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Author(s)	Hoshino, Mitsuo
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## **ON LAMBEK TORSION THEORIES, III**

## MITSUO HOSHINO

## (Received September, 1993)

In this note, developing our previous work [8] with S. Takashima, we will characterize rings R for which every finitely generated submodule of the injective envelope  $E(_RR)$  is torsionless. Those characterizations would yield recent results of Gómez Pardo and Guil Asensio [6, Theorems 1.5 and 2.2]. Also, we will provide a necessary and sufficient condition for an extension ring Q of a ring R to be a quasi-Frobenius maximal two-sided quotient ring of R.

Throughout this note, R stands for an associative ring with identity, modules are unitary modules, and torsion theories are Lambek torsion theories. Sometimes, we consider right R-modules as left  $R^{op}$ -modules, where  $R^{op}$  denotes the opposite ring of R, and we use the notation  $_{R}X$  (resp.  $X_{R}$ ) to stress that the module X considered is a left (resp. right) R-module. We denote by Mod R the category of left R-modules and by ()\* both the R-dual functors. For a module X, we denote by E(X) its injective envelope and by  $\varepsilon_{X}: X \to X^{**}$  the usual evaluation map. A module X is called torsionless (resp. reflexive) if  $\varepsilon_{X}$  is a monomorphism (resp. an isomorphism). For an  $X \in Mod R$ , we denote by  $\tau(X)$  its Lambek torsion submodule. Namely,  $\tau(X)$  is a submodule of X such that  $Hom_{R}(\tau(X), E(_{R}R)) = 0$ and  $X/\tau(X)$  is cogenerated by  $E(_{R}R)$ . A module X is called torsion (resp. torsionfree) if  $\tau(X) = X$  (resp.  $\tau(X) = 0$ ). A submodule Y of a module X is called a dense (resp. closed) submodule if X/Y is torsion (resp. torsionfree).

Here we recall some definitions. Let Y be a submodule of a module X. Then X is called a rational extension of Y if  $\operatorname{Hom}_R(X/Y, E(X)) = 0$ . Let Q be an extension ring of R, i.e., Q is a ring containing R as a subring with common identity. Then Q is called a left (resp. right) quotient ring of R if  $_RQ$  (resp.  $Q_R$ ) is a rational extension of  $_RR$  (resp.  $R_R$ ). A left quotient ring Q of R is called a maximal left quotient ring of R if  $E(_RQ)/Q$  is torsionfree. As an extension ring of R, a maximal left quotient ring of R is isomorphic to the biendomorphism ring of  $E(_RR)$  (see, e.g., Lambek [10] for details). An extension ring Q of R is called a maximal two-sided quotient ring of R if it is both a maximal left quotient ring of R and a maximal right quotient ring of R. A ring homomorphism  $R \to Q$  is called a left (resp. right) flat epimorphism if the induced functor  $_RQ\otimes_{R^-}$  (resp.  $-\otimes_RQ_R$ ) is a localization functor of Mod R (resp. Mod  $R^{op}$ ),

i.e.,  $Q_R$  (resp.  $_RQ$ ) is flat and  $Q \otimes_R Q \simeq Q$  canonically (see, e.g., Silver [17], Lazard [11] and Popescu and Spircu [15] for details). A module X is called  $\tau$ -finitely generated if it contains a finitely generated dense submodule. A finitely generated module X is called  $\tau$ -finitely presented (resp.  $\tau$ -coherent) if for every epimorphism (resp. homomorphism)  $\pi: Y \to X$  with Y finitely generated, Ker  $\pi$  is  $\tau$ -finitely generated. A module X is called  $\tau$ -noetherian (resp.  $\tau$ -artinian) if it satisfies the ascending (resp. descending) chain condition on closed submodules. Finally, a ring R is called left (resp. right)  $\tau$ -noetherian if  $_RR$  (resp.  $R_R$ ) is  $\tau$ -noetherian, left (resp. right)  $\tau$ -artinian if  $_RR$  (resp.  $R_R$ ) is  $\tau$ -artinian, and left (resp. right)  $\tau$ -coherent if  $_RR$  (resp.  $R_R$ ) is  $\tau$ -coherent.

1.  $\tau$ -absolutely pure and  $\tau$ -semicompact rings. In this section, we characterize rings R for which every finitely generated submodule of  $E(_RR)$  is torsionless.

Lemma 1.1 (Hoshino [7, Theorem A]). For a ring R the following are equivalent.

(a)  $\tau(X) = \text{Ker } \varepsilon_X \text{ for every finitely presented } X \in \text{Mod } R.$ 

(a)<sup>op</sup>  $\tau(M) = \text{Ker } \varepsilon_M$  for every finitely presented  $M \in \text{Mod } R^{\text{op}}$ .

