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CIRCLES IN A COMPLEX PROJECTIVE SPACE

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0. Introduction

The study of circles is one of the interesting objects in differential geometry. A curve $\gamma(s)$ on a Riemannian manifold M parametrized by its arc length s is called a *circle*, if there exists a field of unit vectors Y_s along the curve which satisfies, together with the unit tangent vectors $X_s=\dot{\gamma}(s)$, the differential equations: $\nabla_s X_s = kY_s$ and $\nabla_s Y_s=-kX_s$, where k is a positive constant, which is called the curvature of the circle $\gamma(s)$ and ∇_s denotes the covariant differentiation along $\gamma(s)$ with respect to the Riemannian connection ∇ of M. For given a point $x\in M$, orthonormal pair of vectors u, $v\in T_xM$ and for any given positive constant k, we have a unique circle $\gamma(s)$ such that $\gamma(0)=x$, $\dot{\gamma}(0)=u$ and $(\nabla_s\dot{\gamma}(s))_{s=0}=kv$. It is known that in a complete Riemannian manifold every circle can be defined for $-\infty<s<\infty$ (cf. [6]).

The study of global behaviours of circles is very interesting. However there are few results in this direction except for the global existence theorem. In general, a circle in a Riemannian manifold is not closed. Here we call a circle $\gamma(s)$ closed if there exists s_0 with $\gamma(s_0)=\gamma(0)$, $X_{s_0}=x_0$ and $Y_{s_0}=Y_0$. Of course, any circles in Euclidean m-space E^m are closed. And also any circles in Euclidean m-sphere $S^m(c)$ are closed. But in the case of a real hyperbolic space $H^m(c)$, there exist many open circles. It is well-known that in $H^m(c)$ circles with curvature not exceeding $\sqrt{|c|}$ are open and circles with curvature greater than $\sqrt{|c|}$ are closed (cf. [3]). That is, in $H^m(c)$ the answer to the question "Is a circle $\gamma(s)$ closed?" depends on its curvature.

In this paper, we are concerned with circles in an n-dimensional complex projective space $\boldsymbol{CP}^n(c)$ of constant holomorphic sectional curvature c. In Section 1, by using Submanifold Theory we give an interesting family of open circles and closed circles with the same curvature $\sqrt{c}/2\sqrt{2}$ in $\boldsymbol{CP}^n(c)$. In his paper [5], Naitoh

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constructed several parallel isometric immersions of symmetric spaces M of rank 2 into $\boldsymbol{CP}^n(c)$. We can obtain a family of circles as the image of geodesics on M through these immersions. This interesting fact leads us to the study of circles in a complex projective space.

The main purpose of this paper is to classify all circles in a complex projective space. Our idea is based on the fact that a complex projective space is a base manifold of the principal S^1-fiber bundle $\pi: S^{2n+1} \longrightarrow \boldsymbol{CP}^n$. We show that horizontal lifts of circles on $\boldsymbol{CP}^n$ into S^{2n+1} are helices of order 2, 3 or 5. This gives us explicit expressions of circles on $\boldsymbol{CP}^n$. As a consequence of our results we can answer to the question "When is a given circle $\gamma(s)$ closed ?". In particular, we can conclude that there exist open circles and closed circles with any given curvature in a complex projective space. Here we note that there exist closed circles with the same curvature but with different prime periods in $\boldsymbol{CP}^n$. Moreover, we find that every circle in a complex projective space is a simple curve.

1. Circles and parallel submanifolds

We shall start with a congruence theorem. We denote by $M(c)$ a complex space form, that is, a Kaehler manifold of constant holomorphic sectional curvature c. We say that two circles γ and δ are *congruent*, if there exists a holomorphic isometry φ of $M(c)$ such that $\gamma=\varphi\circ\delta$. It is well-known that any isometry φ of a non-flat complex space form $M(c)$ is holomorphic or anti-holomorphic. We are taking account of the orientation of circles.

