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ON THE CLASSIFICATION OF ESSENTIALLY EFFECTIVE $SL(2; C)\times SL(2; C)$-ACTIONS ON ALGEBRAIC THREEFOLDS

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0. Introduction

The main purpose of this paper is to prove the following:

Theorem. *Let V be a non-singular irreducible 3-dimensional complete variety endowed with an essentially effective regular action of the algebraic group $G=SL(2; \boldsymbol{C})\times SL(2; \boldsymbol{C})$. Then V is isomorphic to one of the following five types of varieties:*

i) $\boldsymbol{P}^3(\boldsymbol{C})$.

ii) *The projective bundle $Proj(\mathcal{O}_{\boldsymbol{P}^1}(d)\oplus\mathcal{O}_{\boldsymbol{P}^1}(d)\oplus\mathcal{O}_{\boldsymbol{P}^1}(0))$, $d\in\boldsymbol{Z}$, associated with the vector bundle $\mathcal{O}_{\boldsymbol{P}^1}(d)\oplus\mathcal{O}_{\boldsymbol{P}^1}(d)\oplus\mathcal{O}_{\boldsymbol{P}^1}(0)$ over $\boldsymbol{P}^1(\boldsymbol{C})$.*

iii) *$\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times K$, where K is an arbitrary non-singular complete curve.*

iv) *The hyperquadric $\{(x:y:z:u:v)\in\boldsymbol{P}^4(\boldsymbol{C});\ xu-yz=v^2\}$.*

v) *The projective bundle $Proj(pr_1{}^*(\mathcal{O}_{\boldsymbol{P}^1}(a))\oplus pr_2{}^*(\mathcal{O}_{\boldsymbol{P}^1}(b)))$, $a,b\in\boldsymbol{Z}$, associated with the vector bundle $pr_1{}^*(\mathcal{O}_{\boldsymbol{P}^1}(a))\oplus pr_2{}^*(\mathcal{O}_{\boldsymbol{P}^1}(b))$ over $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$, where pr_i: $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\rightarrow\boldsymbol{P}^1(\boldsymbol{C})$ denotes the canonical projection to the i-th factor.*

(See Theorem 4.1 for the corresponding G-actions and more details.)

The theorem above is, in some sense, regarded as a study of rational (or ruled) algebraic threefolds from a group-theoretical viewpoint. This may be understood, if we observe the following fact:

Fact: Let V be a non-singular irreducible 2-dimensional complete variety endowed with an essentially effective regular action of the algebraic group $SL(2; \boldsymbol{C})$. Then V is isomorphic to one of the following three types of varieties:

i) $\boldsymbol{P}^2(\boldsymbol{C})$.

ii) $F_n=Proj(\mathcal{O}_{\boldsymbol{P}^1}(n)\oplus\mathcal{O}_{\boldsymbol{P}^1}(0))$, $n=1,2,\cdots$

iii) *$\boldsymbol{P}^1(\boldsymbol{C})\times K$, where K is a non-singular complete curve.*

Note that $\boldsymbol{P}^2(\boldsymbol{C})$, $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$, $F_n(n>1)$ are, as is well-known, the relatively

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minimal models of rational algebraic surfaces.

NOTATIONS AND CONVENTIONS

(0.1) $\boldsymbol{Z}$ = the set of all integers,
$\boldsymbol{Z}_+$ = the set of all positive integers,
$\boldsymbol{C}$ = the complex number field,
$\boldsymbol{C}^*$ = the set of all non-zero complex numbers.

(0.2) All varieties and algebraic groups are defined over $\boldsymbol{C}$.

(0.3) Assume that an algebraic group G acts on varieties V and V' regularly. A regular mapping $f: V \to V'$ is said to be G-equivariant, if the equality $f(g \cdot p) = g \cdot f(p)$ holds for every $(g,p) \in G \times V$.

(0.4) A closed subgroup of an algebraic group G is always understood to be an algebraic subgroup of G, ("closed" means "Zariski closed").

(0.5) For every subgroup H of a group G, we denote by $N_G(H)$ the normalizer of H in G.

(0.6) We denote by $\boldsymbol{G}_m$ a 1-dimensional algebraic torus, which is, as a complex Lie group, the multiplicative group $\boldsymbol{C}^*$.

(0.7) An algebraic group G is said to act essentially effectively on a variety V if the group of the elements in G which act identically on V is finite.

In concluding this introduction, I wish to thank all those people who encouraged me and gave me suggestions, and in particular Professors S. Kobayashi, S.S. Roan, and I. Satake who helped me again and again during the preparation of this paper.

Added in proof. After the completion of this work, we learned that several related topics had been studied by Popov [3, 4] from a different viewpoint.

1. Basic theorems

In this section, we shall give two basic tools in handling non-homogeneous algebraic group actions, (cf. (1.1.2) and (1.2.1)).

(1.1) Here, we briefly discuss the notion of "dominant G-equivariant completion," which plays a crucial role in our study of almost-homogeneous $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})$-actions.

DEFINITION 1.1.1. Let U be an irreducible variety on which a connected linear algebraic group G acts regularly. Then a variety V with a regular G-action is said to be a *G-equivariant completion* of U, if the following two conditions are satisfied:

i) U is (embedded as) a G-invariant open dense subset of V.

ii) V is a complete variety.

A G-equivariant completion V of U is said to be *dominant* if the following two conditions are satisfied:

i) V is a normal variety.

ii) $V-U$ is a disjoint union of (a finite number of) 1-codimensional G-orbits in V.

We now quote the following:

Theorem 1.1.2 ([2; Corollary (1.1.3)]). *Let U be an irreducible variety on which a connected linear algebraic group G acts regularly. Assume that there exists a dominant G-equivariant completion V' of U. Then,*

i) *For any G-equivariant completion V of U, the identity mapping id_U: U (as a subset of V')$\to U$(as a subset of V) extends to a G-equivariant birational surjective regular map: $V'\to V$.*

ii) *In particular, any other dominant G-equivariant completion V'' of U is G-equivariantly isomorphic to V', where the isomorphism between V' and V'' is a canonical extension of the identity automorphism of U.*

(1.2) We next consider algebraic group actions with equidimensional orbits.

Theorem 1.2.1. *Let V be an n-dimensional irreducible complete normal variety on which a connected linear algebraic group G acts regularly, satisfying the following two conditions:*

(1) *All orbits in V have the same dimension r.*

(2) *There exists a finite subset $\{p_i;\ i=1,2,\cdots,k\}$ of V such that, for every $p\in V$, the isotropy subgroup G_p of G at p is conjugate to some G_{p_i}.*

Then we have:

(3) *$G_{p_1}, G_{p_2}, \cdots, G_{p_k}$ are all conjugate.*

(4) *The quotient V/G exists as an $(n\text{-}r)$-dimensional complete normal variety.*

(5) *V is G-equivariantly isomorphic to $G/G_{p_1}\times V/G$.*

Proof of (1.2.1). Step 1: For simplicity, we set $H_i:=G_{p_i}$, $i=1,2,\cdots,k$, and let V_i be the fixed point set in V of the H_i-action, where H_i acts on V as a subgroup of G. Then

$$\text{(a)}\qquad V_i=\{p\in V;\ G_p\supseteq H_i\}\ .$$

On the other hand, by Closed Orbit Lemma (Borel [1; p.98]), the condition (1) implies that every G-orbit in V is a complete variety. Hence, for every $p\in V$,

$$\text{(b)}\qquad \begin{aligned} G_p &= \text{a parabolic subgroup of } G\\ &= N_G(G_p) = \text{a connected } ((\dim G)-r)\text{-dimensional group.}\end{aligned}$$

In particular, for $i=1,2,\cdots,k$,

(b)′ $H_i = N_G(H_i) =$ a connected ((dim G)$-r$)-dimensional group.

