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Contribution to the problem of stability

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I. INTRODUCTION

1. H. Hopf and E. Pannwitz[1]) introduced the notion of *stability*[2]) and raised several questions, the most important of which reads: "Can we characterize the stability in terms of the homology theory ?".

Confining themselves to the homogeneous n-complex K^n, they obtained the following theorems:

THEOREM *A*. A linear graph K^1 is stable if and only if it has no free side.

THEOREM *B*. A cyclic[3]) complex K^n is stable for any dimension n.

THEOREM *C*. For $n \geqq 3$ a stable complex K^n is cyclic, provided that it is simply connected.

Recently Professor A. Komatu has reasonably conjectured that to these theorems can be given the following complete and unified form:

THEOREM *D*. For a locally finite homogeneous n-complex K^n $(n \neq 2)$ stability is equivalent to the cyclicity in the sense of local coefficients[4]).

The main purpose of this paper is to prove THEOREM D by generalizing the Hopf-Pannwitz's lemmas based on ordinary coefficients to those based on local coefficients.

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1) H. Hopf and E. Pannwitz, Über stetige Deformationen von Komplexen in sich, Math. Ann., 108, 1932.

2) A topological space R is called stable if for every deformation f_t of R through itself no point of R can get rid of the covering by the image $f_1(R)$, or more simply, if R can never be deformed into its proper subspace.

3) An n-complex K^n is called cyclic, if each n-simplex σ_i^n is contained in at least one n-cycle with suitable coefficients.

4) See, IV. § 8.

In CHAPTER II preliminary considerations will be made concerning the theory of local coefficients in a form convenient for our purpose.

In CHAPTER III a certain cocyle $v^n(f_t)$ which plays a fundamental role in our arguments and which essentially gives the *degree of mapping* in local coefficients and some of its properties will be investigated.

In CHAPTER IV the proof of the THEOREM D will be given.

The THEOREM D for $n=2$ is difficult and the problem still remains open.

II. PRELIMINARIES FROM THE HOMOLOGY THEORY WITH LOCAL COEFFICIENTS

2. Let K^n be a locally finite homogeneous n-complex. The r-simplexes of K^n will be denoted by σ_i^r $(i=0, 1, 2, \ldots.)$.

We choose a point q_0 in σ_0^n [5]) and take this point as the origin of the fundamental group $G=\{x_j\}$ $(j=0, 1, 2, \ldots..)$ of K^n, where $x_0=e=$ the unit element of G. We choose further for each point p of K^n a definite path $[p]$ from q_0 to p in such a way, that for any two points p_1, p_2 of a simplex σ_i^r the closed path [6]) $\langle p_1p_2\rangle=[p_1]\,\overline{p_1p_2}\,[p_2]^{-1}$ is homotopic to 0.

We then define the universal covering complex $\tilde{K}^n$ of K^n as usual by means of the above paths and the fundamental group G. The simplex of $\tilde{K}^n$ over σ_i^r determined by all $[p]$'s such that $p\in\sigma_i^r$ will be denoted by σ_{0i}^r, and the one determined by $\langle p\rangle$'s such that $p\in\sigma_i^r$ and $\langle p\rangle=x_j[p]$ by $\sigma_{ij}^r=x_j\sigma_{0i}^r$, where x_j is a given element of G. Then $\tilde{K}^n$ becomes a complex with automorphisms G in the sense of B. Eckmann [7]).

Concerning incidence numbers on $\tilde{K}^n$ we have

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5) σ_i^r is open.

6) Generally we use this notation $\langle p_1\,p_2\rangle$ when p_1, p_2 are contained *in the closure of a single simplex* to denote the closed path $[p_1]\,\overline{p_1p_2}\,[p_2]^{-1}$. By the conditions for $[p]$ $\langle p_1p_2\rangle$ belongs to a definite homotopy class independent of the choice of p_1, p_2, as long as p_1, p_2 belong to definite simplexes σ_h^s, σ_k^t respectively (where, of course, σ_h^s, σ_k^t are incident to a single simplex σ_i^r). Therefore we often use this notation $\langle p_1\,p_2\rangle$ not only to denote the curve itself but also to denote the homotopy class (i.e., the element of the fundamental group G) determined by the curve.

