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Osaka J. Math.
28 (1991), 367-384

ANALYTIC SEMIGROUPS GENERATED BY ELLIPTIC OPERATORS IN L^1 AND PARABOLIC EQUATIONS

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(Received July 19, 1990)

0. Introduction

Parabolic equations in L^p spaces have been studied both by potential theory and by abstract methods mainly when $p>1$. In this paper we want to continue our previous researches on the L^1 case ([4], [5]) by using a semigroup approach.

Let Ω be an open bounded subset of $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$. We denote by E a second order elliptic operator in Ω and by A_1 the L^1 realization of E with homogeneous Dirichlet boundary conditions. Then it is known (see Amann [1], Pazy [11] and Tanabe [14]) that A_1 is the infinitesimal generator of an analytic semigroup in $L^1(\Omega)$. We set $X=L^1(\Omega)$ and denote by $S(t)$ the semigroup generated by A_1.

In this paper we establish some new properties for the semigroup $S(t)$. Moreover we give a characterization in term of Besov spaces for the interpolation spaces $D_{A_1}(\theta, 1)$, between the domain of A_1 and $L^1(\Omega)$, defined as (see Butzer and Berens [2] and Peetre [12])

$$D_{A_1}(\theta, 1) = \{u\in X: \int_0^{+\infty} ||A_1S(t)u||_X\, t^{-\theta}dt)<+\infty\}. \tag{0.1}$$

This characterization allows us to find new regularity results for the solutions of the following Cauchy problem

$$\begin{cases} u'(t) = A_1 u(t)+f(t) \\ u(0) = u_0 \end{cases} \tag{0.2}$$

where $f\in L^1(0, T; X)$ and $u_0\in X$. For the connection between the regularity properties of solutions of (0.2) and the interpolation spaces $D_{A_1}(\theta, 1)$ we refer to [4].

The plan of the paper is as follows. In section 2 we prove that the semigroup $S(t)$ satisfies the following estimates, for some $M', M''>0$ and $\omega\in\boldsymbol{R}$,

$$\sqrt{t}\,||D_iS(t)||_{L(X)}\leq M' \exp(\omega t) \qquad i = 1, \cdots, n \tag{0.3}$$

(*) The work of the author is partially supported by M.P.I. 40% and by G.N.A.F.A.

and

$$(0.4) \qquad t\,\|D_{ih}\,S(t)\|_{L(X)} \leq M''\exp(\omega t) \qquad i, h = 1, \cdots, n$$

where we have set $D_i = \partial/\partial x_i$ and $D_{ih} = D_i D_h$. Properties (0.3) and (0.4) give precise information about the behavior at $t=0$ of the spatial derivatives of semigroup $S(t)$ (and hence about the solutions of (0.2)).

In section 3 we use these estimates and prove, in a very direct way and without using the reiteration property, the following characterization of the interpolation spaces $D_{A_1}(\theta, 1)$, for each $0<\theta<1$

$$(0.5) \quad D_{A_1}(\theta, 1) = \begin{cases} W^{2\theta,1}(\Omega), & \text{if } 0<\theta<1/2 \\ u \in B^{1,1}(\Omega)\colon \int_\Omega (d(x, \partial\Omega))^{-1} |u(x)|\, dx < +\infty, & \text{if } \theta = 1/2. \\ W^{2\theta,1}(\Omega) \cap W_0^{1,1}(\Omega), & \text{if } 1/2<\theta<1 \end{cases}$$

Here $W^{2\theta,1}(\Omega)$ denotes the Sobolev space of fractional order, $B^{1,1}(\Omega)$ denotes the Besov space and $d(x, \partial\Omega)$ the distance from x to $\partial\Omega$. This characterization has been given by Grisvard [6] for the case $p>1$. If the operator E has C^∞ coefficients and $\theta \neq 1/2$ the characterization (0.5) can be deduced by a result of Guidetti, [8], obtained by complex interpolation methods.

Finally in section 4 we obtain a quite complete description of the regularity of the solutions of the following problem (for which (0.2) is the abstract version)

$$(0.6) \qquad \begin{cases} u_t(t, x) = Eu(t, x) + f(t, x), \; t>0, \; x \in \Omega \\ u(t, x) = 0, \; t>0, \; x \in \partial\Omega \\ u(0, x) = u_0(x), \; x \in \Omega \end{cases}$$

where $f \in L^1(]0, T[\times\Omega)$ and $u_0 \in L^1(\Omega)$.

These results for parabolic second order differential equations extend to the case $p=1$ the classical theory for parabolic equations developed by Ladyzenskaja, Solonnikov and Ura'lceva [10] and others, for the case $p>1$.

1. The spaces $D_A(\theta, p)$ and $(D(A), X)_{\theta,p}$

In this section we recall some definitions and properties concerning interpolation spaces which are needed in the sequel.

a) *The spaces* $D_A(\theta, p)$

Let X be a Banach space with norm $\|.\|$ and let $A\colon D(A) \subseteq X \to X$ be a linear closed operator which generates an analytic semigroup $\exp(tA)$ in X. By this we mean that there exists $\omega \in \boldsymbol{R}$, $\varphi \in]\pi/2, \pi[$ and $M>0$ such that the set $Z_\varphi = \{z\colon |\arg(z-\omega)| < \varphi\} \cup \{\omega\}$ belongs to the resolvent set of A. Moreover for each $z \in Z_\varphi$ we have

(1.1) $$|z-\omega|\ \|R(z, A)\,x\| \leq M\|x\|$$

where $R(z, A)=(z-A)^{-1}$. For convenience we assume that A satisfies (1.1) with $\omega=0$ (so that $\exp(tA)$ is a bounded semigroup). This can be always be achieved by replacing A by $A-\omega I$ and $\exp(tA)$ by $\exp(-\omega t)\exp(tA)$.

In what follows we denote by $D_A(\theta, p)$ (for $0<\theta<1$ and $1\leq p<\infty$) the space of all elements $x\in X$ satisfying

$$H_{\theta,p}(x) = \left(\int_0^{+\infty} (t^{1-\theta}\|A\exp(tA)\,x\|)^p\, t^{-1}dt\right)^{1/p} < +\infty\,.$$

It can be seen that $D_A(\theta, p)$ are Banach spaces under the norm $|||x|||_{\theta,p}=\|x\|+H_{\theta,p}(x)$. Moreover

$$D(A)\hookrightarrow D_A(\theta, p)\hookrightarrow X\,.$$

The spaces $D_A(\theta, p)$ were introduced by Butzer and Berens [2] and by Peetre [12]. We refer to [2 Chapter 3.2] for a more detailed description of the properties of these spaces.

b) *The spaces* $(X, D(A))_{\theta,p}$

For our pourposes it is convenient to incorporate the spaces $D_A(\theta, p)$ in the theory of intermediate spaces. Let X, X_1 and X_2 be Banach spaces such that $X_i\hookrightarrow X$, $i=1, 2$. We denote tbe elements of X and X_i by x and x_i and their norm by $\|.\|$ and $\|x_i\|_i$, respectively.