In order to state the congruence theorem we introduce an important invariant for circles. Let $\gamma=\gamma(s)$ be a circle in $M(c)$ satisfying the equations $\nabla_s X_s = kY_s$ and $\nabla_s Y_s = -kX_s$. We call $\langle X_s, JY_s\rangle$ the *complex torsion* of the circle γ, where J is the complex structure of $M(c)$. Note that the complex torsion of the circle γ is constant. In fact,

$$\begin{aligned}\nabla_s\langle X_s, JY_s\rangle &= \langle \nabla_s X_s, JY_s\rangle + \langle X_s, J\nabla_s Y_s\rangle \\ &= k\cdot\langle Y_s, JY_s\rangle - k\cdot\langle X_s, JX_s\rangle = 0.\end{aligned}$$

The congruence theorem for circles in a complex space form $M(c)$ is stated as follows (see, Theorem 5.1 in [4]):

Proposition 1. *Two circles in $M(c)$ are congruent if and only if they have the same curvatures and the same complex torsions.*

In the following, we give a nice family of circles in $\boldsymbol{CP}^n$. In his paper [5], Naitoh constructed parallel immersions of symmetric spaces $M(=S^1\times S^{n-1}$, $SU(3)/SO(3)$, $SU(3)$, $SU(6)/Sp(3)$, $E_6/F_4)$ of rank 2 into $\boldsymbol{CP}^n(4)$. These examples have various geometric properties. We here pay attention to the property "any geodesic γ of these submanifolds is a circle with the same curvature $1/\sqrt{2}$ in

$\boldsymbol{CP}^n(4)$". In this section, we make use of his parallel isometric imbedding of $S^1\times S^{n-1}/\phi$ into $\boldsymbol{CP}^n(4)$. Here the identification ϕ is defined by

$$\phi((e^{i\theta},\ a_1,\ \cdots,\ a_n))=(-e^{i\theta},\ -a_1,\ \cdots,\ -a_n),$$

where $\sum a_j^2=1$. He defined the isometric imbedding $h:\ S^1\times S^{n-1}/\phi\longrightarrow \boldsymbol{CP}^n(4)$ by

$$h(e^{i\theta}\ ;\ a_1,\ \cdots,\ a_n)=\pi\left(\begin{pmatrix}(1/3)(e^{-2i\theta/3}+2a_1e^{i\theta/3})\\ (\sqrt{2}/3)(e^{-2i\theta/3}-a_1e^{i\theta/3})\\ (2/\sqrt{6})ia_2e^{i\theta/3}\\ \vdots\\ (2/\sqrt{6})ia_ne^{i\theta/3}\end{pmatrix}\right),$$

where $\pi:\ S^{2n+1}\longrightarrow \boldsymbol{CP}^n(4)$ is the Hopf fibration and the metric on $S^1\times S^{n-1}/\phi$ is given by the following;

$$(1.1)\qquad \langle A+\xi,\ B+\eta\rangle=\frac{2}{9}\langle A,\ B\rangle_{S^1}+\frac{2}{3}\langle \xi,\ \eta\rangle_{S^{n-1}}$$

for $A,\ B\in TS^1$ and $\xi,\ \eta\in TS^{n-1}$, where $\langle\ ,\ \rangle_{S^1}$ and $\langle\ ,\ \rangle_{S^{n-1}}$ denote the canonical metric on S^1 and S^{n-1}, respectively.

We denote by σ the second fundamental form of the imbedding h. As was calculated in [5], σ is expressed as follows for any unit vector $w\in TS^{n-1}$;

$$(1.2)\qquad \begin{cases}\sigma(u,\ u)=-1/\sqrt{2}\cdot Ju,\\ \sigma(w,\ w)=1/\sqrt{2}\cdot Ju,\\ \sigma(u,\ w)=1/\sqrt{2}\cdot Jw,\end{cases}$$

where $u\in TS^1$ is the normalized vector of $\partial/\partial\theta$.

A circle γ is said to be closed if there exists a positive s_0 with

$$(1.3)\qquad \gamma(s_0)=\gamma(0),\ \dot\gamma(s_0)=\dot\gamma(0)\ \text{and}\ (\nabla_s\dot\gamma(s))_{s=s_0}=(\nabla_s\dot\gamma(s))_{s=0}.$$

We call l the prime period of a closed circle γ if it is the minimum positive number satisfying (1.3). As we mentioned before every circle on spheres and Euclidean spaces is closed. Moreover the prime period of closed circles of curvature k is as follows (see, page 169 in [6] and [3]):

(i) $2\pi/\sqrt{k^2+c}$ on $S^m(c)$,
(ii) $2\pi/k$ on R^m,
(iii) $2\pi/\sqrt{k^2+c}$ on $H^m(c)$, when $k>\sqrt{|c|}$.