Now, comparing (a) with (b) and (b)′, we obtain:

(a)′ $V_i = \{p \in V;\ G_p = H_i\}$,

which enables us to define a regular mapping τ_i: $G/H_i \times V_i \rightarrow V$ by

(c) $\tau_i(g\cdot H_i, p) = g\cdot p$, for every $(g\cdot H_i, p) \in G/H_i \times V_i$.

Note that:

(c-i): Image $\tau_i = \{p \in V;\ G_p$ is conjugate to $H_i.\}$.

(c-ii): Image τ_i is a closed subvariety of V. (Because $G/H_i \times V_i$ is a complete variety.)

Step 2: *Proof of* (3): In view of (c-i), we obtain:

(c-i)′: $V = \bigcup_{i=1}^{k}$ Image τ_i, (cf. (2)),

(c-i)″: For every $i, j \in \{1,2,\cdots,k\}$, either Image τ_i=Image τ_j or (Image τ_i) $\cap$ (Image τ_j)$=\phi$.

Therefore, (c-i)′, (c-i)″, (c-ii), and the irreducibility of V imply

$$\text{Image } \tau_1 = \text{Image } \tau_2 = \cdots = \text{Image } \tau_k = V\,,$$

and hence, by (c-i), the subgroups $H_1, H_2, \cdots, H_k$ of G are all conjugate, and this finishes the proof of (3).

Step 3: *Proof of* (4): First we consider the surjective regular mapping

$$\tau_1\colon\ G/H_1 \times V_1 \rightarrow V\,.$$

Since (a)′ and (b)′ imply the injectivity of τ_1, Zariski's Main Theorem asserts that:

(d) τ_1 is an (algebraic) isomorphism: $G/H_1 \times V_1 \xrightarrow{\cong} V$.

In particular, by the normality of V,

(*) V_1 is a normal variety.

Now, let pr_1: $G/H_1 \times V_1 \rightarrow V_1$ denote the canonical projection to the second factor. Then, in view of (c), we have:

(**) For every $p \in V_1$, $(pr_1 \circ \tau_1^{-1})^{-1}(p) = G\cdot p =$ a single G-orbit.

Thus, by (*) and (**), (cf. Borel [1;p.179]), the quotient V/G exists as a normal $(n-r)$-dimensional complete variety and is identified with V_1. This finishes the proof of (4).

Step 4: *Proof of* (5): Since V_1 is identified with V/G, noting that $\tau_1(g\cdot(g'\cdot H), p) = g\cdot(g'\cdot p) = g\cdot\tau_1(g'\cdot H, p)$ for all $g \in G$ and $(g'\cdot H, p) \in G/H_1 \times V_1$, we can regard the isomorphism τ_1 in (d) as a G-equivariant isomorphism: $G/H_1 \times V/G \xrightarrow{\cong} V$. This finishes the proof of (5), and hence that of Theorem (1.2.1).

2. Closed subgroups of codimension $\leqq 3$ of the group $SL(2; \boldsymbol{C}) \times SL(2; \boldsymbol{C})$

In this section, we shall make a rough classification of the closed subgroups of codimension$\leqq 3$ of the algebraic group $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})$.

NOTATION. For any linear algebraic group G, the identity component of G (resp. the dimension of a maximal torus of G) is denoted by G^0 (resp. rank (G)).

DEFINITION 2.1. i) We define closed subgroups T and B of $SL(2;\boldsymbol{C})$ by

$$T = \{f = (f_{ij}) \in SL(2; \boldsymbol{C});\ f_{21} = f_{12} = 0\}\ ,$$
$$B = \{f = (f_{ij}) \in SL(2; \boldsymbol{C});\ f_{21} = 0\}\ .$$

ii) We define closed subgroups D and D' of $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})$ by

$$D = \{(f, f) \in SL(2; \boldsymbol{C}) \times SL(2; \boldsymbol{C});\ f \in SL(2; \boldsymbol{C})\}\ ,$$
$$D' = \{(f, \pm f) \in SL(2; \boldsymbol{C}) \times SL(2; \boldsymbol{C});\ f \in SL(2; \boldsymbol{C})\}\ .$$

iii) For each $(a, b) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}$ with $a \geqq b$, we define a closed subgroup $S(a; b)$ of $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})$ by

$$S(a; b) = \{(g(r; u), g(s; v));\ r, s \in \boldsymbol{C}^*,\ r^a = s^b,\ u, v \in \boldsymbol{C}\}\ ,$$

where $g(r;u)$ and $g(s;v)$ are matrices in $SL(2;\boldsymbol{C})$ given by

$$g(r; u) = \begin{pmatrix} r & u \\ 0 & r^{-1} \end{pmatrix}, \quad g(s; v) = \begin{pmatrix} s & v \\ 0 & s^{-1} \end{pmatrix}.$$

REMARK 2.1.1. i) Note that $T \cong \boldsymbol{G}_m$ (=1-dimensional algebraic torus). And its normalizer in $SL(2;\boldsymbol{C})$ is written in the form

$$N_{SL(2;C)}(T) = J \cdot T, \qquad \text{where } J = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}.$$

ii) B is a Borel subgroup of $SL(2;\boldsymbol{C})$, and hence

$$N_{SL(2;C)}(B) = B\,.$$

iii) In view of the equality $N_{SL(2;C) \times SL(2;C)}(D) = D'$, we have:

D and D' are the only closed subgroups of $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})$ that have the identity component D.

iv) Let $(\alpha, \beta) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}$ be such that $\alpha \geqq \beta$ and $g.c.d.(\alpha, \beta) = 1$. Then $N_{SL(2;C) \times SL(2;C)}(S(\alpha; \beta)) = B \times B$, and one can immediately check that:

$S((m\cdot\alpha);(m\cdot\beta))$; $m=1,2,\cdots$ are the only closed subgroups of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ that have the identity component $S(\alpha;\beta)$.

In terms of the notation defined above, the main thing we want to prove in this §2 is now summarized as follows:

Theorem 2.2. i) *Any algebraic group automorphism of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ (which is, from now on, simply called "an automorphism of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$") coincides, up to inner automorphisms, with one of the following*:

$$(1)\quad id_{SL(2;C)\times SL(2;C)}: SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C}) \to SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C}) \qquad (f,g)\mapsto(f,g)\,,$$

$$(2)\quad \textit{transposition } \sigma : SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C}) \to SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C}) \qquad (f,g)\mapsto(g,f)\,.$$

ii) *For any subgroups K_1 and K_2 of $SL(2;\boldsymbol{C})$, we write*:

$$K_1\times K_2 = \{(k,k')\in SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C});\ k\in K_1,\ k'\in K_2\}\,.$$

Then any q-codimensional ($q=1$, 2) parabolic subgroup of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ is, by some automorphism of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$, mapped onto

(1) (In the case $q=1$): $SL(2;\boldsymbol{C})\times B$.