In what follows we set for $t>0$ and $x\in X_1+X_2$

(1.2) $$K(t, x) = \inf_{x=x_1+x_2} (\|x_1\|_1+t\,\|x_2\|_2)\,.$$

Moreover we denote, for $\theta\in\,]0, 1[$ and $p\in[1, +\infty[$

(1.3) $$(X_1, X_2)_{\theta,p} = \{x = x_1+x_2\colon \|x\|_{\theta,p}<+\infty\}$$

where

(1.4) $$\|x\|_{\theta,p} = \left(\int_0^{+\infty} (t^{-\theta}K(t, x))^p\, t^{-1}dt\right)^{1/p}$$

It can be seen that $(X_1, X_2)_{\theta,p}$ are Banach spaces under the norm $\|x\|_{\theta,p}$; moreover we have

$$X_1\cap X_2\hookrightarrow (X_1, X_2)_{\theta,p}\hookrightarrow X_1+X_2\,.$$

The spaces $(X_1, X_2)_{\theta,p}$ where introduced by Peetre in [12] and are extensively studied. We refer to [2, Chapter 3.2] for a detailed description of the properties of these spaces. Here we are interested in the case where $X_1=X$ and $X_2=D(A)$ where $D(A)$ is the domain of a linear closed operator which generates an analytic semigroup in X. In this case the following results can be proved.

Theorem 1.1. *Let $A: D(A)\subseteq X\to X$ be the infinitesimal generator of a bounded analytic semigroup on X. Then we have*

$$D_A(\theta, p)\simeq(X, D(A))_{\theta,p}$$

Proof. For a proof see e.g. [2, Theorems 3.4.2 and 3.5.3]. ■

The following result turns to be useful in many applications.

Theorem 1.2. *Let A and B generate bounded analytic segmigroups in X. If $D(A)\simeq D(B)$ then we have*

$$D_A(\theta, p)\simeq D_B(\theta, p)\,.$$

Proof. The result is an immediate consequence of Theorem 1.1 and of the definitions (1.2), (1.3) and (1.4). ■

2. Analytic semigroups generated by elliptic operators in Ω

Let $\Omega\subseteq \boldsymbol{R}^n$ be a bounded set of class C^2 and let E be the second order elliptic operator geven by

$$Eu=\sum_{i,j=1}^{n} D_j(a_{ij}(x)\, D_i u)+\sum_{i=1}^{n} b_i(x)\, D_i u+c(x)\, u\,. \tag{2.1}$$

Here we have set $D_i=\partial/\partial x_i$; moreover a_{ij}, b_i and c are given functions satisfying

$$a_{ij}\in C_1(\bar{\Omega});\quad b_i,\ c\in C(\bar{\Omega})\,.$$

Moreover let A: $D(A)\subseteq L^1(\Omega)\to L^1(\Omega)$ be the operator defined by

$$\begin{cases} D(A)=\{u\in C^2(\bar{\Omega}): u(x)=0,\ x\in\partial\Omega\} \\ Au=Eu\,. \end{cases} \tag{2.2}$$

We denote by A_1 the closure of A in $L^1(\Omega)$

$$A_1=\bar{A}\,. \tag{2.3}$$

In what follows we set $X=L^1(\Omega)$ and denote by $\|\cdot\|_1$ the norm in X. Then we have (see [1], [11])

Theorem 2.1. *There exist ω', $M'\in R$ and $\varphi'\in]\pi/2, \pi[$ such that setting*

$$Z_{\varphi'}=\{z\colon |\arg(z-\omega')|<\varphi'\}\cup\{\omega'\}$$

we have that $Z_{\varphi'}$ belongs to the resolvent set of A_1. Moreover for each $z\in Z_{\varphi'}$ we have

(2.4) $$|z-\omega'|\ ||R(z, A_1)||_{L(X)} \leq M'$$

where $R(z, A_1)=(z-A_1)^{-1}$.

The following theorem establishes further properties of the resolvent operator.

Theorem 2.2. *There exist* $\omega \geq \omega'$, $M \geq M'$ *and* $\varphi \in]\pi/2, \varphi']$ *such that for each* z *verifying* $|\arg(z-\omega)|<\varphi$ *we have*

(2.5) $$|z-\omega|^{1/2}\ ||D_i R(z, A_1)||_{L(X)} \leq M .$$

Proof. Assertion (2.5) can be proved using the results of [13] and an argument similar to the one used in [3, Lemma 4.3]. ■

In what follows we assume that A_1 satisfies (2.5) with $\omega=0$ (if this is not the case then A_1 is replaced by $A_1-\omega I$). As a consequence of (2.4) (with $\omega'=0$) we have that A_1 generates a bounded analytic semigroup $S(t)$. Then there exist M_0 and M_1 such that

(2.6) $$||S(t)||_{L(X)} \leq M_0 ,$$

(2.7) $$t\ ||A^1 S(t)||_{L(X)} \leq M_1 .$$

Moreover from (2.5) we can establish further properties for the semigroup $S(t)$. We have

Theoerm 2.3. *There exists* M_2 *verifying*

(2.8) $$t^{1/2}\ ||D_i S(t)||_{L(X)} \leq M_2 .$$

Proof. Let φ be given by Theorem 2.2 and set $\Gamma=\Gamma^- \cup \Gamma^0 \cup \Gamma^+$, where

$$\Gamma^\pm = \{z = \pm r \exp(i\varphi),\ r \geq 1\}$$

oriented so that Im z increases, and

$$\Gamma^0 = \{z = \exp(i\psi),\ -\varphi \leq \psi \leq \varphi\}$$

oriented so that ψ increases. We have for $t \geq 0$

$$S(t) = \frac{1}{2\pi i} \int_{+\Gamma} \exp(zt)\, R(z, A_1)\, dz$$

Setting $z'=zt$ we get

$$S(t) = \frac{1}{2\pi i} \int_{+\Gamma} \exp(z')\, R(z'/t, A_1)\, t^{-1} dz'$$

Therefore from (2.5) (with $\omega=0$) we get

$$\|D_i S(t)\|_{L(X)} \leq \text{const} \int_\Gamma \exp(\operatorname{Re} z') \, |tz|^{-1/2} \, d|z'| \leq \text{const} \, t^{-1/2}$$

and the result is proved. ■

To study the spaces $D_{A_1}(\theta, 1)$ we use a further property of the semgiroup $S(t)$ which is established by the following lemma. Using Theorem 1.2 we assume for simplicity that the operator E takes the form

$$Eu = \sum_{i,j=1}^{n} a_{ij} D_{ij} u + \gamma u \tag{2.9}$$

with $\gamma \in \boldsymbol{R}$ (here $D_{ij} = D_i D_j$).

