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## Contribution to the problem of stability

By Tatsuji KUDO

### I. INTRODUCTION

1. H. Hopf and E. Pannwitz<sup>1)</sup> introduced the notion of *stability*<sup>2)</sup> and raised several questions, the most important of which reads: "Can we characterize the stability in terms of the homology theory?"

Confining themselves to the homogeneous  $n$ -complex  $K^n$ , they obtained the following theorems:

THEOREM A. A linear graph  $K^1$  is stable if and only if it has no free side.

THEOREM B. A cyclic<sup>3)</sup> complex  $K^n$  is stable for any dimension  $n$ .

THEOREM C. For  $n \geq 3$  a stable complex  $K^n$  is cyclic, provided that it is simply connected.

Recently Professor A. Komatu has reasonably conjectured that to these theorems can be given the following complete and unified form:

THEOREM D. For a locally finite homogeneous  $n$ -complex  $K^n$  ( $n \neq 2$ ) stability is equivalent to the cyclicity in the sense of local coefficients<sup>4)</sup>.

The main purpose of this paper is to prove THEOREM D by generalizing the Hopf-Pannwitz's lemmas based on ordinary coefficients to those based on local coefficients.

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<sup>1)</sup> H. Hopf and E. Pannwitz, Über stetige Deformationen von Komplexen in sich, Math. Ann., 108, 1932.

<sup>2)</sup> A topological space  $R$  is called stable if for every deformation  $f_t$  of  $R$  through itself no point of  $R$  can get rid of the covering by the image  $f_1(R)$ , or more simply, if  $R$  can never be deformed into its proper subspace.

<sup>3)</sup> An  $n$ -complex  $K^n$  is called cyclic, if each  $n$ -simplex  $\sigma_i^n$  is contained in at least one  $n$ -cycle with suitable coefficients.

<sup>4)</sup> See, IV. § 8.

In CHAPTER II preliminary considerations will be made concerning the theory of local coefficients in a form convenient for our purpose.

In CHAPTER III a certain cocycle  $v^n(f_i)$  which plays a fundamental role in our arguments and which essentially gives the *degree of mapping* in local coefficients and some of its properties will be investigated.

In CHAPTER IV the proof of the THEOREM D will be given.

The THEOREM D for  $n = 2$  is difficult and the problem still remains open.

## II. PRELIMINARIES FROM THE HOMOLOGY THEORY WITH LOCAL COEFFICIENTS

2. Let  $K^n$  be a locally finite homogeneous  $n$ -complex. The  $r$ -simplexes of  $K^n$  will be denoted by  $\sigma_i^r$  ( $i = 0, 1, 2, \dots$ ).

We choose a point  $q_0$  in  $\sigma_0^n$  <sup>5)</sup> and take this point as the origin of the fundamental group  $G = \{x_j\}$  ( $j = 0, 1, 2, \dots$ ) of  $K^n$ , where  $x_0 = e =$  the unit element of  $G$ . We choose further for each point  $p$  of  $K^n$  a definite path  $[p]$  from  $q_0$  to  $p$  in such a way, that for any two points  $p_1, p_2$  of a simplex  $\sigma_i^r$  the closed path <sup>6)</sup>  $\langle p_1 p_2 \rangle = [p_1] \overline{p_1 p_2} [p_2]^{-1}$  is homotopic to 0.

We then define the universal covering complex  $\tilde{K}^n$  of  $K^n$  as usual by means of the above paths and the fundamental group  $G$ . The simplex of  $\tilde{K}^n$  over  $\sigma_i^r$  determined by all  $[p]$ 's such that  $p \in \sigma_i^r$  will be denoted by  $\sigma_{0i}^r$ , and the one determined by  $\langle p \rangle$ 's such that  $p \in \sigma_i^r$  and  $\langle p \rangle = x_j [p]$  by  $\sigma_{ij}^r = x_j \sigma_{0i}^r$ , where  $x_j$  is a given element of  $G$ . Then  $\tilde{K}^n$  becomes a complex with automorphisms  $G$  in the sense of B. Eckmann <sup>7)</sup>.

Concerning incidence numbers on  $\tilde{K}^n$  we have

<sup>5)</sup>  $\sigma_i^r$  is open.

