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Osaka University
0. Introduction

Parabolic equations in $L^p$ spaces have been studied both by potential theory and by abstract methods mainly when $p>1$. In this paper we want to continue our previous researchs on the $L^1$ case ([4], [5]) by using a semigroup approach.

Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. We denote by $E$ a second order elliptic operator in $\Omega$ and by $A_1$ the $L^1$ realization of $E$ with homogeneous Dirichlet boundary conditions. Then it is known (see Amann [1], Pazy [11] and Tanabe [14]) that $A_1$ is the infinitesimal generator of an analytic semigroup in $L^1(\Omega)$. We set $X=L^1(\Omega)$ and denote by $S(t)$ the semigroup generated by $A_1$.

In this paper we establish some new properties for the semigroup $S(t)$. Moreover we give a characterization in term of Besov spaces for the interpolation spaces $D_{A_1}(\theta, 1)$, between the domain of $A_1$ and $L^1(\Omega)$, defined as (see Butzer and Berens [2] and Peetre [12])

\begin{equation}
D_{A_1}(\theta, 1) = \{ u \in X: \int_0^{+\infty} ||A_1 S(t) u||_X t^{-\theta} dt < +\infty \}.
\end{equation}

This characterization allows us to find new regularity results for the solutions of the following Cauchy problem

\begin{equation}
\begin{cases}
 u'(t) = A_1 u(t) + f(t) \\
 u(0) = u_0
\end{cases}
\end{equation}

where $f \in L^1(0, T; X)$ and $u_0 \in X$. For the connection between the regularity properties of solutions of (0.2) and the interpolation spaces $D_{A_1}(\theta, 1)$ we refer to [4].

The plan of the paper is as follows. In section 2 we prove that the semigroup $S(t)$ satisfies the following estimates, for some $M', M'' > 0$ and $\omega \in \mathbb{R}$,

\begin{equation}
\sqrt{t} \| D_i S(t) \|_{L(X)} \leq M' \exp(\omega t) \quad i = 1, \ldots, n
\end{equation}

(*) The work of the author is partially supported by M.P.I. 40% and by G.N.A.F.A.
and

\[ (0.4) \quad t \| D_{ih} S(t) \|_{L^\infty} \leq M'' \exp (\omega t) \quad i, h = 1, \ldots, n \]

where we have set \( D_i = \partial / \partial x_i \) and \( D_{ih} = D_i D_h \). Properties (0.3) and (0.4) give precise information about the behavior at \( t=0 \) of the spatial derivatives of semigroup \( S(t) \) (and hence about the solutions of (0.2)).

In section 3 we use these estimates and prove, in a very direct way and without using the reiteration property, the following characterization of the interpolation spaces \( D_{A_i}(\theta, 1) \), for each \( 0 < \theta < 1 \)

\[
(0.5) \quad D_{A_i}(\theta, 1) = \begin{cases} 
  W^{2s, 1}(\Omega), & \text{if } 0 < \theta < 1/2 \\
  W^{2s, 1}(\Omega) \cap W^{1, 1}_b(\Omega), & \text{if } 1/2 < \theta < 1
\end{cases}
\]

Here \( W^{2s, 1}(\Omega) \) denotes the Sobolev space of fractional order, \( B^{1, 1}(\Omega) \) denotes the Besov space and \( d(x, \partial \Omega) \) the distance from \( x \) to \( \partial \Omega \). This characterization has been given by Grisvard [6] for the case \( p > 1 \). If the operator \( E \) has \( C^\infty \) coefficients and \( \theta = 1/2 \) the characterization (0.5) can be deduced by a result of Guidetti, [8], obtained by complex interpolation methods.

Finally in section 4 we obtain a quite complete description of the regularity of the solutions of the following problem (for which (0.2) is the abstract version)

\[
(0.6) \quad \begin{cases} 
  u_t(t, x) = E u(t, x) + f(t, x), t > 0, x \in \Omega \\
  u(t, x) = 0, t > 0, x \in \partial \Omega \\
  u(0, x) = u_0(x), x \in \Omega
\end{cases}
\]

where \( f \in L^1([0, T] \times \Omega) \) and \( u_0 \in L^1(\Omega) \).

These results for parabolic second order differential equations extend to the case \( p = 1 \) the classical theory for parabolic equations developed by Ladyzenskaja, Solonnikov and Ura’lceva [10] and others, for the case \( p > 1 \).

1. The spaces \( D_{A}(\theta, p) \) and \((D(A), X)_{\theta, p}\)

In this section we recall some definitions and properties concerning interpolation spaces which are needed in the sequel.

a) The spaces \( D_{A}(\theta, p) \)
Let \( X \) be a Banach space with norm \( \| \cdot \| \) and let \( A: D(A) \subseteq X \rightarrow X \) be a linear closed operator which generates an analytic semigroup \( \exp (tA) \) in \( X \). By this we mean that there exists \( \omega \in \mathbb{R}, \phi \in \pi/2, \pi \) and \( M > 0 \) such that the set \( Z_\phi = \{ z: |\arg(z-\omega)| < \phi \} \cup \{ \omega \} \) belongs to the resolvent set of \( A \). Moreover for each \( z \in Z_\phi \) we have
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where $R(x, A) = (z - A)^{-1}$. For convenience we assume that $A$ satisfies (1.1) with $\omega = 0$ (so that $\exp(tA)$ is a bounded semigroup). This can be always be achieved by replacing $A$ by $A - \omega I$ and $\exp(tA)$ by $\exp(-\omega t) \exp(tA)$.

In what follows we denote by $D_A(\theta, \rho)$ (for $0 < \theta < 1$ and $1 \leq \rho < \infty$) the space of all elements $x \in X$ satisfying

$$H_{\theta, \rho}(x) = \left( \int_0^\infty (t^{1-\theta} ||A \exp(tA)x||^\rho t^{-\theta} dt)^{\frac{1}{\rho}} \right)^{\frac{1}{\theta}} < +\infty .$$

It can be seen that $D_A(\theta, \rho)$ are Banach spaces under the norm $||x||_{\theta, \rho} = ||x|| + H_{\theta, \rho}(x)$. Moreover

$$D(A) \hookrightarrow D_A(\theta, \rho) \hookrightarrow X .$$

The spaces $D_A(\theta, \rho)$ were introduced by Butzer and Berens [2] and by Peetre [12]. We refer to [2, Chapter 3.2] for a more detailed description of the properties of these spaces.

b) The spaces $(X, D(A))_{\theta, \rho}$

For our purposes it is convenient to incorporate the spaces $D_A(\theta, \rho)$ in the theory of intermediate spaces. Let $X, X_1$ and $X_2$ be Banach spaces such that $X_i \subset X$, $i = 1, 2$. We denote the elements of $X$ and $X_i$ by $x$ and $x_i$ and their norm by $||.||$ and $||x_i||$, respectively.

