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UNKNOTTING OPERATIONS INVOLVING TRIVIAL TANGLES

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In this note we study knots and links in the 3-sphere S^3. Our starting point is the first author's observation that any link can be transformed into the unlink (in fact unknot) by repeatedly altering a diagram of the link as shown in Fig. 1 (Theorem 3 (1)). We generalize this unknotting operation to n-string trivial tangles as shown in Fig. 2 and call such a diagrammatic change an $H(n)$-move.

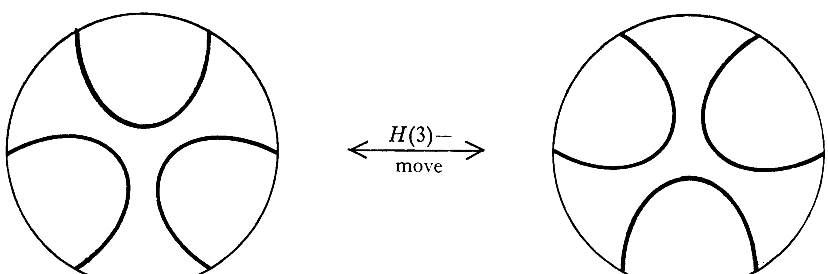

Fig. 1

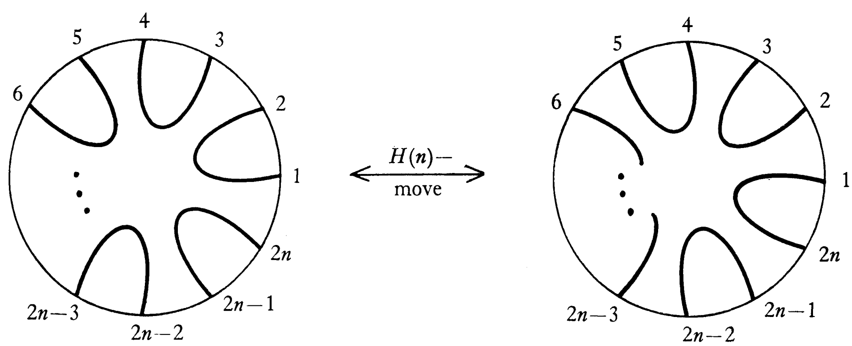

Fig. 2

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We will show that given any integer n ($\geqq 2$), any knot can be transformed into a trivial knot by a finite sequence of $H(n)$-moves (Theorem 1). Hence, we can define a numerical invariant $u_n(K)$ for a knot K much like the ordinary unknotting number. For any knot K, these integers have a descending property $u_2(K)\geqq u_3(K)\geqq u_4(K)\geqq\cdots$ and the limit $\lim_{n\to\infty} u_n(K)=1$ (Theorems 5 and 6). Therefore, we can define $h(K)$ as the minimum integer n satisfying $u_n(K)=1$. We give estimates for $h(K)$ in terms of the unknotting number, the minimum crossing number, the genus, and the minimum number of generators of the first homology group of the p-fold cyclic branched covering space of K (Theorem 7).

When considering oriented links, we can define an $SH(n)$-move as a special type of $H(n)$-move preserving the orientation of arcs outside the trivial tangle (Fig. 3), and prove similar results (Theorems 1*, 3*, 4*, 5*, 6*, and 7*).

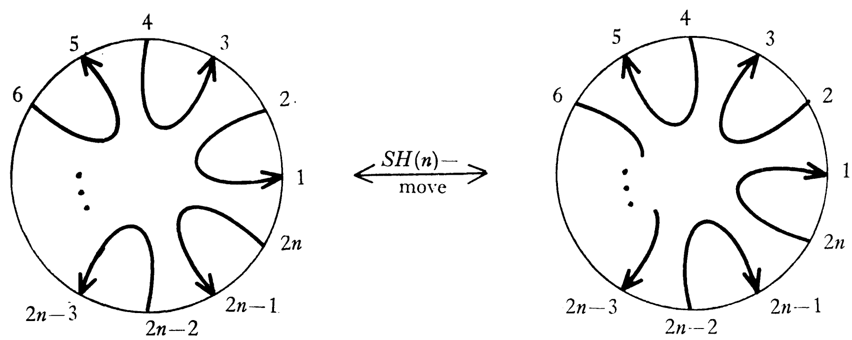

Fig. 3

1. Definitions and Main Theorems

An $H(n)$-move is a local move on a link diagram as indicated in Fig. 2. Furthermore, we (arbitrarily) require an $H(n)$-move to preserve the number of components. If a diagram of a link L' is a result of an $H(n)$-move on a diagram of a link L, then we say that L' is obtained from L by an $H(n)$-move.

