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THE GREEN CORRESPONDENCE AND ORDINARY INDUCTION OF BLOCKS IN FINITE GROUP MODULAR REPRESENTATION THEORY

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Abstract

The first step in the fundamental Clifford theoretic approach to general block theory of finite groups reduces to: H is a subgroup of the finite group G and e is a central idempotent of H such that $e({}^{g}e) = 0$ for all $g \in G - H$. Then $Tr_H^G(e)$ is a central idempotent of G and induction from H to G, Ind_H^G, is part of a Morita equivalence between the categories of e-modules and of $Tr_H^G(e)$-modules. Let W be an indecomposable e-module, so that $V = Ind_H^G(W)$ is an indecomposable $Tr_H^G(e)$-module. We present results that relate the Green correspondents of W and V via induction and restriction.

1. Introduction and results

Our notation and terminology are standard and tend to follow [1] and [5]. All rings have identities and are Noetherian and all modules over a ring are unitary and finitely generated left modules.

Let R be a ring. Then R-mod will denote the abelian category of left R-modules. Let U and V be left R-modules. Then $U|V$ in R-mod signifies that U is isomorphic to a direct summand of V in R-mod. Also if R has the unique decomposition property (cf. [1, p. 37]), then U is a component of V if U is indecomposable in R-mod and $U|V$.

In this paper, G denotes a finite group, p is a prime integer and let $(\mathcal{O}, K, k = \mathcal{O}/J(\mathcal{O}))$ be a p-modular system that is "large enough" for all subgroups of G (i.e., $\mathcal{O}$ is a complete discrete valuation ring, $k = \mathcal{O}/J(\mathcal{O})$ is an algebraically closed field of characteristic p and the field of fractions K of $\mathcal{O}$ is of characteristic zero and is a splitting field for all subgroups of G).

Let A be a finitely generated $\mathcal{O}$-algebra. Then A has the unique decomposition property by the Krull-Schmidt theorem ([1, I, Theorem 11.4] or [5, Theorem 4.4]). Also the natural ring epimorphism —: $\mathcal{O} \to \mathcal{O}/J(\mathcal{O}) = k$ induces an $\mathcal{O}$-algebra epimorphism —: $A \to A/(J(\mathcal{O})A) = \overline{A}$.

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Let $H < G$ and let e be an idempotent of $Z(\mathcal{O}H)$. We shall need an extension of [2, Remark 1.3]:

Lemma 1. *Let $g \in G$. The following six conditions are equivalent*:
(i) $\overline{e}(k(HgH))\overline{e} = (0)$;
(ii) $\overline{e}({}^g\overline{e}) = (0)$;
(iii) $\overline{e}(k(HgH)) \bigotimes_{kH} \overline{V} = (0)$ *for all modules* $\overline{V}$ *of* $(kH)\overline{e}$*-mod*;
(iv) $e(\mathcal{O}(HgH))e = (0)$;
(v) $e({}^g e) = 0$; *and*
(vi) $e\big(\mathcal{O}(HgH) \bigotimes_{\mathcal{O}H} V\big) = (0)$; *for all modules* V *of* $(\mathcal{O}H)e$*-mod.*

Proof. From [2, Remark 1.3] we conclude that (iv), (v) and (vi) are equivalent and (i), (ii) and (iii) are equivalent. Clearly (vi) implies (i). Assume (i) and note that $e(\mathcal{O}(HgH)e)$ is $\mathcal{O}$-free and $\overline{e(\mathcal{O}(HgH))e} = (0)$. Thus (iv) holds and we are done. $\square$

Let V be an indecomposable $\mathcal{O}G$-module with vertex P and $\mathcal{O}P$-source X. Let K be a subgroup of G such that $N_G(P) \leq K$. Thus the Green correspondent $\mathcal{G}r_K^G(V)$ of V in $\mathcal{O}K$-mod also has vertex P and $\mathcal{O}P$-source X. Let L be a subgroup of K such that $P \leq L$.