In what follows we set for $t > 0$ and $x \in X_1 + X_2$

(1.2) $K(t, x) = \inf_{x = x_1 + x_2} (||x_1||^\theta + t ||x_2||^\rho) .$

Moreover we denote, for $\theta \in ]0, 1[$ and $\rho \in [1, +\infty[$

(1.3) $(X_1, X_2)_{\theta, \rho} = \{x = x_1 + x_2; ||x||_{\theta, \rho} < +\infty\}$

where

(1.4) $||x||_{\theta, \rho} = \left( \int_0^\infty \left( t^{-\theta} K(t, x) \right)^\rho t^{-\theta} dt \right)^{\frac{1}{\rho}}$

It can be seen that $(X_1, X_2)_{\theta, \rho}$ are Banach spaces under the norm $||x||_{\theta, \rho}$; moreover we have

$$X_1 \cap X_2 \hookrightarrow (X_1, X_2)_{\theta, \rho} \hookrightarrow X_1 + X_2 .$$

The spaces $(X_1, X_2)_{\theta, \rho}$ were introduced by Peetre in [12] and are extensively studied. We refer to [2, Chapter 3.2] for a detailed description of the properties of these spaces. Here we are interested in the case where $X_1 = X$ and $X_2 = D(A)$ where $D(A)$ is the domain of a linear closed operator which generates an analytic semigroup in $X$. In this case the following results can
be proved.

**Theorem 1.1.** Let $A: D(A) \subseteq X \to X$ be the infinitesimal generator of a bounded analytic semigroup on $X$. Then we have

$$D_A(\theta, p) = (X, D(A))_{\theta, p}$$

Proof. For a proof see e.g. [2, Theorems 3.4.2 and 3.5.3]. ■

The following result turns to be useful in many applications.

**Theorem 1.2.** Let $A$ and $B$ generate bounded analytic semigroups in $X$. If $D(A) = D(B)$ then we have

$$D_A(\theta, p) = D_B(\theta, p).$$

Proof. The result is an immediate consequence of Theorem 1.1 and of the definitions (1.2), (1.3) and (1.4). ■

2. Analytic semigroups generated by elliptic operators in $\Omega$

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded set of class $C^2$ and let $E$ be the second order elliptic operator given by

$$Eu = \sum_{i,j=1}^{n} D_j(a_{ij}(x) D_i u) + \sum_{i=1}^{n} b_i(x) D_i u + c(x) u.$$  \hspace{1cm} (2.1)

Here we have set $D_i = \partial / \partial x_i$; moreover $a_{ij}$, $b_i$ and $c$ are given functions satisfying

$$a_{ij} \in C_4(\overline{\Omega}); \quad b_i, c \in C(\overline{\Omega}).$$

Moreover let $A: D(A) \subseteq L^1(\Omega) \to L^1(\Omega)$ be the operator defined by

$$\begin{cases}
D(A) = \{u \in C^2(\overline{\Omega}) : u(x) = 0, x \in \partial \Omega\} \\
Au = Eu.
\end{cases} \hspace{1cm} (2.2)$$

We denote by $A_1$ the closure of $A$ in $L^1(\Omega)$

$$A_1 = \overline{A}. \hspace{1cm} (2.3)$$

In what follows we set $X = L^1(\Omega)$ and denote by $||\cdot||_1$ the norm in $X$. Then we have (see [1], [11])

**Theorem 2.1.** There exist $\omega', M' \in \mathbb{R}$ and $\varphi' \in ]\pi/2, \pi[$ such that setting

$$Z_{\varphi'} = \{z : |\arg(z - \omega')| < \varphi' \} \cup \{\omega'\}$$

we have that $Z_{\varphi'}$ belongs to the resolvent set of $A_1$. Moreover for each $z \in Z_{\varphi'}$ we have
(2.4) \[ |z - \omega'| \|R(z, A_1)\|_{L(X)} \leq M' \]

where \( R(z, A_1) = (z - A_1)^{-1} \).

The following theorem establishes further properties of the resolvent operator.

**Theorem 2.2.** There exist \( \omega \geq \omega' \), \( M \geq M' \) and \( \varphi \in ]\pi/2, \varphi' [ \) such that for each \( z \) verifying \( |\arg(z - \omega)| < \varphi \) we have

(2.5) \[ |z - \omega|^{1/2} \|D_z R(z, A_1)\|_{L(X)} \leq M. \]

Proof. Assertion (2.5) can be proved using the results of [13] and an argument similar to the one used in [3, Lemma 4.3]. \( \blacksquare \)

In what follows we assume that \( A_1 \) satisfies (2.5) with \( \omega = 0 \) (if this is not the case then \( A_1 \) is replaced by \( A_1 - \omega I \)). As a consequence of (2.4) (with \( \omega = 0 \)) we have that \( A_1 \) generates a bounded analytic semigroup \( S(t) \). Then there exist \( M_0 \) and \( M_1 \) such that

(2.6) \[ \|S(t)\|_{L(X)} \leq M_0, \]
(2.7) \[ t\|A^1 S(t)\|_{L(X)} \leq M_1. \]

Moreover from (2.5) we can establish further properties for the semigroup \( S(t) \). We have

**Theorem 2.3.** There exists \( M_2 \) verifying

(2.8) \[ t^{1/2} \|D_z S(t)\|_{L(X)} \leq M_2. \]

Proof. Let \( \varphi \) be given by Theorem 2.2 and set \( \Gamma = \Gamma^- \cup \Gamma^0 \cup \Gamma^+ \), where

\[ \Gamma^\pm = \{z = \pm r \exp(i\varphi), r \geq 1\} \]

oriented so that \( \text{Im } z \) increases, and

\[ \Gamma^0 = \{z = \exp(i\psi), -\varphi \leq \psi \leq \varphi\} \]

oriented so that \( \psi \) increases. We have for \( t \geq 0 \)

\[ S(t) = \frac{1}{2\pi i} \int_{\Gamma^+} \exp(zt) R(z, A_1) \, dz \]