(2) (In the case $q=2$): $B\times B$.

iii) *Any 3-codimensional closed subgroup of the algebraic group $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ is mapped (isomorphically) onto one of the following by some automorphism of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$*:

$$B\times T,\ B\times(J\cdot T),\ D,\ D',\ SL(2;\boldsymbol{C})\times(\textit{a finite subgroup of } SL(2;\boldsymbol{C}))\,,$$

$$S(a;b),\quad \textit{where } (a,b)\in\boldsymbol{Z}_+\times\boldsymbol{Z} \textit{ is such that } a\geqq b\,.$$

In order to prove Theorem (2.2), we first consider those closed subgroups of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ which have maximal rank ($=2$).

Proposition 2.3. i) *Let G be a q-codimensional ($q=1,2$) parabolic subgroup of the algebraic group $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$. Assume that rank$(G)=2$. Then G is, by some automorphism of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$, mapped onto*

(1) (*In the case $q=1$*): $SL(2;\boldsymbol{C})\times B$.

(2) (*In the case $q=2$*): $B\times B$.

ii) *Let G be a 3-codimensional closed subgroup of the algebraic group $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$. Assume that rank$(G)=2$. Then G is mapped onto one of the following two groups by some automorphism of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$*:

$$B\times T,\ B\times(J\cdot T)\,.$$

Proof of (2.3). We fix a q-codimensional ($q=1,2,3$) closed subgroup

$G\subseteq SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ of rank 2 with Lie algebra $g\subseteq sl(2;\boldsymbol{C})\oplus sl(2;\boldsymbol{C})$. We also fix a common maximal torus H of G and $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ with the corresponding Cartan subalgebra h of $sl(2;\boldsymbol{C})\oplus sl(2;\boldsymbol{C})$. Then

$$(*)\qquad \begin{cases} sl(2;\boldsymbol{C})\oplus sl(2;\boldsymbol{C}) = h+\sum_{i=0}^{3}\boldsymbol{C}\cdot e_{\alpha_i}\,, \\ g = h+\sum_{j=0}^{3-q}\boldsymbol{C}\cdot e_{\beta_j}\,. \end{cases}$$

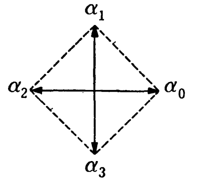
where $A=\{\alpha_i; i=0,1,2,3\}$ is the root system of $sl(2;\boldsymbol{C})\oplus sl(2;\boldsymbol{C})$ relative to h, and $A'=\{\beta_j; j=0,\cdots,3-q\}$ is an additively closed subset of A, i.e., $\beta_j+\beta_{j'}\in A'$ whenever $\beta_j+\beta_{j'}\in A$. First, note the following two facts:

(a) Since the maximal tori in any linear algebraic group are all conjugate, we may assume $H=B\times B$. Therefore, denoting by E_{pr} the 2×2 matrix with the only non-zero element 1 in the (p,r)-*th* entry, we may put:

$$e_{\alpha_0}=E_{12}\oplus 0,\ e_{\alpha_1}=0\oplus E_{12}\ \ e_{\alpha_2}=E_{21}\oplus 0,\ e_{\alpha_3}=0\oplus E_{21}\,.$$

The corresponding roots $\alpha_0,\alpha_1,\alpha_2,\alpha_3$ in the Euclidean 2-space $\boldsymbol{R}^2$ form a square as shown in Figure 1.

Figure 1.



(b) Let $\mathrm{Aut}(A)$ denote the set of all linear transformations of $\boldsymbol{R}^2$ which maps $A=\{\alpha_i; i=0,1,2,3\}$ onto itself. Then, for any $t\in\mathrm{Aut}(A)$, there exists an automorphism τ of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ such that the induced Lie algebra automorphism τ_* of $sl(2;\boldsymbol{C})\oplus sl(2;\boldsymbol{C})$ satisfies

$$\tau_*(h)=h \text{ and } \tau_*(\boldsymbol{C}\cdot e_{\alpha_i})=\boldsymbol{C}\cdot e_{t(\alpha_i)} \text{ for } i=0,1,2,3\,.$$

In view of (*), (a), and (b) above, we now complete the proof of (2.3) as follows:

Case (1): Let $q=1$ and G be a parabolic subgroup of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$. Since A' is an additively closed subset of A with 3 elements, it follows that $A'=\{t(\alpha_0),\ t(\alpha_1),\ t(\alpha_2)\}$ for some $t\in\mathrm{Aut}(A)$. Hence,

$$G(=G^0)=\tau(SL(2;\boldsymbol{C})\times B)\,.$$

Case (2): Let $q=2$ and G be a parabolic subgroup of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$. Since A' is an additively closed subset of A with 2 elements, it follows that:

(2-i) either $A'=\{t(\alpha_0),\ t(\alpha_2)\}$ for some $t\in\mathrm{Aut}(A)$,

(2-ii) or $A'=\{t(\alpha_0),\ t(\alpha_1)\}$ for some $t\in\mathrm{Aut}(A)$.

But, in the case of (2-i), g is not a parabolic subalgebra of $sl(2;\boldsymbol{C})\oplus sl(2;\boldsymbol{C})$. Hence only (2-ii) can happen. Thus, one immediately obtains:

$$G(=G^0)=\tau(B\times B).$$

Case (3): Let $q=3$. (i.e., G is a 3-codimensional closed subgroup of the algebraic group $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$.) Then $A'=\{t(\alpha_0)\}$ for some $t\in \mathrm{Aut}(A)$. Hence,

$$G^0=\tau(B\times T).$$

Since $N_{SL(2;C)\times SL(2;C)}(B\times T)=N_{SL(2;C)}(B)\times N_{SL(2;C)}(T)=B\times(J\cdot T)$, (cf. (2.1.1)), it finally follows that:

$$G=\text{either } \tau(B\times T) \text{ or } \tau(B\times(J\cdot T)).$$

Thus, the above three cases (1), (2) and (3) complete the proof of (2.3).

(2.4) Proof of Theorem 2.2

Here, we use the notation "u.dim." For any linear algebraic group K, we denote by $u.\dim(K)$ the common dimension of all maximal connected unipotent subgroups of K.

Proof of i). i) is a standard fact.

Proof of ii). Since, in this case, G is a parabolic subgroup of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$, the equality $\mathrm{rank}(G)=\mathrm{rank}(SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C}))=2$ holds. Hence, ii) is straightforward from Proposition (2.3).

