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Contribution to the problem of stability

By Tatsuji KUDO

I. INTRODUCTION

1. H. Hopf and E. Pannwitz¹⁾ introduced the notion of *stability*²⁾ and raised several questions, the most important of which reads: "Can we characterize the stability in terms of the homology theory?"

Confining themselves to the homogeneous n -complex K^n , they obtained the following theorems:

THEOREM A. A linear graph K^1 is stable if and only if it has no free side.

THEOREM B. A cyclic³⁾ complex K^n is stable for any dimension n .

THEOREM C. For $n \geq 3$ a stable complex K^n is cyclic, provided that it is simply connected.

Recently Professor A. Komatu has reasonably conjectured that to these theorems can be given the following complete and unified form:

THEOREM D. For a locally finite homogeneous n -complex K^n ($n \neq 2$) stability is equivalent to the cyclicity in the sense of local coefficients⁴⁾.

The main purpose of this paper is to prove THEOREM D by generalizing the Hopf-Pannwitz's lemmas based on ordinary coefficients to those based on local coefficients.

¹⁾ H. Hopf and E. Pannwitz, Über stetige Deformationen von Komplexen in sich, Math. Ann., 108, 1932.

²⁾ A topological space R is called stable if for every deformation f_t of R through itself no point of R can get rid of the covering by the image $f_1(R)$, or more simply, if R can never be deformed into its proper subspace.

³⁾ An n -complex K^n is called cyclic, if each n -simplex σ_i^n is contained in at least one n -cycle with suitable coefficients.

⁴⁾ See, IV. § 8.

In CHAPTER II preliminary considerations will be made concerning the theory of local coefficients in a form convenient for our purpose.

In CHAPTER III a certain cocycle $v^n(f_i)$ which plays a fundamental role in our arguments and which essentially gives the *degree of mapping* in local coefficients and some of its properties will be investigated.

In CHAPTER IV the proof of the THEOREM D will be given.

The THEOREM D for $n=2$ is difficult and the problem still remains open.

II. PRELIMINARIES FROM THE HOMOLOGY THEORY WITH LOCAL COEFFICIENTS

2. Let K^n be a locally finite homogeneous n -complex. The r -simplexes of K^n will be denoted by σ_i^r ($i=0, 1, 2, \dots$).

We choose a point q_0 in σ_0^n ⁵⁾ and take this point as the origin of the fundamental group $G = \{x_j\}$ ($j=0, 1, 2, \dots$) of K^n , where $x_0 = e =$ the unit element of G . We choose further for each point p of K^n a definite path $[p]$ from q_0 to p in such a way, that for any two points p_1, p_2 of a simplex σ_i^r the closed path ⁶⁾ $\langle p_1 p_2 \rangle = [p_1] \overline{p_1 p_2} [p_2]^{-1}$ is homotopic to 0.

We then define the universal covering complex \tilde{K}^n of K^n as usual by means of the above paths and the fundamental group G . The simplex of \tilde{K}^n over σ_i^r determined by all $[p]$'s such that $p \in \sigma_i^r$ will be denoted by σ_{0i}^r , and the one determined by $\langle p \rangle$'s such that $p \in \sigma_i^r$ and $\langle p \rangle = x_j [p]$ by $\sigma_{ij}^r = x_j \sigma_{0i}^r$, where x_j is a given element of G . Then \tilde{K}^n becomes a complex with automorphisms G in the sense of B. Eckmann ⁷⁾.

Concerning incidence numbers on \tilde{K}^n we have

⁵⁾ σ_i^r is open.

⁶⁾ Generally we use this notation $\langle p_1 p_2 \rangle$ when p_1, p_2 are contained in the closure of a single simplex to denote the closed path $[p_1] \overline{p_1 p_2} [p_2]^{-1}$. By the conditions for $[p]$ $\langle p_1 p_2 \rangle$ belongs to a definite homotopy class independent of the choice of p_1, p_2 , as long as p_1, p_2 belong to definite simplexes σ_h^r, σ_k^r respectively (where, of course, σ_h^r, σ_k^r are incident to a single simplex σ_i^r). Therefore we often use this notation $\langle p_1 p_2 \rangle$ not only to denote the curve itself but also to denote the homotopy class (i. e., the element of the fundamental group G) determined by the curve.