Theorem 2.4. *For each $T>0$ there exists $M_3 = M_3(T)$ such that for $t \in [0, T]$ we have*

$$t \, \|D_{ij} S(t)\|_{L(X)} \leq M_3 .$$

Proof. Since $\partial\Omega$ is of class C^2 for each $x_0 \in \partial\Omega$ there exists an open ball V_0 centereed in x_0 such that $V_0 \cap \partial\Omega$ can be represented in the form

$$x_l = g_0(x_1, \cdots, x_{l-1}, x_{l+1}, \cdots, x_n) .$$

Now cover $\partial\Omega$ by a finite number of balls $V_h (h=1, \cdots, m-1)$ and add an open set V_m such that $\bar{V}_m \subseteq \Omega$ so as to obtain a covering of Ω. Moreover denote by $\{\varphi_h\}$ a partition of unity subordinate to this covering. Furthermore fix $\sigma > 0$ and denote by u the solution of the problem

$$\begin{cases} u'(t) = A_1 u(t) \\ u(0) = S(\sigma) u_0 . \end{cases} \tag{2.10}$$

Setting $u_h = \varphi_h u$ we see that u_h satisfies the problem

$$\begin{cases} u_h'(t) = \varphi_h A_1 u(t) = A_1 u_h(t) + B_h u(t) \\ u_h(0) = u_{0,h} \end{cases} \tag{2.11}$$

where

$$u_{0,h} = \varphi_h S(\sigma) u_0$$

and

$$B_h u = -\sum_{i,j=1}^{n} a_{ij} \left[D_i(u D_j \varphi_h) + D_i \varphi_h D_j u \right] . \tag{2.12}$$

Now let $h=m$; since $\bar{V}_m \subseteq \Omega$ and $u_m = 0$ on $\Omega \setminus \bar{V}_m$ we have

$$D_k u_m(t) = S(t) D_k u_{0,m} + \int_0^t S(t-s) B_{k,m} u(s) \, ds$$

where

$$B_{k,m}\,u=\sum_{i,j=1}^{n}(D_k\,a_{ij})\,D_{ij}\,u_m+D_k\,B_m\,u\,. \tag{2.13}$$

Therefore using (2.8) and interpolatory estimates for $\|D_i u\|_1$ we get

$$\|D_{lk}\,u_m(t)\|_1\le\frac{\text{const}}{\sqrt{t}}\,\|D_k\,u_{0,m}\|_1+\int_0^t\frac{\text{const}}{\sqrt{t-s}}\,[\sum_{i,j=1}^{n}\|D_{ij}\,u(s)\|_1+\|u(s)\|_1]\,ds.$$

Now we have from (2.6) and (2.8)

$$\|D_k\,u_{0,m}\|_1\le c\,(\|u_0\|_1+\frac{1}{\sqrt{\sigma}}\,\|u_0\|_1)$$

and

$$\|u(s)\|_1\le M_0\,\|u_0\|_1$$

so that

$$\|D_{lk}\,u_m(t)\|_1\le\frac{c(T)}{\sqrt{t\sigma}}\,\|u_0\|_1+\int_0^t\frac{\text{const}}{\sqrt{t-s}}\sum_{i,j=1}^{n}\|D_{ij}\,u(s)\|_1\,ds$$

and hence

$$\sum_{i,j=1}^{n}\|D_{ij}\,u_m(t)\|_1\le c(T)\,[\frac{\|u_0\|_1}{\sqrt{t\sigma}}+\int_0^t\frac{1}{\sqrt{t-s}}\,\sum_{i,j=1}^{n}\|D_{ij}\,u(s)\|_1\,ds]\,. \tag{2.14}$$

Further fix $h\in[0,m-1]$. Using local transformation of variables we may assume that $V_h\cap\partial\Omega$ can be represented by $x_n=0$ (and that for $x\in V_h\cap\Omega$ we have $x_n>0$). Therefore for $k\neq n$ we have that the function $w_k=D_k\,u_h$ satisfies

$$w_k(t)=S(t)\,D_k\,u_{0,h}+\int_0^t S(t-s)\,B_{k,h}\,u(s)\,ds$$

where $B_{k,h}$ is given by (2.13) with m replaced by h. Hence by a computation similar to the one used above we find for $(l,k)\neq(n,n)$

$$\|D_{lk}\,u_h(t)\|_1\le c(T)\,[\frac{\|u_0\|_1}{\sqrt{t\sigma}}+\int_0^t\frac{1}{\sqrt{t-s}}\,\sum_{i,j=1}^{n}\|D_{ij}\,u(s)\|_1\,ds]\,. \tag{2.15}$$

Moreover for $(l,k)=(n,n)$ we have from (2.11)

$$\begin{aligned}\|D_{nn}\,u_h(t)\|_1&=\|\frac{1}{a_{nn}(\cdot)}\,[A_1\,u_h(t)-\sum_{(i,j)\neq(n,n)}a_{ij}(\cdot)\,D_{ij}\,u_h(t)]\|_1=\\&\|\frac{1}{a_{nn}(\cdot)}\,[\varphi_h\,A_1\,u(t)-B_h\,u(t)-\sum_{(i,j)\neq(n,n)}a_{ij}(\cdot)\,D_{ij}\,u_h(t)]\|_1\,.\end{aligned} \tag{2.16}$$

Hence from (2.15) and (2.16) we find that there exists a constant (again denoted by $c(T)$) verifying

$$\sum_{i,j=1}^{n} \|D_{ij}\, u_h(t)\|_1 \leq c(T)\, \{\frac{\|u_0\|}{\sqrt{t\sigma}} + \int_0^t \frac{1}{\sqrt{t-s}}\, [\sum_{i,j=1}^{n} \|D_{ij}\, u(s)\|_1 + \|A_1\, u(t)\|_1]\, ds\}$$

so that from (2.14) we get

$$\sum_{i,j=1}^{n} \|D_{ij}\, u(t)\|_1 \leq c(T)\, \{\frac{\|u_0\|_1}{\sqrt{t\sigma}} + \int_0^t \frac{1}{\sqrt{t-s}}\, [\sum_{i,j=1}^{n} \|D_{ij}\, u(s)\|_1 + \|A_1\, u(t)\|_1]\, ds\}\,. \tag{2.17}$$