Lemma 2. *Let U be an indecomposable direct summand of $Res_K^G(V)$ in $\mathcal{O}K$-mod such that $Res_L^K(U)$ has a component W in $\mathcal{O}L$-mod with vertex P. Then $U \cong \mathcal{G}r_K^G(V)$ in $\mathcal{O}K$-mod.*

Proof. Assume that U is not isomorphic to $\mathcal{G}r_K^G(V)$ in $\mathcal{O}K$-mod. Then, as in [1, III, Lemma 5.3], there is an $x \in G - K$ and a subgroup $A \leq K \cap (P^x)$ such that A is a vertex of U. Since $W | Res_L^K(U)$ in $\mathcal{O}L$-mod, [1, III, Lemma 4.1] implies that there is a $y \in K$ such that W is $L \cap (A^y)$-projective. But then there is a $z \in L$ such that $P^z \leq L \cap (A^y)$. Here $A^y \leq K \cap (P^{(xy)})$, so that $P^z = L \cap (A^y) = P^{(xy)}$. Thus $xyz^{-1} \in N_G(P) \leq K$ and $x \in K$. This contradiction establishes the lemma. $\square$

The following two propositions are the main results of this paper. For the remainder of this paper, we assume that $e({}^g e) = 0$ for all $g \in G - H$. Hence $E = Tr_H^G(e)$ is an idempotent in $Z(\mathcal{O}G)$ and the functors Ind_H^G: $(\mathcal{O}H)e$-mod $\to (\mathcal{O}G)E$-mod and $e\,Res_H^G$: $(\mathcal{O}G)E$-mod $\to$ $(\mathcal{O}H)e$-mod demonstrate a Morita equivalence between $(\mathcal{O}H)e$-mod and $(\mathcal{O}G)E$-mod as is well-known (cf. [4, Case 1], [5, Theorem 9.9] or [2, Proposition 1.2]).

Let W be an indecomposable $(\mathcal{O}H)e$-module with vertex P and $\mathcal{O}P$-source X. Then $V = Ind_H^G(W)$ is an indecomposable $(\mathcal{O}G)E$-module and P is a vertex of V and X is an $\mathcal{O}P$-source of V (cf. [1, III,Corollary 4.7]). Here $P \leq N_H(P) \leq N_G(P)$.

Let $b \in Bl((\mathcal{O}H)e)$ be such that $bW = W$. Then $Tr_H^G(b) = B \in Bl((\mathcal{O}G)E)$ and $BV = V$. Also $b\,Res_H^G(V) \cong W$ in $\mathcal{O}H$-mod and $b({}^g b) = 0$ for all $g \in G - H$.

Under these conditions, we have the Green correspondents $\mathcal{G}r_{N_G(P)}^G(V)$ and $\mathcal{G}r_{N_H(P)}^H(W)$ of V in $\mathcal{O}N_G(P)$-mod and of W in $\mathcal{O}N_H(P)$-mod, resp., where both indecomposable modules $\mathcal{G}r_{N_G(P)}^G(V)$ and $\mathcal{G}r_{N_H(P)}^H(W)$ have P as a vertex and $\mathcal{O}P$-source X.

Proposition 3. *Let e_P be the unique block of $\mathcal{O}N_H(P)$ such that $e_P\,\mathcal{G}r_{N_H(P)}^H(W) = \mathcal{G}r_{N_H(P)}^H(W)$. Then*
(a) $e_P({}^x e_P) = 0$ *for all $x \in N_G(P) - N_H(P)$, $E_P = Tr_{N_H(P)}^{N_G(P)}(e_P)$ is a block of $\mathcal{O}N_G(P)$ and the conclusions of* [6, Theorem 1] *and* [2, Theorem 1.6] *hold.*
(b) $E_P\,\mathcal{G}r_{N_G(P)}^G(V) = \mathcal{G}r_{N_G(P)}^G(V)$, $Ind_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_H(P)}^H(W)\big) \cong \mathcal{G}r_{N_G(P)}^G(V)$ *in $\mathcal{O}N_G(P)$-mod and $e_P\,Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big) \cong \mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod; and*
(c) *exactly one component of $Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big)$ in $\mathcal{O}N_H(P)$-mod is isomorphic to $\mathcal{G}r_{N_H(P)}^H(W)$.*