Our aim here is to point out that the feature is not the same on a complex projective space.

Theorem 1. *For any unit vector* $X=au+v\in T_x(S^1\times S^{n-1}/\phi)\simeq T_{x_1}S^1\oplus$

$T_{x_2}S^{n-1}$ at a point x, we denote by γ_X the geodesic along X on $S^1\times S^{n-1}/\phi$. Then the circle $h\circ\gamma_X$ on $\boldsymbol{CP}^n(4)$ satisfies the following properties:

(1) *The curvature of $h\circ\gamma_X$ is $1/\sqrt{2}$.*

(2) *The complex torsion of $h\circ\gamma_X$ is $4\alpha^3-3\alpha$ for $-1\leq\alpha\leq 1$.*

(3) *The circle $h\circ\gamma_X$ is closed if and only if $\alpha=0$ or $\sqrt{(1-\alpha^2)/3\alpha^2}$ is rational.*

(4) *When $\alpha=0$, the prime period of the closed circle is $(2\sqrt{6}/3)\pi$.*

(5) *When $\alpha\neq 0$ and $\sqrt{(-\alpha^2)/3\alpha^2}$ is rational, we denote by p/q the irreducible fraction defined by $\sqrt{(1-\alpha^2)/3\alpha^2}$. Then the prime period l of a closed circle $h\circ\gamma_X$ is as follows;*

(a) *When pq is even, l is the least common multiple of $(2\sqrt{2}/3|\alpha|)\pi$ and $(2\sqrt{2}/\sqrt{3(1-\alpha^2)})\pi$. In particular, when $\alpha=\pm 1$, then $l=(2\sqrt{2}/3)\pi$.*

(b) *When pq is odd, l is the least common multiple of $(\sqrt{2}/3|\alpha|)\pi$ and $(\sqrt{2}/\sqrt{3(1-\alpha^2)})\pi$.*

Proof. Assertion (1) is a direct consequence of [5]. We shall show (2), (3), (4) and (5).

(2) For any unit vector $X=\alpha u+v\in T_{x_1}S^1\oplus T_{x_2}S^{n-1}$ we find

$$\begin{aligned}&(\text{the complex torsion of the circle } h\circ\gamma_X)\\&=-\sqrt{2}\langle JX,\ \sigma(X,\ X)\rangle\\&=-\sqrt{2}\langle \alpha Ju+Jv,\ \alpha^2\cdot\sigma(u,\ u)+2\alpha\cdot\sigma(u,\ v)+\sigma(v,\ v)\rangle,\end{aligned}$$

which, together with (1.2), $\langle u,\ v\rangle=0$ and $\|v\|^2=1-\alpha^2$, yields the conclusion.

(3) Since h is injective (see, Lemma 6.2 in [5]), the circle $h\circ\gamma_X$ is closed if and only if γ_X is closed. Let δ_X denote a covering geodesic on $S^1\times S^{n-1}$ of γ_X. As $S^1\times S^{n-1}\to\times S^{n-1}/\phi$ is a double covering, γ_X is closed if and only if δ_X is closed. We take a point $y=(y_1,\ y_2)$ on the geodesic δ_X. By the definition (1.1) of the metric on $S^1\times S^{n-1}$, if y moves along δ_X with velocity 1, the point y_1 moves with velocity $|\alpha|$ along S^1 of radius, say r, and the point y_2 moves with velocity $\sqrt{1-\alpha^2}$ along the great circle S^1 (in the direction of v) of radius $\sqrt{3}r$ on S^{n-1}, where $r=\sqrt{2}/3$. Hence the geodesic δ_X is closed if and only if $\alpha=0$ or the ratio $(2\pi r/|\alpha|)$: $(2\sqrt{3}\pi r/\sqrt{1-\alpha^2})=\sqrt{(1-\alpha^2)/3\alpha^2}$ is rational.

(4), (5): Identification ϕ tells that in the case of $\alpha=0$ and the case (a)

$$\begin{aligned}&(\text{the prime period of the closed geodesic } \gamma_X)\\&=(\text{the prime period of the closed geodesic } \delta_X).\end{aligned}$$

And also in the case (b)

$$\begin{aligned}&(\text{the prime period of the closed geodesic } \gamma_X)\\&=(1/2)(\text{the prime period of the closed geodesic } \delta_X).\end{aligned}$$

On the other hand, since h is an isometric imbedding, (the prime period of the closed geodesic γ_X)=(the prime period of the closed circle $h\circ\gamma_X$). Therefore we get the conclusion.

