

Title	On one-sided QF-2 rings. II
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Citation	Osaka Journal of Mathematics. 1980, 17(2), p. 433-438
Version Type	VoR
URL	<a href="https://doi.org/10.18910/10992">https://doi.org/10.18910/10992</a>
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## ON ONE-SIDED QF-2 RINGS II

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(Received July 19, 1979)

We have studied the extending property on direct sums of indecomposable modules in [4]. We shall apply those results to projective modules and give characterizations of semi-perfect rings whose projective modules have the extending property of simple module. We shall deal with the dual concept of [5].

### 1. Preliminaries

Throughout this paper we shall denote a ring with identity by  $R$  and every  $R$ -module  $M$  is a right unitary  $R$ -module. By  $S(M)$  we denote the *socle* of  $M$ . We shall recall the definition of extending property of simple module. If for every simple submodule  $A_\alpha$  of  $S(M)$  there exists a direct summand  $M_\alpha$  of  $M$  such that  $S(M_\alpha) = A_\alpha$ , we say  $M$  have the *extending property of simple module*. Let  $\{N_\beta\}_I$  be a set of submodules of  $M$ . If  $\bigcap_{I_1} N_\gamma \supsetneq \bigcap_{I_2} N_\delta$  for subset  $I_1 \subsetneq I_2$ ,  $\bigcap_I N_\delta$  is called *irredundant*.

In this paper we shall study the dual properties to those in [5] and so we shall first introduce the dual condition to (\*\*\*) in [2] and [3].

(\*\*)\* *Every indecomposable projective module contains a unique minimal submodule and is uniform.*

If further every indecomposable left projective module contains a unique minimal submodule, we call  $R$  a QF-2 ring following Thrall [7]. Hence, if  $R$  satisfies (\*\*)\*, we call  $R$  a *right QF-2 ring* in this note.

Let  $M$  be an  $R$ -module. If  $M$  is a homomorphic image of projective module with non-essential kernel, we call  $M$  a *non-cosmall module* [3] and [6]. Every epimorphism onto non-cosmall module has the non-essential kernel [3]. We have dealt with conditions on non-small modules in [5]. We shall consider the dual or similar conditions to them.

(\*1)\* *Every non-cosmall module which is contained in a projective module contains a non-zero projective summand (dual to (\*1) in [5]).*

And

(\*\*2) For every finitely generated projective  $P$  with essential socle  $S(P)$ ,  $P/T$  contains a non-zero projective summand for any submodule  $T \subseteq S(P)$ .

They are weaker conditions than the following:

(\*)<sup>\*</sup> Every non-cosmall module contains a non-zero projective summand [4].

## 2. Right QF-2 rings

We are only interested in right QF-2 rings in this note and so from now on we always assume that  $R$  satisfies (\*\*)<sup>\*</sup> unless otherwise stated. Furthermore, we assume  $R$  is semi-perfect [1] and we shall denote the Jacobson radical of  $R$  and primitive idempotents by  $J$  and  $e$ , respectively. Let  $P$  be projective. Then  $P = \sum \oplus P_\alpha$ ; the  $P_\alpha$  is indecomposable. Hence,  $S(P)$  is essential in  $P$  by (\*\*)<sup>\*</sup> (see [8]).

**Lemma 1.** *Let  $R$  be a right QF-2 and semi-perfect ring and  $e$  a primitive idempotent. Let  $eR \supset eJ^n \cong eJ^{n+k}$  be projectives. Then  $eJ^n \cong eJ^{n+k}$  if  $J$  is nil or  $eR$  is injective.*

*Proof.* Since  $eJ^n$  is projective and  $S(eJ^n)$  is simple,  $eJ^n \cong fR$  for some idempotent  $f$ . If  $eJ^n \cong eJ^{n+k}$ ,  $fR \cong ff^k$ . This isomorphism is induced by an element in  $fff$ . If  $J$  is nil, we have a contradiction. If  $eR$  is injective, the isomorphism  $eJ^n \cong eJ^{n+k}$  is extended to one on  $eR$ . Hence,  $eJ^n = eJ^{n+k}$ , a contradiction.