7) B. Eckmann. Proc. Nat. Acad. 33. 9., 1947.

$$[\overset{r}{\sigma}_{ji} : \overset{r-1}{\sigma}_{kh}] = [x_j \overset{r}{\sigma}_{0i} : x_k \overset{r-1}{\sigma}_{0h}] = [x\, x_j \overset{r}{\sigma}_{0i} : x\, x_k \overset{r-1}{\sigma}_{0h}]$$

$$= [x_k^{-1} x_j \overset{r}{\sigma}_{0i} : \overset{r-1}{\sigma}_{0h}] = [\overset{r}{\sigma}_{0i} : x_j^{-1} x_k \overset{r-1}{\sigma}_{0h}] \text{ where } x \varepsilon G .$$

Further we have, concerning the relations between incidence numbers on $\tilde{K}^n$ and K^n, $[\overset{r}{\sigma}_{0i} : x_j \overset{r-1}{\sigma}_{0k}] \neq 0$ if and only if $[\overset{r}{\sigma}_i \ \overset{r-1}{\sigma}_k] \neq 0$ and $x_j = \langle p_1 p_2 \rangle$ where $p_1 \varepsilon \overset{r}{\sigma}_i$, $p_2 \varepsilon \overset{r-1}{\sigma}_k$.

3. Given a homorphism $\varphi : G \rightarrow A(J)$ of the fundamantal group G into the group of automorphisms of a coefficient group J we define $\partial^\varphi (\delta^\varphi)$ as follows:

$$\partial^\varphi (\alpha \overset{r}{\sigma}_i) = \sum_k [\overset{r}{\sigma}_i : \overset{r-1}{\sigma}_k] (\alpha\, x_{ik}) \overset{r-1}{\sigma}_k \text{ [8]}$$

$$(\delta^\varphi (\alpha \overset{r}{\sigma}_i) = \sum_k [\overset{r}{\sigma}_i : \overset{r+1}{\sigma}_k] (\alpha\, x_{ik}) \overset{r+1}{\sigma}_k), \quad \text{where} \quad \alpha \varepsilon J , \ x_{ik} = \langle p_1 p_2 \rangle ,$$

$p_1 \varepsilon \overset{r}{\sigma}_i$, $p_2 \varepsilon \overset{r-1}{\sigma}_k$ $(p_2 \varepsilon \overset{r+1}{\sigma}_k)$.

Then $\partial^\varphi (\delta^\varphi)$ has the property $\partial^\varphi \partial^\varphi = 0$ $(\delta^\varphi \delta^\varphi = 0)$, and the homology (cohomology) theory can be developed with $\partial^\varphi (\delta^\varphi)$ as boundary (coboundary) operator in the usual way. The corresponding r-dimensional infinite chain group, cycle group, boundary group (finite cochain group, cocycle group, coboundary group) of K^n will be denoted by $C_r(K^n, J^\varphi)$, $Z_r(K^n, J^\varphi)$, $B_r(K^n, J^\varphi)$, $(\mathfrak{C}^r(K^n, J^\varphi)$, $\mathfrak{Z}^r(K^n, J^\varphi)$, $\mathfrak{B}^r(K^n, J^\varphi))$ respectively.

If φ is a zero homomorphism, $\partial^\varphi (\delta^\varphi)$ reduces to the ordinary boundary (coboundary) operator ∂ (δ), and we shall omit the symbol φ in such cases.

We recall here the duality between homology and cohomology.