Theorem 1. *Given any integer $n(\geqq 2)$, any knot can be transformed into a trivial knot by a finite sequence of $H(n)$-moves, i.e. an $H(n)$-move is a kind of unknotting operation.*

DEFINITION. For a knot K in S^3 and an integer n ($\geqq 2$), $u_n(K)$ is defined to be the minimum number of $H(n)$-moves which can transform K into a trivial knot.

Theorem 2. *For a knot K, we have*
(1) $u_2(K)\leqq 2u(K)$, *and*

(2) $u_2(K) \leqq cr(K)-2$,
where $u(K)$ *is the (ordinary) unknotting number and* $cr(K)$ *is the minumum crossing number.*

Theorem 3. *For a knot* K, *we have*
(1) $u_3(K) \leqq u(K)$, *and*
(2) $u_3(K) \leqq g(K)$,
where $g(K)$ *is the genus of* K.

Let $mg(K, p)$ denote the minimum number of generators of the first integral homology group of the p-fold cyclic branched covering space of K.

Theorem 4. *For a knot* K *and an integer* p $(\geqq 2)$, *we have* $u_n(K) \geqq mg(K, p)/(n-1)(p-1)$.

The sequence of these integers $u_n(K)$ for a knot K has the following descending properties.

Theorem 5. *For a knot* K, *we have* $u_n(K) \geqq u_{n+1}(K)$.

Theorem 6. *For a knot* K, *we have* $\lim_{n \to \infty} u_n(K) = 1$.

DEFINITION. For a knot K in S^3, $h(K)$ is defined to be the minimum integer n satisfying $u_n(K)=1$.

Theorem 7. *For a knot* K, *we have*
(1) $h(K) \leqq cr(K)-1$,
(2) $h(K) \leqq 2u(K)+1$,
(3) $h(K) \leqq 2g(K)+1$,
(4) $h(K) \leqq (n-1)u_n(K)+1$,
(5) $h(K) \geqq mg(K, p)/(p-1)+1$,
(6) $h(K) \geqq 2u_2(K)/3+1$, *and*
(7) $h(K) \geqq u_3(K)+1$.

Theorem 8. *The invariant* $h(K)$ *minus one is subadditive with respect to connected sum* $\sharp$, *i.e.* $h(K_1 \sharp K_2) \leqq h(K_1)+h(K_2)-1$.

DEFINITION. An $SH(n)$-move is an $H(n)$-move on an oriented link diagram which preserves the orientation of arcs outside the trivial tangle. (We can easily see that the orientation of arcs within the trivial tangle must be as indicated in Fig. 3.)

We remaek that if n is even then this move must change the number of components. This is because the move is realized by $n-1$ fissions and/or fusions as in Fig. 4 and each fission or fusion must change the number of components by one. Therefore, we shall consider only $SH(2n+1)$-moves, since we wish to preserve the number of components.

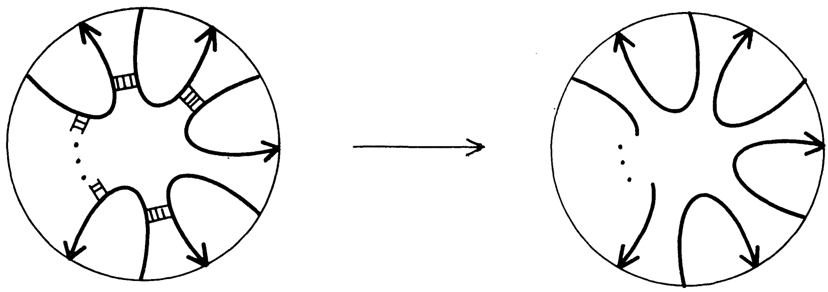

Fig. 4

Theorem 1*. *Given any positive integer n, any oriented knot can be transformed into a trivial knot by a finite sequence of $SH(2n+1)$-moves, i.e. an $SH(2n+1)$-move is a kind of unknotting operation.*

DEFINITION. For an oriented knot K in S^3 and a positive integer n, $su_{2n+1}(K)$ is defined to be the minimum number of $SH(2n+1)$-moves which can transform K into a trivial knot.

Theorem 3*. *For an oriented knot K, we have*
(1) $su_3(K) \leqq u(K)$, *and*
(2) $su_3(K) \leqq g(K)$.