**Theorem 1.** *Let  $R$  be a semi-perfect and right QF-2 ring with nil Jacobson radical. Then the following conditions are equivalent.*

- 1)  $R$  satisfies (\*1)<sup>\*</sup>.
- 2) Let  $\{P_\alpha\}_I$  be a set of direct summands of a projective  $P$  such that  $P = P_\alpha \oplus P_{\alpha'}$  and  $S(P_{\alpha'})$  is simple. If  $\bigcap_I S(P_\alpha)$  is irredundant,  $\bigcap_K P_\alpha$  is a direct summand of  $P$  for any finite subset  $K$  of  $I$ .
- 3) i) For some primitive idempotent  $e$ , there exists a positive integer  $t(e)$  such that  $eR/eJ^{t(e)}$  is a serial module,  $eB (= eJ^s, s \leq t(e))$  is projective for any  $eR \supset eB \supset eJ^{t(e)}$  and  $Z(eC) = eC$  and  $eC \subseteq eJ^{t(e)}$  for every non-projective right ideal  $eC$  in  $eR$ .  
 ii)  $\{eJ^s\}_{e, s=0}^{t(e)}$  is the representative set of indecomposable projectives, where  $Z(\ )$  means the singular submodule (dual to [5], Theorem 2).

*Proof.* 1)  $\rightarrow$  2). Let  $K = \{1, 2, \dots, n\}$  be a finite subset of  $I$  and put  $P(n) = \bigcap_{i=1}^n P_i$ . We shall show  $P(n)$  is a direct summand of  $P$  by the induction on  $n$ . If  $n=1$ , it is clear by the assumption. Put  $P = P_n \oplus P_n'$  with  $P_n'$  indecomposable and  $\pi_n: P \rightarrow P_n'$  the projection. We note  $S(\bigcap P_\alpha) = \bigcap S(P_\alpha)$ . Since

$S(P(n-1)) = \bigcap_{i=1}^{n-1} S(P_i) \not\subseteq S(P_n)$ ,  $\pi_n(S(P(n-1))) \neq 0$ . Hence,  $\pi_n(P(n-1))$  is non-cosmall module in  $P_n'$ . Then there exists an indecomposable summand  $P_0$  of  $\pi_n(P(n-1))$  by 1). Since  $S(P_n')$  is simple,  $\pi_n(P(n-1)) = P_0$ . Therefore,  $P(n-1) = P_0' \oplus \ker \pi_n | P(n-1) = P_0' \oplus P(n)$ , where  $P_0' \approx P_0$ . Since  $P = P(n-1) \oplus P'$ ,  $P(n)$  is a direct summand of  $P$ .

2)  $\rightarrow$  3). Let  $e$  be a primitive idempotent. We assume  $eA$  is projective and  $eB (\subset eA)$  is non-cosmall for right ideals  $eA$  and  $eB$ . Then there exists a projective module  $P$  such that  $0 \leftarrow eB \xleftarrow{f} P \leftarrow K \leftarrow 0$  is exact and  $S(P) \not\subseteq K$  by the definition (see [3], Proposition 3.1). If  $S(P)$  is simple,  $K=0$  and  $eB$  is projective. We assume  $P = P_1 \oplus \sum_{\alpha} P_{\alpha}$  such that the  $P_{\alpha}$  is indecomposable and  $S(P_1)$  is a simple module not contained in  $K$ . We put  $Q = P \oplus eA$  and  $P' = \{x + f(x) | x \in P\} \subset Q$ . Then  $S(P') = (S(P) \cap K) \oplus S((1+f)(P_1))$  and  $S(P) = S(P_1) \oplus (S(P) \cap K)$ . Since  $S(P) \cap S(P')$  is irredundant,  $P \cap P' = K$  is a direct summand of  $Q$  and hence of  $P$ . Accordingly,  $eB$  is projective. Now if  $eJ$  is non-cosmall,  $eJ$  is projective from the above. Hence,  $eJ$  contains a unique maximal submodule  $eJ^2$ , since  $eJ$  is indecomposable by (\*\*)\*. Repeating those arguments, we obtain a unique chain  $eR \supset eJ \supset eJ^2 \supset \dots \supset eJ^t$  of projectives and  $eB$  is cosmall for any  $eB \subseteq eJ^t$  by Lemma 1. Hence,  $eB = Z(eB)$  by [3], Proposition 3.2. The remaining part is clear from the construction of  $eJ^t$ .

3)  $\rightarrow$  1). Let  $P$  be a projective module which contains a non-cosmall module  $M$ . Then  $P = \sum \oplus e_i J^{t_{ij}}$ . Let  $\pi_{ij}: P \rightarrow e_i J^{t_{ij}}$  be the projection. Since  $M \neq Z(M)$ ,  $\pi_{kl}(M) \not\subseteq Z(e_k J^{t_{kl}}) \subseteq e_k R$  for some  $k, l$ . Hence,  $\pi_{kl}(M)$  is projective and so  $M = \ker \pi_{kl} | M \oplus M'$ ;  $M' \approx \pi_{kl}(M)$ .