<sup>6)</sup> Generally we use this notation  $\langle p_1 p_2 \rangle$  when  $p_1, p_2$  are contained in the closure of a single simplex to denote the closed path  $[p_1] \overline{p_1 p_2} [p_2]^{-1}$ . By the conditions for  $[p]$   $\langle p_1 p_2 \rangle$  belongs to a definite homotopy class independent of the choice of  $p_1, p_2$ , as long as  $p_1, p_2$  belong to definite simplexes  $\sigma_h^r, \sigma_k^r$  respectively (where, of course,  $\sigma_h^r, \sigma_k^r$  are incident to a single simplex  $\sigma_i^r$ ). Therefore we often use this notation  $\langle p_1 p_2 \rangle$  not only to denote the curve itself but also to denote the homotopy class (i. e., the element of the fundamental group  $G$ ) determined by the curve.

<sup>7)</sup> B. Eckmann. Proc. Nat. Acad. 33. 9., 1947.

$$\begin{aligned} [\sigma_{jt}^r : \sigma_{kh}^{r-1}] &= [x_j \sigma_{0i}^r : x_k \sigma_{0h}^{r-1}] = [x x_j \sigma_{0i}^r : x x_k \sigma_{0h}^{r-1}] \\ &= [x_k^{-1} x_j \sigma_{0i}^r : \sigma_{0h}^{r-1}] = [\sigma_{0i}^r : x_j^{-1} x_k \sigma_{0h}^{r-1}] \text{ where } x \in G. \end{aligned}$$

Further we have, concerning the relations between incidence numbers on  $\tilde{K}^n$  and  $K^n$ ,  $[\sigma_{0i}^r : x_j \sigma_{0k}^{r-1}] \neq 0$  if and only if  $[\sigma_i^r : \sigma_k^{r-1}] \neq 0$  and  $x_j = \langle p_1 p_2 \rangle$  where  $p_1 \in \sigma_i^r$ ,  $p_2 \in \sigma_k^{r-1}$ .

3. Given a homomorphism  $\varphi : G \rightarrow A(J)$  of the fundamental group  $G$  into the group of automorphisms of a coefficient group  $J$  we define  $\partial^\varphi(\delta^\varphi)$  as follows :

$$\partial^\varphi(\alpha \sigma_i^r) = \sum_k [\sigma_i^r : \sigma_k^{r-1}] (\alpha x_{ik}) \sigma_k^{r-1} \delta$$

$$(\delta^\varphi(\alpha \sigma_i^r)) = \sum_k [\sigma_i^r : \sigma_k^{r+1}] (\alpha x_{ik}) \sigma_k^{r+1}, \text{ where } \alpha \in J, x_{ik} = \langle p_1 p_2 \rangle,$$

$$p_1 \in \sigma_i^r, p_2 \in \sigma_k^{r-1} (p_2 \in \sigma_k^{r+1}).$$

Then  $\partial^\varphi(\delta^\varphi)$  has the property  $\partial^\varphi \partial^\varphi = 0$  ( $\delta^\varphi \delta^\varphi = 0$ ), and the homology (cohomology) theory can be developed with  $\partial^\varphi(\delta^\varphi)$  as boundary (coboundary) operator in the usual way. The corresponding  $r$ -dimensional infinite chain group, cycle group, boundary group (finite cochain group, cocycle group, coboundary group) of  $K^n$  will be denoted by  $C_r(K^n, J^\varphi)$ ,  $Z_r(K^n, J^\varphi)$ ,  $B_r(K^n, J^\varphi)$ ,  $(\mathbb{C}^r(K^n, J^\varphi), \mathfrak{B}^r(K^n, J^\varphi), \mathfrak{B}^r(K^n, J^\varphi))$  respectively.

If  $\varphi$  is a zero homomorphism,  $\partial^\varphi(\delta^\varphi)$  reduces to the ordinary boundary (coboundary) operator  $\partial$  ( $\delta$ ), and we shall omit the symbol  $\varphi$  in such cases.

We recall here the duality between homology and cohomology.