Now we have fiom (2.7) and (2.10)

$$\|A_1\, u(t)\|_1 \leq M_1\, \|u_0\|_1 \frac{1}{t+\sigma} \leq M_1\, \|u_0\|_1 \frac{1}{\sqrt{2t\sigma}}$$

and finally from (2.17) we find that there exists a constant (again denoted by $c(T)$) such that

$$\sum_{i,j=1}^{n} \|D_{ij}\, u(t)\|_1 \leq c(T)\, \{\frac{\|u_0\|_1}{\sqrt{t\sigma}} + \int_0^t \frac{1}{\sqrt{t-s}} \sum_{i,j=1}^{n} \|D_{ij}\, u(s)\|_1\, ds\}\,.$$

Hence using Gronwall's generalized inequality (see e.g. [9, Chapter 7.1]) we get (for some constant depending on T)

$$\sum_{i,j=1}^{n} \|D_{ij}\, u(t)\|_1 \leq c(T) \frac{\|u_0\|_1}{\sqrt{t\sigma}}$$

so that the result follows by taking $\sigma=t$. ■

3. Characterization of interpolation spaces between $D(A_1)$ and $L_1(\Omega)$

Let A_1 be given by (2.1)—(2.3). Then we have the following result.

Theoerm 3.1. *For each $\theta\in]0,1[$ and $1\leq p<\infty$ we have*

$$(L^1, D(A_1))_{\theta,p} \cong (L^1, W^{2,1} \cap W_0^{1,1})_{\theta,p}$$

where $L^1=L^1(\Omega)$, $W^{2,1}=W^{2,1}(\Omega)$ and $W_0^{1,1}=W_0^{1,1}(\Omega)$.

Proof. From Theorem 1.2 it suffices to prove the theorem in the case where A_1 is given by (2.2)–(2.3) where E is given by (2.9) and satisfies (2.5) with $\omega=0$. Now we have

$$W^{2,1} \cap W_0^{1,1} \hookrightarrow D(A_1)\,,$$

therefore using (1.2)–(1.4) we obtain

$$(L^1, W^{2,1} \cap W_0^{1,1})_{\theta,p} \hookrightarrow (L^1, D(A_1))_{\theta,p}\,. \tag{3.1}$$

Conversely let $u\in(L^1, D(A_1))_{\theta,p}$ and set for $t\in[0, 1]$

$$u = u-S(t)\,u+S(t)\,u = \int_0^t A_1\,S(s)\,u ds+S(t)\,u = v_1+v_2\,. \tag{3.2}$$

We have

$$\|v_1\|_1 \le \int_0^t \|A_1\,S(s)\,u\|_1\,ds\,,$$

moreover $v_2 \in W^{2,1} \cap W_0^{1,1}$ and

$$\begin{aligned}\|v_2\|_{W^{2,1}} &= \|S(t)\,u\|_1+\sum_{i,j=1}^n \|D_{ij}\,[S(t)\,u-S(1)\,u+S(1)\,u]\,\|_1\\ &\le M_0\|u\|_1+\sum_{i,j=1}^n \|D_{ij}\int_t^1 S(s/2)\,A_1 S(s/2)\,u\,ds\|_1+M_3\,\|u\|_1\\ &\le \text{const}\,[\|u\|_1+\int_t^1 s^{-1}\,\|A_1 S(s/2)\,u\|_1\,ds]\end{aligned}$$

where we used (2.6) and Theorem 2.4. Therefore we obtain for $t \in [0, 1]$

$$\begin{aligned}K(t, u) &= \inf_{u=u_1+u_2} (\|u_1\|_1+t\,\|u_2\|_{W^{2,1}})\\ &\le \|v_1\|_1+t\,\|v_2\|_{W^{2,1}}\\ &\le \text{const}\,[t\,\|u\|_1+\int_0^t \|A_1 S(s)\,u\|_1\,ds+t\int_t^1 s^{-1}\,\|A_1 S(s/2)\,u\|_1\,ds]\,.\end{aligned}$$

Now we have $K(t, u) \le \|u\|_1$ (choosing $u_1=u$ and $u_2=0$) and hence

$$K(t, u) \le \text{const}\,[\min(1, t)\,\|u\|_1+\int_0^t \|A_1 S(s)\,u\|_1\,ds+t\int_t^1 s^{-1}\,\|A_1 S(s/2)\,u\|_1\,ds]\,.$$

Therefore for each $\theta \in]0, 1[$ and $1 \le p < \infty$ we get

$$\begin{aligned}&\int_0^{+\infty} (t^{-\theta}\,K(t, u))^p t^{-1}\,dt \le \text{const}\,[\int_0^{+\infty} (t^{-\theta} \min(1, t))^p t^{-1}\,dt\,\|u\|_1^p+\\ &\int_0^{+\infty} t^{-1}\,dt\,(t^{-\theta}\int_0^t \|A_1\,S(s)\,u\|_1\,ds)^p+\int_0^{+\infty} t^{-1}\,dt (t^{1-\theta}\int_t^{+\infty} s^{-1}\,\|A_1 S(s)\,u\|_1\,ds)^p]\,,\end{aligned}$$

so that using Hardy inequality (see e.g. [2. Lemma 3.4.7])

$$\int_0^{+\infty} (t^{-\theta}\,K(t, u))^p\,t^{-1}\,dt \le \text{const}\,[\|u\|_1^p+\int_0^{+\infty} (s^{1-\theta}\,\|A_1 S(s)\,u\|_1)^p s^{-1}\,ds]\,,$$

and hence from Theorem 1.1

$$(L^1, D(A_1))_{\theta,p} \hookrightarrow (L^1, W^{2,1} \cap W_0^{1,1})_{\theta,p}\,. \tag{3.3}$$

Hence the desired result follows combining (3.1) and (3.3). ■

Corollary 3.1. *For each $\theta \in]0, 1[$ and $1 \le p < \infty$ we have*

$$D_{A_1}(\theta, p) \cong (L^1, W^{2,1} \cap W_0^{1,1})_{\theta,p}$$

Proof. The result follows from Theorems 1.1 and 3.1. ■

In view of the study of parabolic equations in $L^1(\Omega)$ (see sect. 4 below) it is convenient to consider the case $p=1$.