**Corollary.** *Let  $R$  be semi-perfect. Then  $R$  satisfies (\*)<sup>\*</sup> if and only if  $R$  is right QF-2 and QF-3 and satisfies (\*1)<sup>\*</sup>.*

Proof. In the above proof the implication 1)  $\rightarrow$  2) is valid without the assumption on  $J$ . Hence, we obtain the corollary by the implication 2)  $\rightarrow$  3), Lemma 1 and [3], Theorems 1.3 and 3.6.

As the dual to Theorem 2' in [5] we have

**Theorem 1'.** *Let  $R$  be as before. Then the following conditions are equivalent.*

- 1)  $R$  is right hereditary.
- 2) Let  $P$  be projective and  $P_i$  direct summands of  $P$  for  $i=1, 2$ . Then  $P_1 \cap P_2$  is a direct summand of  $P$ .
- 3) i) For some primitive idempotent  $e$ ,  $eR$  is uni-serial and  $eB$  is projective for any right ideal  $eB \subseteq eR$ . ii)  $\{eB\}_{e,B}$  is the representative set of indecomposable projectives.

*In this case  $R$  is right artinian.*

Proof. 1)  $\rightarrow$  2). We can use the same argument as before.

2)  $\rightarrow$  1). Let  $P$  be projective and  $A$  a submodule of  $P$ . Let  $P_1 \xrightarrow{f} A \rightarrow 0$  be an exact sequence with  $P_1$  projective. We put  $F = P_1 \oplus P$  and  $P'_1 = \{x + f(x) \mid x \in P_1\}$ . Then  $F = P'_1 \oplus P$  and so  $K = \ker f = P_1 \cap P'_1$  is a direct summand of  $F$ . Hence,  $K$  is a direct summand of  $P_1$ . Therefore,  $A$  is projective and  $R$  is hereditary.

1)  $\rightarrow$  3). It is clear from Theorem 1.

3)  $\rightarrow$  1). We know from 3) that  $R$  is right artinian and  $Z(R) = 0$ . Hence, every right ideal  $A$  contains a projective summand by Theorem 1. Since  $R$  is noetherian,  $A$  is projective.

**Theorem 2.** *Let  $R$  be a right QF-2 and semi-perfect ring. Then the following conditions are equivalent.*

- 1)  $R$  satisfies  $(**2)$ .
- 2) Every projective module has the extending property of simple module.
- 3) i) For some primitive idempotent  $e$  there exists a chain of projective right ideals  $eA_i$  such that  $eR = eA_1 \supset eA_2 \supset \dots \supset eA_t$  and  $\text{Hom}_R(S(eA_i), S(eA_j))$  is extended to  $\text{Hom}_R(eA_i, eA_j)$  for any pair  $i \geq j$ , (see [4], Theorem 2).
- ii)  $\{eA_i\}_{e,i}$  is the representative set of indecomposable projective such that  $S(eR) \cong S(e'R)$  if  $e \neq e'$ .

Proof. 1)  $\rightarrow$  2). Let  $P$  be projective and  $P = \sum_I \oplus P_\alpha$ ; the  $P_\alpha$  is uniform. Let  $S$  be a simple submodule of  $S(P)$ . Then there exists a finite subset  $K = \{1, 2, \dots, n\}$  of  $I$  such that  $S \subset S(\sum_K \oplus P_i)$ . If  $n = 1$ , it is clear. Hence, we assume  $S \subsetneq S(\sum_K \oplus P_i)$  and put  $P(n) = \sum_{i=1}^n \oplus P_i$ . Then  $P^{(n)}/S = P_0 \oplus Q$  and  $P_0$  is projective by 1). Considering an epimorphism  $P^{(n)} \rightarrow P/S \rightarrow P_0$ , we obtain  $P^{(n)} = P'_0 \oplus L$ ;  $P'_0 \cong P_0$  and  $L \supset S$ . Since  $L = \sum_{i=1}^{n-1} \oplus P_i'$ , we can use the induction argument.

2)  $\rightarrow$  3). Let  $eR$  and  $fR$  be uniform projectives with isomorphic socle. Then there exists a monomorphism  $f: eR \rightarrow fR$  (or  $fR \rightarrow eR$ ) by [4], Corollary 8, i.e.  $eR <^* fR$  or  $fR <^* eR$  (see [4]). Let  $eR$  be a maximal one among uniform projectives  $P$  with isomorphic socle with respect to the relation  $<^*$ . Then those  $P$  are isomorphic to right ideals  $eA$  in  $eR$ . Since the relation  $<^*$  is linear on  $\{eA\}$ , taking repeatedly maximal ones, we get a chain of projective right ideals  $eR = eA_1 \supset eA_2 \supset \dots \supset eA_t$ . The second condition is clear by [4], Corollary 8.