LEMMA 3.1. The groups $C_r(K^n, \mathfrak{R}_1)$ and $\mathfrak{C}^r(K^n, I)$[9] are character groups of each other. And the annihilators of the groups $Z_r(K^n, \mathfrak{R}_1)$, $B_r(K^n, \mathfrak{R}_1)$, $\mathfrak{Z}^r(K^n, I)$ and $\mathfrak{B}^r(K^n, I)$ are respectively the groups $\mathfrak{B}^r(K^n, I)$, $\mathfrak{Z}^r(K^n, I)$, $B_r(K^n, \mathfrak{R}_1)$ and $Z_r(K^n, \mathfrak{R}_1)$.

4. Given an abelian group L, we consider the group ring $L \circ G$ of G with coefficients in L. In $L \circ G$ the right multiplication of an element x $(x \varepsilon G)$ can be regarded as an automorphism of $L \circ G$. Thus we obtain a homomorphism $\psi^\circ : G \rightarrow A(L \circ G)$. An element of $L \circ G$

8) $(\alpha\, x_{ik})$ is the element of J obtained from α as the result of the automorphism x_{ik}.
9) I = group of integers; $\mathfrak{R}_1$ = group of real numbers mod 1.

has the form $\sum_i \alpha_i x_i$, where $\alpha_i \varepsilon L$ and the summation is finite. If we remove the last restriction of finiteness, we obtain a group $L * G$ and a homomorphism $\psi^* : G \to A(L * G)$.

LEMMA 4. 1.[10] The homology (cohomology) theory on $\widetilde{K}^n$ with coefficients in L is identical with the homology (cohomology) theory with local coefficients in $(L * G)^{\psi^*}$ $(L \circ G)^{\psi^0})$.

Proof: In order to prove the statement for homology it is sufficient to verifiy the identity $\underline{\kappa}\,\partial = \partial\,\underline{\kappa}$, where $\underline{\kappa}$ is the isomorphism between $C_r(\widetilde{K}^n, L)$ and $C_r(K^n, L * G)$,[11] determined by

$$\underline{\kappa} : \sum_{ij} \alpha^{ij} \overset{r}{\sigma}_{ji} \longleftrightarrow \sum_i (\sum_j \alpha^{ij} x_j) \overset{r}{\sigma}_i .$$

$$\underline{\kappa}\,\partial\ (\sum_{ij} \alpha^{ij} \overset{r}{\sigma}_{ji}) = \underline{\kappa} \sum_{ijhk} \alpha^{ij} [\overset{r}{\sigma}_{ji} : \overset{r-1}{\sigma}_{kh}]\, \overset{r-1}{\sigma}_{kh}$$

$$= \underline{\kappa} \sum_{ijhk} \alpha^{ij} [\overset{r}{\sigma}_{0i} : x_j^{-1} x_k \overset{r-1}{\sigma}_{0h}]\, x_k \overset{r-1}{\sigma}_{0h}$$

$$= \underline{\kappa} \sum_i \sum_{x_{ih}=x_j^{-1}x_k} \alpha^{ij} [\overset{r}{\sigma}_i : \overset{r-1}{\sigma}_h]\, x_k \overset{r-1}{\sigma}_{0h} = \sum_i \sum_h [\overset{r}{\sigma}_i : \overset{r-1}{\sigma}_h] (\sum_j \alpha^{ij} x_j)\, x_{ih} \overset{r-1}{\sigma}_h)$$

$$= \partial (\sum_i (\sum_j \alpha^{ij} x_j)\, \overset{r}{\sigma}_i) = \partial\,\underline{\kappa} (\sum_{ij} \alpha^{ij} \overset{r}{\sigma}_{ji}) .$$

5. Let $v^r = \sum_i t^i \overset{r}{\sigma}_i$ be an element of $\mathfrak{C}^r(K^n, I \circ G)$ and let $c^r = \sum s^i \overset{r}{\sigma}_i$ be an element of $C_r(K^n, J^\varphi)$. We define the Kronecker index of c^r with v^r by

$KI\ (c^r, v^r) = \sum_{ij} n^{ij} (s^i x_j^{-1})$ [8] where $t^i = \sum_j n^{ij} x_j$.

