar{a}_r^a \bar{a}_s^a) \cup (ab) e$. By using Theorem 8, we have $(\bigcup_{r \neq s} \bar{a}_r^a \bar{a}_s^a)^{(p)} \le (\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(p)} \le (\bigcup_{r \neq s} \bar{a}_r^a \bar{a}_s^a \cup (ab) e)^{(p)} = (\bigcup_{r \neq s} \bar{a}_r^a \bar{a}_s^a)^{(p)} \cup ((ab) e)^{(p)} = (\bigcup_{r \neq s} \bar{a}_r^k \bar{a}_s^a)^{(p)} \cup (ab)^{(p+1)} = (\bigcup_{r \neq s} \bar{a}_r^a \bar{a}_s^a)^{(p)}$ (because $(ab)^{(k)} = o$). This completes the proof.

Theorem 19. *If* $\bigcup_{i=1}^{n}{}^{(k)} a_i = e$, *then* $e^{[k-1]} (\bigcup_{r \neq s} \bar{a}_r \bar{a}_s) = o$.

Proof. By using Theorem 9, we have $e^{[k-1]} (\bigcup_{r \neq s} \bar{a}_r \bar{a}_s) \le (\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(k)} = o$.

Theorem 20. *If* $a \cup^{(k)} b = e$, *then* $\bar{a}^{[k]} \bar{b} = o$

Proof. We shall show that $\bar{a}^{[p]} \bar{b} \le (ab)^{(p)}$. Since $\bar{a}\bar{b} = \overline{ab}$ (by Theorem 6), we have $\bar{a}^{[0]} \bar{b} = \bar{a}\bar{b} = \overline{ab} = (ab)^{(0)}$. Let us assume that $a^{[p-1]} b \le (ab)^{(p-1)}$. Then we have

$$\begin{aligned}\bar{a}^{[p]}\bar{b} &= (\bar{a}\bar{a}^{[p-1]})\bar{b} \\ &\leq (\bar{a}\bar{b})a^{[p-1]} \cup (\bar{a}^{[p-1]}\bar{b})\bar{a} && \text{(by using M5)} \\ &\leq (\bar{a}\bar{b})e^{[p-1]} \cup (\bar{a}\bar{b})^{(p-1)}e && \text{(by the assumption)} \\ &\leq (\bar{a}\bar{b})^{(p)} \cup (\bar{a}\bar{b})^{(p)} && \text{(by Theorem 9)} \\ &= (ab)^{(p)}\,.\end{aligned}$$

Putting $p=k$, we obtain $\bar{a}^{[k]}\bar{b} \leq (ab)^{(k)}=o$.

5. Applications

(1) Application to groups

Let G be any group and let $A_1, A_2, \cdots, A_n$ be a finite number of subgroups of G. The following notations will be used:
$[A_1, A_2]$; the commutator subgroup of A_1 and A_2,
$\{A_1, A_2, \cdots, A_n\}$; the subgroup which is generated by $A_1, A_2, \cdots, A_n$,
$\bar{A}_1$; the normal subgroup which is generated by A_1,
$\{[A_r, A_s]\}$; the subgroup which is generated by all commutator subgroups $[A_r, A_s]$ $r \neq s$, $r, s=1, 2, \cdots, n$,
$A_1^{(p)}$; the commutator subgroup $[[\cdots\overbrace{[[\bar{A}_1, G], G], \cdots], G}^{p}]$,
$A_1^{[p]}$; the commutator subgroup $[[\cdots\overbrace{[[A_1, A_1], A_1], \cdots], A_1}^{p}]$,
$A_1 \wedge A_2$; the intersection of A_1 and A_2.

Lemma 10. *Let A, B and C be any subgroups of a group G. Then A, B and C have the following properties*:

(1) $[A, B]=[B, A]$,
(2) $[A, B] \subseteq \{A, B\}$,
(3) *If* $[B, C] \subseteq B$, *then* $[A, \{B, C\}]=\{[A, B], [A, C], [[A, B], C]\}$,
(4) *If A, B and C are normal subgroups of G, then* $[[A, B], C] \subseteq [[B, C]A][[C, A]B]$,
(5) *If* $B \subseteq \{A, C\}$ *and* $[A, C] \subseteq A$, *then* $B \subseteq \{A \wedge \{B, C\}, C \wedge \{A, B\}\}$.