LEMMA 3.1. The groups  $C_r(K^n, \mathfrak{R}_1)$  and  $\mathbb{C}^r(K^n, I)^9$  are character groups of each other. And the annihilators of the groups  $Z_r(K^n, \mathfrak{R}_1)$ ,  $B_r(K^n, \mathfrak{R}_1)$ ,  $\mathfrak{B}^r(K^n, I)$  and  $\mathfrak{B}^r(K^n, I)$  are respectively the groups  $\mathfrak{B}^r(K^n, I)$ ,  $\mathfrak{B}^r(K^n, I)$ ,  $B_r(K^n, \mathfrak{R}_1)$  and  $Z_r(K^n, \mathfrak{R}_1)$ .

4. Given an abelian group  $L$ , we consider the group ring  $L \circ G$  of  $G$  with coefficients in  $L$ . In  $L \circ G$  the right multiplication of an element  $x$  ( $x \in G$ ) can be regarded as an automorphism of  $L \circ G$ . Thus we obtain a homomorphism  $\psi^0 : G \rightarrow A(L \circ G)$ . An element of  $L \circ G$

8)  $(\alpha x_{ik})$  is the element of  $J$  obtained from  $\alpha$  as the result of the automorphism,  $x_{ik}$ .

9)  $I =$  group of integers;  $\mathfrak{R}_1 =$  group of real numbers mod 1.

has the form  $\sum_i \alpha_i x_i$ , where  $\alpha_i \in L$  and the summation is finite. If we remove the last restriction of finiteness, we obtain a group  $L * G$  and a homomorphism  $\psi^* : G \rightarrow A(L * G)$ .

LEMMA 4. 1. <sup>10)</sup> The homology (cohomology) theory on  $\tilde{K}^n$  with coefficients in  $L$  is identical with the homology (cohomology) theory with local coefficients in  $(L * G)^{\psi^*} (L \circ G)^{\psi^0}$ .

Proof: In order to prove the statement for homology it is sufficient to verify the identity  $\underline{\kappa} \partial = \partial \underline{\kappa}$ , where  $\underline{\kappa}$  is the isomorphism between  $C_r(\tilde{K}^n, L)$  and  $C_r(K^n, L * G)$ , <sup>11)</sup> determined by

$$\begin{aligned} \underline{\kappa} : \sum_{i,j} \alpha^{ij} \sigma_{ji}^r &\longleftrightarrow \sum_i (\sum_j \alpha^{ij} x_j) \sigma_i^r. \\ \underline{\kappa} \partial (\sum_{i,j} \alpha^{ij} \sigma_{ji}^r) &= \underline{\kappa} \sum_{ijhk} \alpha^{ij} [\sigma_{ji}^r : \sigma_{kh}^{r-1}] \sigma_{kh}^{r-1} \\ &= \underline{\kappa} \sum_{ijhk} \alpha^{ij} [\sigma_{0i}^r : x_j^{-1} x_k \sigma_{0h}^{r-1}] x_k \sigma_{0h}^{r-1} \\ &= \underline{\kappa} \sum_i \sum_{x_{ih} = x_j^{-1} x_k} \alpha^{ij} [\sigma_i^r : \sigma_h^{r-1}] x_k \sigma_{0h}^{r-1} = \sum_i \sum_h [\sigma_i^r : \sigma_h^{r-1}] (\sum_j \alpha^{ij} x_j) x_{ih} \sigma_h^{r-1} \\ &= \partial (\sum_i (\sum_j \alpha^{ij} x_j) \sigma_i^r) = \partial \underline{\kappa} (\sum_{i,j} \alpha^{ij} \sigma_{ji}^r). \end{aligned}$$

5. Let  $v^r = \sum_i t^i \sigma_i^r$  be an element of  $\mathbb{C}^r(K^n, I \circ G)$  and let  $c^r = \sum_i s^i \sigma_i^r$  be an element of  $C_r(K^n, J^\varphi)$ . We define the Kronecker index of  $c^r$  with  $v^r$  by

$$KI(c^r, v^r) = \sum_{i,j} n^{ij} (s^i x_j^{-1})^\varphi \quad \text{where } t^i = \sum_j n^{ij} x_j.$$

LEMMA 5.1.  $KI(\partial^\varphi c^r, v^{r-1}) = KI(c^r, \delta^\varphi v^{r-1})$ .

LEMMA 5.2. Let  $v_1^r, v_2^r \in \mathbb{B}^r(K^n, I \circ G)$ , and  $z^r \in Z_r(K^n, J^\varphi)$ .