REMARKS.

(1) Since the complex torsion $4a^3-3a(-1\leq a\leq 1)$ takes an arbitrary value in [-1, 1], Theorem 1 asserts that for any circle δ of curvature $1/\sqrt{2}$ there exists our circle $h\circ\gamma_X$ which is congruent to δ in $\boldsymbol{CP}^n(4)$.

(2) The proof of Theorem 1 asserts that every circle with curvature $1/\sqrt{2}$ in $\boldsymbol{CP}^n(4)$ is a simple curve.

(3) The statement (4) and (5) in Theorem 1 tells us that there exist circles with the same curvature $1/\sqrt{2}$ but with different prime periods in $\boldsymbol{CP}^n(4)$.

(4) By Theorem 1 we can obtain the fact "Complex torsion tells whether a circle with curvature $1/\sqrt{2}$ is closed or open".

2. Complex torsion of circles

Let N be the outward unit normal on the unit sphere $S^{2n+1}(1)\subset\boldsymbol{R}^{2n+2}\subset\boldsymbol{C}^{n+1}$ We denote by J the natural complex structure on $\boldsymbol{C}^{n+1}$. In the following we mix the complex structures of $\boldsymbol{C}^{n+1}$ and $\boldsymbol{CP}^n(4)$.

In this section, we shall investigate circles in $\boldsymbol{CP}^n(4)$ by making use of the Hopf fibration $\pi: S^{2n+1}(1)\longrightarrow\boldsymbol{CP}^n(4)$. For the sake of simplicity we identify a vector field X on $\boldsymbol{CP}^n(4)$ with its horizontal lift X^* on $S^{2n+1}(1)$.

The relation between the Riemannian connection ∇ of $\boldsymbol{CP}^n(4)$ and the Riemannian connection $\tilde{\nabla}$ of $S^{2n+1}(1)$ is as follows:

$$\tilde{\nabla}_X Y=\nabla_X Y+\langle X, JY\rangle JN \tag{2.1}$$

for any vector fields X and Y on $\boldsymbol{CP}^n(4)$, where $\langle\ ,\ \rangle$ is the natural metric on $\boldsymbol{C}^{n+1}$. In fact, denoting by $\bar{\nabla}$ the canonical Riemannian connection of $\boldsymbol{C}^{n+1}$, we have $\bar{\nabla}_X N=X$ and get

$$\begin{aligned}\nabla_X Y&=\tilde{\nabla}_X Y-\langle\tilde{\nabla}_X Y, JN\rangle JN=\tilde{\nabla}_X Y+\langle Y, \tilde{\nabla}_X(JN)\rangle JN\\&=\tilde{\nabla}_X Y+\langle Y, \bar{\nabla}_X(JN)\rangle JN=\tilde{\nabla}_X Y+\langle Y, JX\rangle JN.\end{aligned}$$

The following is a fundamental result about circles in a complex projective space.

Theeorem 2. *Let γ denote a circle with curvature k and complex torsion τ in $\boldsymbol{CP}^n(4)$ satisfying: $\nabla_s X_s=kY_s$ and $\nabla_s Y_s=-kX_s$. Then a horizontal lift $\tilde{\gamma}$ of γ in $S^{2n+1}(1)$ is a helix of order 2, 3 or 5 corresponding to $\tau=0$, $\tau=\pm 1$ or $\tau\neq 0, \pm 1$, respectively. Moreover it satisfies the following differential equations:*

$$\begin{cases}\tilde{\nabla}_s X_s= & kY_s,\\ \tilde{\nabla}_s Y_s= -kX_s & +\tau JN,\\ \tilde{\nabla}_s JN= & -\tau Y_s \quad +\sqrt{1-\tau^2}Z_s,\\ \tilde{\nabla}_s Z_s= & -\sqrt{1-\tau^2}JN \quad +kW_s,\\ \tilde{\nabla}_s W_s= & -kZ_s,\end{cases} \tag{2.2}$$

where $Z_s=1/\sqrt{1-\tau^2}\cdot(JX_s+\tau Y_s)$, $W_s=1/\sqrt{1-\tau^2}\cdot(JY_s-\tau X_s)$.