LEMMA 5. 1. $KI(\partial^\varphi c^r, v^{r-1}) = KI(c^r, \delta^\varphi v^{r-1})$.

LEMMA 5. 2. Let $v_1{}^r, v_2{}^r \varepsilon\, \mathfrak{Z}^r(K^n, I \circ G)$, and $z^r \varepsilon\, Z_r(K^n, J^\varphi)$. If $\overset{r}{v}_1 \backsim \overset{r}{v}_2$[12], then $KI\ (z^r, \overset{r}{v}_1) = KI\ (z^r, \overset{r}{v}_2)$.

We need further the

LEMMA 5. 4. Let $v^r = \sum_{ij} (n^{ij} x_j) \overset{r}{\sigma}_i$. If for every $z^r = \sum s^i \overset{r}{\sigma}_i$

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[10] This lemma is due to Prof. A. Komatu.

[11] We shall always omit the symbol $\psi^*(\psi^0)$ in $(L * G)^{\psi^*}$ $((L \circ G)^{\psi^0})$.

[12] $\backsim$ = "is cohomologous to".

$= \sum_i (\sum_j \rho^{ij} x_j) \sigma_i^r$ from $Z_r(K^n, \Re_1 * G)$ $KI(z^r, v^r) = 0$, then $v^r \backsim 0$.

Proof: $KI(z^r, v^r) = \sum_{ij} n^{ij} (\sum_h \rho^{ih} x_h) x_j^{-1} = \sum_{jh} (\sum_i n^{ij} \rho^{ih}) x_h x_j^{-1}$

$= \sum_k \sum_{x_h x_j^{-1} = x_k} (\sum_i n^{ij} \rho^{ih}) x_k = 0$, hence $\sum_{x_h x_j^{-1} = e,\, i} n^{ij} \rho^{ih} = \sum_{ij} n^{ij} \rho^{ij} = 0$.

Therefore

$$KI(\underline{\kappa}^{-1} z^r, \bar{\kappa}^{-1} v^r) \text{ [13]} = \sum_{ij} n^{ij} \rho^{ij} = 0 .$$

By LEMMA 4.1 $\underline{\kappa}^{-1} z^r$ can be any infinite cycle from $Z_r(\tilde{K}^n, \Re_1)$, and consequently by LEMMA 3.1 $\bar{\kappa}^{-1} v^r \varepsilon \mathfrak{B}^r(\tilde{K}^n, I)$. Applying LEMMA 4.1 again we conclude that $v^r \varepsilon \mathfrak{B}^r(K^n, I \circ G)$.

III. DEFORMATIONS

6. By a deformation f_t of a topological space R we mean a continuous mapping of the product-space $R \times \langle 0, 1 \rangle$ into R subject to the conditions that for each $t\,(0 \leq t \leq 1)$ and for each $p \varepsilon R$ the inverse image $\overset{-1}{f_t}(p)$ is compact and $\overset{-1}{f_0}(p) = p$.

7. Let K^n be as in § 2-5. We say that a deformation f_t is q_0-regular if for every i the image $f_1(\partial \sigma_i^n)$ does not contain q_0, and that two q_0-regular deformations f_t and g_t are q_0-equivalent if there exists a homotopy $h_t(p)$, $0 \leq t \leq 1$, $p \varepsilon K^n$ and $h_t(p) \varepsilon K^n$, such that $h_0 = f_1$, $h_1 = g_1$, and $h_t(\partial \sigma_i^n)$ never contains q_0.

LEMMA 7.1. For any deformation f_t of K^n we can find a (uniquely determined) deformation $\tilde{f}_t$ of the universal covering complex $\tilde{K}^n$ such that

$$\pi \tilde{f}_t = f_t \pi . \qquad \text{[14]}$$

It is clear that if f_t is q_0-regular, then $\tilde{f}_t$ is $\tilde{q}_0$-regular, where $\tilde{q}_0$ is the point over q_0 belonging to σ_{00}^n. Hence the degree n^{ij} of $f_1(x_j \sigma_{i0}^n)$ at $\tilde{q}_0$ is well defined. We put $t^i = \sum_j n^{ij} x_j$ and define an element $v^n =$

[13] $\bar{\kappa}$ is analogously defined as $\underline{\kappa}$.