Theorem 3.2. *For each $\theta\in]0,1[$ we have $D_{A_1}(\theta,1)\simeq\overset{0}{B}{}^{2\theta,1}(\Omega)$, where*

$$\overset{0}{B}{}^{\theta,1}(\Omega)=\begin{cases} W^{2\theta,1}(\Omega)\,, & \text{if } 0<\theta<1/2\\ u\in B^{1,1}(\Omega)\colon \int_\Omega (d(x,\partial\Omega))^{-1}|u(x)|\,dx<+\infty\,, & \text{if } \theta=1/2\\ W^{2\theta,1}(\Omega)\cap W_0^{1,1}(\Omega)\,, & \text{if } 1/2<\theta<1\,. \end{cases}$$

Here $W^{2\theta,1}(\Omega)$ denotes the Sobolev space of fractional order, $B^{1,1}(\Omega)$ denotes the Besov space and $d(x,\partial\Omega)$ the distance from x to $\partial\Omega$.

Proof. The result follows from Theorems 1.1 and 3.1 and from the characterization of the spaces $(L^1, W^{2,1}\cap W_0^{1,1})_{\theta,1}$ (see Proposition 1 of the Appendix). ■

REMARK. In the case $\Omega=\boldsymbol{R}^n$ the results of Theorem 3.2 where presented in [5].

4. Parabolic second order equations in L^1

Let E be the operator given by (2.1) and consider the problem

$$\text{(4.1)}\qquad \begin{cases} u_t(t,x)=Eu(t,x)+f(t,x),\ t>0,\ x\in\Omega\\ u(t,x)=0,\ t>0,\ x\in\partial\Omega\\ u(0,x)=u_0(x),\ x\in\Omega\,. \end{cases}$$

Regularity results for parabolic equations with f in $L^p(0,T;L^q(\Omega))$ and u_0 in $L^q(\Omega)$ are well known in the literature if $1<p,q<\infty$. In this section we study in a quite complete way also the case $p=q=1$ by using the abstract results of [4, sect. 8] and Theorem 3.2.

To state our results it is convenient to introduce some notation and definitions. Let Y be a Banach space and let $a<b$ be real numbers. We shall be concerned with the following spaces of Y-valued functions defined on $[a,b]$

$L^1(a,b;Y)$ is the space of measurable functions u such that $\|u(\cdot)\|_Y$ is integrable in $]a,b[$,

$C(a,b;Y)$ is the space of continuous functions on $[a,b]$,

$W^{1,1}(a,b;Y)$ is the space of functions u of $L^1(a,b;Y)$ having distributional derivative in $L^1(a,b;Y)$,

$$L^1_+(a,b;Y)=\{u\in L^1(\varepsilon,b;Y)\,,\quad \text{for each}\quad a<\varepsilon<b\}\,,$$

$$W^{1,1}_+(a,b;Y)=\{u\in W^{1,1}(\varepsilon,b;Y)\,,\quad \text{for each}\quad a<\varepsilon<b\}\,,$$

$W^{\theta,1}(a,b;Y)$, $0<\theta<1$, is the Sobolev space of functions u of $L^1(a,b;Y)$

such that

$$\int_0^T dt \int_0^T ds \, \|u(t)-u(s)\|_Y \, |t-s|^{-1-\theta} < +\infty \, .$$

Finally $\overset{0}{B}{}^{2\theta,1}(\Omega)$ is the Besov space introduced in Theorem 3.2 and $D(A_1)$ is the domain of the operator A_1 given by (2.2)–(2.3), i.e.

$$D(A_1) = \{u \in L^1(\Omega) \colon Eu \in L^1(\Omega)\}$$

where Eu is understood in the sense of distributions.

The following theorems describe the regularity of the solutions of (4.1) when the regularity of f and u_0 increases.

Theorem 4.1. *Let $f \in L^1(]0, T[\times\Omega)$ and $u_0 \in L^1(\Omega)$. Then* (4.1) *admits a unique generalized solution u and we have*

(i) $u(t, \cdot) \in C(0, T; L^1(\Omega)) \cap L^1(0, T; \overset{0}{B}{}^{2\beta,1}(\Omega)) \cap W^{\beta,1}(0, T; L^1(\Omega))$, *for each* $0<\beta<1$,
$u(t, \cdot) \in W^{\beta-\alpha,1}(0, T; \overset{0}{B}{}^{2\alpha,1}(\Omega))$, *for each* $0<\alpha<\beta<1$.

Proof. The result follows from [4, Th. 28] and Theorem 3.2. ■

Theorem 4.2. *Let $f(t, \cdot) \in L^1(0, T; \overset{0}{B}{}^{2\theta,1}(\Omega))$, for some $0<\theta<1$. Then for each $u_0 \in L^1(\Omega)$* (4.1) *admits a unique solution u and we have*

i) $u(t, \cdot) \in C(0, T; L^1(\Omega)) \cap L^1_+(0, T; D(A_1)) \cap W^{1,1}_+(0, T; L^1(\Omega))$,

ii) $u(t, \cdot) \in L^1(0, T; \overset{0}{B}{}^{2\beta,1}(\Omega)) \cap W^{\beta,1}(0, T; L^1(\Omega)) \cap W^{\beta-\alpha,1}(0, T; \overset{0}{B}{}^{2\alpha,1}(\Omega))$, *for each* $0<\alpha<\beta<1$.

If in addition $u_0 \in \overset{0}{B}{}^{2\gamma,1}(\Omega)$, for some $0<\gamma<1$, then we have for $\delta = \min(\theta, \gamma)$

iii) $u(t, \cdot) \in C(0, T; \overset{0}{B}{}^{2\delta,1}(\Omega)) \cap W^{\alpha,1}(0, T; \overset{0}{B}{}^{2\beta,1}(\Omega))$, *for each* $0<\alpha, \beta<1$, $\alpha+\beta = 1+\delta$,

iv) $Eu(t, \cdot) \in L^1(0, T; \overset{0}{B}{}^{2\delta,1}(\Omega)) \cap W^{\delta,1}(0, T; L^1(\Omega)) \cap W^{\delta-\alpha,1}(0, T; \overset{0}{B}{}^{2\alpha,1}(\Omega))$, *for each* $0<\alpha<\delta<1$,

v) $u_t(t, \cdot) \in L^1(0, T; \overset{0}{B}{}^{2\delta,1}(\Omega))$.

Proof. The assertions follow from [4, Th. 29] and Theorem 3.2. ■

Theorem 4.3. *Let $f(t, \cdot) \in W^{\theta,1}(0, T; L^1(\Omega))$, for some $0<\theta<1$. Then for each $u_0 \in L^1(\Omega)$ there exists a unique solution u of* (4.1) *and we have*

i) $u(t, \cdot) \in C(0, T; L^1(\Omega)) \cap L^1_+(0, T; D(A_1)) \cap W^{1,1}_+(0, T; L^1(\Omega))$,

ii) $u(t, \cdot)\in L^1(0, T; \overset{0}{B}{}^{2\beta,1}(\Omega))\cap W^{\beta,1}(0, T; L^1(\Omega))\cap W^{\beta-\alpha,1}(0, T; \overset{0}{B}{}^{2\alpha,1}(\Omega))$, *for each* $0<\alpha<\beta<1$.