Setting \( z' = zt \) we get

\[ S(t) = \frac{1}{2\pi i} \int_{\Gamma^+} \exp(z't) R(z'/t, A_1) \, dz' \]

Therefore from (2.5) (with \( \omega = 0 \)) we get
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\begin{equation}
||D_i S(t)||_{L^\infty} \leq \text{const} \int \exp(\text{Re } z') |t z|^{-1/2} d |z'| \leq \text{const} t^{-1/2}
\end{equation}
and the result is proved. ■

To study the spaces \( D_{A_1}(\theta, 1) \) we use a further property of the semigroup \( S(t) \) which is established by the following lemma. Using Theorem 1.2 we assume for simplicity that the operator \( E \) takes the form

\begin{equation}
Eu = \sum_{i,j=1}^n a_{ij} D_{ij} u + \gamma u
\end{equation}

with \( \gamma \in \mathbb{R} \) (here \( D_{ij} = D_i D_j \)).

**Theorem 2.4.** For each \( T > 0 \) there exists \( M_3 = M_3(T) \) such that for \( t \in [0, T] \) we have

\begin{equation}
t ||D_{ij} S(t)||_{L^\infty} \leq M_3.
\end{equation}

**Proof.** Since \( \partial \Omega \) is of class \( C^2 \) for each \( x_0 \in \partial \Omega \) there exists an open ball \( V_0 \) centered in \( x_0 \) such that \( V_0 \cap \partial \Omega \) can be represented in the form

\begin{equation}
x_i = g_6(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).
\end{equation}

Now cover \( \partial \Omega \) by a finite number of balls \( V_h \) for \( h = 1, \ldots, m-1 \) and add an open set \( V_m \) such that \( V_m \subseteq \Omega \) so as to obtain a covering of \( \Omega \). Moreover denote by \( \{ \varphi_h \} \) a partition of unity subordinate to this covering. Furthermore fix \( \sigma > 0 \) and denote by \( u \) the solution of the problem

\begin{equation}
\begin{cases}
u'(t) = A_1 u(t) \\
u(0) = S(\sigma) u_0.
\end{cases}
\end{equation}

Setting \( u_h = \varphi_h u \) we see that \( u_h \) satisfies the problem

\begin{equation}
\begin{cases}
u_h'(t) = \varphi_h A_1 u(t) = A_1 u_h(t) + B_h u(t) \\
u_h(0) = u_{0,h}
\end{cases}
\end{equation}

where

\begin{equation}
u_{0,h} = \varphi_h S(\sigma) u_0
\end{equation}

and

\begin{equation}
B_h u = - \sum_{i,j=1}^n a_{ij} [ D_i (u D_j \varphi_h) + D_i \varphi_h D_j u ].
\end{equation}

Now let \( h = m \); since \( V_m \subseteq \Omega \) and \( u_m = 0 \) on \( \Omega \setminus V_m \) we have

\begin{equation}
D_h u_m(t) = S(t) D_h u_{0,m} + \int_0^t S(t-s) B_{h,m} u(s) ds
\end{equation}
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where

\begin{equation}
B_{k,m} u = \sum_{i,j=1}^{s} (D_k a_{ij}) D_{ij} u_m + D_k B_m u.
\end{equation}

Therefore using (2.8) and interpolatory estimates for \( ||D_k u||_1 \) we get

\[ ||D_k u_m(t)||_1 \leq \frac{\text{const}}{\sqrt{t}} ||D_k u_{0,m}||_1 + \int_0^t \frac{\text{const}}{\sqrt{t-s}} \left[ \sum_{i,j=1}^{s} ||D_{ij} u(s)||_1 + ||u(s)||_1 \right] ds. \]

Now we have from (2.6) and (2.8)

\[ ||D_k u_{0,m}||_1 \leq c \left( ||u_0||_1 + \frac{1}{\sqrt{\sigma}} ||u_0||_1 \right) \]

and

\[ ||u(s)||_1 \leq M_0 ||u_0||_1 \]

so that

\[ ||D_k u_m(t)||_1 \leq c(T) \left[ ||u_0||_1 + \int_0^t \frac{1}{\sqrt{t-s}} \sum_{i,j=1}^{s} ||D_{ij} u(s)||_1 ds \right] \]

and hence

\begin{equation}
\sum_{i,j=1}^{s} ||D_{ij} u_m(t)||_1 \leq c(T) \left[ ||u_0||_1 + \int_0^t \frac{1}{\sqrt{t-s}} \sum_{i,j=1}^{s} ||D_{ij} u(s)||_1 ds \right].
\end{equation}

Further fix \( h \in [0, m-1] \). Using local transformation of variables we may assume that \( V_h \cap \partial \Omega \) can be represented by \( x_n=0 \) (and that for \( x \in V_h \cap \Omega \) we have \( x_n > 0 \)). Therefore for \( k=n \) we have that the function \( w_h = D_k u_h \) satisfies

\[ w_h(t) = S(t) D_h u_{0,h} + \int_0^t S(t-s) B_{h,k} u(s) ds \]

where \( B_{h,k} \) is given by (2.13) with \( m \) replaced by \( h \). Hence by a computation similar to the one used above we find for \( (l, k) \neq (n, n) \)

\begin{equation}
||D_{l,k} u_h(t)||_1 \leq c(T) \left[ \frac{||u_0||_1}{\sqrt{t\sigma}} + \int_0^t \frac{1}{\sqrt{t-s}} \sum_{i,j=1}^{s} ||D_{ij} u(s)||_1 ds \right].
\end{equation}

Moreover for \( (l, k) = (n, n) \) we have from (2.11)

\begin{equation}
||D_{n,n} u_h(t)||_1 = \left| \frac{1}{a_{nn}(\cdot)} \left[ A_1 u_h(t) - \sum_{G_i \cap \Omega \neq 0} a_{ij}(\cdot) D_{ij} u_h(t) \right] \right|_1 =
\end{equation}

\[ \left| \frac{1}{a_{nn}(\cdot)} [\varphi_h A_1 u(t) - B_n u(t) - \sum_{G_i \cap \Omega \neq 0} a_{ij}(\cdot) D_{ij} u_h(t)] \right|_1. \]