Proof. From [1, III, Theorem 7.8], we conclude that $Br_P(e)\bar{e}_P = \bar{e}_P$. Let $x \in N_G(P) - N_H(P)$. Then $\bar{e}_P({}^x\bar{e}_P) = \bar{e}_P Br_P(e)({}^x Br_P(e))({}^x\bar{e}_P) = \bar{e}_P Br_P(e)({}^x Br_P(e))({}^x\bar{e}_P) = \bar{e}_P Br_P(e({}^x e))^x\bar{e}_P = 0$. We conclude from Lemma 1 that $e_P({}^x e_P) = 0$ for all $x \in N_G(P) - N_H(P)$. Then (a) follows from [2, Proposition 1.2]. Here $W | Ind_{N_H(P)}^H\big(\mathcal{G}r_{N_H(P)}^H(W)\big)$ in $\mathcal{O}H$-mod. Thus

$$V \cong Ind_H^G(W) | Ind_{N_H(P)}^G\big(\mathcal{G}r_{N_H(P)}^H(W)\big)$$

in $\mathcal{O}G$-mod. Since $Ind_{N_H(P)}^G\big(\mathcal{G}r_{N_H(P)}^H(W)\big) \cong Ind_{N_G(P)}^G\big(Ind_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_H(P)}^H(W)\big)\big)$ in $\mathcal{O}G$-mod and $Ind_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_H(P)}^H(W)\big)$ is indecomposable in $\mathcal{O}N_G(P)$-mod with vertex P and $\mathcal{O}P$-source X by [2, Theorem 1.6 (c)], we conclude from [1, III, Theorem 5.6 (iii)] that $\mathcal{G}r_{N_G(P)}^G(V) \cong Ind_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_H(P)}^H(W)\big)$ in $\mathcal{O}N_G(P)$-mod. But then [2, Proposition 1.2] completes our proof of (b).

Clearly

$$Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big) = e_P\,Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big) \oplus (1 - e_P)\,Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big)$$

in $\mathcal{O}N_H(P)$-mod. Let $\mathcal{U}$ be a component of $(1 - e_P)Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big)$ in $\mathcal{O}N_H(P)$-mod such that $\mathcal{U} \cong \mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod. Let e_P^* be the unique block of $Z(\mathcal{O}N_H(P))$ such that $e_P^*\mathcal{U} = \mathcal{U}$. Since $e_P^*(1 - e_P) = e_P^*$, we have $e_P e_P^* = 0$. This contradiction completes our proof of Proposition 3. □

For our next result, we shall investigate a more general situation than in Proposition 3. Consequently we assume that K is a subgroup of G such that $N_G(P) \leq K$. Then $N_H(P) \leq K \cap H \leq H$, $\mathcal{G}r_K^G(V)$ is an indecomposable $\mathcal{O}K$-module with vertex P

and $\mathcal{O}P$-source X and $\mathcal{G}r^H_{K\cap H}(W)$ is an indecomposable $\mathcal{O}(K\cap H)$-module with vertex P and $\mathcal{O}P$-source X.

Proposition 4. (a) *Let U be a component of $Res^H_{K\cap H}(W)$ such that $Ind^K_{K\cap H}(U)$ has a component with vertex P. Then $U \cong \mathcal{G}r^H_{K\cap H}(W)$ in $\mathcal{O}(K\cap H)$-mod*;
(b) *in an indecomposable decomposition of $Ind^K_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big)$ in $\mathcal{O}K$-mod, exactly one component has P as a vertex and it is isomorphic to $\mathcal{G}r^G_K(V)$ in $\mathcal{O}K$-mod and all of the remaining components have a proper subgroup of P as a vertex*;
(c) *let Y be a component of $Res^G_K(V)$ such that $Res^K_{K\cap H}(Y)$ has a component with vertex P. Then $Y \cong \mathcal{G}r^G_K(V)$ in $\mathcal{O}K$-mod*; *and*
(d) *in an indecomposable decomposition of $Res^K_{K\cap H}\big(\mathcal{G}r^G_K(V)\big)$ in $\mathcal{O}(K\cap H)$-mod, exactly one component is isomorphic to $\mathcal{G}r^H_{K\cap H}(W)$.*

Proof. For (a), assume that $U \not\cong \mathcal{G}r^H_{K\cap H}(W)$ in $\mathcal{O}(K\cap H)$-mod. Then [1, III, Lemma 5.3] implies that there is an $x \in H-(K\cap H)$ and a vertex A of U such that $A \leq (K\cap H)\cap(P^x)$. Let Y be a component of $Ind^K_{K\cap H}(U)$ with P as a vertex. Then, as $Ind^K_{K\cap H}(U)$ is A-projective, there is a $k \in K$ such that $P^k \leq A$. But then $P^k = A = P^x$ and so $xk^{-1} \in N_G(P) \leq K$. This contradiction establishes (a).