Proof of iii). Step 1: First, note that $\dim G^0=\dim G=3$. Now, by Chevalley decomposition, G^0 is expressible as

(♯) $\quad G^0=H\cdot U,\ H\cap U=\{e\}$, (semi-direct product),

where H=a connected reductive closed subgroup of G^0,
U=a connected unipotent normal closed subgroup of G^0.
Since H satisfies

$$3-\dim H+u.\dim(H)=\dim U+u.\dim(H)\leqq u.\dim(G)$$
$$\leqq u.\dim(SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C}))=2,$$

H cannot be $\{e\}$. Therefore, from the inequality

$$1\leqq \mathrm{rank}(H)=\mathrm{rank}(G)\leqq \mathrm{rank}(SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C}))=2,$$

it follows that:

Case 1: either H is isomorphic to $\boldsymbol{G}_m$,
Case 2: or H is isogenous to $SL(2;\boldsymbol{C})$,
Case 3: or $\mathrm{rank}(H)=2$.

Step 2: (Case 1): In this case, by (♯) above, G^0 is a connected solvable subgroup of $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})$, and hence, for some $s \in SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})$, we have the inclusion $s \cdot G^0 \cdot s^{-1} \subseteq B \times B$. Therefore, without loss of generality, we may assume that:

(♯♯) $\quad G^0 \subseteq B \times B$.

Now, consider the following algebraic group homomorphism:

$$\tau\colon\ G^0(\hookrightarrow B \times B) \to \boldsymbol{G}_m \times \boldsymbol{G}_m$$
$$g = (g', g'') \mapsto ((g')_{11}, (g'')_{11}),$$

where $g' = ((g')_{ij}) \in B(\hookrightarrow SL(2;\boldsymbol{C}))$ and $g'' = ((g'')_{ij}) \in B$. Then, by the equality $\mathrm{rank}(G^0) = \mathrm{rank}(H) = 1$, the image $\tau(G^0)$ is a 1-dimensional torus subgroup of $\boldsymbol{G}_m \times \boldsymbol{G}_m$, i.e., for some $(\alpha, \beta) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}$ with $\alpha \geqq \beta$ and $g.c.d.(\alpha, \beta) = 1$,

$$\tau(G^0) = \text{either } \{(t^\beta, t^\alpha);\ t \in \boldsymbol{G}_m\} \text{ or } \{(t^\beta, t^\alpha);\ t \in \boldsymbol{G}_m\}.$$

Hence, in view of (♯♯) above, we have:

$$S(\alpha; \beta) \subseteq \text{either } G^0 \text{ or } \sigma(G^0), \qquad (\text{cf. (iii) of (2.1)}),$$

where σ is the automorphism of $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})$ defined in i) of (2.2). Comparing their dimensions, we immediately obtain:

$$S(\alpha; \beta) = \text{either } G^0 \text{ or } \sigma(G^0)\, (= (\sigma(G))^0).$$

Thus, by iv) of (2.1.1), either G or $\sigma(G)$ is written in the form $S((m \cdot \alpha); (m \cdot \beta))$ for some $m \in \boldsymbol{Z}_+$. This shows that, in Case 1, there exists an automorphism of $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})$ which maps G onto

$$S(a; b) \qquad \text{for some } (a, b) \in \boldsymbol{Z}_+ \times \boldsymbol{Z} \text{ with } a \geqq b.$$

Step 3: (Case 2): In this case, $\dim H = 3$ ($= \dim G$), and hence we have the equality $H = G^0$. Let $\gamma\colon SL(2;\boldsymbol{C}) \to H(=G^0)$ be the isogeny the existence of which is the assumption of Case 2. Now, denoting by π_1 (resp. π_2) the canonical projection: $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C}) \to SL(2;\boldsymbol{C})$ to the first (resp. second) factor, we define the following algebraic group homomorphisms:

$$\gamma_1 \overset{\text{defn}}{=\!=} \pi_1 \circ \gamma\colon SL(2;\boldsymbol{C}) \to SL(2;\boldsymbol{C}),$$
$$\gamma_2 \overset{\text{defn}}{=\!=} \pi_2 \circ \gamma\colon SL(2;\boldsymbol{C}) \to SL(2;\boldsymbol{C}).$$

Note that $G^0 = \{(\gamma_1(s), \gamma_2(s)) \in SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C});\ s \in SL(2;\boldsymbol{C})\}$. Since any algebraic group homomorphism: $SL(2;\boldsymbol{C}) \to SL(2;\boldsymbol{C})$ is either trivial or an inner automorphism of $SL(2;\boldsymbol{C})$, γ_1 and γ_2 above are, up to inner automorphisms of $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})$, given by

either 1) $\gamma_1 = id_{SL(2;C)}$, $\quad \gamma_2 = \text{trivial}$,

or 2) $\gamma_1 = \text{trivial}$, $\quad \gamma_2 = id_{SL(2;C)}$, or $\quad$ 3) $\gamma_1 = id_{SL(2;C)}$, $\quad \gamma_2 = id_{SL(2;C)}$.

Hence G^0 is conjugate to either $SL(2;\boldsymbol{C})\times\{e\}$ or $\{e\}\times SL(2;\boldsymbol{C})$ or D, (cf. (ii) of (2.1)). Since $\sigma(\{e\}\times SL(2;\boldsymbol{C}))=SL(2;\boldsymbol{C})\times\{e\}$, (cf. (i) of (2.2)), in view of iii) of (2.1.1), we obtain:

In Case 2, there exists an automorphism of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ which maps G onto one of the following: $SL(2;\boldsymbol{C})\times$(a finite subgroup of $SL(2;\boldsymbol{C})$), D, D'.

Step 4: (Case 3): Since rank$(H)=2$, ii) of Proposition (2.3) immediately implies that:

In Case 3, there exists an automorphism of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ which maps G onto either $B\times T$ or $B\times(J\cdot T)$.

Thus, the last three steps finish the proof of iii) of (2.2), and hence we completed that of Theorem (2.2).

3. Examples of dominant $SL(2;C)\times SL(2;C)$-equivariant completions

In this section, several examples of dominant $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$-equivariant completions will be given for later purpose.

(3.1) Dominant equivariant completions of the homogeneous spaces $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})/(B\times T)$ and $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})/(B\times(J\cdot T))$

(3.1.1) Actions of $SL(2;\boldsymbol{C})$ on $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$ and $\boldsymbol{P}^2(\boldsymbol{C})$.

(i) First note that $G=SL(2;\boldsymbol{C})$ acts on $\boldsymbol{P}^1(\boldsymbol{C})$ via the canonical homomorphism: $SL(2;\boldsymbol{C})\to PGL(2;\boldsymbol{C})$. In terms of this action, we can identify $\boldsymbol{P}^1(\boldsymbol{C})$ with the homogeneous space $SL(2;\boldsymbol{C})/B$, (cf. (2.1)). Now, $G=SL(2;\boldsymbol{C})$ acts on $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$ by

$$\begin{array}{ccccc} G=SL(2;\boldsymbol{C}) & \times & (\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})) & \to & \boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C}) \\ g\,, & & (b,c) & \mapsto & (g\cdot b, g\cdot c)\,. \end{array}$$

Let $q'=((1:0),(0:1))\in\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$ and let $q''=((1:0),(1:0))\in\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$. Then

$$\begin{aligned} G\cdot q' &= \{(b,c)\in\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C});\ b\neq c\} \\ &= \text{an open dense orbit in } \boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\,, \\ G\cdot q'' &= \{(b,c)\in\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C});\ b=c\} \\ &= \text{a 1-codimensional orbit in } \boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\,, \\ \boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C}) &= (G\cdot q')\cup(G\cdot q'')\,. \end{aligned}$$

Since the isotropy subgroup $G_{q'}$ of G at q' is given by

$$(*) \qquad G_{q'} = T\,, \qquad \text{(cf. (2.1))},$$

it follows that $P^1(C)\times P^1(C)$ is a dominant $SL(2;C)$-equivariant completion of the homogeneous space $SL(2;C)/T$.