Following [8], we call a ring R  $\tau$ -absolutely pure if it satisfies the equivalent conditions in Lemma 1.1. We call a homomorphism  $\pi: X \to Y$  a  $\tau$ -epimorphism if Cok  $\pi$  is torsion. Then we call a module X  $\tau$ -semicompact if for every inverse system of  $\tau$ -epimorphisms  $\{\pi_{\lambda}: X \to Y_{\lambda}\}_{\lambda \in \Lambda}$  with each  $Y_{\lambda}$  torsionless, the induced homomorphism  $\lim_{t \to \infty} \pi_{\lambda}: X \to \lim_{t \to \infty} Y_{\lambda}$  is a  $\tau$ -epimorphism. Finally, we call a ring R left (resp. right)  $\tau$ -semicompact if  $_{R}R$  (resp.  $R_{R}$ ) is  $\tau$ -semicompact.

**REMARKS.** (1) The  $\tau$ -semicompactness is just the *R*-linear compactness, in the sense of Gómez Pardo [5], relative to Lambek torsion theory.

(2) Let Mod  $R/\tau$  denote the quotient category of Mod R over the full subcategory Ker(Hom<sub>R</sub>(-,  $E(_RR)$ )). Assume that the image of  $_RR$  in Mod  $R/\tau$  is linearly compact in the sense of Gómez Pardo [5]. Then R is left  $\tau$ -semicompact.

**Theorem 1.2.** For a ring R the following are equivalent.

(a) Every finitely generated submodule of E(RR) is torsionless.

(b)  $\tau(X) = \text{Ker } \varepsilon_X$  for every finitely generated  $X \in \text{Mod } R$ .

(c)  $\operatorname{Ext}_{R}^{1}(X,R)$  is torsion for every finitely generated  $X \in \operatorname{Mod} R$ .

(d) R is  $\tau$ -absolutely pure and right  $\tau$ -semicompact.

Proof. (a)  $\Leftrightarrow$  (b). See Hoshino [7, Lemma 5].

(b)  $\Rightarrow$  (c). This is due essentially to Ohtake [14, Lemma 2.3]. Let  $0 \rightarrow Y \rightarrow F \rightarrow X \rightarrow 0$  be an exact sequence in Mod R with F finitely generated free

and let  $\pi: Y^* \to \operatorname{Ext}^1_R(X, R)$  denote the canonical epimorphism. Let  $h \in Y^*$  and form a push-out diagram:

$$0 \to Y \to F \to X \to 0$$
$${}^{h} \downarrow \qquad \parallel$$
$$0 \to R \stackrel{\phi}{\to} Z \to X \to 0.$$

Since Z is finitely generated, Ker  $\varepsilon_Z$  is torsion. Thus  $\phi^{**} \circ \varepsilon_R = \varepsilon_Z \circ \phi$  is monic, so is  $\phi^{**}$ . Hence  $(\operatorname{Cok}\phi^*)^* \simeq \operatorname{Ker}\phi^{**} = 0$ . Since  $\pi(h)R_R$  is an epimorphic image of  $\operatorname{Cok}\phi^*$ ,  $(\pi(h)R_R)^* = 0$  and thus  $\operatorname{Ext}_R^1(X,R)$  is torsion.

(c)  $\Rightarrow$  (b). Let  $X \in Mod R$  be finitely generated. Let Y be a submodule of Ker  $\varepsilon_X$  and let  $j: Y \to X$  denote the inclusion. Then  $j^*=0$  and  $Y^*$  embeds in  $Ext_R^1(X/Y, R)$ . Thus  $Y^*$  is torsion, so that  $Y^*=0$ . Hence Ker  $\varepsilon_X$  is torsion and  $\tau(X) = Ker \varepsilon_X$ .

(c)  $\Leftrightarrow$  (d). This is easily deduced from [8, Lemma 2.7].

REMARK. The equivalence (a)  $\Leftrightarrow$  (d) of Theorem 1.2 would yield a result of Gómez Pardo and Guil Asensio [6, Theorem 2.2].

**Corollary 1.3** (cf. Sumioka [20, Theorem 1]). Let R be left perfect. Then the following are equivalent.

(a) Every finitely generated submodule of E(R) is torsionless.

(b) R contains a faithful and injective left ideal.

Proof. (a)  $\Rightarrow$  (b). By Storrer [18] *R* contains an idempotent *e* with *ReR* a minimal dense right ideal. It is obvious that <sub>R</sub>*Re* is faithful. Since by Theorem 1.2 Ext<sup>1</sup><sub>R</sub>(*X*,*Re*)  $\simeq$  Ext<sup>1</sup><sub>R</sub>(*X*,*R*) $\otimes_R Re = 0$  for every finitely generated  $X \in \text{Mod } R$ , <sub>R</sub>*Re* is injective.