Proof. From (2.1) we can easily see that the first and the second equalities of (2.2) hold. Next we shall compute $\tilde{\nabla}_s JN$. Note that $\|JN\|=1$. We have

$$\tilde{\nabla}_s JN=\bar{\nabla}_s(JN)-\langle\bar{\nabla}_s(JN), N\rangle N=JX_s=-\tau Y_s+(JX_s+\tau Y).$$

Since $\|JX_s+\tau Y_s\|^2=1-\tau^2$, when $\tau\neq\pm1$, the third equality of (2.2) holds. The same calcultion as above leads us to other equalities.

Complex torsion plays an important role in the study of circles. We consider the cases that the complex torsion is 0, ±1 and otherwise one by one.

Proposition 2. *Let γ be a circle with curvature k in $\boldsymbol{CP}^n(4)$. Suppose that the complex torsion τ of γ is 0. Then γ is a simple closed curve with prime period $2\pi/\sqrt{k^2+1}$ which lies on some real 2-dimensional totally geodesic $\boldsymbol{RP}^2(1)$ in $\boldsymbol{CP}^n(4)$.*

Proof. Let $\tilde{\gamma}$ be a horizontal lift of γ. By hypothesis $\tilde{\gamma}$ satisfies the following (see, (2.2)):

(2.3) $\tilde{\nabla}_s X_s=kY_s$ and $\tilde{\nabla}_s Y_s=-kX_s$.

It is well- known that the curve $\tilde{\gamma}$ is a small circle with curvature k in $S^{2n+1}(1)$ so that $\tilde{\gamma}$ is a simple closed curve and its prime period is $2\pi/\sqrt{k^2+1}$. Now we shall show that there exists some horizontal totally geodesic S^2 in $S^{2n+1}(1)$ which contains $\tilde{\gamma}$. So we must solve the differential equation (2.3) with initial condition

(2.4) $\quad\tilde{\gamma}(0)=x$, $X_0=u$ and $Y_0=v$.

Rewriting the differential equation (2.3) in $\boldsymbol{C}^{n+1}$, we find that it is equivalent to $\tilde{\gamma}^{(3)}+(k^2+1)\dot{\tilde{\gamma}}=0$. So we obtain

(2.5) $\quad\tilde{\gamma}(s)=A+B\cdot\exp(i\sqrt{k^2+1}s)+C\cdot\exp(-i\sqrt{k^2+1}s)$.

On the other hand the initial condition (2.4) is rewritten as:

(2.6) $\quad\tilde{\gamma}(0)=x$, $\dot{\tilde{\gamma}}(0)=u$ and $\ddot{\tilde{\gamma}}(0)+\tilde{\gamma}(0)=kv$.

It follows from (2.5) and (2.6) that

$$\tilde{\gamma}(s)=\frac{k}{k^2+1}(kx+v)+\frac{\cos(\sqrt{k^2+1}s)}{k^2+1}(x-kv)+\frac{\sin(\sqrt{k^2+1}s)}{\sqrt{k^2+1}}u$$

which implies that the curve $\tilde{\gamma}$ lies on the linear subspace $\boldsymbol{R}^3$ (spanned by $\{x, u, v\}$) which passes the origin of $\boldsymbol{C}^{n+1}$, so that the $\tilde{\gamma}$ lies (as a circle) on a horizontal totally geodesic $S^2(1)$ of $S^{2n+1}(1)$. Therefore we conclude that our circle $\gamma(=\pi(\tilde{\gamma}))$ lies on some totally geodesic $\boldsymbol{RP}^2(1)(=\pi(S^2(1))$ in $\boldsymbol{CP}^n(4)$.

Next, we shall investigate the case of $\tau=\pm 1$.