Theorem 4*. *For an oriented knot K, a positive integer n, and an integer $p(\geqq 2)$, we have $su_{2n+1}(K) \geqq mg(K,p)/2n(p-1)$*

Theorem 5*. *For an oriented knot K, we have*

$$su_{2n+1}(K) \geqq su_{2n+3}(K) .$$

Theorem 6*. *For an oriented knot K, we have*

$$\lim_{n\to\infty} su_{2n+1}(K) = 1 .$$

DEFINITION. For an oriented knot K in S^3, $sh(K)$ is defined to be the minimum integer n satisfying $su_n(K)=1$.

Theorem 7*. *For an oriented knot K, we have*
(1) $sh(K) \leqq 2u(K)+1$,
(2) $sh(K) \leqq 2g(K)+1$,
(3) $sh(K) \leqq 2n\, su_{2n+1}(K)+1$,
(4) $sh(K) \geqq mg(K,p)/(p-1)+1$, *and*

(5) $sh(K) \geqq 2g^*(K)+1$
where $g^*(K)$ is the 4-ball genus of K.

Theorem 8*. *The invariant $sh(K)$ minus one is subadditive with respect to connected sum $\#$, i.e. $sh(K_1 \# K_2) \leqq sh(K_1)+sh(K_2)-1$.*

2. Examples

In this section we give examples of non-prime knots with $h(K)=n$ or $sh(K)=n$ for each integer n ($\geqq 2$).

That $h(K)$ and $sh(K)$ are not smaller than n, as asserted in each of these examples, follows from Theorems 7 (5) and 7* (4) using $mg(K, 2)=n-1$.

2.1. Fujitsugu Hosokawa has pointed out examplse with $h(K)=2$: the connected sum of a $(p+1)$-twisted knot and a -p-twisted knot (see Fig. 5).

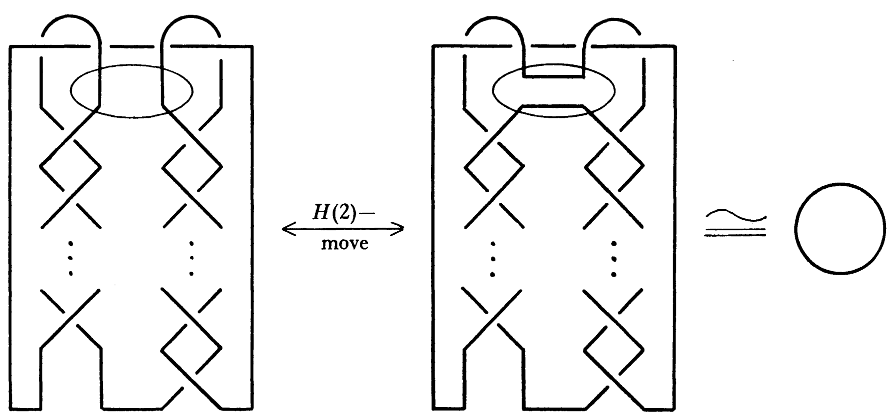

Fig. 5

2.2. The connected sum of a $(p, 2)$-torus knot and a $(-p, 2)$-torus knot has $h(K)=sh(K)=3$ (see Fig. 6).

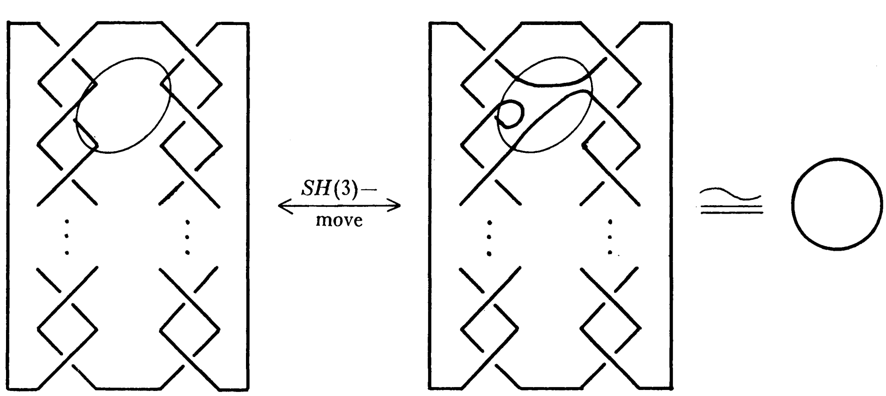

Fig. 6

2.3. For an even integer $n(\geqq 4)$, the connected sum of a $(p, 2)$-torsu knot and a pretzel knot $K(p, p, \cdots, p)$ with p-half-twisted $n-1$ strands has $h(K)=n$ (see Fig. 7).