If  $v_1^r \infty v_2^r$  <sup>12)</sup>, then  $KI(z^r, v_1^r) = KI(z^r, v_2^r)$ .

We need further the

LEMMA 5.4. Let  $v^r = \sum_{i,j} (n^{ij} x_j) \sigma_i^r$ . If for every  $z^r = \sum_i s^i \sigma_i^r$

<sup>10)</sup> This lemma is due to Prof. A. Komatu.

<sup>11)</sup> We shall always omit the symbol  $\psi^* (\psi^0)$  in  $(L * G)^{\psi^*} ((L \circ G)^{\psi^0})$ .

<sup>12)</sup>  $\infty =$  "is cohomologous to".

$= \sum_i (\sum_j \rho^{ij} x_j) \sigma_i^r$  from  $Z_r(K^n, \mathfrak{R}_1 * G)$   $KI(z^r, v^r) = 0$ , then  $v^r \infty 0$ .

Proof:  $KI(z^r, v^r) = \sum_{i,j} n^{ij} (\sum_h \rho^{ih} x_h) x_j^{-1} = \sum_{j,h} (\sum_i n^{ij} \rho^{ih}) x_h x_j^{-1}$   
 $= \sum_k \sum_{x_h x_j^{-1} = x_k} (\sum_i n^{ij} \rho^{ih}) x_k = 0$ , hence  $\sum_{i,j} n^{ij} \rho^{ih} = \sum_{i,j} n^{ij} \rho^{ij} = 0$ .

Therefore

$$KI(\underline{\kappa}^{-1} z^r, \bar{\kappa}^{-1} v^r)^{13)} = \sum_{i,j} n^{ij} \rho^{ij} = 0.$$

By LEMMA 4.1  $\underline{\kappa}^{-1} z^r$  can be any infinite cycle from  $Z_r(\tilde{K}^n, \mathfrak{R}_1)$ , and consequently by LEMMA 3.1  $\bar{\kappa}^{-1} v^r \in \mathfrak{B}^r(\tilde{K}^n, I)$ . Applying LEMMA 4.1 again we conclude that  $v^r \in \mathfrak{B}^r(K^n, I \circ G)$ .

### III. DEFORMATIONS

6. By a deformation  $f_t$  of a topological space  $R$  we mean a continuous mapping of the product-space  $R \times \langle 0, 1 \rangle$  into  $R$  subject to the conditions that for each  $t$  ( $0 \leq t \leq 1$ ) and for each  $p \in R$  the inverse image  $f_t^{-1}(p)$  is compact and  $f_0^{-1}(p) = p$ .

7. Let  $K^n$  be as in § 2-5. We say that a deformation  $f_t$  is  $q_0$ -regular if for every  $i$  the image  $f_1(\partial \sigma_i^n)$  does not contain  $q_0$ , and that two  $q_0$ -regular deformations  $f_t$  and  $g_t$  are  $q_0$ -equivalent if there exists a homotopy  $h_t(p)$ ,  $0 \leq t \leq 1$ ,  $p \in K^n$  and  $h_t(p) \in K^n$ , such that  $h_0 = f_1$ ,  $h_1 = g_1$ , and  $h_t(\partial \sigma_i^n)$  never contains  $q_0$ .

LEMMA 7.1. For any deformation  $f_t$  of  $K^n$  we can find a (uniquely determined) deformation  $\tilde{f}_t$  of the universal covering complex  $\tilde{K}^n$  such that

$$\pi \tilde{f}_t = f_t \pi. \quad 14)$$

It is clear that if  $f_t$  is  $q_0$ -regular, then  $\tilde{f}_t$  is  $\tilde{q}_0$ -regular, where  $\tilde{q}_0$  is the point over  $q_0$  belonging to  $\sigma_{00}^n$ . Hence the degree  $n^{ij}$  of  $f_1(x_j \sigma_{i0}^n)$  at  $\tilde{q}_0$  is well defined. We put  $t^i = \sum_j n^{ij} x_j$  and define an element  $v^n =$

13)  $\bar{\kappa}$  is analogously defined as  $\underline{\kappa}$ .