Proposition 3 (c.f. [1]). *Let γ be a circle with curvature k in $\boldsymbol{CP}^n(4)$. Suppose that the complex torsion τ of γ is 1 or -1. Then the circle γ is a simple closed curve with prime period $2\pi/\sqrt{k^2+4}$ which lies on some totally geodesic $\boldsymbol{CP}^1(4)$ in $\boldsymbol{CP}^n(4)$.*

Proof We consider the case of $\tau=1$, that is, $Y_s=-JX_s$. Let $\tilde{\gamma}$ be a horizontal lift of γ. Note that $\tilde{\gamma}$ is a helix of order 3 with the first curvature k and the second curvature 1 in $S^{2n+1}(1)$. From (2.2) $\tilde{\gamma}$ satisfies the differential equation $\tilde{\nabla}_s X_s=-kJX_s$, so that

$$(2.7) \qquad \ddot{\tilde{\gamma}}+kJ\dot{\tilde{\gamma}}+\tilde{\gamma}=0.$$

Since the characteristic equation for (2.7) is $t^2+\mathrm{kit}+1=0$, under the initial condition $\tilde{\gamma}(0)=x$, $\dot{\tilde{\gamma}}(0)=u$, the solution $\tilde{\gamma}$ for (2.7) is expressed as follows:

$$(2.8) \qquad \tilde{\gamma}(s)=1/(1+\alpha^2)\cdot(e^{\alpha is}+\alpha^2 e^{\beta is})x+\alpha/(1+\alpha^2)\cdot(-e^{\alpha is}+e^{\beta is})Ju,$$

where $\alpha+\beta=-k$, $\alpha\beta=-1$. This expression shows that $\tilde{\gamma}$ lies on the linear subspace $\boldsymbol{C}^2$ (spanned by $\{x, Jx, u, Ju\}$) which passes the origin of $\boldsymbol{C}^{n+1}$, so that $\tilde{\gamma}$ lies on a totally geodesic $S^3(1)$ of $S^{2n+1}(1)$. And hence we can see that our circle $\gamma(=\pi(\tilde{\gamma}))$ lies on some totally geodesic $\boldsymbol{CP}^1(4)(=\pi(S^3(1))$ in $\boldsymbol{CP}^n(4)$. Namely γ is a circle with curvature k in $S^2(4)(=\boldsymbol{CP}^1(4))$. We can therefore conclude that the circle γ is a simple closed curve with prime period $2\pi/\sqrt{k^2+4}$. Of course our conclusion also holds in the case of $\tau=-1$.

REMARK.

(5) Although every circle γ with complex torsion $\tau=1$ or -1 in $\boldsymbol{CP}^n(4)$ is closed, its horizontal lift $\tilde{\gamma}$ in $S^{2n+1}(1)$ is not necessarily closed. The expression (2.8) asserts that $\tilde{\gamma}$ is closed if and only if the ratio α/β is rational and the prime period of the closed $\tilde{\gamma}$ is the least common multiple of $2\pi/|\alpha|$ and $2\pi/|\beta|$.

The rest of this paper is devoted to study in the case of $\tau\neq 0, \pm 1$. We establish the following:

Theorem 3. *Let γ be a circle with curvature k and with complex torsion τ in $\boldsymbol{CP}^n(4)$. Suppose that $\tau\neq 0, \pm 1$. Let a, b and d $(a<b<d)$ be nonzero real solutions for the following cubic equation;*

(2.9) $\lambda^3-(k^2+1)\lambda+k\tau=0.$

Then the following hold ;
(1) *The circle γ is closed if and only if one of the three ratios a/b, b/d and d/a is rational. Moreover, the prime period of γ is the least common multiple of $2\pi/(b-a)$ and $2\pi/(d-a)$.*
(2) *Every circle with complex torsion $\tau\neq 0, \pm 1$ is a simple curve which lies on some totally geodesic $\boldsymbol{CP}^2(4)$ in $\boldsymbol{CP}^n(4)$.*

Proof. First of all we note that the cubic equation (2.9) has three distinct (nonzero) real solutions, because $-1<\tau<1$. Let $\tilde{\gamma}$ be a horizontal lift of γ. Then (2.2) shows that $\tilde{\gamma}$ is a helix of order 5 with the first curvature k, the second curvature $|\tau|$, the third curvature $\sqrt{1-\tau^2}$ and the fourth curvature k. We shall solve the differential equation (2.2). It follows from the first and the second equations in (2.2) that

(2.10) $\tilde{\gamma}^{(3)}+(1+k^2)\dot{\tilde{\gamma}}-\tau kJ\tilde{\gamma}=0.$

On the other hand the real solutions a, b and d $(a<b<d)$ for (2.9) satisfy

(2.11) $a+b+d=0,\ ab+bd+da=-k^2-1,\ abd=-\tau k.$

Since these three purely imaginary numbers ai, bi and di satisfy the characteristic equation $t^3+(1+k^2)t-\tau ki=0$ for (2.10), the curve $\tilde{\gamma}$ is expressed as:

(2.12) $\tilde{\gamma}(s)=Ae^{ais}+Be^{bis}+De^{dis}.$

Under the initial condition (2.6), we find from (2.6), (2.11) and (2.12) that

$$(2.13)\quad \begin{cases} A=\dfrac{1}{(a-b)(d-a)}\{-(1+bd)x+aJu+kv\}, \\ B=\dfrac{1}{(b-d)(a-b)}\{-(1+da)x+bJu+kv\}, \\ D=\dfrac{1}{(d-a)(b-d)}\{-(1+ab)x+dJu+kv\}. \end{cases}$$

It follows from (2.12) and (2.13) that $\tilde{\gamma}$ lies on the linear subspace $\boldsymbol{C}^3$ (spanned by $\{x, Jx, u, Ju, v, Jv\}$) which passes the origin of $\boldsymbol{C}^{n+1}$, so that $\tilde{\gamma}$ lies on a totally geodesic $S^5(1)$ of S^{2n+1}. Hence we can see that our circle $\gamma(=\pi(\tilde{\gamma}))$ lies on some totally geodesic $\boldsymbol{CP}^2(4)(=\pi(S^5(1))$ in $\boldsymbol{CP}^n(4)$.

Next we suppose that there exists some s_0 such that $\gamma(s_0)=\gamma(0)$. Namely, there exists some $\rho\in[0, 2\pi)$ such that $\tilde{\gamma}(s_0)=e^{\rho i}\tilde{\gamma}(0)$. This is equivalent to the following

(2.14) $A+B\cdot\exp((b-a)is_0))+D\cdot\exp((d-a)is_0))=x\cdot\exp((\rho-\mathrm{a}s_0)i).$

Now we shall show that the existence of some s_0 with $\gamma(s_0)=\gamma(0)$ is equivalent to

$b/d \in \boldsymbol{Q}$. Since the complex torsion is not ± 1, the vectors x, Ju and v are linearly independent. Substituting (2.13) into the left-hand side of (2.14), we get the following:

(2.15) $(b-d)(1+bd)+(d-a)(1+da)\cdot\exp((b-a)is_0)$
$+(a-b)(1+ab)\cdot\exp((d-a)is_0)=-(a-b)(b-d)(d-a)\cdot\exp((\rho-as_0)i)$,

(2.16) $b-d+(d-a)\cdot\exp((b-a)is_0)+(a-b)\cdot\exp((d-a)is_0))=0$,

(2.17) $a(b-d)+b(d-a)\cdot\exp((b-a)is_0)+d(a-b)\cdot\exp((d-a)is_0)=0$.

From (2.16) and (2.17) we find

$$(b-a)(d-a)\cdot\exp((b-a)is_0)+(d-a)(a-b)\cdot\exp((d-a)is_0)=0,$$

which implies that $\exp((b-d)is_0)=1$, that is, $(b-d)s_0/2\pi \in \boldsymbol{Z}$. Similarly we find that $(d-a)s_0/2\pi \in \boldsymbol{Z}$ and $(b-a)s_0/2\pi \in \boldsymbol{Z}$. This yields that $(b-a)/(d-a) \in \boldsymbol{Q}$. Then we get

$$\frac{b-a}{d-a}=\frac{b+(b+d)}{d+(b+d)}=2-3\frac{d}{b+2d},$$

which shows that $b/d \in \boldsymbol{Q}$. Thus we know that the existence of some s_0 with $\gamma(s_0)=\gamma(0)$ implies $b/d \in \boldsymbol{Q}$. Next we suppose that $b/d \in \boldsymbol{Q}$. Then the above discussion tells us that $(b-a)/(d-a) \in \boldsymbol{Q}$. We here denote by s_0 the least common multiple of $2\pi/(d-a)$ and $2\pi/(b-a)$. Then s_0 satisfies (2.16) and (2.17). Moreover, (2.15) holds when $\rho \equiv as_0 (\mathrm{mod}\ 2\pi)$. Hence $\gamma(s_0)=\gamma(0)$.

We here remark that "one of a/b, b/d and d/a is rational" is equivalent to "each of a/b, b/d and d/a is rational".