If in addition $u_0\in\overset{0}{B}{}^{2\gamma,1}(\Omega)$, *for some* $0<\gamma<1$, *then we have, for* $\delta=\min(\theta, \gamma)$

iii) $u(t, \cdot)\in C(0, T; \overset{0}{B}{}^{2\delta,1}(\Omega))\cap W^{\alpha,1}(0, T; \overset{0}{B}{}^{2\beta,1}(\Omega))$, *for each* $0<\alpha, \beta<1$, $\alpha+\beta=1+\delta$,

iv) $u_t(t, \cdot)\in L^1(0, T; \overset{0}{B}{}^{2\delta,1}(\Omega))\cap W^{\delta,1}(0, T; L^1(\Omega))\cap W^{\delta-\alpha,1}(0, T; \overset{0}{B}{}^{2\alpha,1}(\Omega))$, *for each* $0<\alpha<\delta<1$,

v) $Eu(t, \cdot)\in W^{\delta,1}(0, T; L^1(\Omega))$.

Proof. The assertions follow from [4, Th. 30] and Theorem 3.2. ∎

Appendix

We want to give here the proof concerning the characterization of the intermediate spaces $(L^1(\Omega), W^{2,1}(\Omega)\cap W_0^{1,1}(\Omega))_{\theta,1}$, for $0<\theta<1$, which has been used in section 3. If Ω is of class C^2 using local change of coordinates it suffices to consider the case $\Omega=\boldsymbol{R}^n_+$ where

$$\boldsymbol{R}^n_+ = \{x = (x', x_n): x'\in\boldsymbol{R}^{n-1}, x_n>0\}.$$

If $\theta\neq 1/2$ this characterization can be deduced from kown results (see e.g. [2, Th. 4.3.6]) but we give here a direct proof for all $0<\theta<1$ in order to make the paper self-contained.

In what follows we denote by $B^{r,1}(\boldsymbol{R}^n_+)$, for $0<r\leq 1$, the Besov spaces defined as

$$B^{r,1}(\boldsymbol{R}^n_+) = \{u\in L^1(\boldsymbol{R}^n_+): H_r(u) = \int_{\boldsymbol{R}^n_+} dy \int_{\boldsymbol{R}^n_+} dx\, \left|u(x)+u(y)-2u\left(\frac{x+y}{2}\right)\right| |x-y|^{-n-r}<+\infty\}$$

endowed with the norm

$$\|u\|_{B^{r,1}} = \|u\|_1^+ + H_r(u)$$

where $\|\cdot\|_1^+$ denotes the norm in $L^1(\boldsymbol{R}^n_+)$, whereas for $1<r<2$ we define

$$B^{r,1}(\boldsymbol{R}^n_+) = \{u\in W^{1,1}(\boldsymbol{R}^n_+): D_j u\in B^{r-1}(\boldsymbol{R}^n_+)\}$$

with the norm

$$\|u\|_{B^{r,1}} = \|u\|_1^+ + \sum_{i=1}^n H_{r-1}(D_i u).$$

It is known that if $r\neq 1$ we have $B^{r,1}(\boldsymbol{R}^n_+)=W^{r,1}(\boldsymbol{R}^n_+)$, the usual Sobolev spaces of fractional order.

Proposition 1. *We have* $(L^1(\boldsymbol{R}^n_+),\ W^{2,1}(\boldsymbol{R}^n_+)\cap W^{1,1}_0(\boldsymbol{R}^n_+))_{\theta,1}=\overset{0}{B}{}^{2\theta,1}(\boldsymbol{R}^n_+)$, *where*

$$\overset{0}{B}{}^{2\theta,1}(\boldsymbol{R}^n_+)=\begin{cases} W^{2\theta,1}(\boldsymbol{R}^n_+)\,, & \textit{if}\quad 0<\theta<1/2\\ u\in B^{1,1}(\boldsymbol{R}^n_+)\colon \int_{\boldsymbol{R}^n_+}(x_n)^{-1}|u(x)|\,dx<+\infty\,, & \textit{if}\quad \theta=1/2\\ W^{2\theta,1}(\boldsymbol{R}^n_+)\cap W^{1,1}_0(\boldsymbol{R}^n_+)\,, & \textit{if}\quad 1/2<\theta<1\,. \end{cases}$$

In proving Proposition 1 we need some preliminary result. Set

$$N_+(t,u)=\sup_{0<|y|<t,\,y_n>0}\|u(\cdot)+u(\cdot+2y)-2u(\cdot+y)\|_1^+$$

and

$$|||u|||^+_{\theta,1}=\int_0^{+\infty}t^{-1-2\theta}\,N_+(t,u)\,dt+\|u\|_1^++\int_{\boldsymbol{R}^n_+}(x_n)^{-2\theta}|u(x)|\,dx\,.$$

Then for each $\theta\in]0,1/2]$ it is easily checked that

$$\text{(1)}\qquad \int_{\boldsymbol{R}^n_+}dy\int_{\boldsymbol{R}^n_+}dx\,\Big|u(x)+u(y)-2u\Big(\frac{x+y}{2}\Big)\Big|\,|x-y|^{-n-2\theta}\le\text{const}\int_0^{+\infty}t^{-1-2\theta}N_+(t,u)\,dt\,.$$

Moreover we have the following result.

Lemma 1. *Let us denote by* $X_{\theta,1}$ *the Banach space corresponding to the norm* $|||\cdot|||^+_{\theta,1}$. *Then*

$$X_{\theta,1}=\overset{0}{B}{}^{2\theta,1}(\boldsymbol{R}^n_+)\,.$$

Proof. Given $u\in L^1(\boldsymbol{R}^n_+)$, let us introduce the function $U\in L^1(\boldsymbol{R}^n)$ defined as

$$U(x)=\begin{cases} u(x)\,, & \text{if}\quad x_n>0\\ -u(x',-x)\,, & \text{if}\quad x_n\le 0\,. \end{cases}$$