14)  $\pi$  is the projection of  $\tilde{K}^n$  on  $K^n$ . For the proof of the LEMMA, see, A. Komatu, Zur Topologie der Abbildungen von Komplexen, Jap. Journ. Math., vol. XVII, 1941, Satz 1.4.

$v^n(f_t) = \sum_i t^i \sigma_i^n \in \mathfrak{C}^n(K^n, I \circ G)$  <sup>15)</sup>  $= \mathfrak{B}^n(K^n, I \circ G)$ . Evidently if two  $q_0$ -regular deformations  $f_t, g_t$  are  $q_0$ -equivalent then  $v^n(f_t) = v^n(g_t)$ .

LEMMA 7.2. For every  $q_0$ -regular deformation  $f_t$  the corresponding  $v^n(f_t)$  belongs to a definite cohomology class independent of the special  $f_t$ .

Proof : For any  $z^n = \sum \rho^{i,j} \sigma_{ji}^n \in Z_n(\tilde{K}^n, \mathfrak{R}_1)$ ,  $KI(z^n, \bar{\kappa}^{-1} v^n)$  is the degree of  $\tilde{f}_1(z^n)$  at  $\tilde{q}_0$ . We know that the degree of mapping of an ordinary cycle is invariant under deformation.

Consequently we have

$$KI(z^n, \bar{\kappa}^{-1} v^n(f_t) - \bar{\kappa}^{-1} v^n(g_t)) = 0.$$

By LEMMA 3.1 this implies that

$$\bar{\kappa}^{-1} v^n(f_t) - \bar{\kappa}^{-1} v^n(g_t) \in \mathfrak{B}^n(\tilde{K}^n, I).$$

Hence by LEMMA 4.1

$$v^n(f_t) - v^n(g_t) \in \mathfrak{B}^n(K^n, I \circ G).$$

LEMMA 7.3. If  $v^n \infty v^n(f_t)$ , there exists a deformation  $g_t$  such that  $v^n = v^n(g_t)$ , ( $n > 1$ ).

Proof: We may assume that  $f_1$  is a simplicial transformation of a subdivision  $K_1^n$  of  $K^n$  into  $K^n$ . Following Hopf <sup>16)</sup> we take a point  $p$  on  $\sigma_j^{n-1}$  and join the image point  $f_1(p)$  to a point  $q$  of  $\sigma_0^n$  different from  $q_0$  by a curve  $w$  disjoint from  $q_0$  such that  $\overline{q \cdot q} w^{-1} f_1(p) p [p]^{-1} = x_k$ , where  $\overline{f_1(p) p}$  is the inverse path of  $f_t(p)$  ( $0 \leq t \leq 1$ ),

We want to modify  $f_t$  within a small neighborhood of  $p$  so as to obtain a deformation  $g_t$  such that

$$v^n(g_t) - v^n(f_t) = \delta(x_k \sigma_j^{n-1}).$$

Before we carry out this modification, we make several conventions about notations.

Let  $K_\nu^n$  ( $\nu = 2, 3, 4, 5$ ) be successive subdivisions of  $K_1^n$ , and let

<sup>15)</sup> Since  $f_1^{-1}(p_0)$  is compact there are at most a finite number of  $n^{i,j}$  which are different from zero.

<sup>16)</sup> H. Hopf, Über wesentliche Abbildungen von Komplexen, Recueil Math., vol. 37, 1930, Beweis von Satz IIIa.

$s_\nu(p)$  ( $\nu = 2, 3, 4, 5$ ) be the stars of  $p$  corresponding to these subdivisions. We suppose that the subdivision  $K_1^n$  be sufficiently fine, so that every point of the image  $f_1(s_2(p))$  can be joined to  $f_1(p)$  by a segment and that  $q$  does not lie on any of these segments. We suppose further that the stars  $s_\nu(p)$  are all similar and similarly placed with  $p$  as center of similarity.

Given a point  $p'$  in  $s_\nu(p) - \overline{s_{\nu+1}(p)}$  ( $\nu = 2, 3, 4$ ) we denote by  $\lambda_\nu(p')$  the ratio  $\overline{p_\nu p'} : \overline{p_\nu p_{\nu+1}}$  where  $p_\nu$  is the point of intersection of  $\overline{pp'}$  with  $\partial s_\nu(p)$ .