(b)  $\Rightarrow$  (a). Obvious.

**Corollary 1.4.** Let R be  $\tau$ -absolutely pure, left and right  $\tau$ -semicompact. Then both Ker  $\varepsilon_X$  and Cok  $\varepsilon_X$  are torsion for every finitely generated  $X \in Mod R$ .

Proof. Let  $X \in Mod R$  be finitely generated. By Theorem 1.2 Ker  $\varepsilon_X$  is torsion. We know from the argument of Jans [9, Theorem 1.1] that  $\operatorname{Cok} \varepsilon_X \simeq \operatorname{Ext}^1_R(M, R)$  with  $M \in \operatorname{Mod} R^{\operatorname{op}}$  finitely generated. Thus again by Theorem 1.2 Cok  $\varepsilon_X$  is torsion.

REMARK. Assume that R is a maximal left quotient ring of itself, i. e., E(R)/R is torsionfree. Then  $\operatorname{Ext}_{R}^{1}(X,Y)=0$  for all torsion  $X \in \operatorname{Mod} R$  and reflexive

 $Y \in Mod R$ . Thus Corollary 1.4 would yield a result of Gómez Pardo and Guil Asensio [6, Theorem 1.5].

**Corollary 1.5.** Let R be  $\tau$ -absolutely pure and left  $\tau$ -semicompact. Then every finitely generated  $X \in Mod R$  is  $\tau$ -semicompact.

Proof. Let  $X \in Mod R$  be finitely generated. Since every factor module of a  $\tau$ -semicompact module is  $\tau$ -semicompact, we may assume that X is free. Then the argument of [8, Lemma 2.7] applies.

2. Flat epimorphic extension rings. Throughout this section, Q stands for an extension ring of R.

The following lemmas seem to be known (cf. Silver [7], Lazard [11], Popescu and Spircu [15], Morita [13] and so on). However, for the benefit of the reader, we include proofs.

Lemma 2.1. The following are equivalent.

(1) The inclusion  $R \rightarrow Q$  is a left flat epimorphism.

(2)  $Q \otimes_{\mathbb{R}} X = 0$  for every submodule X of  $_{\mathbb{R}}Q/\mathbb{R}$ .

Proof. (1)  $\Rightarrow$  (2). Obvious.

 $(2) \Rightarrow (1)$ . Let  $\pi: Q \otimes_R Q \to Q$  denote the multiplication map. Then  $_Q \text{Ker } \pi \simeq_Q Q \otimes_R (Q/R) = 0$ . Next, let  $F_1 \to F_0 \to X \to 0$  be an exact sequence in Mod R with each  $F_i$  finitely generated free and put  $Y = \text{Im}(F_1 \to F_0)$ . We have a sequence of embeddings  $\text{Tor}_1^R(Q,X) \subseteq \text{Tor}_1^R(Q/R,X) \subseteq (Q/R) \otimes_R Y$ . Let us form a pull-back diagram:

$$(Q/R) \otimes_R F_1 \twoheadrightarrow (Q/R) \otimes_R Y$$

$$\forall \qquad \forall$$

$$Z \qquad \twoheadrightarrow \operatorname{Tor}_1^R(Q,X).$$

Since  $(Q/R) \otimes_R F_1$  is isomorphic to a finite direct sum of copies of  ${}_RQ/R$ , it follows by induction that  $Q \otimes_R Z = 0$ . Thus, since  $Q \otimes_R Q \simeq Q$  canonically,  $\operatorname{Tor}_1^R(Q,X) \simeq Q \otimes_R \operatorname{Tor}_1^R(Q,X) = 0$ .

Lemma 2.2. The following are equivalent.

- (1) Q is a left quotient ring of R.
- (2) (a)  ${}_{Q}Q \otimes_{R}(Q/R)$  is torsion. (b)  ${}_{O}\text{Tor}^{R}_{1}(Q,X)$  is torsion for every  $X \in \text{Mod } R$ .

Proof. Note that  $\operatorname{Hom}_{O}(Q \otimes_{R}(Q/R), E(_{O}Q)) \simeq \operatorname{Hom}_{R}(Q/R, \operatorname{Hom}_{O}(_{O}Q_{R}, E(_{O}Q))))$ 

and that  $\operatorname{Hom}_{Q}(\operatorname{Tor}_{1}^{R}(Q, X), E(Q)) \simeq \operatorname{Ext}_{R}^{1}(X, \operatorname{Hom}_{Q}(Q_{R}, E(Q)))$  for every  $X \in \operatorname{Mod} R$ . (1)  $\Rightarrow$  (2). Obvious.

 $(2) \Rightarrow (1)$ . It follows that  $_{R}\text{Hom}_{Q}(_{Q}Q_{R}, E(_{Q}Q))$  is injective. Thus  $E(_{R}Q)$  embeds in  $\text{Hom}_{Q}(_{Q}Q_{R}, E(_{Q}Q))$ . It then follows that  $\text{Hom}_{R}(Q/R, E(_{R}Q)) = 0$ .