Hence from (2.15) and (2.16) we find that there exists a constant (again denoted by \( c(T) \)) verifying
\[ \sum_{i,j=1}^{n} \| D_{ij} u(t) \|_1 \leq c(T) \left\{ \frac{\| u_0 \|}{\sqrt{t^\sigma}} + \int_{0}^{t} \frac{1}{\sqrt{t-s}} \left[ \sum_{i,j=1}^{n} \| D_{ij} u(s) \|_1 + \| A_1 u(t) \|_1 \right] \, ds \right\} \]

so that from (2.14) we get

(2.17) \[ \sum_{i,j=1}^{n} \| D_{ij} u(t) \|_1 \leq c(T) \left\{ \frac{\| u_0 \|}{\sqrt{t^\sigma}} + \int_{0}^{t} \frac{1}{\sqrt{t-s}} \left[ \sum_{i,j=1}^{n} \| D_{ij} u(s) \|_1 + \| A_1 u(t) \|_1 \right] \, ds \right\}. \]

Now we have from (2.7) and (2.10)

\[ \| A_1 u(t) \|_1 \leq M_1 \frac{1}{t+\sigma} \leq M_1 \frac{1}{\sqrt{2t^\sigma}} \]

and finally from (2.17) we find that there exists a constant (again denoted by \( c(T) \)) such that

\[ \sum_{i,j=1}^{n} \| D_{ij} u(t) \|_1 \leq c(T) \left\{ \frac{\| u_0 \|}{\sqrt{t^\sigma}} + \int_{0}^{t} \frac{1}{\sqrt{t-s}} \sum_{i,j=1}^{n} \| D_{ij} u(s) \|_1 \, ds \right\}. \]

Hence using Gronwall's generalized inequality (see e.g. [9, Chapter 7.1]) we get (for some constant depending on \( T \))

\[ \sum_{i,j=1}^{n} \| D_{ij} u(t) \|_1 \leq c(T) \frac{\| u_0 \|}{\sqrt{t^\sigma}} \]

so that the result follows by taking \( \sigma = t \). \( \Box \)

3. Characterization of interpolation spaces between \( D(A_1) \) and \( L_{1}(\Omega) \)

Let \( A_1 \) be given by (2.1)—(2.3). Then we have the following result.

Theorem 3.1. For each \( \theta \in [0,1[ \) and \( 1 \leq p < \infty \) we have

\[ (L^1, D(A_1))_{\theta,p} \cong (L^1, W^{2,1} \cap W^{1,1}_{0})_{\theta,p} \]

where \( L^1 = L^1(\Omega) \), \( W^{2,1} = W^{2,1}(\Omega) \) and \( W^{1,1} = W^{1,1}_{0}(\Omega) \).

Proof. From Theorem 1.2 it suffices to prove the theorem in the case where \( A_1 \) is given by (2.2)—(2.3) where \( E \) is given by (2.9) and satisfies (2.5) with \( \omega = 0 \). Now we have

\[ W^{2,1} \cap W^{1,1}_{0} \hookrightarrow D(A_1), \]

therefore using (1.2)—(1.4) we obtain

(3.1) \[ (L^1, W^{2,1} \cap W^{1,1}_{0})_{\theta,p} \cong (L^1, D(A_1))_{\theta,p}. \]

Conversely let \( u \in (L^1, D(A_1))_{\theta,p} \) and set for \( t \in [0,1) \)
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\[ (3.2) \quad u = u - S(t) u + S(t) u = \int_0^t A_s S(s) u \, ds + S(t) u = v_1 + v_2. \]

We have

\[ ||v_1||_1 \leq \int_0^t ||A_s S(s) u||_1 \, ds, \]

moreover \( v_2 \in W^{2,1} \cap W_0^{1,1} \) and

\[ ||v_2||_{W^{2,1}} = ||S(t) u||_1 + \sum_{i,j=1}^n ||D_{ij} [S(t) u - S(1) u + S(1) u]||_1 \]

\[ \leq M_0 ||u||_1 + \sum_{i,j=1}^n ||D_{ij} \int_0^t (s/2) A_s S(s/2) u \, ds||_1 + M_3 ||u||_1 \]

\[ \leq \text{const} \left[ ||u||_1 + \int_0^t s^{-1} ||A_s S(s/2) u||_1 \, ds \right] \]

where we used (2.6) and Theorem 2.4. Therefore we obtain for \( t \in [0, 1] \)

\[ K(t, u) = \inf \left( ||u||_1 + t ||u_2||_{W^{2,1}} \right) \]

\[ \leq ||v_1||_1 + t ||v_2||_{W^{2,1}} \]

\[ \leq \text{const} \left[ t ||u||_1 + \int_0^t ||A_s S(s) u||_1 \, ds + t \int_0^1 s^{-1} ||A_s S(s/2) u||_1 \, ds \right]. \]

Now we have \( K(t, u) \leq ||u||_1 \) (choosing \( u_1 = u \) and \( u_2 = 0 \)) and hence

\[ K(t, u) \leq \text{const} \left[ \min(1, t) ||u||_1 + \int_0^t ||A_s S(s) u||_1 \, ds + t \int_0^1 s^{-1} ||A_s S(s/2) u||_1 \, ds \right]. \]

Therefore for each \( \theta \in [0, 1[ \) and \( 1 \leq p < \infty \) we get

\[ \int_0^{+\infty} (t^{-\theta} K(t, u))^p t^{-1} \, dt \leq \text{const} \left[ \int_0^{+\infty} (t^{-\theta} \min(1, t))^p t^{-1} \, dt \right] ||u||_p^p + \int_0^{+\infty} t^{-1} \, dt \left( t^{-\theta} \int_0^{+\infty} ||A_s S(s) u||_1 \, ds \right)^p + \int_0^{+\infty} t^{-1} \, dt \left( t^{-\theta} \int_0^{+\infty} s^{-1} ||A_s S(s) u||_1 \, ds \right)^p, \]

so that using Hardy inequality (see e.g. [2. Lemma 3.4.7])

\[ \int_0^{+\infty} (t^{-\theta} K(t, u))^p t^{-1} \, dt \leq \text{const} \left[ ||u||_p^p + \int_0^{+\infty} (s^{1-\theta} ||A_s S(s) u||_1)^p s^{-1} \, ds \right], \]

and hence from Theorem 1.1

\[ (3.3) \]

\[ (L^1, D(A_t))_{\theta, p} \hookrightarrow (L^1, W^{2,1} \cap W_0^{1,1})_{\theta, p}. \]

Hence the desired result follows combining (3.1) and (3.3). \( \Box \)

**Corollary 3.1.** For each \( \theta \in [0, 1[ \) and \( 1 \leq p < \infty \) we have

\[ D(A_t, \theta, p) = (L^1, W^{2,1} \cap W_0^{1,1})_{\theta, p}. \]

Proof. The result follows from Theorems 1.1 and 3.1. \( \Box \)
In view of the study of parabolic equations in $L^1(\Omega)$ (see sect. 4 below) it is convenient to consider the case $p=1$.