Furthermore set, for $\theta\in]0,1[$

$$|||U|||_{\theta,1}=\|U\|_1+\int_0^{+\infty}t^{-1-2\theta}\,N(t,U)\,dt$$

where $\|\cdot\|_1$ denotes the norm in $L^1(\boldsymbol{R}^n)$ and

$$N(t,U)=\sup_{0<|y|<t}\|U(\cdot)+U(\cdot+2y)-2U(\cdot+y)\|_1\,.$$

Then (see [2, Prop. 4.3.5])

$$\text{(2)}\qquad |||\cdot|||_{\theta,1}\cong\|\cdot\|_{B^{2\theta,1}}$$

where $B^{2\theta,1}=B^{2\theta,1}(\boldsymbol{R}^n)$. Moreover one easily obtains, for each $\theta\in]0,1[$ (here by

c, c', c'', c_i, we denote various constants)

$$(3) \qquad |||U|||_{\theta,1} \leq c\, |||u|||^{+}_{\theta,1} \leq c' \left[|||U|||_{\theta,1} + \int_{R^n_+} (x_n)^{-2\theta} |u(x)| dx\right]$$

and

$$(4) \qquad ||U||_{B^{2\theta,1}} \leq c'' \left[||u||_{B^{2\theta,1}_+} + \int_{R^n_+} (x_n)^{-2\theta} |u(x)| dx\right]$$

where $B^{2\theta,1}_+ = B^{2\theta,1}(\boldsymbol{R}^n_+)$. Now let $\theta < 1/2$; we have (see [7, Th. 1.4.4.4])

$$(5) \qquad \int_{R^n_+} (x_n)^{-2\theta} |u(x)| dx \leq \text{const}\, ||u||_{W^{2\theta,1}_+}.$$

Therefore from (1), (2), (3) and (4) we get, for $\theta \leq 1/2$

$$|||u|||^{+}_{\theta,1} \leq c_1 \left[||U||_{B^{2\theta,1}} + \int_{R^n_+} (x_n)^{-2\theta} |u(x)| dx\right] \leq c_2 \left[||u||_{B^{2\theta,1}_+} + \int_{R^n_+} (x_n)^{-2\theta} |u(x)| dx\right]$$
$$\leq c_3\, |||u|||^{+}_{\theta,1}$$

which, together with (5), proves the assertion if $\theta \leq 1/2$.

Finally let $\theta > 1/2$. If $u \in W^{2\theta,1}(\boldsymbol{R}^n_+) \cap W^{1,1}_0(\boldsymbol{R}^n_+)$ then $U \in W^{2\theta,1}(\boldsymbol{R}^n)$ and (5) holds (see [7, Th. 1.4.4.4]). Therefore from (2), (3), (4) and (5)

$$|||u|||^{+}_{\theta,1} \leq c_1 \left[|||U|||_{W^{2\theta,1}} + \int_{R^n_+} (x_n)^{-2\theta} |u(x)| dx\right] \leq c_4\, ||u||_{W^{2\theta,1}_+}.$$

Conversely let $u \in X_{\theta,1}$; from (2) and (3) we get

$$||U||_{W^{2\theta,1}} \leq c_5\, |||U|||_{\theta,1} \leq c_6\, |||u|||^{+}_{\theta,1}$$

so that $u \in W^{2\theta,1}(\boldsymbol{R}^n_+)$ and

$$||u||_{W^{2\theta,1}_+} \leq ||U||_{W^{2\theta,1}} \leq c_6\, |||u|||^{+}_{\theta,1}.$$

Finally the assertion $u \in W^{1,1}_0(\boldsymbol{R}^n_+)$ follows from the fact that $u \in W^{1,1}(\boldsymbol{R}^n_+)$ and

$$\int_{R^n_+} (x_n)^{-2\theta} |u(x)| dx < +\infty$$

implies that $u(x', 0) = 0$. ■

Proof of Proposition 1. For simplicity in notation we restrict ourseleves to the case $n=2$. The method of the proof will lead the way for all $n \geq 1$.

In what follows we denote by Q_t, for $t > 0$, the subset of $\boldsymbol{R}^2_+$ defined as

$$Q_t = \{x \in \boldsymbol{R}^2_+ : 0 \leq x_i \leq \frac{t}{4\sqrt{2}}, i = 1, \cdots, 2\},$$

moreover we set $c = (4\sqrt{2})^4$. Furthermore, given $u \in L^1(\boldsymbol{R}^2_+)$, we denote by v_1 and v_2 the functions defined as

$$v_1=\int_{Q_t}dy\int_{Q_t}u(x+2(y+z))\,dz=\frac{1}{16}\prod_i\int_{x_i}^{x_i+t/2\sqrt{2}}dy_i\int_{y_i}^{y_i+t/2\sqrt{2}}u(z)\,dz_i$$

and

$$v_2=\int_{Q_t}dy\int_{Q_t}2u(x+y+z)\,dz=2\prod_i\int_{x_i}^{x_i+t/4\sqrt{2}}dy_i\int_{y_i}^{y_i+t/4\sqrt{2}}u(z)\,dz_i\,.$$

Moreover set $w_1=ct^{-4}(v_1-v_2)$, $w_2=ct(t+x_2)^{-5}(v_1-v_1)$ and $u_1=u+w_1-w_2$, $u_2=-w_1+w_2$. Then we have that $u=u_1+u_2$ with $u_1\in L^1(\boldsymbol{R}^2_+)$ and $u_2\in W^{2,1}(\boldsymbol{R}^2_+)\cap W^{1,1}_0(\boldsymbol{R}^2_+)$. Furthermore, using the fact that $y_2+z_2\leq t(2\sqrt{2})^{-1}$, we get

$$\text{(6)}\qquad \|u+w_1\|_1^+\leq\frac{c}{t^4}\int_{\boldsymbol{R}^2}dx\int_{Q_t}dy\int_{Q_t}|u(x)+u(x+2(y+z))-2u(x+y+z)|\,dz \leq N_+(t,u)$$

and

$$\|w_2\|_1^+\leq c't\int_{Q_t}dy\int_{Q_t}dz\int_{\boldsymbol{R}}dx_1\{\int_{y_2+z_2}^{+\infty}\frac{|u(x)|}{|t+x_2-(y_2+z_2)|^5}dx_2+\int_{2(y+z_2)}^{+\infty}\frac{|u(x)|}{|t+x_2-2(y_2+z_2)|^5}dx_2\}$$
$$\leq c't\int_{Q_t}dy\int_{Q_t}dz\int_{\boldsymbol{R}}dx_1\,[\int_{y_2+z_2}^{t}\frac{|u(x)|}{t^5}dx_2+\int_t^{+\infty}\frac{|u(x)|}{x_2^5}dx_2]$$

where c' denotes a constant. Therefore setting

$$L(t,u)=\int_{\boldsymbol{R}}dx_1\,[\int_0^t|u(x)|\,dx_2+t^5\int_t^{+\infty}\frac{|u(x)|}{x_2^5}dx_2]$$