The next lemma generalizes results of Cateforis [2, Proposition 2.2] and Masaike [12, Proposition 3] (cf. also Morita [13, Theorem 7.2]).

Lemma 2.3. The following are equivalent.

- (1) The inclusion  $R \rightarrow Q$  is a left flat epimorphism.
- (2) (a) Q is a left quotient ring of R.
  - (b)  ${}_{O}Q \otimes_{R} X$  is torsionfree for every submodule X of  ${}_{R}Q$ .

Proof. (1)  $\Rightarrow$  (2). By Lemma 2.2 (a) follows. It is obvious that (b) holds.

 $(2) \Rightarrow (1)$ . Let Y be a submodule of  ${}_{R}Q/R$ . Since  ${}_{R}Y$  is torsion, so is  ${}_{O}Q \otimes_{R}Y$ . Next, let us form a pull-back diagram:

$$0 \to R \xrightarrow{\phi} Q \to Q / R \to 0$$
$$\parallel \quad \mho \quad \mho$$
$$0 \to R \xrightarrow{\phi} X \to \quad Y \to 0,$$

where  $j: R \to Q$  is an inclusion. Since  ${}_{Q}Q \otimes_{R} j$  is a split monomorphism, so is  ${}_{Q}Q \otimes_{R} \phi$ . Thus  ${}_{Q}Q \otimes_{R} Y$  is torsionfree, so that  $Q \otimes_{R} Y = 0$ . By Lemma 2.1 the assertion follows.

Lemma 2.4. The following are equivalent.

- (1) (a) Q is a maximal left quotient ring of R.
  (b) E(QQ) is an injective cogenerator in Mod Q.
- (2) (a)  $_{R}Q/R$  is torsion. (b)  $Q \otimes_{R} X = 0$  for every torsion  $X \in Mod R$ .

Proof.  $(1) \Rightarrow (2)$ . Obvious.

 $(2) \Rightarrow (1)$ . By Lemma 2.1 the inclusion  $R \to Q$  is a left flat epimorphism. Thus by Lemma 2.2 Q is a left quotient ring of R. Next, let  $X \in Mod Q$  be torsion. Then  $_RX$  is torsion and thus  $_QX \simeq _QQ \otimes _RX = 0$ . Hence  $E(_QQ)$  is an injective cogenerator in Mod Q, so that Q is a maximal left quotient ring of R.

3. Flatness of the injective envelope. Throughout this section, Q stands for a left quotient ring of R.

**Lemma 3.1.** Let R be left  $\tau$ -noetherian and let  $X \in \text{Mod } R$  be flat. Then  ${}_{O}Q \otimes_{\mathbb{R}} X$  is torsionfree.

Proof. Let *I* be a dense left ideal of *R*. By Faith [4, Proposition 3.1] *I* contains a finitely generated subideal *J* with I/J torsion. Then R/J is finitely presented torsion, so that  $\operatorname{Hom}_{R}(R/J, Q \otimes_{R} X) \simeq \operatorname{Hom}_{R}(R/J, Q) \otimes_{R} X = 0$ . Thus  $\operatorname{Hom}_{R}(R/I, Q \otimes_{R} X) = 0$ . Hence  $_{R}Q \otimes_{R} X$  is torsionfree, so is  $_{O}Q \otimes_{R} X$ .

**Corollary 3.2.** Let R be left  $\tau$ -noetherian. Let  $n \ge 1$  and let  $X \in \text{Mod } R$  with weak dim<sub>R</sub> $X \le n$ . Then  $\text{Tor}_n^R(Q, X) = 0$ .

Proof. Let  $\dots \to F_1 \to F_0 \to X \to 0$  be an exact sequence in Mod R with each  $F_i$  free and put  $Y = \operatorname{Cok}(F_{n+1} \to F_n)$ . Then Y is flat and thus by Lemma 3.1  ${}_{Q}Q \otimes_{R} Y$  is torsionfree. On the other hand, by Lemma 2.2  ${}_{Q}\operatorname{Tor}_{n}^{R}(Q, X)$  is torsion. It follows that  $\operatorname{Tor}_{n}^{R}(Q, X) = 0$ .

**Lemma 3.3.** Let  $X \in \text{Mod } Q$  with  ${}_{Q}Q \otimes_{R} X$  torsionfree. Then  ${}_{Q}Q \otimes_{R} X \simeq_{Q} X$  canonically.

Proof. Let  $\pi: Q \otimes_R X \to X$  denote the canonical epimorphism. Then <sub>R</sub>Ker  $\pi \simeq_R(Q/R) \otimes_R X$  is torsion, so is <sub>o</sub>Ker  $\pi$ . It follows that Ker  $\pi = 0$ .