(ii) Secondly, $G=SL(2;C)$ acts on $P^2(C)$ via the following algebraic group homomorphism:

$$G = SL(2; C) \to PGL(3; C)$$
$$\begin{pmatrix} r & t \\ s & u \end{pmatrix} \mapsto \begin{pmatrix} r^2, & t^2, & rt \\ s^2, & u^2, & su \\ 2rs, & 2tu, & ru+st \end{pmatrix}.$$

Since the 2-sheeted ramified covering

$$f\colon P^1(C)\times P^1(C) \to P^2(C)$$
$$(x\colon y), (v\colon w) \mapsto (xv\colon yw\colon xw+yv)$$

is G-equivariant in terms of the actions defined above, (cf. (i)), it immediately follows that:

$$G\cdot f(q') = f(G\cdot q') = \text{an open dense orbit in } P^2(C),$$
$$G\cdot f(q'')= f(G\cdot q'')= \text{a 1-codimensional orbit in } P^2(C),$$
$$P^2(C) = (G\cdot f(q'))\cup(G\cdot f(q'')).$$

Furthermore, the isotropy subgroup $G_{f(q')}$ of G at $f(q')$ is written as

$$G_{f(q')} = \{g\in G;\, g\cdot q'\in f^{-1}(f(q'))\} = J\cdot T, \quad \text{(cf. (2.1.1))}.$$

Thus, $P^2(C)$ is a dominant $SL(2;C)$-equivariant completion of the homogeneous space $SL(2;C)/J\cdot T$.

(3.1.2) EXAMPLE 1. We define an $SL(2;C)\times SL(2;C)$-action on $P^1(C)\times P^1(C)\times P^1(C)$ by

$$(SL(2;C)\times SL(2;C))\times(P^1(C)\times P^1(C)\times P^1(C)) \to (P^1(C)\times P^1(C)\times P^1(C))$$
$$(h,\ g),\qquad (a,\quad b,\quad c)\quad \mapsto\quad (h\cdot a,\ g\cdot b,\ g\cdot c).$$

Since $(SL(2;C)\times SL(2;C))/(B\times T)\cong(SL(2;C)/B)\times(SL(2;C)/T)\cong P^1(C)\times(SL(2;C)/T)$, and since $P^1(C)\times P^1(C)$ is a dominant equivariant completion of $SL(2;C)/T$, (cf. (i) of (3.1.1)), we immediately obtain:

$P^1(C)\times P^1(C)\times P^1(C)$ endowed with this action is a dominant $SL(2;C)\times SL(2;C)$-equivariant completion of the homogeneous space $(SL(2;C)\times SL(2;C))/(B\times T)$.

(3.1.3) EXAMPLE 2. We define an $SL(2;C)\times SL(2;C)$-action on $P^1(C)\times P^2(C)$ by

$$(SL(2; C)\times SL(2; C))\times(P^1(C)\times P^2(C)) \to P^1(C)\times P^2(C)$$
$$(h,\ g),\qquad (a,\quad b)\quad \to (h\cdot a,\ g\cdot b),$$

where $SL(2;\boldsymbol{C})$ acts on $\boldsymbol{P}^2(\boldsymbol{C})$ as in (ii) of (3.1.1). Since $(SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C}))/(B\times(J\cdot T))\cong(SL(2;\boldsymbol{C})/B)\times(SL(2;\boldsymbol{C})/J\cdot T)\cong\boldsymbol{P}^1(\boldsymbol{C})\times(SL(2;\boldsymbol{C})/J\cdot T)$, and since $\boldsymbol{P}^2(\boldsymbol{C})$ is a dominant equivariant completion of $SL(2;\boldsymbol{C})/J\cdot T$, (cf. (ii) of (3.1.1)), we obtain:

$\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^2(\boldsymbol{C})$ endowed with the action just above is a dominant $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$-equivariant completion of the homogeneous space $(SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C}))/(B\times(J\cdot T))$.

(3.2) Dominant equivariant completions of the homogeneous spaces $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})/D$ and $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})/D'$

(3.2.1) EXAMPLE 3. We define a non-singular hyperquadric W in $\boldsymbol{P}^4(\boldsymbol{C})$ by

$$W=\{(x:y:z:u:v)\in\boldsymbol{P}^4(\boldsymbol{C});\ xu-yz=v^2\}\ .$$

Now, $G=SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ acts on W by

$$\begin{array}{ccccc} SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C}) & \times & W & \to & W \\ (h,\ k)\,, & & (x:y:z:u:v) & \mapsto & (x':y':z':u':v)\,, \end{array}$$

where $\quad h\cdot\begin{pmatrix} x & z \\ y & u\end{pmatrix}\cdot k^{-1}=\begin{pmatrix} x' & z' \\ y' & u'\end{pmatrix}.$

Then, letting $p=(1:0:0:1:1)\in W$ and $p'=(1:0:0:0:0)\in W$, we have:

$$\begin{aligned} G\cdot p &= \{(x:y:z:u:v)\in W;\ v\neq 0\} \\ &= \text{an open dense orbit in } W, \\ G\cdot p' &= \{(x:y:z:u:v)\in W;\ v=0\} \\ &= \text{a 1-codimensional orbit in } W, \\ W &= (G\cdot p)\cup(G\cdot p')\,. \end{aligned}$$

Furthermore, a computation shows that the isotropy subgroup of G at p is exactly D. Thus,

the hyperquadric W with the above action is a dominant $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$-equivariant completion of the homogeneous space $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})/D$.