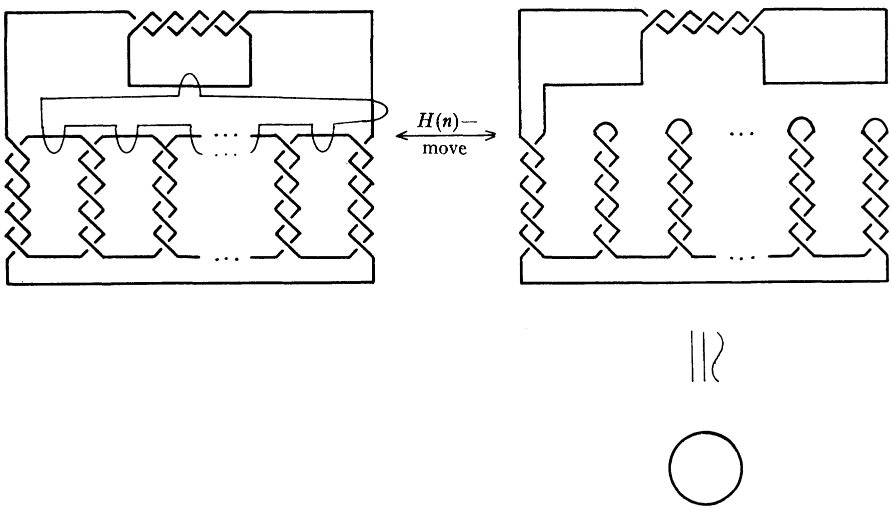

Fig. 7

2.4. For an odd integer n $(\geqq 5)$, the connected sum of a $(p, 2)$-torus knot, a $(-p, 2)$-torus knot, and a pretzel knot $K(p, p, \cdots, p)$ with p-half-twisted $n-2$ strands has $h(K)=sh(K)=n$ (see Fig. 8).

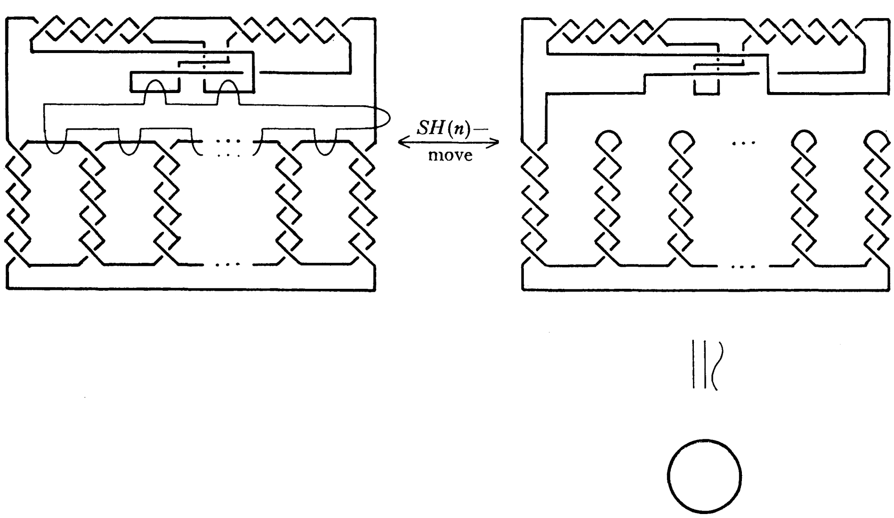

Fig. 8

REMARK. It is easy to see examples of prime knots with $h(K)=n$ or

$sh(K) = n$ as follows. For an odd integer $n(\geqq 3)$, a pretzel knot $K(p, p, \cdots, p)$ with p-half-twisted n strands has $h(K) = sh(K) = n$. For an even integer $n\ (\geqq 2)$, a pretzel knot $K(p, p, \cdots, p, 2p)$ with p-half-twisted $n-1$ strands and $2p$-half-twisted strand has $h(K) = n$.

3. Proofs

Lemma 1. *Any knot can be transformed into a trivial knot by a finite sequence of $H(2)$-moves.*

Proof. Given a diagram of the knot K, orient the diagram and perform an $H(2)$-move at a crossing as in Fig. 9.