**Proposition 3.4.** Let R be left  $\tau$ -noetherian. Then every  $X \in \text{Mod } Q$  with  $_{R}X$  flat is flat. In particular,  $E(_{Q}Q)$  is flat whenever  $E(_{R}R)$  is.

Proof. Let  $X \in Mod Q$  with  $_{R}X$  flat. Then by Lemmas 3.1 and 3.3  $_{O}Q \otimes_{R}X \simeq_{O}X$  canonically. Since both  $- \otimes_{Q}Q_{R}$  and  $- \otimes_{R}X$  are exact, so is  $- \otimes_{Q}X$ .

**Proposition 3.5.** For a ring R the following are equivalent.

(1) Arbitrary direct products of copies of E(R) are flat.

(2) R is  $\tau$ -absolutely pure and right  $\tau$ -coherent.

Proof. (1)  $\Rightarrow$  (2). By Hoshino and Takashima [8, Lemma 1.4] R is  $\tau$ -absolutely pure. Next, let  $0 \rightarrow M \rightarrow F \rightarrow R$  be an exact sequence in Mod  $R^{\circ p}$  with F finitely generated free. By Colby and Rutter [3, Theorem 1.3] M contains a finitely generated submodule N with  $(M/N) \otimes_R E(R) = 0$ . It suffices to show that M/N is torision. For an  $L \in Mod R^{\circ p}$ , there exists a natural homomorphism

$$\theta_L: L \otimes_R E(R) \to \operatorname{Hom}_R(L^*, E(R))$$

such that  $\theta_L(x \otimes y)(\alpha) = \alpha(x)y$  for  $x \in L$ ,  $y \in E({}_RR)$  and  $\alpha \in L^*$ . Now, let L be a cyclic submodule of M/N and let  $\pi: R \to L$  be epic in Mod  $R^{\text{op}}$ . Since  $\theta_L \circ (\pi \otimes_R E({}_RR))$ 

= Hom<sub>R</sub>( $\pi^*, E(_R R)$ )  $\circ \theta_R$  is epic, so is  $\theta_L$ . Note that  $L \otimes_R E(_R R) = 0$ . Thus Hom<sub>R</sub>( $L^*, E(_R R)$ ) = 0 and hence  $L^* = 0$ . It follows that M/N is torsion. (2)  $\Rightarrow$  (1). See Hoshino and Takashima [8, Proposition 1.6].

4. Quasi-Frobenius qoutient rings. In this section, we provide a necessary

4. Quasi-Frobenius quatient rings. In this section, we provide a necessary and sufficient condition for an extension ring Q of R to be a quasi-Frobenius maximal two-sided quatient ring of R.

**Lemma 4.1.** Let R be left  $\tau$ -noetherian and let Q be a maximal left quotient ring of R. Assume that weak dim  ${}_{R}Q \leq 1$ . Then the inclusion  $R \rightarrow Q$  is a ring epimorphism.

Proof. We claim that  $(Q/R) \otimes_R Q = 0$ . Let *I* be a dense left ideal of *R*. By Faith [4, Proposition 3.1] *I* contains a finitely generated subideal *J* with I/Jtorsion. Note that *J* is also a dense left ideal of *R*. It follows that  $(Q/R)_R$  is an epimorphic image of the direct sum  $\oplus \operatorname{Hom}_R(R/J,Q/R)_R$ , where *J* runs over all finitely generated dense left ideals of *R*. Let *J* be a finitely generated dense left ideal of *R*. Since  $\operatorname{Hom}_R(R/J,Q/R)_R \simeq \operatorname{Ext}_R^1(R/J,R)$ , we have only to show that  $\operatorname{Ext}_R^1(R/J,R) \otimes_R Q = 0$ . For an  $X \in \operatorname{Mod} R$ , there exists a natural homomorphism

$$\delta_X : X^* \otimes_R Q \to \operatorname{Hom}_R(X, Q)$$

such that  $\delta_X(\alpha \otimes q)(x) = \alpha(x)q$  for  $\alpha \in X^*$ ,  $q \in Q$  and  $x \in X$ . As we remarked in [8], there exists an epimorphism  $\pi: X \to J$  with X finitely presented and Ker  $\pi$ torsion. Note that by Auslander [1, Proposition 7.1]  $\delta_X$  is monic. Since  $\pi^*$  is an isomorphism,  $\operatorname{Hom}_R(\pi, Q) \circ \delta_J = \delta_X \circ (\pi^* \otimes_R Q)$  is monic, so is  $\delta_J$ . Next, let  $j: J \to R$ denote the inclusion. Since  $\operatorname{Hom}_R(j,Q)$  is an isomorphism, so is  $\operatorname{Hom}_R(j,Q) \circ \delta_R$  $= \delta_J \circ (j^* \otimes_R Q)$ . Thus  $\delta_J$  is epic. Hence  $\delta_J$  is an isomorphism, so is  $j^* \otimes_R Q$ . It follows that  $\operatorname{Ext}^1_R(R/J, R) \otimes_R Q \simeq \operatorname{Cok}(j^* \otimes_R Q) = 0$ .