(3.2.2) EXAMPLE 4. Note that $G=SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ acts on $\boldsymbol{P}^3(\boldsymbol{C})$ by

$$\begin{array}{ccccc} SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C}) & \times & \boldsymbol{P}^3(\boldsymbol{C}) & \to & \boldsymbol{P}^3(\boldsymbol{C}) \\ (h,\ \ k)\,, & & (x:y:z:u) & \mapsto & (x':y':z':u')\,, \end{array}$$

where $\quad h\cdot\begin{pmatrix} x & z \\ y & u\end{pmatrix}\cdot k^{-1}=\begin{pmatrix} x' & z' \\ y' & u'\end{pmatrix}.$

Then, letting $q=(1:0:0:1)\in\boldsymbol{P}^3(\boldsymbol{C})$ and $q'=(1:0:0:0)\in\boldsymbol{P}^3(\boldsymbol{C})$, we have:

$$\begin{aligned} G\cdot q &= \{(x:y:z:u)\in\boldsymbol{P}^3(\boldsymbol{C});\ xu\neq yz\} \\ &= \text{an open dense orbit in } \boldsymbol{P}^3(\boldsymbol{C})\,, \end{aligned}$$

$$\begin{aligned} G \cdot q' &= \{(x:y:z:u) \in \boldsymbol{P}^3(\boldsymbol{C});\ xu = yz\} \\ &= \text{a 1-codimensional orbit in } \boldsymbol{P}^3(\boldsymbol{C}), \\ \boldsymbol{P}^3(\boldsymbol{C}) &= (G \cdot q) \cup (G \cdot q'). \end{aligned}$$

Since a computation shows that the isotropy subgroup of G at q is exactly D', we obtain:

$\boldsymbol{P}^3(\boldsymbol{C})$ with the above action is a dominant $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})$-equivariant completion of the homogeneous space $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})/D'$.

(3.3) Dominant equivariant completion of the homogeneous space $SL(2;\boldsymbol{C}) \times SL(2;\boldsymbol{C})/S(a;b)$.

DEFINITION 3.3.1. We shall define a canonical $SL(2;\boldsymbol{C})$-action on the line bundle $\mathcal{O}_{\boldsymbol{P}^1}(d)$, $d \in \boldsymbol{Z}$, over $\boldsymbol{P}^1(\boldsymbol{C})$.

Let π: $\boldsymbol{C}^2 - \{0\} \to \boldsymbol{P}^1(\boldsymbol{C})$ be the canonical projection, and let σ: $Q_0(\boldsymbol{C}^2) \to \boldsymbol{C}^2$ be the blowing-up of the origin 0 of $\boldsymbol{C}^2$. Then

$$Q_0(\boldsymbol{C}^2) - \sigma^{-1}(0) = \boldsymbol{C}^2 - \{0\},$$

and under this identification, the mapping π extends to

$$\bar{\pi}\colon Q_0(\boldsymbol{C}^2) \to \boldsymbol{P}^1(\boldsymbol{C}).$$

In terms of this mapping, we can regard $Q_0(\boldsymbol{C}^2)$ as the line bundle $\mathcal{O}_{\boldsymbol{P}^1}(-1)$ over $\boldsymbol{P}^1(\boldsymbol{C})$. Note that:

(1) The matrix $SL(2;\boldsymbol{C})$-action on $\boldsymbol{C}^2$ canonically induces an $SL(2;\boldsymbol{C})$-action on $Q_0(\boldsymbol{C}^2)$ $(=\mathcal{O}_{\boldsymbol{P}^1}(-1))$, and under this action, $Q_0(\boldsymbol{C}^2)$ $(=\mathcal{O}_{\boldsymbol{P}^1}(-1))$ consists of two orbits $\sigma^{-1}(0)$ (=the zero-section of $\mathcal{O}_{\boldsymbol{P}^1}(-1)$) and $Q_0(\boldsymbol{C}^2) - \sigma^{-1}(0)$.

Now, to each $p \in \boldsymbol{P}^1(\boldsymbol{C})$, let l_p denote the corresponding line through 0 in $\boldsymbol{C}^2$, (l_p is canonically identified with the fibre of $\mathcal{O}_{\boldsymbol{P}^1}(-1)$ over p), and we fix a base $\boldsymbol{e}_p$ of this fibre l_p. For instance, if $p_0 = (1:0) \in \boldsymbol{P}^1(\boldsymbol{C})$, we set:

(2) $\qquad \boldsymbol{e}_{p_0} = (1, 0) \in l_{p_0}$.

In terms of this notation, the fibre of $\mathcal{O}_{\boldsymbol{P}^1}(d)(=(\mathcal{O}_{\boldsymbol{P}^1}(-1))^{\otimes -d})$ over p is expressed as $(l_p)^{\otimes -d} = \boldsymbol{C} \cdot (\boldsymbol{e}_p)^{\otimes -d}$. Hence

(3) we can define a canonical $SL(2;\boldsymbol{C})$-action on $\mathcal{O}_{\boldsymbol{P}^1}(d)$, setting $g \cdot (\lambda \cdot (\boldsymbol{e}_p)^{\otimes -d}) \overset{\text{defn}}{=\!=} \lambda \cdot (g \cdot \boldsymbol{e}_p)^{\otimes -d}$ for all $g \in SL(2;\boldsymbol{C})$ and all $\lambda \in \boldsymbol{C}$.

Here, in view of (1) above, we have:

(4) If $d \neq 0$, then $\mathcal{O}_{\boldsymbol{P}^1}(d)$ is a disjoint union of two orbits, one of which is the zero-section of $\mathcal{O}_{\boldsymbol{P}^1}(d)$, and the other is its complement.

(3.3.2) EXAMPLE 5. Let pr_i: $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C}) \to \boldsymbol{P}^1(\boldsymbol{C})$ be the projection to the *i-th* factor, ($i=1,2$), and let a, b be integers with $a>0$ and $a\geqq b$. Regarding $E \overset{\text{defn}}{=\!=} \mathcal{O}_{P^1}(a)\times\mathcal{O}_{P^1}(b)$ as the vector bundle $pr_1^*(\mathcal{O}_{P^1}(a))\oplus pr_2^*(\mathcal{O}_{P^1}(b))$ of rank 2 over $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$, we have the identification:

$$Proj(E)=\begin{cases}\text{the quotient by } \boldsymbol{C}^* \text{ of the set}\\ \mathcal{O}_{P^1}(a)\times\mathcal{O}_{P^1}(b)\text{-(zero section of } \mathcal{O}_{P^1}(a))\times\text{(zero section of } \mathcal{O}_{P^1}(b))\end{cases}$$

where the complex scalar multiplication by $\boldsymbol{C}^*$ on E is written as

$$(5)\qquad \begin{array}{ccc}\boldsymbol{C}^*\times(\mathcal{O}_{P^1}(a)\times\mathcal{O}_{P^1}(b)) & \rightarrow & \mathcal{O}_{P^1}(a)\times\mathcal{O}_{P^1}(b)\\ c\,,\qquad (q,\quad r) & \mapsto & (c\cdot q,\ c\cdot r)\,.\end{array}$$

and ρ: $Proj(E)\rightarrow\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$ denotes the associated projective bundle of the vector bundle E. In terms of this identification, we have the following canonical quotient map by $\boldsymbol{C}^*$:

$$\begin{aligned}\varphi\colon\ &\mathcal{O}_{P^1}(a)\times\mathcal{O}_{P^1}(b)-\text{(zero section of } \mathcal{O}_{P^1}(a))\times\text{(zero section of } \mathcal{O}_{P^1}(b))\\ &\rightarrow Proj(E)\,.\end{aligned}$$

Now, $G=SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ acts on E by

$$(6)\qquad \begin{array}{ccc}(SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C}))\times(\mathcal{O}_{P^1}(a)\times\mathcal{O}_{P^1}(b)) & \rightarrow & (\mathcal{O}_{P^1}(a)\times\mathcal{O}_{P^1}(b))\\ (h,\ \ k)\,,\qquad\qquad (q,\quad r) & \rightarrow & (h\cdot q,\ k\cdot r)\,,\end{array}$$

where $SL(2;\boldsymbol{C})$ acts on $\mathcal{O}_{P^1}(a)$ and $\mathcal{O}_{P^1}(b)$ as in (3) of (3.3.1). Since this G-action on E commutes with the $\boldsymbol{C}^*$-action, (cf. (5), (6)),

(7) we can canonically define an action of $G=SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ on $Proj(E)$ so that the quotient map φ is G-equivariant.