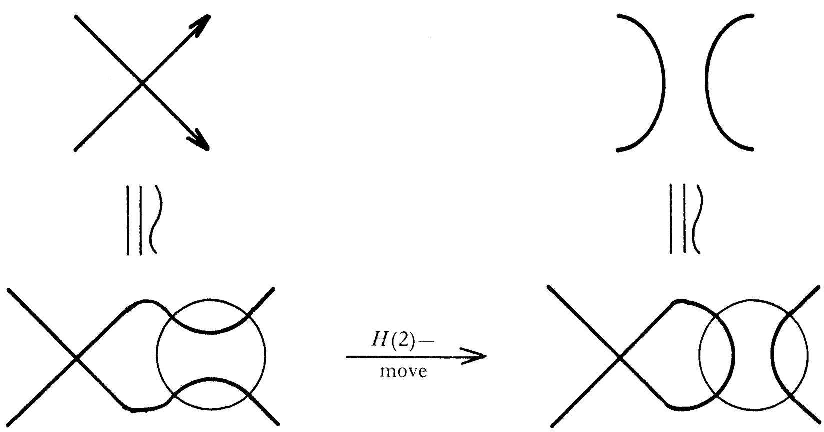

Fig. 9

This produces a knot with one fewer crossings. Thus, $cr(K)-2$ $H(2)$-moves suffice to transform K to a trivial knot, completing the proof.

Lemma 2. *An $H(n)$-move can be realized by an $H(n+1)$-move.*

Proof. Fig. 10 illustrates how this can be accomplished.

Theorem 1 now follows immediately from Lemmas 1 and 2.

Proof of Theorem 2. The proof of (2) is contained in the proof of Lemma 1. On the other hand, an (oridnary) unknotting operation is realized by two $H(2)$-moves: first eliminate the crossing as in Fig. 9, then reintroduce it but with the opposite sign.

Proof of Theorem 3. (1) An (ordinary) unknotting operation is realized by one $H(3)$-move as in Fig. 11. Thus, we have $u_3(K) \leqq u(K)$. (2) Consider a Seifert surface of K as seen in Fig. 12. A single $H(3)$-move can be used to

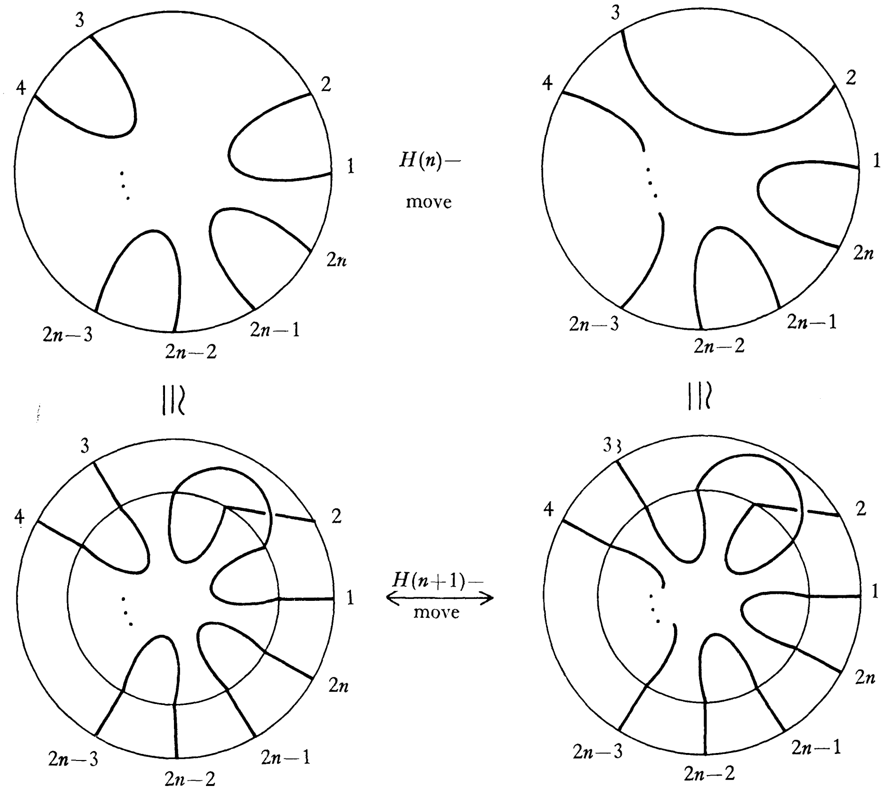
3
4
2
1
$H(n)-$
move
$2n$
$2n-3$
$2n-2$
$2n-1$
3
4
2
1
$2n$
$2n-3$
$2n-2$
$2n-1$
$\cong$
$\cong$
3
4
2
1
$H(n+1)-$
move
$2n$
$2n-3$
$2n-2$
$2n-1$
3
4
2
1
$2n$
$2n-3$
$2n-2$
$2n-1$