Let us now return to the original question. Our modification shall be achieved after the following three steps.

FIRST STEP: Let  $\phi_t(p')$  ( $1 \leq t \leq 2$ ) be a deformation of  $K^n$  under which each point  $p'$  of  $K^n$  moves along a segment at a constant velocity to  $\phi_2(p')$ , where  $\phi_2(p')$  is given by:

$$\phi_2(p') = p' \quad \text{for} \quad p' \in K^n - s_2(p),$$

$$\phi_2(p') = p \quad \text{for} \quad p' \in \overline{s_3(p)},$$

$$\phi_2(p') = \text{the point } p'' \text{ on } \overline{pp'} \text{ such that}$$

$$\overline{p_2 p''} = \overline{p_2 p} \lambda_2(p').$$

We define

$$f_t(p') \quad \text{for} \quad (1 \leq t \leq 2) \text{ by}$$

$$f_t(p') = f_1(\phi_t(p')) \quad (1 \leq t \leq 2).$$

Then we have

$$f_2(s_3(p)) = f_1(p).$$

SECOND STEP: Let the curve  $w$  be given with respect to the parameter  $t$  ( $2 \leq t \leq 3$ ) by  $w(t)$ .

We define  $f_t(p')$  ( $2 \leq t \leq 3$ ) as follows:

$$f_t(p') = f_2(p') \quad \text{for} \quad p' \in K^n - s_3(p),$$

$$f_t(p') = w(t) \quad \text{for} \quad p' \in \overline{s_4(p)},$$

$$f_t(p') = w(2 + \lambda_3(p')(t-2)) \quad \text{for} \quad p' \in s_3(p) - \overline{s_4(p)}.$$

Then we have



$$f_3(s_4(p)) = w(3) = q.$$

THIRD STEP : Let  $\tau^n$  be a subsimplex of  $\sigma_0^n$  containing  $q_0$  and having  $q$  as a vertex, and let the boundary simplex of  $\tau^n$  opposite to  $q$  be  $\tau^{n-1}$ . We suppose that  $\tau^n$  is given the orientation induced from the one of  $\sigma_0^n$ , and that  $\tau^{n-1}$  is oriented in such a way that

$$\partial \tau^n = + \tau^{n-1} + \dots \dots \dots$$

Now we consider  $s_5(p)$ . Let  $t^{n-1}$  be the  $(n-1)$ -simplex which is the common part of  $\sigma_j^{n-1}$  with  $s_5(p)$ , and  $t_i^n$  the  $n$ -simplex which is the common part of  $\sigma_i^n$  and  $s_5(p)$ . Let further  $e^i$  be the vertex of  $t_i^n$  in  $\sigma_i^n$ .

We map the simplex  $t_i^n$  affinely on  $\tau^n$  in such a way that the point  $e^i$  is mapped into  $q$  and the simplex  $t^{n-1}$  is mapped positively on  $\tau^{n-1}$ , where the positive orientation of  $t^{n-1}$  is the one induced from  $\sigma_j^{n-1}$ . We name this mapping  $f_4(p')$ , and extend this outside of  $s_5(p)$  requiring :

$$\begin{aligned} f_4(p') &= f_3(p') && \text{for } p' \in K^n - s_4(p), \\ f_4(p') &= \text{the point } p'' \text{ on } \overline{f_4(p_4) f_4(p_5)} \text{ such that} \\ \overline{f_4(p_4) p''} (= \overline{q p''}) &= \lambda_4(p') \overline{f_4(p_4) f_4(p_5)}, && \text{for } p' \in s_4(p) - \overline{s_5(p)}. \end{aligned}$$

We then define  $f_t(p')$  for  $(3 \leq t \leq 4)$  by

$$f_t(p') = f_3(p') = f_4(p') \text{ for } p' \in K^n - s_4(p), \quad f_t(p') = \text{the point } p'' \text{ on } \overline{q f_4(p')} \text{ such that}$$

$$\overline{p q''} = t \overline{q f_4(p')} \text{ for } p' \in s_4(p).$$

Thus we obtain a deformation  $f_t(p)$  ( $0 \leq t \leq 4$ ) which is a modification of  $f_t(p)$  ( $0 \leq t \leq 4$ ) within a small neighborhood of  $p$ .