In case Q = R, the next theorem is due to Faith [4, Corollary 5.4].

**Theorem 4.2.** For an extension ring Q of R the following are equivalent. (1) Q is a quasi-Frobenius maximal two-sided quotient ring of R.

- (2) (a) R is left  $\tau$ -noetherian.
  - (b)  $_{R}Q/R$  is torsion.
    - (c)  $Q_R$  is injective.

Proof. (1)  $\Rightarrow$  (2). Obvious. (2)  $\Rightarrow$  (1). For an  $X \in Mod R$ , there exists a natural homomorphism M. HOSHINO

$$\theta_X: Q \otimes_R X \to \operatorname{Hom}_R(X^*, Q)$$

such that  $\theta_X(q \otimes x)(\alpha) = q\alpha(x)$  for  $q \in Q$ ,  $x \in X$  and  $\alpha \in X^*$ . Since  $Q_R$  is injective,  $\theta_X$ is an isomorphism for every finitely presented  $X \in Mod R$ . Let *I* be a dense left ideal of *R*. By Faith [4, Proposition 3.1] *I* contains a finitely generated subideal *J* with I/J torsion. Then R/J is finitely presented torsion, so that  $Q \otimes_R (R/J)$  $\simeq Hom_R((R/J)^*, Q) = 0$ . Thus  $Q \otimes_R (R/I) = 0$ . It follows that  $Q \otimes_R X = 0$  for every torsion  $X \in Mod R$ . Hence by Lemma 2.4, *Q* is a maximal left quotient ring of *R*, and E(QQ) is an injective cogenerator in Mod *Q*. Thus by Lemma 2.1  $Q_R$  is flat as well as injective, so that  $E(R_R)$  is flat. Hence by Hoshino and Takashima [8, Proposition 1.7] and Masaike [12, Proposition 2] *Q* is a right quotient ring of *R*. It follows that *Q* is a right selfinjective maximal right quotient ring of *R*. On the other hand, since *R* is left  $\tau$ -noetherian, so is *Q*. Thus *Q* is left noetherian. Hence by Faith [4, Theorem 2.1] *Q* is quasi-Frobenius.

**Corollary 4.3.** Let R be left and right noetherian and let Q be a maximal left quotient ring of R. Then the following are equivalent.

(1) Q is a quasi-Frobenius maximal two-sided quotient ring of R.

(2)  $_{R}Q$  is flat and inj dim  $_{R}Q \leq 1$ .

Proof. (1)  $\Rightarrow$  (2). By Lemma 2.3 <sub>R</sub>Q is flat. Also, <sub>R</sub>Q is injective by Lambek [10, §5].

 $(2) \Rightarrow (1)$ . By Lemmas 4.1 and 2.2 Q is a right quotient ring of R. Next, we claim that  ${}_{R}Q$  is injective. Since

$$\operatorname{Tor}_{2}^{R}(E(R_{R}), X) \simeq \operatorname{Hom}_{R}(\operatorname{Ext}_{R}^{2}(X, R), E(R_{R}))$$
$$\simeq \operatorname{Hom}_{R}(\operatorname{Ext}_{R}^{2}(X, R), \operatorname{Hom}_{Q}(_{R}Q_{Q}, E(Q_{Q})))$$
$$\simeq \operatorname{Hom}_{Q}(\operatorname{Ext}_{R}^{2}(X, R) \otimes_{R}Q, E(Q_{Q}))$$
$$\simeq \operatorname{Hom}_{Q}(\operatorname{Ext}_{R}^{2}(X, Q), E(Q_{Q}))$$
$$= 0$$

for every finitely generated  $X \in Mod R$ , we have weak dim  $E(R_R) \leq 1$ . Thus by Hoshino [7, Propositions F and C] every finitely generated submodule of E(RR) is torsionless. Let  $X \in Mod R$  be finitely generated. Since by Theorem 1.2  $X/\tau(X)$  is torsionless, there exists an exact sequence  $0 \to X/\tau(X) \to F \to Y \to 0$  in Mod R with F free. Thus  $\operatorname{Ext}^1_R(X,Q) \simeq \operatorname{Ext}^1_R(X/\tau(X),Q) \simeq \operatorname{Ext}^2_R(Y,Q) = 0$ . Hence  $_RQ$ is injective and by Theorem 4.2 the assertion follows.