We now want to show that $Proj(E)$ with this G-action is a dominant G-equivariant completion of the homogeneous space $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})/S(a;b)$. For this purpose, let p be the point of E defined by

$$p=((e_{p_0})^{\otimes -a},\ (e_{p_0})^{\otimes -b})\in\mathcal{O}_{P^1}(a)\times\mathcal{O}_{P^1}(b)$$

in terms of the notation in (2) of (3.3.1). Then

$$\rho(\varphi(p))=(p_0,\,p_0)=((1:0),\ (1:0))\in\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\,.$$

In view of the G-equivariance of ρ which follows at once from our definition of the G-action on $Proj(E)$, (cf. (7)), we infer that the isotropy subgroup $G_{\varphi(p)}$ of G at $\varphi(p)$ is contained in the isotropy subgroup of G at $((1:0),\ (1:0))$, i.e.,

$$G_{\varphi(p)}\subseteq B\times B,\qquad \text{(cf. (2.1))}.$$

Now, for every $h=(h_{ij})\in B(\subseteq SL(2;\boldsymbol{C}))$, we have $h_{21}=0$, and hence

$$\begin{aligned}h\cdot(e_{p_0})^{\otimes\gamma}&=(h\cdot e_{p_0})^{\otimes\gamma}=((h_{11},\ 0))^{\otimes\gamma}\\ &=(h_{11}\cdot e_{p_0})^{\otimes\gamma}=(h_{11})^{\gamma}\cdot(e_{p_0})^{\otimes\gamma}\,,\qquad \text{for every } \gamma\in\boldsymbol{Z}\,.\end{aligned}$$

Therefore,

$$\begin{aligned} G_{\varphi(p)} &= \{g(=(h,k))\in B\times B;\ g\cdot\varphi(p)=\varphi(p)\} \\ &= \left\{(h,k)\in B\times B;\ \begin{array}{l}\text{For some } c\in \boldsymbol{C}^*,\ h\cdot(e_{p_0})^{\otimes -a} = c\cdot(e_{p_0})^{\otimes -a} \\ \text{and } k\cdot(e_{p_0})^{\otimes -b} = c\cdot(e_{p_0})^{\otimes -b}\,.\end{array}\right\} \\ &= \{\ (h,k)=((h_{ij}),(k_{ij}))\in B\times B;\ (h_{11})^a=(k_{11})^b\ \} = S(a;b)\,. \end{aligned}$$

Thus, we obtain:

(8) The orbit $G\cdot\varphi(p)$ is regarded as the homogeneous space $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})/S(a:b)$.

Now, combining (5) and (6), one can canonically define an action of $H:=\boldsymbol{C}^*\times SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})=\boldsymbol{C}^*\times G$ on $E=\mathcal{O}_{\boldsymbol{P}^1}(a)\times\mathcal{O}_{\boldsymbol{P}^1}(b)$. Note that

$$\varphi^{-1}(G\cdot\varphi(p)) = H\cdot p\,,$$

which is, by $a\neq 0$ (cf. (4)), exactly

$$E-\{((\text{zero section of } \mathcal{O}_{\boldsymbol{P}^1}(a))\times\mathcal{O}_{\boldsymbol{P}^1}(b))\cup(\mathcal{O}_{\boldsymbol{P}^1}(a)\times(\text{zero section of } \mathcal{O}_{\boldsymbol{P}^1}(b)))\}\,.$$

Regarding $E':=(\text{zero section of } \mathcal{O}_{\boldsymbol{P}^1}(a))\times\mathcal{O}_{\boldsymbol{P}^1}(b)$ and $E'':=\mathcal{O}_{\boldsymbol{P}^1}(a)\times(\text{zero section of } \mathcal{O}_{\boldsymbol{P}^1}(b))$ as line subbundles of E with $E=E'\oplus E''$, we decompose $E-$(zero section of E) into three H-orbits:

$$H\cdot p,\ E'-(\text{zero section of } E'),\ E''-(\text{zero section of } E'').$$

Therefore, $Proj(E)$ is a disjoint union of the corresponding three G-orbits:

$$G\cdot\varphi(p),\ Proj(E')\ (\cong \boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})),\ Proj(E'')\ (\cong \boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C}))\,.$$

Finally, in view of this fact and (8) above, we obtain:

(9) $Proj(E)\ (=Proj(pr_1{}^*(\mathcal{O}_{\boldsymbol{P}^1}(a))\oplus pr_2{}^*(\mathcal{O}_{\boldsymbol{P}^1}(b))))$ endowed with the $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$-action defined in (7) is a dominant $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$-equivariant completion of the homogeneous space $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})/S(a;b)$.

4. The classification of essentially effective $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$-actions on algebraic threefolds

Let V be a variety endowed with a regular action $\gamma\colon G\times V\to V$ of an algebraic group G. (We sometimes denote such a V by the pair $(V;\gamma)$.) Then, to every algebraic group automorphism h of G, we associate a regular G-action $\gamma^h\colon G\times V\to V$ by

$$\gamma^h(g,y)=\gamma(h(g),y), \qquad \text{for all } (g,y)\in G\times V\,.$$

In this last section, we prove the following main theorem:

Theorem 4.1. *Let V be a non-singular irreducible 3-dimensional complete variety endowed with an essentially effective regular action γ of the algebraic group*

$G=SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$. *Then, for some algebraic group automorphism* h *of* G, *the space* $(V;\gamma^h)$ *is G-equivariantly isomorphic to one of the following*:

(1) $\boldsymbol{P}^3(\boldsymbol{C})$ *endowed with the G-action* ν *defined in* [2; Theorem (4.2.4)].

(2) $Proj(\mathcal{O}_{\boldsymbol{P}^1}(d)\oplus\mathcal{O}_{\boldsymbol{P}^1}(d)\oplus\mathcal{O}_{\boldsymbol{P}^1}(0))$ *endowed with the G-action* $\bar{\sigma}\times\bar{\tau}$ *defined in* [2; Theorem (4.2.4)], *where* $d\in\boldsymbol{Z}$ *is arbitrary, and* $Proj(\mathcal{O}_{\boldsymbol{P}^1}(d)\oplus\mathcal{O}_{\boldsymbol{P}^1}(d)\oplus\mathcal{O}_{\boldsymbol{P}^1}(0))$ *denotes the projective bundle over* $\boldsymbol{P}^1(\boldsymbol{C})$ *associated with the vector bundle* $\mathcal{O}_{\boldsymbol{P}^1}(d)\oplus\mathcal{O}_{\boldsymbol{P}^1}(d)\oplus\mathcal{O}_{\boldsymbol{P}^1}(0)$.