We shall show that this deformation  $f_t(p)$  ( $0 \leq t \leq 4$ ) (rewritten as  $g_t(p)$  ( $0 \leq t \leq 1$ )) is the one with the required property.

Consider the covering deformation  $\tilde{g}_t$  of  $g_t$ . Then  $\tilde{g}_t$  is the modification of  $\tilde{f}_t$  within a neighborhood of the inverse image  $\pi^{-1}(p)$ , but, as can be easily seen, it has no influence upon the values of  $n^{th}(f_t)$  except at the point  $x_k \tilde{p}$ . As for the influence at  $x_k \tilde{p}$  Hopf's calculation

shows that the difference  $n^{in}(g) - n^{in}(f) =$  the incidence number  $[\sigma_{kj}^n : \sigma_{kj}^{n-1}]$ . From this, after a simple calculation, we obtain

$$v^n(g) - v^n(f) = \delta(x_k \sigma_j^{n-1}).$$

Now let  $v^n \infty v^n(f_t)$ , then there exists an  $(n-1)$ -cochain

$$\sum m^{jk} x_k \sigma_j^{n-1} \text{ such that } v^n - v^n(f_t) = \delta(\sum m^{jk} x_k \sigma_j^{n-1}).$$

Repeating the above process for  $x_k \sigma_j^{n-1}$ 's successively we can conclude that there exists a deformation  $g_t$  such that

$$v^n(g_t) - v^n(f_t) = \delta(\sum m^{jk} x_k \sigma_j^{n-1}).$$

Comparing the last two identities we obtain

$$v^n(g_t) = v^n.$$

### III. STABILITY

8. Let  $K^n$  be as in § 2-7.  $K^n$  is called  $*$ -cyclic if each  $n$ -simplex  $\sigma_i^n$  is contained in an  $n$ -cycle (finite or infinite) with suitable local coefficients.

**THEOREM B'.** A  $*$ -cyclic complex  $K^n$  is stable for any dimension  $n$ .

**Proof:** If  $K^n$  were not stable, then there exists a deformation  $f_t$  such that a point of  $K^n$  (for example  $q_0$ ) is not contained in  $f_2(K^n)$ .  $f_t$  is then  $q_0$ -regular and  $v^n(f_t)$  is well defined and  $= 0$ . On the other hand by assumption there exists an  $n$ -cycle

$$z^n = \sum s^i \sigma_i^n \in Z_n(K^n, J^q) \text{ with } s_0 \neq 0.$$

Now by LEMMA 7.2  $v^n(f_t) \infty v^n(I)$ , where  $I$  is the identity deformation. By LEMMA 5.2 we have

$$0 = KI(z^n, v^n(f_t)) = KI(z^n, v^n(I)) = s_0.$$

This is a contradiction.

**THEOREM C'.** For  $n \geq 3$  a stable complex is  $*$ -cyclic.

**Proof:** Let  $K^n$  be not  $*$ -cyclic, then at least one  $n$ -simplex (for example  $\sigma_0^n$ ) is never contained in an  $n$ -cycle  $z^n \in Z_n(K^n, \mathfrak{B}_1 * G)$ . Con-

sequently  $KI(z^n, v^n) = 0$ . Hence by LEMMA 5.3  $v^n(\mathbf{I}) \infty 0$ , and by LEMMA 7.3 there exists a deformation  $f_t$  such that  $0 = v^n(f_t)$ .

We may assume that  $f_1$  (and hence  $\tilde{f}_1$ ) is simplicial, and the inverse image  $\tilde{f}_1^{-1}(\tilde{q}_0)$  in  $\sigma_{j_i}^n$  consists of a finite number of points  $\tilde{p}_{j_1}, \tilde{p}_{j_2}, \dots, \tilde{p}_{j_{s_j}}$  with projections  $p_{j_1}, p_{j_2}, \dots, p_{j_{s_j}}$  in  $\sigma_i^n$ .