3)  $\rightarrow$  2). It is clear from [4], Corollary 8.

2)  $\rightarrow$  1). Let  $P = P_1 \oplus P_2 \oplus \dots \oplus P_n$  be projective and the  $P_i$  uniform. Let  $T \subsetneq$

$S(P)$  and  $T=S_1\oplus S_2\oplus\cdots\oplus S_i$ ; the  $S_j$  is simple. Then there exists a direct summand  $P_1'$  of  $P$  such that  $S(P_1')=S_1$ . Let  $P=P_1'\oplus K_1$ . Then  $T=S_1\oplus \pi_1(T)$ ;  $\pi_1: P\rightarrow K_1$ . Hence,  $S(K_1)\cong \pi_1(T)$  and  $P/T\approx P_1'/S_1\oplus K_1/\pi_1(T)$ . Repeating the same argument on  $K_1/\pi_1(T)$ , finally we obtain  $P/T\approx P_1'/S_1\oplus\cdots\oplus P_i'/S_i'\oplus K_i$  and  $K_i$  is projective, since  $\pi_j(T)=0$  for some  $j\leq n$ .

**3. Corollaries and examples**

We shall consider some special cases of rings.

**Corollary 1.** *If  $R$  is a right QF-2 and semi-perfect ring with  $Z(R)\supset J$ , then  $R$  satisfies  $(*1)^*$ .*

Proof. It is clear from the proof of the implication  $3)\rightarrow 1)$  in Theorem 1.

**Corollary 2.** *If  $R$  is a right QF-2 and semi-perfect ring with  $J^2=0$ , then  $R$  satisfies  $(*1)^*$ .*

Proof. Let  $R=\sum\oplus e_iR\oplus\sum\oplus f_jR$ , where the  $e_i$  and the  $f_j$  are primitive and the  $f_jR$  is simple. Then  $S(R)=\sum\oplus e_iJ\oplus\sum\oplus f_jR$ . If  $e_iJf_j\neq 0$ ,  $e_iJ\approx f_jR$ . Hence,  $e_iJ=Z(e_iJ)$  or  $e_iJ$  is projective. Accordingly,  $R$  satisfies  $(*1)^*$  by Theorem 1.

**Corollary 3.** *Let  $R$  be a right QF-2 and semi-perfect ring with nil Jacobson radical. Then  $Z(R)=0$  and  $(*1)^*$  is satisfied if and only if  $R$  is a right generalized uniserial and right artinian hereditary ring.*

Proof. It is clear from Theorem 1.

EXAMPLES 1. Let  $K\subset L$  be fields and put

$$R = \begin{pmatrix} K & 0 & L \\ 0 & L & L \\ 0 & 0 & L \end{pmatrix}.$$

Then  $R$  is a right QF-2 and hereditary artinian ring. Hence,  $R$  satisfies  $(*1)^*$ . If  $[L:K]=\infty$ ,  $R$  is not left artinian and does not satisfy  $(**2)$ .

2. Let  $C=K\oplus M$ ;  $M=K$ , be the trivial extension and put

$$R = \begin{pmatrix} C & C \\ 0 & C \end{pmatrix} \text{ ([5], Example 2).}$$

Then  $R$  is QF-2 and  $e_{11}R$  is injective and projective. Hence,  $R$  satisfies  $(**2)$  by Theorem 2. Put  $P=e_1R\oplus e_1R\oplus e_2R$ , where  $e_i=e_{ii}$ . We have a homomorphism  $e_1R$  to  $e_1R$  by a multiplication of  $m(m\in M)$  from the left side and a

monomorphism  $\rho$  of  $e_2R$  into  $e_1R$ . We take an epimorphism

$$(1, m, \rho): P \rightarrow e_1R.$$

Then its kernel  $N_1 = \{(x, y, z) \in P, \lambda + my + \rho(z) = 0\}$  is a direct summand of  $P$ . Put  $N_2 = \{(0, y, z) \in P\}$  and  $N_3 = \{(x, 0, z) \in P\}$ . Then  $N_1 \cap N_2 \cap N_3 = 0$ . However,  $N_1 \cap N_2 = \{(0, 0), (a, b), (0, mb) \mid a \in M, b \in C\} \approx e_1J$  is not projective. Hence,  $R$  does not satisfy (\* 1).

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