**Theorem 3.2.** For each $\theta \in [0,1[$ we have $D_{t\lambda}(\theta,1) \approx \mathring{H}^{0,1}(\Omega)$, where

$$
\mathring{H}^{0,1}(\Omega) = \left\{ \begin{array}{ll}
W^{2,1}(\Omega), & \text{if } 0 < \theta < 1/2 \\
\frac{1}{\theta} \int_{\Omega} (d(x, \partial\Omega))^{-1} |u(x)| \, dx < +\infty, & \text{if } \theta = 1/2 \\
W^{2,1}(\Omega) \cap W^{1,1}_{\theta,1}(\Omega), & \text{if } 1/2 < \theta < 1.
\end{array} \right.
$$

Here $W^{2,1}(\Omega)$ denotes the Sobolev space of fractional order, $B^{1,1}(\Omega)$ denotes the Besov space and $d(x, \partial\Omega)$ the distance from $x$ to $\partial\Omega$.

Proof. The result follows from Theorems 1.1 and 3.1 and from the characterization of the spaces $(L^1, W^{2,1} \cap W^{1,1}_{\theta,1})$ (see Proposition 1 of the Appendix). ■

**Remark.** In the case $\Omega = \mathbb{R}^n$ the results of Theorem 3.2 were presented in [5].

4. Parabolic second order equations in $L^1$

Let $E$ be the operator given by (2.1) and consider the problem

$$
\begin{cases}
\begin{align*}
& u_{t}(t,x) = Eu(t,x) + f(t,x), \quad t > 0, x \in \Omega \\
& u(t,x) = 0, \quad t > 0, x \in \partial\Omega \\
& u(0,x) = u_0(x), \quad x \in \Omega.
\end{align*}
\end{cases}
$$

(4.1)

Regularity results for parabolic equations with $f$ in $L^p(0,T; L^q(\Omega))$ and $u_0$ in $L^q(\Omega)$ are well known in the literature if $1 < p, q < \infty$. In this section we study in a quite complete way also the case $p=q=1$ by using the abstract results of [4, sect. 8] and Theorem 3.2.

To state our results it is convenient to introduce some notation and definitions. Let $Y$ be a Banach space and let $a < b$ be real numbers. We shall be concerned with the following spaces of $Y$-valued functions defined on $[a,b]$:

- $L^1(a,b; Y)$ is the space of measurable functions $u$ such that $|u(\cdot)|_Y$ is integrable in $[a,b]$,
- $C(a,b; Y)$ is the space of continuous functions on $[a,b]$,
- $W^{1,1}(a,b; Y)$ is the space of functions $u$ of $L^1(a,b; Y)$ having distributional derivative in $L^1(a,b; Y)$,
- $L^1_\varepsilon(a,b; Y) = \{ u \in L^1(a,b; Y) \},$ for each $a < \varepsilon < b$,
- $W^{1,1}_\varepsilon(a,b; Y) = \{ u \in W^{1,1}(a,b; Y) \},$ for each $a < \varepsilon < b$,
- $W^{0,1}(a,b; Y), 0 < \theta < 1,$ is the Sobolev space of functions $u$ of $L^1(a,b; Y)$.
such that
\[
\int_0^T dt \int_0^T ds \| u(t) - u(s) \|_{L^1} | t - s |^{-1-\theta} < +\infty .
\]

Finally \( \dot{B}^{1,1}_\gamma (\Omega) \) is the Besov space introduced in Theorem 3.2 and \( D(A_1) \) is the domain of the operator \( A_1 \) given by (2.2)-(2.3), i.e.
\[
D(A_1) = \{ u \in L^1(\Omega) : Eu \in L^1(\Omega) \}
\]
where \( Eu \) is understood in the sense of distributions.

The following theorems describe the regularity of the solutions of (4.1) when the regularity of \( f \) and \( u_0 \) increases.

**Theorem 4.1.** Let \( f \in L^1([0, T] \times \Omega) \) and \( u_0 \in L^1(\Omega) \). Then (4.1) admits a unique generalized solution \( u \) and we have

(i) \( u(t, \cdot) \in C(0, T; L^1(\Omega)) \cap L^1(0, T; \dot{B}^{1,1}_\gamma (\Omega) \cap W^{1,1}(0, T; L^1(\Omega))) \),

for each \( 0 < \beta < 1 \),

\( u(t, \cdot) \in W^{\beta-\gamma,1}(0, T; \dot{B}^{2,1}_\gamma(\Omega)) \), for each \( 0 < \alpha < \beta < 1 \).

Proof. The result follows from [4, Th. 28] and Theorem 3.2.

**Theorem 4.2.** Let \( f(t, \cdot) \in L^1(0, T; \dot{B}^{1,1}_\gamma(\Omega)) \), for some \( 0 < \theta < 1 \). Then for each \( u_0 \in L^1(\Omega) \) (4.1) admits a unique solution \( u \) and we have

i) \( u(t, \cdot) \in C(0, T; L^1(\Omega)) \cap L^1(0, T; D(A_1)) \cap W^{1,1}(0, T; L^1(\Omega)) \),

ii) \( u(t, \cdot) \in L^1(0, T; \dot{B}^{1,1}_\gamma(\Omega) \cap W^{1,1}(0, T; L^1(\Omega)) \cap W^{\beta-\gamma,1}(0, T; \dot{B}^{2,1}_\gamma(\Omega)) \),

for each \( 0 < \alpha < \beta < 1 \).