There are at most a finite number of  $i$ 's such that the system  $\{p_{j_1}, p_{j_2}, \dots, p_{j_{s_j}}\}$  is not vacuous, and any two systems  $\{p_{j_1}, p_{j_2}, \dots, p_{j_{s_j}}\}$  and  $\{p_{k_1}, p_{k_2}, \dots, p_{k_{s_k}}\}$  are disjoint when  $j \neq k$ . Therefore, since  $n \geq 3$ , we can choose a finite number of disjoint (topological) subsimplexes  $\tau_j^n$  of  $\sigma_i^n$  such that non-vacuous system  $\{p_{j_1}, p_{j_2}, \dots, p_{j_{s_j}}\}$  is contained in  $\tau_j^n$ .

Now since the closed paths  $f_1(\overline{p_{j_{\nu'}} p_{j_{\nu''}}})$  ( $\nu', \nu'' = 1, 2, \dots, s_k$ ) are easily seen to be contractible to a point, and (the degree of  $f_1(\tau_j^n)$  at  $q_0 = n^{i,j}(f_t) = 0$ , and since  $n \geq 3$ , we can apply Hopf's lemmas <sup>17)</sup> to obtain a deformation  $g_t$  such that:

$$g_t \text{ is a modification of } f_t \text{ within } \tau_j^n, \text{ and } g_t^{-1}(q_0) \cap \tau_j^n = 0.$$

Since  $\{\tau_j^n\}$ 's are disjoint and  $g_t$  coincides with  $f_t$  outside of  $\tau_j^n$ , we can repeat the above process for the remaining  $\tau_k^n$ 's and arrive at a deformation  $g_t'$  such that:

$$g_t' \text{ is a modification of } f_t \text{ within } \sigma_i^n, \text{ and } g_t'^{-1}(q_0) \cap \sigma_i^n = 0.$$

We then repeat the above total process for the remaining  $\sigma_h^n$ 's and finally obtain a deformation  $g_t''$  such that:

$$g_t''^{-1}(q_0) = 0.$$

COROLLARY TO THEOREM C'. If an  $n$ -simplex  $\sigma_0^n$  of  $K^n$  is never contained in an  $n$ -cycle from  $Z_n(K^n, \mathfrak{R}_1 * G)$ , then there exists a deformation  $g_t$  such that  $\sigma_0^n \cap g_t(K^n) = 0$ .

THEOREM A'. A linear graph  $K^1$  is stable if and only if it is  $*$ -cyclic.

Proof: We have only to show that a stable  $K^1$  is  $*$ -cyclic. By THEOREM A  $K^1$  has no free side. Let  $\sigma_0^1 = \overline{q_1 q_2}$  be any 1-simplex of

<sup>17)</sup> H. Hopf and E. Pannwitz, l. c., p. 453.

$K^1$ . Then there are two possibilities :

1°.  $K^1 - \sigma_0^1$  is connected.

2°.  $K^1 - \sigma_0^1$  is not connected and consists of two disjoint linear graphs  $K_1^1$  and  $K_2^1$ . In the case 1°, it is evident that  $\sigma_0^1$  is contained in a 1-cycle. In the case 2°,  $K_i^1$  ( $i=1, 2$ ) contains a subcomplex  $c_i^1$  which is homeomorphic with either a circle or a ray.

Let  $w_i$  be a broken line joining  $q_1$  to a point of  $c_i^1$  in  $K_i^1$ , and let  $p_i$  be the first point on  $w_i$  which is contained in  $c_i^1$ .

For simplicity we suppose that  $p_i$  is the end point of  $w_i$  and, when  $c_i^1$  is homeomorphic to a ray,  $p_i$  is the origin of the ray.

Let  $L^1 = c_1^1 + w_1 + \sigma_0^1 + w_2 + c_2^1$ , then we can easily construct a 1-cycle  $z^1$  with local coefficients having  $L^1$  as its carrier.

In this  $z^1$   $\sigma_0^1$  appears with a non-vanishing coefficient.

Combining THEOREM A', THEOREM B' and THEOREM C' we have

**THEOREM D.** For a locally finite homogeneous  $n$ -complex  $K^n$  ( $n \neq 2$ ) stability is equivalent to  $*$ -cyclicity.

Finally we state the following theorem which can be easily proved using LEMMA 4.1 and COROLLARY to THEOREM C' :

**THEOREM E.** For  $n \neq 2$  a locally finite homogeneous  $n$ -complex  $K^n$  is stable if and only if its universal covering complex  $\tilde{K}^n$  is stable.

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