[14] π is the projection of $\tilde{K}^n$ on K^n. For the proof of the LEMMA, see, A. Komatu, Zur Topologie der Abbildungen von Komplexen, Jap. Journ. Math., vol. XVII, 1941, Satz 1.4.

$v^n(f_t) = \sum_i t^i \overset{n}{\sigma}_i \varepsilon \mathfrak{C}^n(K^n, I \circ G)$ [15] $= \mathfrak{Z}^n(K^n, I \circ G)$. Evidently if two q_0-regular deformations f_t, g_t are q_0-equivalent then $v^n(f_t) = v^n(g_t)$.

LEMMA 7.2. For every q_0-regular deformation f_t the corresponding $v^n(f_t)$ belongs to a definite cohomology class independent of the special f_t.

Proof : For any $z^n = \sum \rho^{ij} \overset{n}{\sigma}_{ji} \varepsilon Z_n(\widetilde{K}^n, \mathfrak{R}_1)$, $KI(z^n, \bar{\kappa}^{-1} v^n)$ is the degree of $\widetilde{f}_1(z^n)$ at $\widetilde{q_0}$. We know that the degree of mapping of an ordinary cycle is invariant under deformation.

Consequently we have

$$KI(z^n, \bar{\kappa}^{-1} v^n(f_t) - \bar{\kappa}^{-1} v^n(g_t)) = 0 .$$

By LEMMA 3.1 this implies that

$$\bar{\kappa}^{-1} v^n(f_t) - \bar{\kappa}^{-1} v^n(g_t) \;\varepsilon\; \mathfrak{B}^n(\widetilde{K}^n, I) .$$

Hence by LEMMA 4.1

$$v^n(f_t) - v^n(g_t) \;\varepsilon\; \mathfrak{B}^n(K^n, I \circ G) .$$

LEMMA 7.3. If $v^n \infty v^n(f_t)$, there exists a deformation g_t such that $v^n = v^n(g_t)$, $(n > 1)$.

Proof: We may assume that f_1 is a simplicial transformation of a subdivision K_1^n of K^n into K^n. Following Hopf [16]) we take a point p on $\overset{n-1}{\sigma_j}$ and join the image point $f_1(p)$ to a point q of $\sigma_0{}^n$ different from q_0 by a curve w disjoint from q_0 such that $\overline{q_0 q}\, w^{-1} \widehat{f_1(p)\, p}\, [p]^{-1} = x_k$, where $\widehat{f_1(p)\, p}$ is the inverse path of $f_t(p)$ $(0 \leq t \leq 1)$,

We want to modify f_t within a small neighborhood of p so as to obtain a deformtion g_t such that

$$v^n(g_t) - v^n(f_t) = \delta(x_k\, \sigma_j{}^{n-1}) .$$

Before we carry out this modification, we make several conventions about notations.

Let K_ν^n $(\nu = 2, 3, 4, 5)$ be successive subdivisions of K_1^n, and let

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15) Since $f_1^{-1}(p_0)$ is compact there are at most a finite number of n^{ij} which are different from zero.

16) H. Hopf, Über wesentliche Abbildungen von Komplexen, Recueil Math., vol. 37, 1930, Beweis von Satz IIIa.

$s_\nu(p)$ $(\nu=2,3,4,5)$ be the stars of p corresponding to these subdivisions. We suppose that the subdivision K_1^n be sufficiently fine, so that every point of the image $f_1(s_2(p))$ can be joined to $f_1(p)$ by a segment and that q does not lie on any of these segments. We suppose further that the stars $s_\nu(p)$ are all similar and similarly placed with p as center of similarity.