For (b), [1, III, Lemma 5.4] yields:

$$Ind^H_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big) \cong W \oplus \left(\bigoplus_{i\in I} \mathcal{U}_i\right) \quad \text{in} \quad \mathcal{O}H\text{-mod} \tag{1.1}$$

where I is a finite set and for each $i \in I$, $\mathcal{U}_i$ is an indecomposable $\mathcal{O}H$-module having a proper subgroup of P as a vertex.

Thus:

$$Ind^G_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big) \cong V \oplus \left(\bigoplus_{i\in I} Ind^G_H(\mathcal{U}_i)\right) \quad \text{in} \quad \mathcal{O}G\text{-mod}. \tag{1.2}$$

Clearly $Ind^G_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big) \cong Ind^G_K\big(Ind^K_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big)\big)$ in $\mathcal{O}G$-mod and all components of $Ind^K_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big)$ are P-projective. Let T be a component of $Ind^K_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big)$ in $\mathcal{O}K$-mod such that $V|Ind^G_K(T)$ in $\mathcal{O}K$-mod. Then P must be a vertex of T and $T \cong \mathcal{G}r^G_K(V)$ in $\mathcal{O}K$-mod. Let T_1 be a component of $Ind^K_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big)$ with P as a vertex and such that $(T\oplus T_1)|Ind^K_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big)$ in $(\mathcal{O}K)$-mod. Then $Ind^G_K(T_1)$ has a component with P as a vertex by [1, III, Corollary 4.7]. Thus (1.1) and (1.2) imply that $V|Ind^G_K(T_1)$ and (1.1) and (1.2) yield a contradiction. Thus (b) is proved.

Clearly (c) follows from Lemma 2.

For (d), note that

$$\mathcal{G}r^H_{K\cap H}(W)|Res^H_{K\cap H}(W)|Res^H_{K\cap H}\big(Res^G_H(V)\big) = Res^K_{K\cap H}\big(Res^G_K(V)\big)$$

in $\mathcal{O}(K \cap H)$-mod. Thus $Res_K^G(V)$ has a component T in $\mathcal{O}K$-mod such that $\mathcal{G}r_{K\cap H}^H(W) | Res_{K\cap H}^K(T)$ in $\mathcal{O}(K \cap H)$-mod. Now (c) implies that $T \cong \mathcal{G}r_K^G(V)$ in $\mathcal{O}K$-mod and so $\mathcal{G}r_{K\cap H}^H(W) | Res_{K\cap H}^K\big(\mathcal{G}r_K^G(V)\big)$ in $\mathcal{O}(K \cap H)$-mod.

Let r be the number of components in an indecomposable decomposition of $Res_{K\cap H}^K(\mathcal{G}r_K^G(V))$ in $\mathcal{O}(K \cap H)$-mod that are isomorphic to $\mathcal{G}r_{K\cap H}^H(W)$. Thus there are at least r components in an indecomposable decomposition of $Res_{N_H(P)}^K(\mathcal{G}r_K^G(V))$ that are isomorphic to $\mathcal{G}r_{N_H(P)}^{K\cap H}(\mathcal{G}r_{K\cap H}^H(W)) \cong \mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod. But

$$Res_{N_H(P)}^K\big(\mathcal{G}r_K^G(V)\big) = Res_{N_H(P)}^{N_G(P)}\Big(Res_{N_G(P)}^K\big(\mathcal{G}r_K^G(V)\big)\Big)$$

in $\mathcal{O}N_H(P)$-mod and

$$Res_{N_G(P)}^K\big(\mathcal{G}r_K^G(V)\big) \cong \mathcal{G}r_{N_G(P)}^G(V) \oplus \left(\bigoplus_{i\in I} \mathcal{U}_i\right)$$