(3) $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times K$, (*where* K *is an arbitrary complete non-singular curve*), *endowed with the G-action which factors to the product of the standard homogeneous one on* $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$ *and the trivial one on* K.

(4) $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$ *with the G-action defined in Example* 1 *of* (3.1.2).

(5) $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^2(\boldsymbol{C})$ *with the G-action defined in Example* 2 *of* (3.1.3).

(6) *The hyperquadric* $\{(x:y:z:u:v)\in\boldsymbol{P}^4(\boldsymbol{C});\, xu-yz=v^2\}$ *with the G-action defined in Example* 3 *of* (3.2.1).

(7) $\boldsymbol{P}^3(\boldsymbol{C})$ *with the G-action defined in Example* 4 *of* (3.2.2).

(8) $Proj(pr_1{}^*(\mathcal{O}_{\boldsymbol{P}^1}(a))\oplus pr_2{}^*(\mathcal{O}_{\boldsymbol{P}^1}(b)))$ *with the G-action defined in Example* 5 *of* (3.3.2), (*cf.* (9) *of* (3.3.2)), *where* $(a, b)\in\boldsymbol{Z}_+\times\boldsymbol{Z}$ *is arbitrarily chosen with* $a\geqq b$, *and* pr_i: $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\to\boldsymbol{P}^1(\boldsymbol{C})$ *denotes the projection to the i-th factor*, $(i=1,2)$.

Proof of (4.1). Let c be the minimal dimension of the G-orbits in V and c' be the maximal dimension of the G-orbits in V. Since the Borel subgroup $B\times B$ (cf. (2.1)) of $G=SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ has codimension 2, and since every orbit of the least dimension is closed, the inequality $c\leqq 2$ holds. On the other hand, we now show that $c\geqq 1$. For contradiction, we assume $c=0$. Then, fixing a point $p\in V^G$, we consider the isotropy representation r_p: $G\to GL(T(V:p))$ of G at p. Since G acts on V essentially effectively, a theorem of A. Bialynicki-Birula (cf. [2; Corollary (2.3.3)]) shows that r_p induces an essentially effective linear G-action on $T(V:p)$. This means that, for a basis for $T(V:p)$, r_p is regarded as a representation: $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})\,(=G)\to GL(3;\boldsymbol{C})$ which induces an essentially effective G-action on $\boldsymbol{C}^3$. But, by standard facts on representations of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$, this cannot happen. Thus, our assumption $c=0$ is wrong, i.e., $c\geqq 1$. Hence $1\leqq c\leqq 2$, and the following three cases are possible:

Case i) $c=1$.

Case ii) $c=c'=2$.

Case iii) $c=2$ and $c'=3$.

First we consider Case i): Since Theorem (2.2) shows that every 1-codimensional parabolic subgroup of G is mapped onto $SL(2;\boldsymbol{C})\times B$ by some automorphism h of G, every 1-dimensional G-orbit W (which is automatically closed by $c=1$) is isomorphic to $\boldsymbol{P}^1(\boldsymbol{C})$ as a variety, and in terms of the G-action

γ^h obtained by modifying the original action γ by $h\in\mathrm{Aut}(G)$, the subgroups $G'=SL(2;\boldsymbol{C})\times\{e\}$ and $G''=\{e\}\times SL(2;\boldsymbol{C})$ of G satisfy the following:

(a) W sits in the fixed point set $V^{G'}$.

(b) W is a single G''-orbit.

Therefore, applying [2; Theorem (4.2.4)] to $m=n=1$, we obtain:

In Case i), for some $h'\in\mathrm{Aut}(G)$, the space $(V;\gamma^{h'})$ is G-equivariantly isomorphic to

(1)′ either $(\boldsymbol{P}^3(\boldsymbol{C});\nu)$

(2)′ or $(Proj(\mathcal{O}_{\boldsymbol{P}^1}(d)\oplus\mathcal{O}_{\boldsymbol{P}^1}(d)\oplus\mathcal{O}_{\boldsymbol{P}^1}(0));\bar{\sigma}\times\bar{\tau})$ for some $d\in\boldsymbol{Z}$.

Secondly, we consider Case ii): By combining (i) and (ii-2) of Theorem (2.2), we see that every 2-codimensional parabolic subgroup of $G=SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ is conjugate to $B\times B$. On the other hand, by $c=c'=2$, the isotropy subgroup of G at any point of V has the same codimension 2 and is parabolic. Hence, applying Theorem (1.2.1) to $k=1$, we have a G-equivariant isomorphism:

$$(V;\gamma)\cong(G/(B\times B))\times(V/G)$$

where the quotient V/G exists as a 1-dimensional normal (and hence non-singular) complete variety. Since $G/(B\times B)$ is G-equivariantly isomorphic to $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$ with the standard G-action, we obtain:

(3)′ In Case ii), $(V;\gamma)$ is G-equivariantly isomorphic to $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times K$, (where K is a non-singular complete curve), endowed with the G-action specified in (3) above.

Lastly, we consider Case iii): By $c'=3$, V contains an open dense (3-dimensional) G-orbit, which we denote by U. Then (iii) of Theorem (2.2) asserts that, for some automorphism h of G, our U endowed with the G-action γ^h (obtained by modifying our original γ by h) is G-equivariantly isomorphic to one of the following:

$$G/(B\times T),\ G/(B\times(J\cdot T)),\ G/D,\ G/D',$$
$$G/S(a;b),\qquad \text{where } (a,b)\in\boldsymbol{Z}_+\times\boldsymbol{Z} \text{ is such that } a\geqq b.$$

(Note that $G/(SL(2;\boldsymbol{C})\times$(a finite subgroup of $SL(2;\boldsymbol{C})$)) is not included in the above, because G must act essentially effectively on V and hence on U.) On the other hand, by $c=2$, the space $(V;\gamma^h)$ is a dominant G-equivariant completion of $(U;\gamma^h)$. Therefore, by (ii) of Theorem (1.1.2), (cf. Example 1 of (3.1.2), Example 2 of (3.1.3), Example 3 of (3.2.1), Example 4 of (3.2.2), (9) of Example 5 of (3.3.2)), in Case iii), for some $h\in\mathrm{Aut}(G)$, the space $(V;\gamma^h)$ is G-equivariantly isomorphic to one of the following:

(4)′ $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$ with the action specified in (4) above.

(5)′ $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^2(\boldsymbol{C})$ with the action specified in (5) above.

(6)′ $\{(x:y:z:u:v)\in\boldsymbol{P}^4(\boldsymbol{C});\ xu-yz=v^2\}$ with the action in (6) above.

(7)′ $\boldsymbol{P}^3(\boldsymbol{C})$ with the action specified in (7) above.

(8)′ $Proj(pr_1{}^*(\mathcal{O}_{\boldsymbol{P}^1}(a))\oplus pr_2{}^*(\mathcal{O}_{\boldsymbol{P}^1}(b)))$ with the action in (8) above.

Thus, (1)′, (2)′, (3)′, ⋯, (8)′ above complete the proof of Theorem (4.1).

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