REMARK. Let R be left noetherian and let  $X \in \text{Mod } R$  be flat. Then Ext<sub>R</sub><sup>i</sup>(Y,R) $\otimes_R X \simeq \text{Ext}_R^i(Y,X)$  for all  $i \ge 0$  and finitely generated  $Y \in \text{Mod } R$ , so that inj dim  $_R X \le \text{inj dim }_R R$ . Thus, together with Lemma 2.3, Corollary 4.3 would yield a result of Sato [16, Theorem].

5. Appendix. Throughout this section, Q stands for an extension ring of R. We make some remarks on submodules of  $Q_R$ .

The argument of Sumioka [19, Proposition 6] suggests the following lemma.

Lemma 5.1. The following are equivalent.

(1) Q is a left quotient ring of R.

(2) (a)  $_{R}Q/R$  is torsion.

(b)  $R \cap I \neq 0$  for every nonzero two-sided ideal I of Q.

Proof. (1)  $\Rightarrow$  (2). Obvious.

 $(2) \Rightarrow (1)$ . Put  ${}_{Q}E = \operatorname{Hom}_{R}({}_{R}Q_{Q}, E({}_{R}R))$ . Then  ${}_{R}E \simeq E({}_{R}R)$  canonically, so that the composite of ring homomorphisms End  $(E({}_{R}R)) \rightarrow \operatorname{End}({}_{Q}E) \rightarrow \operatorname{End}({}_{R}E)$  is an isomorphism. Thus End  $({}_{Q}E) = \operatorname{End}({}_{R}E)$  and hence Biend  $({}_{Q}E) = \operatorname{Biend}({}_{R}E)$ . Let  $\phi: Q \rightarrow \operatorname{Biend}({}_{Q}E)$  denote the canonical ring homomorphism. Since  ${}_{R}E$  is faithful,  $R \cap \operatorname{Ker} \phi = 0$  and thus  $\operatorname{Ker} \phi = 0$ . Since Biend  $({}_{R}E)$  is a maximal left quotient ring of R, the assertion follows.

**Lemma 5.2** (cf. Masaike [12, Proposition 2]). Assume that Q is a right quotient ring of R. Let M be a submodule of  $Q_R$  containing R and put  $I = \{a \in R | aM \subset R\}$ . Then M is torsionless if and only if  $({}_RR/I)^* = 0$ .

Proof. Let  $j: R_R \to M_R$  denote the inclusion. Then j is an essential monomorphism, so that Ker  $\varepsilon_M = 0$  if and only if Ker  $j^{**}=0$ . It suffices to show that Ker  $j^{**}\simeq (_RR/I)^*$ . Identify  $(R_R)^*$  with  $_RR$ . We claim that Im  $j^*=I$ . It is obvious that  $I \subset \text{Im } j^*$ . Conversely, let  $h \in M^*$ . Since  $E(Q_Q)_R \simeq E(R_R)$  is injective, h extends to some  $\phi: Q_R \to E(Q_Q)_R$ . It is easy to see that  $\phi$  is Q-linear. Thus  $h(1)x = \phi(1)x = \phi(x) = h(x) \in R$  for all  $x \in M$  and hence  $j^*(h) = h(1) \in I$ .

For an  $M \in Mod R^{op}$ , there exists a natural homomorphism

 $\eta_M: M \to \operatorname{Hom}_O(\operatorname{Hom}_R(M, Q), Q)$ 

such that  $\eta_M(x)(\alpha) = \alpha(x)$  for  $x \in M$  and  $\alpha \in \operatorname{Hom}_R(M,Q)$ , and for an  $X \in \operatorname{Mod} R$  there exists a natural homomorphism

$$\zeta_X: X^* \to \operatorname{Hom}_O(Q \otimes_R X, Q)$$

such that  $\zeta_X(\alpha)(q \otimes x) = q\alpha(x)$  for  $\alpha \in X^*$ ,  $q \in Q$  and  $x \in X$ . Also, for L,  $M \in Mod \mathbb{R}^{op}$  there exists a natural homomorphism

$$\delta_{L,M}: L \otimes_R M^* \to \operatorname{Hom}_R(M,L)$$

such that  $\delta_{L,M}(x \otimes \alpha)(y) = x\alpha(y)$  for  $x \in L$ ,  $\alpha \in M^*$  and  $y \in M$ . For each  $M \in Mod R^{op}$ , we have a commutative diagram:

which yields the following lemma.

**Lemma 5.3.** Let  $M \in \text{Mod } R^{\text{op}}$ . Assume that both  $\eta_M$  and  $\text{Hom}_Q(\delta_{Q,M}, Q)$  are monic. Then M is torsionless.