Given a point p' in $s_\nu(p)-\overline{s_{\nu+1}(p)}$ $(\nu=2,3,4)$ we denote by $\lambda_\nu(p')$ the ratio $\overline{p_\nu p'}:\overline{p_\nu p_{\nu+1}}$ where p_ν is the point of intersection of $\overline{pp'}$ with $\partial s_\nu(p)$.

Let us now return to the original question. Our modification shall be achieved after the following three steps.

FIRST STEP : Let $\phi_t(p')$ $(1\leq t\leq 2)$ be a deformation of K^n under which each point p' of K^n moves along a segment at a constant velocity to $\phi_2(p')$, where $\phi_2(p')$ is given by:

$$\phi_2(p')=p' \qquad \text{for} \qquad p'\in K^n-s_2(p),$$

$$\phi_2(p')=p \qquad \text{for} \qquad p'\in\overline{s_3(p)},$$

$$\phi_2(p')=\text{the point } p'' \text{ on } \overline{pp'} \text{ such that}$$

$$\overline{p_2p''}=\overline{p_2p}\,\lambda_2(p').$$

We define $f_t(p')$ for $(1\leq t\leq 2)$ by

$$f_t(p')=f_1(\phi_t(p'))\quad(1\leq t\leq 2).$$

Then we have

$$f_2(s_3(p))=f_1(p).$$

SECOND STEP : Let the curve w be given with respect to the parameter t $(2\leq t\leq 3)$ by $w(t)$.

We define $f_t(p')$ $(2\leq t\leq 3)$ as follows:

$$f_t(p')=f_2(p') \qquad \text{for} \quad p'\in K^n-s_3(p),$$

$$f_t(p')=w(t) \qquad \text{for} \quad p'\in\overline{s_4(p)},$$

$$f_t(p')=w(2+\lambda_3(p')(t-2)) \qquad \text{for} \quad p'\in s_3(p)-\overline{s_4(p)}.$$

Then we have

$$f_3(s_4(p)) = w(3) = q .$$

THIRD STEP : Let τ^n be a subsimplex of $\sigma_0{}^n$ containing q_0 and having q as a vertex, and let the boundary simplex of τ^n opposite to q be τ^{n-1}. We suppose that τ^n is given the orientation induced from the one of $\sigma_0{}^n$, and that τ^{n-1} is oriented in such a way that

$$\partial \tau^n = +\tau^{n-1} + \ldots\ldots .$$

Now we consider $s_5(p)$. Let t^{n-1} be the $(n-1)$-simplex which is the common part of σ_j^{n-1} with $s_5(p)$, and $t_i{}^n$ the n-simplex which is the common part of $\sigma_i{}^n$ and $s_5(p)$. Let further e^i be the vertex of $t_i{}^n$ in $\sigma_i{}^n$.

We map the simplex $t_i{}^n$ affinely on τ^n in such a way that the point e^i is mapped into q and the simplex t^{n-1} is mapped positively on τ^{n-1}, where the positive orientation of t^{n-1} is the one induced from σ_j^{n-1}. We name this mapping $f_4(p')$, and extend this outside of $s_5(p)$ requiring:

$$f_4(p') = f_3(p') \qquad \text{for} \quad p' \varepsilon K^n - s_4(p),$$

$$f_4(p') = \text{ the point } p'' \text{ on } \overline{f_4(p_4) f_4(p_5)} \text{ such that}$$

$$\overline{f_4(p_4)\, p''} \ (= \overline{q\, p''}) = \lambda_4(p')\ \overline{f_4(p_4) f_4(p_5)}, \quad \text{for } p' \varepsilon s_4(p) - \overline{s_5(p)} .$$

We then define $f_t(p')$ for $(3 \leq t \leq 4)$ by

$f_t(p') = f_3(p') = f_4(p')$ for $p' \varepsilon K^n - s_4(p)$, $f_t(p') =$ the point p'' on $\overline{q\, f_4(p')}$ such that

$$\overline{p\, q''} = t\, \overline{q\, f_4(p')} \quad \text{for} \quad p' \varepsilon s_4(p) .$$

Thus we obtain a deformation $f_t(p)$ $(0 \leq t \leq 4)$ which is a modification of $f_t(p)$ $(0 \leq t \leq 4)$ within a small neighborhood of p.