If in addition \( u_0 \in \dot{B}^{2,1}_\gamma(\Omega) \), for some \( 0 < \gamma < 1 \), then we have for \( \delta = \min (\theta, \gamma) \)

iii) \( u(t, \cdot) \in C(0, T; \dot{B}^{2,1}_\gamma(\Omega)) \cap W^{\alpha,1}(0, T; \dot{B}^{2,1}_\gamma(\Omega)) \), for each \( 0 < \alpha, \beta < 1 \), \( \alpha + \beta = 1 + \delta \),

iv) \( Eu(t, \cdot) \in L^1(0, T; \dot{B}^{1,1}_\gamma(\Omega) \cap W^{1,1}(0, T; L^1(\Omega)) \cap W^{\beta-\gamma,1}(0, T; \dot{B}^{2,1}_\gamma(\Omega)) \),

for each \( 0 < \alpha < \delta < 1 \),

v) \( u(t, \cdot) \in L^1(0, T; \dot{B}^{2,1}_\gamma(\Omega)) \).

Proof. The assertions follow from [4, Th. 29] and Theorem 3.2.

**Theorem 4.3.** Let \( f(t, \cdot) \in W^{\theta,1}(0, T; L^1(\Omega)) \), for some \( 0 < \theta < 1 \). Then for each \( u_0 \in L^1(\Omega) \) there exists a unique solution \( u \) of (4.1) and we have

i) \( u(t, \cdot) \in C(0, T; L^1(\Omega)) \cap L^1(0, T; D(A_1)) \cap W^{1,1}(0, T; L^1(\Omega)) \),
ii) \( u(t, \cdot) \in L^1(0, T; B^\beta_{2\alpha}(\Omega)) \cap W^{\beta, 1}(0, T; L^1(\Omega)) \cap W^{\beta-\alpha, 1}(0, T; B^{2\alpha}_{2\beta}(\Omega)) \),
for each \( 0 < \alpha < \beta < 1 \).

If in addition \( u_0 \in B^{2\beta}_{2\gamma}(\Omega) \), for some \( 0 < \gamma < 1 \), then we have, for \( \delta = \min(\theta, \gamma) \)

iii) \( u(t, \cdot) \in C(0, T; B^{2\beta}_{2\alpha}(\Omega)) \cap W^{\alpha, 1}(0, T; B^{2\beta}_{2\alpha}(\Omega)) \),
for each \( 0 < \alpha < 1 \), \( \alpha + \beta = 1 + \delta \),

iv) \( u(t, \cdot) \in L^1(0, T; B^{2\beta}_{2\alpha}(\Omega)) \cap W^{\alpha, 1}(0, T; L^1(\Omega)) \cap W^{\beta-\alpha, 1}(0, T; B^{2\alpha}_{2\beta}(\Omega)) \),
for each \( 0 < \alpha < \delta < 1 \),

v) \( E_u(t, \cdot) \in W^{\delta, 1}(0, T; L^1(\Omega)) \).

Proof. The assertions follow from [4, Th. 30] and Theorem 3.2. ■

Appendix

We want to give here the proof concerning the characterization of the intermediate spaces \((L^1(\Omega), W^{2,1}(\Omega) \cap W^{1,1}_0(\Omega))_{\theta, 1}\), for \( 0 < \theta < 1 \), which has been used in section 3. If \( \Omega \) is of class \( C^2 \) using local change of coordinates it suffices to consider the case \( \Omega = \mathbb{R}^n_+ \) where

\[ \mathbb{R}^n_+ = \{ x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0 \} . \]

If \( \theta \neq 1/2 \) this characterization can be deduced from known results (see e.g. [2, Th. 4.3.6]) but we give here a direct proof for all \( 0 < \theta < 1 \) in order to make the paper self-contained.

In what follows we denote by \( B^{r,1}(\mathbb{R}^n_+) \), for \( 0 < r \leq 1 \), the Besov spaces defined as

\[ B^{r,1}(\mathbb{R}^n_+) = \{ u \in L^1(\mathbb{R}^n_+) : H_r(u) = \int_{\mathbb{R}^n_+} dy \int_{\mathbb{R}^n_+} dx \ | u(x) + u(y) - 2u(\frac{x+y}{2}) | \ | x-y|^{-n-r} < +\infty \} \]
endowed with the norm

\[ ||u||_{B^{r,1}} = ||u||^{r} + H_r(u) \]
where \( || \cdot ||^r \) denotes the norm in \( L^1(\mathbb{R}^n_+) \), whereas for \( 1 < r < 2 \) we define

\[ B^{r,1}(\mathbb{R}^n_+) = \{ u \in W^{1,1}(\mathbb{R}^n_+) : D_j u \in B^{r,1}(\mathbb{R}^n_+) \} \]
with the norm

\[ ||u||_{B^{r,1}} = ||u||^r + \sum_{j=1}^{n} H_{r-1}(D_j u) . \]

It is known that if \( r \neq 1 \) we have \( B^{r,1}(\mathbb{R}^n_+) = W^{r,1}(\mathbb{R}^n_+) \), the usual Sobolev spaces of fractional order.
Proposition 1. We have \((L^1(\mathbb{R}^+), W^{2,1}(\mathbb{R}^+))\cap W^{0,1}_0(\mathbb{R}^+))_{\theta,1} = B^{0,1}(\mathbb{R}^+),\)

where

\[
B^{0,1}(\mathbb{R}^+) = \begin{cases} \ W^{0,1}(\mathbb{R}^+), & \text{if } 0 < \theta < 1/2 \\ u \in B^{1,1}(\mathbb{R}^+), \int_{\mathbb{R}^+} (x_n)^{-1} |u(x)| \, dx < +\infty, & \text{if } \theta = 1/2 \\ W^{0,1}(\mathbb{R}^+) \cap W^{0,1}_0(\mathbb{R}^+), & \text{if } 1/2 < \theta < 1. \end{cases}
\]

In proving Proposition 1 we need some preliminary result. Set

\[
N_+(t, u) = \sup_{0 < |y| < t} |u(\cdot) + u(\cdot + 2y) - 2u(\cdot + y)|_1^t
\]

and

\[
|||u|||_{\theta,1}^t = \int_0^{+\infty} t^{-1-2\theta} N_+(t, u) \, dt + ||u||_1^t + \int_{\mathbb{R}^+} (x_n)^{-2\theta} |u(x)| \, dx.
\]

Then for each \(\theta \in [0, 1/2]\) it is easily checked that

\[
\int_{\mathbb{R}^+} dy \int_{\mathbb{R}^+} dx |u(x) + u(y) - 2u(\frac{x+y}{2})| |x-y|^{-n-2\theta} \leq \text{const} \int_0^{+\infty} t^{-1-2\theta} N_+(t, u) \, dt.
\]

Moreover we have the following result.