we obtain

$$\text{(7)}\qquad \|w_2\|_1^+\leq c'L(t,u)\,.$$

Concerning u_2 we have

$$\text{(8)}\qquad \|u_2\|_1^+\leq c'\,\|u\|_1^+\,.$$

Moreover, to estimate $\|D_{h,h}u_2\|_1^+$, let us note that

$$D_{h,h}v_1=\int_{x_i}^{x_i+t/2\sqrt{2}}dy_i\int_{y_i}^{y_i+t/2\sqrt{2}}[u(z_i,x_h+t/\sqrt{2})-2u(z_i,x_h+t/2\sqrt{2})-u(z_i,x_h)]\,dz_i$$

where $i\neq h$. Moreover

$$D_{1,2}v_1=\int_{x_1}^{x_1+t/2\sqrt{2}}dz_1\int_{x_2}^{x_3+t/2\sqrt{2}}dz_2\,[u(z_1,z_2+t/2\sqrt{2})-2u(z_1-t/4\sqrt{2},z_2+t/4\sqrt{2}) +u(z_1-t/2\sqrt{2},z_2)-u(z_1-t/2\sqrt{2},z_2+t/2\sqrt{2}) +2u(z_1-t/4\sqrt{2},z_2+t/4\sqrt{2})-u(z)]\,.$$

Thereofre for each h, k we get

$$(9) \qquad ||D_{h,k}\, w_1||_1^+ \leq c' t^{-2}\, N_+(t, u)\,.$$

Now we have $||D_{1,1}\, w_2||_1^+ \leq ||D_{1,1}\, w_1||_1^+$ so that (9) holds for $h=k=1$ with w_1 replaced by w_2. Furthermore

$$||D_{2,2}\, w_2||_1^+ \leq ct \int_{R_+^2} [\frac{1}{(t+x_2)^5}\, |D_{2,2}(v_1-v_2)(x)| + \frac{1}{(t+x_2)^6}\, |D_2(v_1-v_2)(x)| + \frac{1}{(t+x_2)^7}\, |(v_1-v_2)(x)|]\, dx = I_1+I_2+I_3\,.$$

Now we get

$$I_1+I_3 \leq ||D_{2,2}\, w_1||_1^+ + c' t^{-2}\, ||u||_1^+ \leq c' t^{-2}\, [N_+(t, u) + ||u||_1^+]$$

where we used (9). Furthermore, proceding as in (7), we obtain

$$I_2 \leq c' t^{-2}\, L(t, u)\,.$$

Therefore

$$(10) \qquad ||D_{2,2}\, w_2||_1^+ \leq c' t^{-2}\, \{||u||_1^+ + N_+(t, u) + L(t, u)\}\,.$$

Finally in a similar way we get

$$(11) \qquad ||D_{1,2}\, w_2||_1^+ \leq c' t^{-2}\, \{N_+(t, u) + L(t, u)\}\,.$$

Summarizing using (6)–(11) we obtain that given $u \in L^1(\boldsymbol{R}_+^2)$, we can write $u = u_1 + u_2$ with $u_1 \in L^1(\boldsymbol{R}_+^2)$ and $u_2 \in W^{2,1}(\boldsymbol{R}_+^2) \cap W_0^{1,1}(\boldsymbol{R}_+^2)$ and

$$||u_1||_1^+ \leq N_+(t, u) + c' L(t, u)$$

and

$$||u_2||_2^+ \leq c' t^{-2}\, [(1+t^2)\, ||u||_1 + N_+(t, u) + L(t, u)]$$

where $||\cdot||_2^+$ denotes the norm in $W^{2,1}(\boldsymbol{R}_+^2)$. Therefoie (see (1.2)) there exists c_1 such that

$$(12) \qquad K(t^2, u) \leq c_1\, [N_+(t, u) + \min(1, t^2)\, ||u||_1^+ + L(t, u)]\,.$$

Conversely let $u = u_1 + u_2$ with $u_1 \in L^1(\boldsymbol{R}_+^2)$ and $u_2 \in W^{2,1}(\boldsymbol{R}_+^2) \cap W_0^{1,1}(\boldsymbol{R}_+^2)$. Then we have

$$(13) \qquad \min(1, t^2)\, ||u||_1^+ \leq K(t^2, u)$$

and

$$(14) \qquad N_+(t, u) \leq N_+(t, u_1) + N_+(t, u_2) \leq 4\, ||u_1||_1^+ + t^2\, ||u_2||_2^+ \leq 4\, K(t^2, u)$$

the third estimate following by

$$u(x)-2u(x+y)+u(x+2y) = 2\int_0^{|y|} ds \int_0^s d\sigma \frac{\partial}{\partial s}\frac{\partial}{\partial \sigma} u(x+(s+\sigma))\frac{y}{|y|} .$$

Furthermore

$$\begin{aligned}(15) \qquad L(t,u) &\leq \|u_1\|_1^+ + \int_R dx_1 \Big[\int_0^t dx_2 \int_0^{x_2} dy_2 \int_{y_2}^{+\infty} d\xi_2 \, |D_{22}u_2(x_1,\xi_2)| + \\ & t^5 \int_t^{+\infty} \frac{1}{x_2^5} dx_2 \int_0^{x_2} dy_2 \int_{x_2}^{+\infty} d\xi_2 \, |D_{22}u_2(x_1,\xi_2)|\Big] \\ &\leq \|u_1\|_1^+ + ct^2 \|D_{22}u_2\|_1^+\end{aligned}$$

so that

$$L(t,u) \leq c\,K(t^2,u) .$$

Finally from (12)–(15) we obtain that there exists c_2 such that

$$K(t^2,u) \leq c_1 [N_+(t,u)+\min(1,t^2)\|u\|_1^+ + L(t,u)] \leq c_2 K(t^2,u) .$$

Therefore

$$\begin{aligned}\int_0^{+\infty} t^{-1-\theta} K(t,u)\,dt &= 2\int_0^{+\infty} t^{-1-2\theta} K(t^2,u)\,dt \leq c_1' \Big[\int_0^{+\infty} t^{-1-2\theta} N_+(t,u)\,dt + \|u\|_1^+ \\ &+ \int_0^{+\infty} t^{-1-2\theta} L(t,u)\,dt\Big] \leq c_2' \int_0^{+\infty} t^{-1-\theta} K(t,u)\,dt .\end{aligned}$$

Now

$$\int_0^{+\infty} t^{-1-2\theta} L(t,u)\,dt = \text{const}\int_{R_+^2} (x_2)^{-2\theta} |u(x)|\,dx ,$$

therefore the desired result follows from Lemma 1. ■

AKNOWLEDGEMENT. The author thanks the referee for helpful comments.

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