in $\mathcal{O}N_G(P)$-mod where I is a finite set and if $i \in I$, then $\mathcal{U}_i$ is an indecomposable $\mathcal{O}N_G(P)$-module with a vertex $A_i \leq N_G(P) \cap P^{x_i}$ for some $x_i \in K - N_G(P)$. Let $i \in I$ be such that $Res_{N_H(P)}^{N_G(P)}(\mathcal{U}_i)$ has a component isomorphic to $\mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod. Then there is a $y \in N_G(P)$ such that $P^y \leq A_i$ by [1, III, Lemma 4.6]. Thus $P^y = P \leq A_i = P^{x_i}$ and so $x_i \in N_G(P)$. This contradiction implies that $Res_{N_H(P)}^{N_G(P)}\big(\bigoplus_{i\in I} \mathcal{U}_i\big)$ does not have a component isomorphic to $\mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod. Since, Proposition 3 (c) asserts that exactly one component of $Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big)$ is isomorphic to $\mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod, $r = 1$ and our proof of Proposition 4 is complete. $\square$

We conclude with a discussion of the Brauer block induction (cf. [3, Chapter 5, Section 3]) in the context of Proposition 4 as suggested by the referee. So we assume the context of Proposition 4. Thus $b \in Bl((\mathcal{O}H)e)$, $b({}^g b) = 0$ for all $g \in G - H$, $bW = W$, $B = Tr_H{}^G(b) \in Bl((\mathcal{O}G)E)$, $V = Ind_H{}^G(W)$ and $BW = W$. Here b^G is defined and $b^G = B$ by [3, Chapter 5, Theorem 3.1 (ii)] and [2, Proposition 1.7].

Let B_K be the block idempotent of $\mathcal{O}K$ such that $B_K\, \mathcal{G}r_K^G(V) = \mathcal{G}r_K^G(V)$, let B_P be the block idempotent of $\mathcal{O}N_G(P)$ such that $B_P\, \mathcal{G}r_{N_G(P)}^G(V) = \mathcal{G}r_{N_G(P)}^G(V)$, let $b_{K\cap H}$ be the block idempotent of $\mathcal{O}(K \cap H)$ such that $b_{(K\cap H)}\, \mathcal{G}r_{(K\cap H)}^H(W) = \mathcal{G}r_{(K\cap H)}^H(W)$ and let b_P be the block idempotent of $\mathcal{O}N_H(P)$ such that $b_P\, \mathcal{G}r_{N_H(P)}^H(W) = \mathcal{G}r_{N_H(P)}^H(W)$.

Clearly

$$N_K(P) = N_G(P), \quad N_H(P) = N_{(K\cap H)}(P), \quad \mathcal{G}r_{N_K(P)}^K(\mathcal{G}r_K^G(V)) \cong \mathcal{G}r_{N_G(P)}^G(V)$$

in $\mathcal{O}N_G(P)$-mod and $\mathcal{G}r_{N_H(P)}^{K\cap H}\big(\mathcal{G}r_{K\cap H}^H(W)\big) \cong \mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod. From [3, Chapter 5, Theorem 3.12], we conclude that $(b_P)^{K\cap H}$ is defined and $(b_P)^{(K\cap H)} = b_{(K\cap H)}$ and that $(B_P)^K$ is defined and $(B_P)^K = B_K$. Also from [3, Chapter 5, Theorem 3.1 (ii)], [2, Proposition 1.7] and Proposition 3 (a), we deduce that $(b_P)^{N_G(P)}$ is defined and $(b_P)^{N_G(P)} = B_P$.

Here $(B_P)^K = B_K = ((b_P)^{N_G(P)})^K$ and so [3, Chapter 5, Lemma 3.4] implies that $(b_P)^K = B_K$. Since $(b_P)^{(K\cap H)}$ is defined and $(b_P)^{(K\cap H)} = b_{(K\cap H)}$, the same lemma forces $((b_P)^{(K\cap H)})^K = B_K = (b_{(K\cap H)})^K$. This is the proof given by the referee of:

Proposition 5. *As in* Proposition 4 *and with the notation above,* $(b_{K\cap H})^K$ *is defined and* $(b_{K\cap H})^K = B_K$.

Finally a question:

QUESTION 6. In the situation of Proposition 5, is $b_{(K\cap H)}({}^x(b_{(K\cap H)})) = 0$ for all $x \in K - (K \cap H)$?

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