⁷⁾ B. Eckmann. Proc. Nat. Acad. 33. 9., 1947.

$$\begin{aligned} [\sigma_{jt}^r : \sigma_{kh}^{r-1}] &= [x_j \sigma_{0i}^r : x_k \sigma_{0h}^{r-1}] = [x x_j \sigma_{0i}^r : x x_k \sigma_{0h}^{r-1}] \\ &= [x_k^{-1} x_j \sigma_{0i}^r : \sigma_{0h}^{r-1}] = [\sigma_{0i}^r : x_j^{-1} x_k \sigma_{0h}^{r-1}] \text{ where } x \in G. \end{aligned}$$

Further we have, concerning the relations between incidence numbers on \tilde{K}^n and K^n , $[\sigma_{0i}^r : x_j \sigma_{0k}^{r-1}] \neq 0$ if and only if $[\sigma_i^r : \sigma_k^{r-1}] \neq 0$ and $x_j = \langle p_1 p_2 \rangle$ where $p_1 \in \sigma_i^r$, $p_2 \in \sigma_k^{r-1}$.

3. Given a homomorphism $\varphi : G \rightarrow A(J)$ of the fundamental group G into the group of automorphisms of a coefficient group J we define $\partial^\varphi(\delta^\varphi)$ as follows :

$$\partial^\varphi(\alpha \sigma_i^r) = \sum_k [\sigma_i^r : \sigma_k^{r-1}] (\alpha x_{ik}) \sigma_k^{r-1} \delta$$

$$(\delta^\varphi(\alpha \sigma_i^r)) = \sum_k [\sigma_i^r : \sigma_k^{r+1}] (\alpha x_{ik}) \sigma_k^{r+1}, \text{ where } \alpha \in J, x_{ik} = \langle p_1 p_2 \rangle,$$

$$p_1 \in \sigma_i^r, p_2 \in \sigma_k^{r-1} (p_2 \in \sigma_k^{r+1}).$$

Then $\partial^\varphi(\delta^\varphi)$ has the property $\partial^\varphi \partial^\varphi = 0$ ($\delta^\varphi \delta^\varphi = 0$), and the homology (cohomology) theory can be developed with $\partial^\varphi(\delta^\varphi)$ as boundary (coboundary) operator in the usual way. The corresponding r -dimensional infinite chain group, cycle group, boundary group (finite cochain group, cocycle group, coboundary group) of K^n will be denoted by $C_r(K^n, J^\varphi)$, $Z_r(K^n, J^\varphi)$, $B_r(K^n, J^\varphi)$, $(\mathbb{C}^r(K^n, J^\varphi), \mathfrak{B}^r(K^n, J^\varphi), \mathfrak{B}^r(K^n, J^\varphi))$ respectively.

If φ is a zero homomorphism, $\partial^\varphi(\delta^\varphi)$ reduces to the ordinary boundary (coboundary) operator ∂ (δ), and we shall omit the symbol φ in such cases.

We recall here the duality between homology and cohomology.

LEMMA 3.1. The groups $C_r(K^n, \mathfrak{R}_1)$ and $\mathbb{C}^r(K^n, I)^9$ are character groups of each other. And the annihilators of the groups $Z_r(K^n, \mathfrak{R}_1)$, $B_r(K^n, \mathfrak{R}_1)$, $\mathfrak{B}^r(K^n, I)$ and $\mathfrak{B}^r(K^n, I)$ are respectively the groups $\mathfrak{B}^r(K^n, I)$, $\mathfrak{B}^r(K^n, I)$, $B_r(K^n, \mathfrak{R}_1)$ and $Z_r(K^n, \mathfrak{R}_1)$.

4. Given an abelian group L , we consider the group ring $L \circ G$ of G with coefficients in L . In $L \circ G$ the right multiplication of an element x ($x \in G$) can be regarded as an automorphism of $L \circ G$. Thus we obtain a homomorphism $\psi^0 : G \rightarrow A(L \circ G)$. An element of $L \circ G$

8) (αx_{ik}) is the element of J obtained from α as the result of the automorphism, x_{ik} .

9) $I =$ group of integers; $\mathfrak{R}_1 =$ group of real numbers mod 1.

has the form $\sum_i \alpha_i x_i$, where $\alpha_i \in L$ and the summation is finite. If we remove the last restriction of finiteness, we obtain a group $L * G$ and a homomorphism $\psi^* : G \rightarrow A(L * G)$.

LEMMA 4. 1. ¹⁰⁾ The homology (cohomology) theory on \tilde{K}^n with coefficients in L is identical with the homology (cohomology) theory with local coefficients in $(L * G)^{\psi^*} (L \circ G)^{\psi^0}$.