We shall show that this deformtion $f_t(p)$ $(0 \leq t \leq 4)$ (rewritten as $g_t(p)$ $(0 \leq t \leq 1)$) is the one with the required property.

Consider the covering deformation $\tilde{g}_t$ of g_t. Then $\tilde{g}_t$ is the modification of $\tilde{f}_t$ within a neighborhood of the inverse image $\pi^{-1}(p)$, but, as can be easily seen, it has no influence upon the values of $n^{ih}(f_t)$ except at the point $x_k \tilde{p}$. As for the influence at $x_k \tilde{p}$ Hopf's calculation shows that the difference $n^{ih}(g) - n^{ih}(f) =$ the incidence number $[\sigma^n_{hi} : \sigma^{n-1}_{kj}]$. From this, after a simple calculation, we obtain

$$v^n(g) - v^n(f) = \delta(x_k \sigma_j^{n-1}).$$

Now let $v^n \infty v^n(f_t)$, then there exists an $(n-1)$-cochain

$$\sum m^{jk} x_k \sigma_j^{n-1} \quad \text{such that} \quad v^n - v^n(f_t) = \delta(\sum m^{jk} x_k \sigma_j^{n-1}).$$

Repeating the above process for $x_k \sigma_j^{n-1}$'s successively we can conclude that there exists a deformaion g_t such that

$$v^n(g_t) - v^n(f_t) = \delta(\sum m^{jk} x_k \sigma_j^{n-1}).$$

Comparing the last two identities we obtain

$$v^n(g_t) = v^n .$$

III. STABILITY

8. Let K^n be as in § 2-7. K^n is called $*$-cyclic if each n-simplex σ_i^n is contained in an n-cycle (finite or infinite) with suitable local coefficients.

THEOREM B'. A $*$-cyclic complex K^n is stable for any dimension n.

Proof: If K^n were not stable, then there exists a deformation f_t such that a point of K^n (for example q_0) is not contained in $f_2(K^n)$. f_t is then q_0-regular and $v^n(f_t)$ is well defined and $= 0$. On the other hand by assumption there exists an n-cycle

$$z^n = \sum s^i \sigma_i^n \varepsilon Z_n(K^n, J^\varphi) \text{ with } s_0 \neq 0.$$

Now by LEMMA 7. 2 $v^n(f_t) \infty v^n(\mathbf{1})$, where $\mathbf{1}$ is the identity deformation. By LEMMA 5.2 we have

$$0 = KI(z^n, v^n(f_t)) = KI(z^n, v^n(\mathbf{1})) = s_0 .$$

This is a contradiction.

THEOREM C'. For $n \geq 3$ a stable complex is $*$-cyclic.

Proof: Let K^n be not $*$-cyclic, then at least one n-simplex (for example σ_0^n) is never contained in an n-cycle $z^n \varepsilon Z_n(K^n, \mathfrak{R}_1 * G)$. Consequently $KI(z^n, v^n) = 0$. Hence by LEMMA 5.3 $v^n(\mathit{1}) \infty 0$, and by LEMMA 7.3 there exists a deformation f_t such that $0 = v^n(f_t)$.

We may assume that f_1 (and hence $\tilde{f}_1$) is simplicial, and the invers image $\tilde{f}_1^{-1}(\tilde{q}_0)$ in σ_{ji}^n consists of a finite number of points $\tilde{p}_{j1}, \tilde{p}_{j2}, \ldots, \tilde{p}_{js_j}$ with projections $p_{j1}, p_{j2}, \ldots, p_{js_j}$ in σ_i^n.