Proof. The proofs of (1) and (4) are well-known. For (3), since $[B, C] \subseteq B$, for any elements $b \in B$ and $c \in C$ there exists an element $b' \in B$ such that $bc=cb'$. Therefore the generator of the commutator subgroup $[\{B, C\}, A]$ can be represented in the form $[bc, a]$, where $a \in A$, $b \in B$, $c \in C$. And we have $[bc, a] =[b, a][[b, a], c][c, a]$. Hence $[bc, a]$ belongs to $\{[B, A], [C, A], [[B, A], C]\}$. Thus $[\{B, C\}, A] \subseteq \{[B, A], [C, A], [[B, A], C]\}$. On the other hand, we have $[[b, a], c]=[a, b]\,[bc, a]\,[a, c]$, hence $[[b, a], c]$ belongs to $[\{B, C\}, A]$. The generator of the commutator subgroup $[[B, A], C]$ can be represented in the form $[u_1 u_2 \cdots u_m, c]$, where u_i are of the form $[b_i, a_i]$, $a_i \in A$, $b_i \in B$ $(i=1, 2, \cdots, m)$.

Since $[u_1u_2\cdots u_m, c]=(u_1u_2\cdots u_m)^{-1}c^{-1}u_1u_2\cdots u_mc=(u_1u_2\cdots u_m)^{-1}c^{-1}u_1cc^{-1}u_2c\cdots c^{-1}u_mc$, where $u_1u_2\cdots u_m$ and $c^{-1}u_ic$ belong to $[\{B, C\}, A]$, $[u_1u_2\cdots u_m, c]$ belongs to $[\{B, C\}, A]$, and hence $[[B, A], C]\subseteq[\{B, C\}, A]$. Therefore we obtain $[\{B, C\}, A]=\{[B, A], [C, A], [[B, A], C]\}$. For (5), let b be any element of B. Then there exist two elements $a\in A$ and $c\in C$ such that $b=ac$. Since $a=bc^{-1}$ and $c=a^{-1}b$, we have $a\in A\wedge\{B, C\}$, $c\in C\wedge\{A, B\}$. Thus b belongs to $\{A\wedge\{B, C\}, C\wedge\{A, B\}\}$. Hence we have $B\subseteq\{A\wedge\{B, C\}, C\wedge\{A, B\}\}$. (2) is obvious.

By Lemma 10, the results of the preceding sections are applicable to groups. That is, the results in §§1 and 2 illustrate the properties of the subgroups (general subgroups, normal subgroups, commutator subgroups, etc.) and factor-groups of a group. The results in §§3 and 4 can be applied to the theory of solvable groups and nilpotent groups and theory of direct products, free products, regular products and k-th nilpotent products[3] of the subgroups.

We shall list briefly the applied results.

(1) If $\{A_1, A_2\}=G$, then $[A_1, A_2]=\overline{[A_1, A_2]}=[\bar{A}_1, \bar{A}_2]$.
(2) $\bar{A}=A[A, G]$ for any subgroup A of G.
(3) If $\{A_1, A_2\}=G$, then $\bar{A}_1=A_1[A_1, A_2]$.
(4) If $\{A_1, A_2\}=G$ and N is a normal subgroup of A_1 then $N[A_1, A_2]$ is a normal subgroup of G.
(5) If $\{A_1, A_2,\cdots,A_n\}=G$, then $\{\overline{[A_r, A_s]}\}=\{[\bar{A}_r, \bar{A}_s]\}$.
(6) $\{A_1, A_2, \cdots, A_n\}^{(p)}=\{A_1^{(p)}, A_2^{(p)},\dots,A_n^{(p)}\}$.
(7) $[G^{[p-1]}, A]\subseteq A^{(p)}$ for any subgroup A of G.
(8) If $G=\{A_1, A_2,\cdots,A_n\}$, then $\{[A_r, A_s]\}^{(p)}=\{[\bar{A}_r, \bar{A}_s]\}^{(p)}$
(9) Let $Z_0=1\subseteq Z_1\subseteq Z_2\subseteq\cdots\subseteq Z_n\subseteq\cdots$ be an increasing central chain of G, where 1 is a unit group. Then $(Z_n)^{(n)}=1$, and if $A^{(n)}=1$ for some subgroup A of G then $A\subseteq Z_n$.
(10) The center of any nilpotent group is not the unit group.
(11) If G is a regular product of its subgroups A and B, and A is a normal subgroup of G, then G is a direct product of A and B, and B is a normal subgroup of G.
(12) If G is a k-th nilpotent product of its subgroups A and B and $A=\{A_1, A_2, \cdots, A_n\}$, then $\{[\bar{A}_r, \bar{A}_s]\}^{(p)}=\{[\bar{A}_r^A, \bar{A}_s^A]\}^{(p)}$, where $\bar{A}_i^A$ are the normal subgroups of A which are generated by A_i $(i=1, 2, \cdots, n)$.
(13) If G is a k-th nilpotent product of its subgroups $A_1, A_2, \cdots, A_n$, then $[G^{[k-1]}, \{[A_r, A_s]\}]$ is a unit group.
(14) If G is a k-th nilpotent product of its subgroups A and B, then $[\bar{A}^{[k]}, \bar{B}]$ is a unit group.