Now we shall show that our circle γ satisfying $\nabla_s X_s = kY_s$, $\nabla_s Y_s = -kY_s$ is a simple closed curve under the condition that there exists some s_0 with $\gamma(s_0)=\gamma(0)$. Note that $\tilde{\gamma}(s_0)=\exp(ais_0)\cdot\tilde{\gamma}(0)$. We shall show the following:

(2.18) $X_{s_0}=X_0$ and $Y_{s_0}=Y_0$.

From (2.12) we see

$$\begin{aligned}\dot{\tilde{\gamma}}(s_0)&=i(aA\cdot\exp(ais_0)+bB\cdot\exp(ais_0)+dD\cdot\exp(dis_0))\\&=\exp(ais_0)\cdot i(aA+bB\cdot\exp((b-a)is_0)+dD\cdot\exp((d-a)is_0)\\&=\exp(ais_0)\cdot i(aA+bB+dD)=\exp(ais_0)\,\dot{\tilde{\gamma}}(0),\end{aligned}$$

that is, $X_{s_0}=X_0$. Next, we have

$$\begin{aligned}\ddot{\tilde{\gamma}}(s_0)+\tilde{\gamma}(s_0)&=-a^2A\cdot\exp(ais_0)-b^2B\cdot\exp(bis_0)-d^2D\cdot\exp(dis_0)\\&\quad+A\cdot\exp(ais_0)+B\cdot\exp(bis_0)+D\cdot\exp(dis_0)\\&=\exp(ais_0)\cdot(-a^2A-b^2B-d^2D+A+B+D)\\&=\exp(ais_0)\cdot(\ddot{\tilde{\gamma}}(0)+\tilde{\gamma}(0)).\end{aligned}$$

Hence $Y_{s_0}=Y_0$. Therefore we conclude that our circle is a simple closed curve and its prime period is the least common multiple of $2\pi/(b-a)$ and $2\pi/(d-a)$.

REMARKS.
(6) According to our proof "one of the three ratios a/b, b/d and d/a is rational" is equivalent to "each of a/b, b/d and d/a is rational". This implies that for a given a circle γ with complex torsion $-1<\tau<1$, "a horizontal lift $\tilde{\gamma}$ of the circle γ is closed" is equivalent to "the circle γ is closed". The period of $\tilde{\gamma}$ is the least common multiple of $2\pi/|a|$ and $2\pi/|d|$.
(7) The solution for (2.9) is as follows: $\lambda=\sqrt[3]{\alpha}+\sqrt[3]{\beta}$, $\omega\cdot\sqrt[3]{\alpha}+\omega^2\cdot\sqrt[3]{\beta}$ or $\omega^2\cdot\sqrt[3]{\alpha}+\omega\cdot\sqrt[3]{\beta}$, where $\omega=\exp(2\pi i/3)$ and α, $\beta=1/2\cdot(\tau ki\pm\sqrt{-\tau^2k^2+(4/27)(1+k^2)^3}$.

Let γ be a circle on a Riemannian manifold (M, g) with curvature k. When we change the metric homothetically $g\longrightarrow m^2\cdot g$ for some positive constant m, the curve $\sigma(s)=\gamma(s/m)$ is a circle on $(M, m^2\cdot g)$ with curvature k/m. Under the operation $g\longrightarrow m^2g$, the prime period of a closed curve changes to m-times of the original prime period. We can conclude the following

Theorem. *Let γ be a circle with curvature k and with complex torsion τ on a complex projective space $\boldsymbol{CP}^n(c)$ of holomorphic sectional curvature c. Then the following hold:*

(1) *When $\tau=0$, γ is a simple closed curve with prime period $4\pi/\sqrt{4k^2+c}$.*

(2) *When $\tau=\pm1$, γ is a simple closed curve with prime period $2\pi/\sqrt{k^2+c}$.*

(3) *When $\tau\neq0, \pm1$, we denote by a, b and d $(a<b<d)$ the nonzero solutions for*

$$c\lambda^3-(4k^2+c)\lambda+2\sqrt{c}\,k\tau=0.$$

Then we find the following:

(i) *If one of the three ratios a/b, b/d and d/a is rational, γ is a simple closed curve. Its prime period is the least common multiple of $4\pi/\sqrt{c}(b-a)$ and $4\pi/\sqrt{c}(d-a)$.*

(ii) *If each of the three ratios a/b, b/d and d/a is irrational, γ is a simple open curve.*

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