Fig. 10

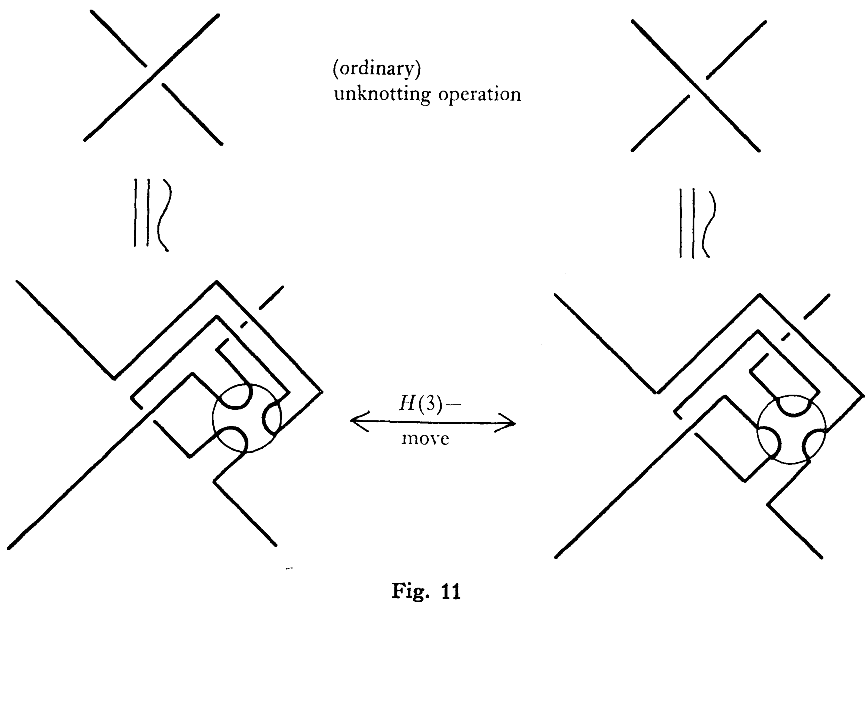
(ordinary)
unknotting operation
$\cong$
$\cong$
$H(3)-$
move

Fig. 11

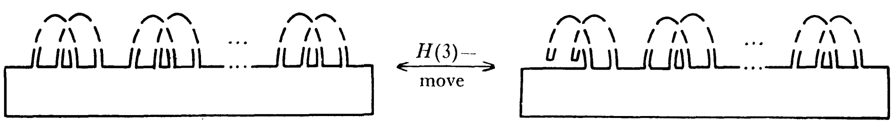

Fig .12

cut one pair of bands. Therefore, we have $u_3(K) \leqq g(K)$, completing the proof.

To prove Theorem 4, we first prove the following lemma. Here $mg(M)$ is the minimum number of generators of to first integral homology group $H_1(M, \boldsymbol{Z})$ of M.

Lemma 3. *Let M_i be a closed connected orientable 3-dimensional manifold and V_i a handlebody of genus g embedded in M_i $(i=1, 2)$. If $M_1-int\ V_1$ and $M_2-int\ V_2$ are homeomorphic, then we have $|mg(M_1)-mg(M_2)| \leqq g$.*

Proof. Glueing g meridinal disks of V_i to $M_i-\mathrm{int}\ V_i$, we can get M_i^* with $H_1(M_i^*, \boldsymbol{Z}) \cong H_i(M_i, \boldsymbol{Z})$ by adding g relators to a presentation of $H_1(M_i-\mathrm{int}\ V_i, \boldsymbol{Z})$. Therefore, we have $mg(M_i) \leqq mg(M_i-\mathrm{int}\ V_i)$. On the other hand, we can get a presentation of $H_1(M_i-\mathrm{int}\ V_i, \boldsymbol{Z})$ by adding g generators corresponding to g meridians of V_i and some relators and by changing the original relators suitably in a presentation of $H_1(M_i, \boldsymbol{Z})$. Therefore, we have $mg(M_i-\mathrm{int}\ V_i) \leqq mg(M_i)+g$. Hence ,we have $mg(M_i-\mathrm{int}\ V_i)-g \leqq mg(M_i) \leqq mg(M_i-\mathrm{int}\ V_i)$. But $M_1-\mathrm{int}\ V_1$ and $M_2-\mathrm{int}\ V_2$ are homeomorphic. Thus, we have $|mg(M_1)-mg(M_2)| \leqq g$, completing the proof.