Lemma 1. Let us denote by \(X_{\theta,1}\) th.: Banach space corresponding to the norm \(|||\cdot|||_{\theta,1}^t\). Then

\[
X_{\theta,1} = B^{0,1}(\mathbb{R}^+).
\]

Proof. Given \(u \in L^1(\mathbb{R}^+),\) let us introduce the function \(U \in L^1(\mathbb{R}^+)^n\) defined as

\[
U(x) = \begin{cases} u(x), & \text{if } x_n > 0 \\ -u(x', -x), & \text{if } x_n \leq 0. \end{cases}
\]

Furthermore set, for \(\theta \in [0, 1]\)

\[
|||U|||_{\theta,1} = ||U||_1 + \int_0^{+\infty} t^{-1-2\theta} N(t, U) \, dt
\]

where \(||\cdot||_1\) denotes the norm in \(L^1(\mathbb{R}^n)\) and

\[
N(t, U) = \sup_{0 < |y| < t} ||U(\cdot) + U(\cdot + 2y) - 2U(\cdot + y)||_1.
\]

Then (see [2, Prop. 4.3.5])

\[
(2) \quad |||\cdot|||_{\theta,1} \leq |||\cdot|||_{\theta}^{0,1}
\]

where \(B^{0,1} = B^{0,1}(\mathbb{R}^+).\) Moreover one easily obtains, for each \(\theta \in [0, 1]\) (here by
c, c', c'', c_i, we denote various constants)

\[(3) \quad |||U|||_{\beta,1} \leq c \quad |||u|||_{\beta,1} \leq c' \quad |||U|||_{\beta,1} + \int_{R^+_n} (x_n)^{-20} |u(x)| \, dx\]

and

\[(4) \quad |||U|||_{\beta,0,1} \leq c'' \quad |||u|||_{\beta,0,1} + \int_{R^+_n} (x_n)^{-20} |u(x)| \, dx\]

where $B^0_{\beta,1} = B^0_{\beta,1}(R^+_n)$. Now let $\theta < 1/2$; we have (see [7, Th. 1.4.4.4])

\[(5) \quad \int_{R^+_n} (x_n)^{-20} |u(x)| \, dx \leq \text{const} \cdot |||u|||_{\beta,0,1}.
\]

Therefore from (1), (2), (3) and (4) we get, for $\theta \leq 1/2$

\[|||u|||_{\beta,1} \leq c_1 \quad |||U|||_{\beta,0,1} + \int_{R^+_n} (x_n)^{-20} |u(x)| \, dx \leq c_2 \quad |||u|||_{\beta,0,1} + \int_{R^+_n} (x_n)^{-20} |u(x)| \, dx\]

\[\leq c_3 \quad |||u|||_{\beta,1},
\]

which, together with (5), proves the assertion if $\theta \leq 1/2$.

Finally let $\theta > 1/2$. If $u \in W^{0,1}_{\beta,1}(R^+_n) \cap W^{1,1}_{0,1}(R^+_n)$ then $U \in W^{0,1}_{\beta,1}(R^n)$ and (5) holds (see [7, Th. 1.4.4.4]). Therefore from (2), (3), (4) and (5)

\[|||u|||_{\beta,1} \leq c_1 \quad |||U|||_{\beta,0,1} + \int_{R^+_n} (x_n)^{-20} |u(x)| \, dx \leq c_4 \quad |||u|||_{\beta,0,1}.
\]

Conversely let $u \in X_{\beta,1}$; from (2) and (3) we get

\[|||U|||_{\beta,0,1} \leq c_5 \quad |||u|||_{\beta,1} \leq c_6 \quad |||u|||_{\beta,1},
\]

so that $u \in W^{0,1}_{\beta,1}(R^+_n)$ and

\[|||u|||_{\beta,0,1} \leq |||U|||_{\beta,0,1} \leq c_6 \quad |||u|||_{\beta,1}.
\]

Finally the assertion $u \in W^{1,1}_{0,1}(R^+_n)$ follows from the fact that $u \in W^{1,1}_{\beta,1}(R^+_n)$ and

\[\int_{R^+_n} (x_n)^{-20} |u(x)| \, dx < +\infty
\]

implies that $u(x', 0) = 0$. □

Proof of Proposition 1. For simplicity in notation we restrict ourselves to the case $n = 2$. The method of the proof will lead the way for all $n \geq 1$.

In what follows we denote by $Q_t$, for $t > 0$, the subset of $R^+_2$ defined as

\[Q_t = \{ x \in R^+_2 : 0 \leq x_i \leq \frac{t}{4\sqrt{2}}, i = 1, \ldots, 2 \},
\]

moreover we set $c = (4\sqrt{2})^4$. Furthermore, given $u \in L^1(R^+_2)$, we denote by $v_1$ and $v_2$ the functions defined as
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$$v_1 = \int_Q dy \int_Q u(x+2(y+z)) \, dx = \frac{1}{16} \prod_i \int_{x_i}^{x_i+t/2\sqrt{2}} dy_i \int_{y_i}^{y_i+t/2\sqrt{2}} u(x) \, dx_i$$
and

$$v_2 = \int_Q dy \int_Q 2u(x+y+z) \, dx = 2\prod_i \int_{x_i}^{x_i+t/4\sqrt{2}} dy_i \int_{y_i}^{y_i+t/4\sqrt{2}} u(x) \, dx_i.$$  