The proofs are obvious by the following correspondences;

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(3) Cf. 1.

(1) ⇔ Theorems 1 and 6, (2) ⇔ Theorem 2, (3) ⇔ Corollary 3,
(4) ⇔ Corollary 4, (5) ⇔ Theorem 7, (6) ⇔ Theorem 8,
(7) ⇔ Theorem 9, (8) ⇔ Theorem 10, (9) ⇔ Theorem 11,
(10) ⇔ Theorem 12, (11) ⇔ Theorem 15, (12) ⇔ Theorem 18,
(13) ⇔ Theorem 19, (14) ⇔ Theorem 20.

(2) Applicaton to commutative rings

Let R be any commutative ring with or without unity quantity and let $A_1, A_2, \cdots, A_n$ be a finite number of subrings of R. The following notations will be used:

$\{A_1, A_2, \cdots, A_n\}$; the subring which is generated by $A_1, A_2, \cdots, A_n$,
A_1A_2; the module-product of A_1 and A_2,
$\bar{A}_1$; the ideal in R which is generated by A_1,
$A^m = \overbrace{AA\cdots A}^{m}$.

It is easily verified that the set $\mathfrak{R}$ consisting of the subrings of a ring R satisfies the conditions M1∼M5 in §1. Hence the results of the preceding sections can be applied to the set $\mathfrak{R}$.

We shall list briefly the appied results.

(1) If $\{A_1, A_2\}=R$, then $A_1A_2=\overline{A_1A_2}=\bar{A}_1\bar{A}_2$.
(2) $\bar{A}=\{A, AR\}$ for any subring A of R.
(3) If $\{A_1, A_2\}=R$, then $\bar{A}_1=\{A_1, A_1A_2\}$.
(4) If $\{A_1, A_2\}=R$ and B is an ideal of A_1, then $\{B, A_1A_2\}$ is an ideals in R.
(5) Let $\{A_rA_s\}$ be a subring which is generated by all module-products A_rA_s, $r \neq s$, $r, s=1, 2, \cdots, n$. If $\{A_1, A_2, \cdots, A_n\}=R$, then $\{\overline{A_rA_s}\}=\{\bar{A}_r\bar{A}_s\}$.
(6) $\{A_1, A_2, \cdots, A_n\}^m=\{A_1^m, A_2^m, \cdots, A_n^m\}$.
(7) If $\{A_1, A_2, \cdots, A_n\}=R$, then $\{A_rA_s\}^m=\{\bar{A}_r\bar{A}_s\}^m$, $r \neq s$, $r, s=1, 2, \cdots, n$.

The proofs are obvious by the following correspondences;

(1) ⇔ Theorems 1 and 6, (2) ⇔ Theorem 2, (3) ⇔ Corollary 3,
(4) ⇔ Corollary 4, (5) ⇔ Theorem 7, (6) ⇔ Theorem 8,
(7) ⇔ Theorem 10.

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References

[1] O.N. Golovin: *Nilpotent prooduct of groups*, Mat. Sb. **27** (1950), 427–454.