Also, for each  $M \in Mod R^{op}$ , we have a commutative diagram with exact rows:

$$R \otimes_{R} M^{*} \to Q \otimes_{R} M^{*} \to (Q/R) \otimes_{R} M^{*} \to 0$$

$$\delta_{R,M} \downarrow \qquad \qquad \qquad \downarrow^{\delta_{Q,M}} \qquad \qquad \qquad \downarrow^{\delta_{Q/R,M}}$$

$$0 \to \operatorname{Hom}_{R}(M,R) \to \operatorname{Hom}_{R}(M,Q) \to \operatorname{Hom}_{R}(M,Q/R).$$

Note that, in case M is finitely generated,  $\operatorname{Hom}_{R}(M,Q/R)$  embeds in a direct sum of copies of  $_{R}Q/R$ . Thus Snake lemma yields the following two lemmas.

**Lemma 5.4.** Assume that  $_{R}Q/R$  is torsion. Then both  $_{R}\text{Ker }\delta_{Q,M}$  and  $_{R}\text{Cok }\delta_{O,M}$  are torsion for every finitely generated  $M \in \text{Mod } R^{\text{op}}$ .

**Lemma 5.5.** Assume that the inclusion  $R \to Q$  is a left flat epimorphism. Then  $\delta_{Q,M} \simeq_Q Q \otimes_R \delta_{Q,M}$  is an isomorphism for every finitely generated  $M \in \text{Mod } R^{\text{op}}$ .

We are now in a position to formulate results of Masaike [12] as follows.

**Proposition 5.6** (Masaike [12]). For an extension ring Q of R the following hold.

(1) If Q is a left quotient ring of R, every finitely generated submodule of  $Q_R$  is torsionless.

(2) If the inclusion  $R \rightarrow Q$  is a left flat epimorphism, every finitely generated submodule of  $Q_R$  embeds in a free module.

(3) Assume that Q is a right quotient ring of R. Then Q is a left quotient ring of R if and only if every finitely generated submodule of  $Q_R$  is torsionless.

Proof. (1) By Lemmas 5.3 and 5.4.

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(2) By Lemma 5.5.

(3) By Lemmas 5.1 and 5.2.

## References

- [1] M. Auslander: Coherent functors, Proc. Conf. Cat. Algebra, 189-231, Springer, Berlin 1966.
- [2] V.C. Cateforis: Two-sided semisimple maximal quotient rings, Trans. Amer. Math. Soc. 149 (1970), 339-349.
- [3] R.R. Colby and E.A. Rutter, Jr.: II-flat and II-projective modules, Arch. Math. 22 (1971), 246-251.
- [4] C. Faith: Rings with ascending condition on annihilators, Nagoya Math. J. 27 (1966), 179-191.
- [5] J.L. Gómez Pardo: Counterinjective modules and duality, J. Pure and Appl. Algebra 61 (1989), 165-179.
- [6] J.L. Gómez Pardo and P.A. Guil Asensio: Morita dualities associated with R-dual functors, J. Pure and Appl. Algebra 93 (1994), 179-194.
- [7] M. Hoshino: On Lambek torsion theories, Osaka J. Math. 29 (1992), 447-453.
- [8] M. Hoshino and S. Takashima: On Lambek torsion theories II, Osaka J. Math. 31 (1994), 729-746.
- [9] J.P. Jans: Duality in noetherian rings, Proc. Amer. Math. Soc. 12 (1961), 829-835.
- [10] J. Lambek: On Utumi's ring of quotients, Canad. J. Math. 15 (1963), 363-370.
- [11] D. Lazard: Epimorphismes plats d'anneaux, C. R. Acad. Sci. Paris 266 (1968), 314-316.
- [12] K. Masaike: On quotient rings and torsionless modules, Sci. Rep. Tokyo Kyoiku Daigaku A11 (1971), 26-30.
- [13] K. Morita: Localization in categories of modules I, Math. Z. 114 (1970), 121-144.
- [14] K. Ohtake: A generalization of Morita duality by Localizations, preprint
- [15] N. Popescu and T. Spircu: Sur les epimorphismes plats d'anneaux, C. R. Acad. Sci. Paris 268 (1969), 376–379.
- [16] H. Sato: On localizations of a 1-Gorenstein ring, Sci. Rep. Tokyo Kyoiku Daigaku A13 (1977), 188-193.
- [17] L. Silver: Noncommutative localizations and applications, J. Algebra 7 (1976), 44-76.
- [18] H.H. Storrer: Rings of quotients of perfect rings, Math. Z. 122 (1971), 151-165.
- [19] T. Sumioka: On finite dimensional QF-3' rings, Proceedings of 10<sup>th</sup> Symposium on Ring Theory, 79-105, Okayama Univ., Okayama, Japan 1978.
- [20] T. Sumioka: On QF-3 and 1-Gorenstein rings, Osaka J. Math. 16 (1979), 395-403.

Institute of Mathematics University of Tsukuba Ibaraki, 305 Japan