Moreover set $w_1 = c(t-v_1 - v_2)$, $w_2 = c(t+x_2)^2(v_1 - w_1)$ and $u_i = u + w_1 - w_2$, $u_2 = -w_1 + w_2$. Then we have that $u = u_1 + u_2$ with $u_1 \in L^1(R_i^2)$ and $u_2 \in W^{2,1}(R_i^2) \cap W^{2,1}(R_i^3)$. Furthermore, using the fact that $y_2 + z_2 \leq t(2\sqrt{2})$, we get

$$||u + w_i|| \leq c^i \int_{R^2} dx \int_{Q^i} dy \int_{Q^i} dx \left\{ \int_{x_2+z_2}^{x_2+z_2+|u(x)|} \right\} dx_2 + \int_{x_2+z_2}^{x_2+z_2+|u(x)|} dx_2 + \int_{x_2+z_2}^{x_2+z_2+|u(x)|} dx_2$$

where $c'$ denotes a constant. Therefore setting

$$L(t, u) = \int_{R} dx_1 \left[ \int_{0}^{t} |u(x)| \, dx_2 + \int_{0}^{t} \frac{|u(x)|}{x_2^2} \right]$$

we obtain

$$||w_2|| \leq c^i L(t, u).$$

Concerning $u_2$ we have

$$||u_2|| \leq c' ||u|| t.$$  

Moreover, to estimate $||D_{i, k} u_2|| t$, let us note that

$$D_{i, k} v_1 = \int_{x_i}^{x_i+t/2\sqrt{2}} dy_i \int_{y_i}^{y_i+t/2\sqrt{2}} [u(z_0, x_0+t/\sqrt{2}) - 2u(z_0, x_0+t/2\sqrt{2}) - u(z_0, x_0)] \, dz_0$$

where $i \neq h$. Moreover

$$D_{i, 2} v_1 = \int_{x_i}^{x_i+t/2\sqrt{2}} dx_i \int_{x_2}^{x_2+t/2\sqrt{2}}$$

$$\int_{x_2}^{x_2+t/2\sqrt{2}} [u(z_1, z_2+t/\sqrt{2}) - 2u(z_1-t/4\sqrt{2}, z_2+t/4\sqrt{2}) + u(z_1-t/2\sqrt{2}, z_2) - u(z_1-t/2\sqrt{2}, z_2+t/2\sqrt{2}) + 2u(z_1-t/4\sqrt{2}, z_2+t/4\sqrt{2}) - u(x)] .$$
Therefore for each $h, k$ we get

$$
||D_{h,k} w_i||^* \leq c't^{-2} N_+ (t,u).
$$

Now we have $||D_{1,1} w_2||^* \leq ||D_{1,1} w_1||^*$ so that (9) holds for $h=k=1$ with $w_1$ replaced by $w_2$. Furthermore

$$
||D_{2,2} w_2||^* \leq c t \int_{R^2_+} \left[ \frac{1}{(t+x_2)^6} |D_{2,2} (v_1-v_2) (x)| + \frac{1}{(t+x_2)^6} |D_2 (v_1-v_2) (x)| \right] dx = I_1 + I_2 + I_3.
$$

Now we get

$$
I_1 + I_2 \leq ||D_{2,2} w_i||^* + c't^{-2} ||u||^* \leq c't^{-2} [N_+ (t,u) + ||u||^*]
$$

where we used (9). Furthermore, proceeding as in (7), we obtain

$$
I_2 \leq c't^{-2} L(t,u).
$$

Therefore

$$
||D_{2,2} w_2||^* \leq c't^{-2} \{ ||u||^* + N_+ (t,u) + L(t,u) \}.
$$

Finally in a similar way we get

$$
||D_{1,1} w_2||^* \leq c't^{-2} \{ N_+ (t,u) + L(t,u) \}.
$$

Summarizing using (6)–(11) we obtain that given $u \in L^1 (R^2_+)$, we can write $u = u_1 + u_2$ with $u_1 \in L^1 (R^2_+)$ and $u_2 \in W^{2,1} (R^2_+) \cap W^{0,1} (R^2_+)$ and

$$
||u_1||^* \leq N_+ (t,u) + c'L(t,u)
$$

and

$$
||u_2||^* \leq c't^{-2} [(1+t^2) ||u||^* + N_+ (t,u) + L(t,u)]
$$

where $||.|^*$ denotes the norm in $W^{2,1} (R^2_+)$. Therefore (see (1.2)) there exists $c_1$ such that

$$
K(\ell, u) \leq c_1 [N_+ (t,u) + \min (1, \ell^2) ||u||^* + L(t,u)].
$$

Conversely let $u = u_1 + u_2$ with $u_1 \in L^1 (R^2_+)$ and $u_2 \in W^{2,1} (R^2_+) \cap W^{0,1} (R^2_+)$. Then we have

$$
\min (1, \ell^2) ||u||^* \leq K(\ell, u)
$$

and

$$
N_+ (t,u) \leq N_+ (t,u_1) + N_+ (t,u_2) \leq 4 ||u_1||^* + \ell^2 ||u_2||^* \leq 4 K(\ell, u)
$$

the third estimate following by
\[ u(x) - 2u(x+y) + u(x+2y) = 2 \int_0^{|y|} ds \int_0^t \frac{\partial}{\partial s} \frac{\partial}{\partial \sigma} u(x+(s+\sigma)) \left| \frac{\sigma}{y} \right| . \]

Furthermore
\begin{align*}
L(t, u) &\leq ||u_1||_1^t + \int_R dx_1 \int_0^t dx_2 \int_0^s dy_2 \int_{\gamma_2} d\xi_2 \left| D_{22} u_2(x_1, \xi_2) \right| + \\
&+ c^2 \int_0^t \frac{1}{3^2} dx_2 \int_0^s dy_2 \int_{\gamma_2} d\xi_2 \left| D_{22} u_2(x_1, \xi_2) \right| \\
&\leq ||u_1||_1^t + c^2 ||D_{22} u_2||_t^t
\end{align*}

so that
\[ L(t, u) \leq c \, K(t^2, u) . \]

Finally from (12)–(15) we obtain that there exists \( c_2 \) such that
\[ K(t^2, u) \leq c_1 \left[ N_+(t, u) + \min (1, t^2) ||u||_t^t + L(t, u) \right] \leq c_2 \, K(t^2, u) . \]

Therefore
\[ \int_0^{+\infty} t^{-1-\theta} K(t, u) \, dt = 2 \int_0^{+\infty} t^{-1-2\theta} K(t^2, u) \, dt \leq c_1 \left[ \int_0^{+\infty} t^{-1-2\theta} N_+(t, u) \, dt + ||u||_t^t \right. \\
+ \left. \int_0^{+\infty} t^{-1-2\theta} L(t, u) \, dt \right] \leq c_2 \int_0^{+\infty} t^{-1-\theta} K(t, u) \, dt . \]

Now
\[ \int_0^{+\infty} t^{-1-2\theta} L(t, u) \, dt = \text{const} \int_{R^2_+} (x_2)^{-2\theta} |u(x)| \, dx , \]
therefore the desired result follows from Lemma 1. 

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