Proof of Theorem 4. Notice that changing K by an $H(n)$-move causes the p-fold cyclic branched covering space of K to change by surgery on a handlebody of genus $(n-1)(p-1)$. Thus, if K is obtained from a trivial knot by a sequence of $H(n)$-moves then its p-fold cyclic branched covering space is obtained from S^3 by a sequence of surgeries on $(n-1)(p-1)$ genus handlebodies. Each of these can introduces at most $(n-1)(p-1)$ generators to the first integeral homology group. Thus, we have $mg(K, p) \leqq (n-1)(p-1)u_n(K)$, completing the proof.

The proof of Theorem 5 is essentialy in Lemma 2.
The proof of Theorem 6 is due to Theorems 5 and 7.

Proof of Theorem 7. Using Theorems 2 and 3, (4) implies (1), (2), and (3). Moreover, (4) (and Theorem 5) imply Theorem 6 and so justify our definition of $h(K)$. Therefore, we show (4) first. (4) An $H(n)$-move is equivalent to attaching to K $n-1$ bands as in Fig. 13. By repeatedly applying $H(n)$-moves to K, and thus by repeatedly attaching bands to K, we may reach a trivial knot. By sliding bands if necessary, we may assume that all the bands from all the $H(n)$-moves are disjoint. Thus, we may think that K is obtained from a trivial

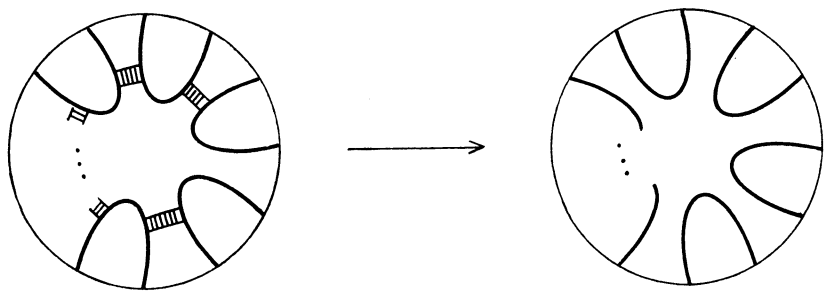

Fig. 13

knot by attaching $(n-1)u_n(K)$ mutually disjoint bands. We may gather one root of each band near one point of the trivial knot. (If necessary, slide a root along the trivial knot and along another band.) The knot K now appears as in Fig. 14 (a) and a single $H(n-1)\,u_n(K)+1)$-move will produce the trivial knot. Hence, we have $h(K) \leqq (n-1)u_n(K)+1$.

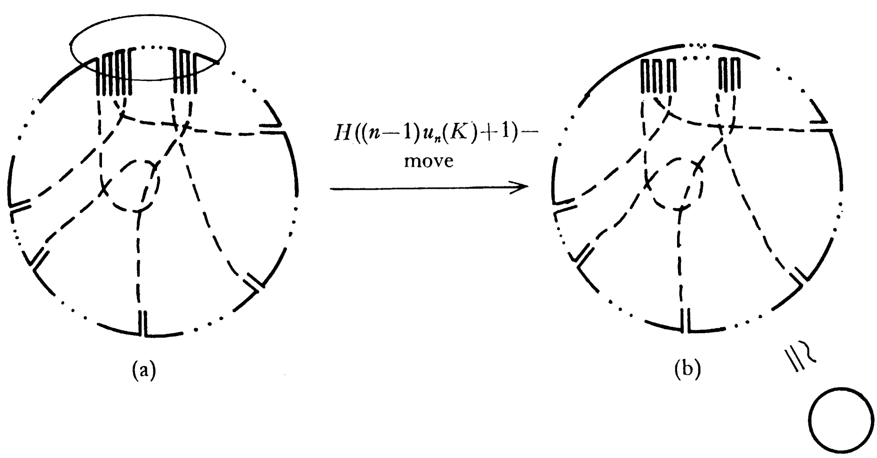

Fig. 14

(5) This follows by letting $n=h(K)$ in Theorem 4. (6 and 7) Suppose $u_n(K)=1$. Then as before K can be obtained from a trivial knot by attaching $n-1$ bands. If any one of these bands is cut, then the number of components either remains one or is changed to two. If any bands of the first type exist, cut them using $H(2)$-moves. (Note that each of these $H(2)$-moves could be accomplished as an $H(3)$-move.) Therefore, assume that only bands of the second type remain. Consider an "innermost" band. This is one whose roots divide the trivial knot into two arcs, one of which contains at most one root of any other band. Moreover, this arc must contain the root of at least one other band. Hence, there exists a pair of bands positioned as in Fig. 15 (a). Now three $H(2)$-moves as shown in Fig. 15 (c–e) can be used to cut both the bands. A single $H(3)$-

move as shown in Fig. 15 (a, f) can accomplish the same goal. Hence, using $H(2)$-moves, either we need one move per band or we need three moves per two bands. Thus, we have $h(K) \geqq 2u_2(K)/3+1$. Using $H(3)$-moves, either one move per band is needed or one move per two bands. Thus, we have $h(K) \geqq u_3(K)+1$. We complete the proof.