There are at most a finite number of i's such that the system $\{p_{j1}, p_{j2}, \ldots, p_{js_j}\}$ is not vacuous, and any two systems $\{p_{j1}, p_{j2}, \ldots, p_{js_j}\}$ and $\{p_{k1}, p_{k2}, \ldots, p_{ks_k}\}$ are disjoint when $j \neq k$. Therefore, since $n \geq 3$, we can choose a finite number of disjoint (topological) subsimplexes τ_j^n of σ_i^n such that non-vacuous system $\{p_{j1}, p_{j2}, \ldots, p_{js_j}\}$ is contained in τ_j^n.

Now since the closed paths $f_1(\overline{p_{j\nu'} p_{j\nu''}})$ (ν', $\nu'' = 1, 2, \ldots, s_k$) are easily seen to be contractible to a point, and (the degree of $f_1(\tau_j^n)$ at $q_0 = n^{ij}(f_t) = 0$, and since $n \geq 3$, we can apply Hopf's lemmas [17]) to obtain a deformation g_t such that:

$$g_t \text{ is a modification of } f_t \text{ within } \tau_j^n, \text{ and } g_1^{-1}(q_0) \frown \tau_j^n = 0.$$

Since $\{\tau_j^n\}$'s are disjoint and g_t coincides with f_t outside of τ_j^n, we can repeat the above process for the remaining τ_k^n's and arrive at a deformation g_t' such that:

$$g_t' \text{ is a modification of } f_t \text{ within } \sigma_i^n, \text{ and } g_1'^{-1}(q_0) \frown \sigma_i^n = 0.$$

We then repeat the above total process for the remaining σ_h^n's and finally obtain a deformation g_t'' such that:

$$g_1''^{-1}(q_0) = 0.$$

COROLLARY TO THEOREM C'. If an n-simplex σ_0^n of K^n is never contained in an n-cycle from $Z_n(K^n, \mathfrak{R}_1 * G)$, then there exists a deformation g_t such that $\sigma_0^n \frown g_1(K^n) = 0$.

THEOREM A'. A linear graph K^1 is stable if and only if it is $*$-cyclic.

Proof : We have only to show that a stable K^1 is $*$-cyclic. By THEOREM A K^1 has no free side. Let $\sigma_0^1 = \overline{q_1 q_2}$ be any 1-simplex of

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[17]) H. Hopf and E. Pannwitz, l. c., p. 458.

K^1. Then there are two possibilities:

1°. $K^1 - \sigma_0^1$ is connected.

2°. $K^1 - \sigma_0^1$ is not connected and consists of two disjoint linear graphs K_1^1 and K_2^1. In the case 1°, it is evident that σ_0^1 is contained in a 1-cycle. In the case 2°, K_i^1 $(i=1, 2)$ contains a subcomplex c_i^1 which is homeomorphic with either a circle or a ray.

Let w_i be a broken line joining q_1 to a point of c_i^1 in K_i^1, and let p_i be the first point on w_i which is contained in c_i^1.

For simplicity we suppose that p_i is the end point of w_i and, when c_i^1 is homeomorphic to a ray, p_i is the origin of the ray.

Let $L^1 = c_1^1 + w_1 + \sigma_0^1 + w_2 + c_2^1$, then we can easily construct a 1-cycle z^1 with local coefficients having L^1 as its carrier.

In this z^1 σ_0^1 appears with a non-vanishing coefficient.

Combining THEOREM A′, THEOREM B′ and THEOREM C′ we have

THEOREM *D*. For a locally finite homogeneous n-complex K^n $(n \neq 2)$ stability is equivalent to $*$-cyclicity.

Finally we state the following theorem which can be easily proved using LEMMA 4.1 and COROLLARY to THEOREM C′ :

THEOREM *E*. For $n \neq 2$ a locally finite homogeneous n-complex K^n is stable if and only if its universal covering complex $\tilde{K}^n$ is stable.

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