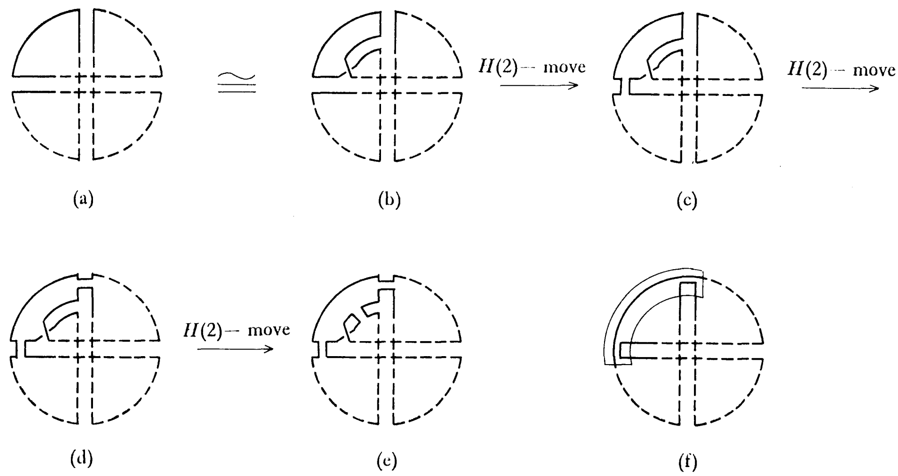

Fig. 15

Proof of Theorem 8. Suppose $h(K_1)=n_1$ and $h(K_2)=n_2$. Then as before K_1 and K_2 can be obtained from a trivial knot by attaching n_1-1 and n_2-1 bands, respectively. Then, the connected sum $K_1 \# K_2$ can be obtained from a trivial knot by attaching n_1+n_2-2 bands. The same argument as in the proof of Theorem 7 (4) shows $h(K_1 \# K_2) \leqq n_1+n_2-1$, completing the proof.

Lemma 4. *Any oriented knot can be deformed into a trivial knot by a finite*

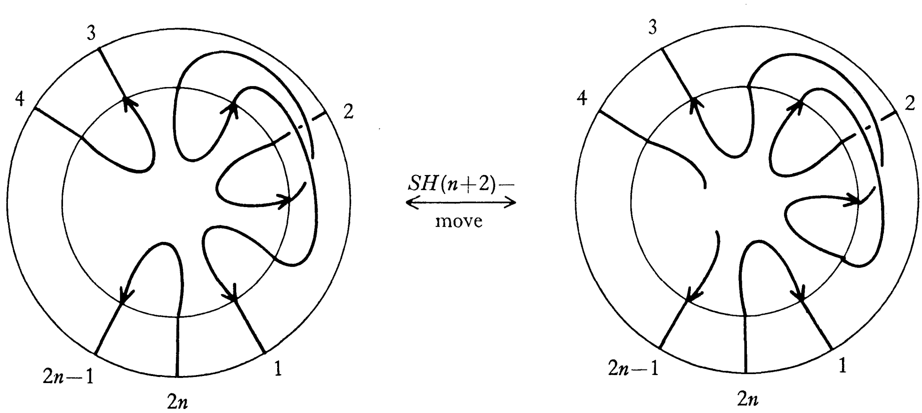

Fig. 16

sequence of SH(3)-moves.

Proof. Notice that the $H(3)$-move illustrated in Fig. 11 can actually be performed as an $SH(3)$-move.

Lemma 5. *An $SH(n)$-move can be realized by an $SH(n+2)$-move.*

Proof. Fig. 16 illustrates how this can be done.

The proofs of Theorems 1*, 3*, 4*, 5*, 6*, 7* (1)–(4), 8* are parallel to those of Theorems 1, 3, 4, 5, 6, 7, 8, and so we omit them.

Proof of Theorem 7* (5). Suppose $sh(K)=n$. From Fig. 4, we see that K with $n-1$ fissions and fusions makes a trivial knot. Hence, K bounds a surface of genus at most $(n-1)/2$ in a 4-ball. Thus, we have $(n-1)/2 \geqq g^*(K)$.

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