Proof: In order to prove the statement for homology it is sufficient to verify the identity $\underline{\kappa} \partial = \partial \underline{\kappa}$, where $\underline{\kappa}$ is the isomorphism between $C_r(\tilde{K}^n, L)$ and $C_r(K^n, L * G)$, ¹¹⁾ determined by

$$\begin{aligned} \underline{\kappa} : \sum_{i,j} \alpha^{ij} \sigma_{ji}^r &\longleftrightarrow \sum_i (\sum_j \alpha^{ij} x_j) \sigma_i^r. \\ \underline{\kappa} \partial (\sum_{i,j} \alpha^{ij} \sigma_{ji}^r) &= \underline{\kappa} \sum_{ijhk} \alpha^{ij} [\sigma_{ji}^r : \sigma_{kh}^{r-1}] \sigma_{kh}^{r-1} \\ &= \underline{\kappa} \sum_{ijhk} \alpha^{ij} [\sigma_{0i}^r : x_j^{-1} x_k \sigma_{0h}^{r-1}] x_k \sigma_{0h}^{r-1} \\ &= \underline{\kappa} \sum_i \sum_{x_{ih} = x_j^{-1} x_k} \alpha^{ij} [\sigma_i^r : \sigma_h^{r-1}] x_k \sigma_{0h}^{r-1} = \sum_i \sum_h [\sigma_i^r : \sigma_h^{r-1}] (\sum_j \alpha^{ij} x_j) x_{ih} \sigma_h^{r-1} \\ &= \partial (\sum_i (\sum_j \alpha^{ij} x_j) \sigma_i^r) = \partial \underline{\kappa} (\sum_{i,j} \alpha^{ij} \sigma_{ji}^r). \end{aligned}$$

5. Let $v^r = \sum_i t^i \sigma_i^r$ be an element of $\mathbb{C}^r(K^n, I \circ G)$ and let $c^r = \sum_i s^i \sigma_i^r$ be an element of $C_r(K^n, J^\varphi)$. We define the Kronecker index of c^r with v^r by

$$KI(c^r, v^r) = \sum_{i,j} n^{ij} (s^i x_j^{-1})^\varphi \quad \text{where } t^i = \sum_j n^{ij} x_j.$$

LEMMA 5.1. $KI(\partial^\varphi c^r, v^{r-1}) = KI(c^r, \delta^\varphi v^{r-1})$.

LEMMA 5.2. Let $v_1^r, v_2^r \in \mathbb{B}^r(K^n, I \circ G)$, and $z^r \in Z_r(K^n, J^\varphi)$.

If $v_1^r \infty v_2^r$ ¹²⁾, then $KI(z^r, v_1^r) = KI(z^r, v_2^r)$.

We need further the

LEMMA 5.4. Let $v^r = \sum_{i,j} (n^{ij} x_j) \sigma_i^r$. If for every $z^r = \sum_i s^i \sigma_i^r$

¹⁰⁾ This lemma is due to Prof. A. Komatu.

¹¹⁾ We shall always omit the symbol $\psi^* (\psi^0)$ in $(L * G)^{\psi^*} ((L \circ G)^{\psi^0})$.

¹²⁾ $\infty =$ "is cohomologous to".

$= \sum_i (\sum_j \rho^{ij} x_j) \sigma_i^r$ from $Z_r(K^n, \mathfrak{R}_1 * G)$ $KI(z^r, v^r) = 0$, then $v^r \infty 0$.

Proof: $KI(z^r, v^r) = \sum_{i,j} n^{ij} (\sum_h \rho^{ih} x_h) x_j^{-1} = \sum_{j,h} (\sum_i n^{ij} \rho^{ih}) x_h x_j^{-1}$
 $= \sum_k \sum_{x_h x_j^{-1} = x_k} (\sum_i n^{ij} \rho^{ih}) x_k = 0$, hence $\sum_{i,j} n^{ij} \rho^{ih} = \sum_{i,j} n^{ij} \rho^{ij} = 0$.

Therefore

$$KI(\underline{\kappa}^{-1} z^r, \bar{\kappa}^{-1} v^r)^{13)} = \sum_{i,j} n^{ij} \rho^{ij} = 0.$$

By LEMMA 4.1 $\underline{\kappa}^{-1} z^r$ can be any infinite cycle from $Z_r(\tilde{K}^n, \mathfrak{R}_1)$, and consequently by LEMMA 3.1 $\bar{\kappa}^{-1} v^r \in \mathfrak{B}^r(\tilde{K}^n, I)$. Applying LEMMA 4.1 again we conclude that $v^r \in \mathfrak{B}^r(K^n, I \circ G)$.

III. DEFORMATIONS

6. By a deformation f_t of a topological space R we mean a continuous mapping of the product-space $R \times \langle 0, 1 \rangle$ into R subject to the conditions that for each t ($0 \leq t \leq 1$) and for each $p \in R$ the inverse image $f_t^{-1}(p)$ is compact and $f_0^{-1}(p) = p$.

7. Let K^n be as in § 2-5. We say that a deformation f_t is q_0 -regular if for every i the image $f_1(\partial \sigma_i^n)$ does not contain q_0 , and that two q_0 -regular deformations f_t and g_t are q_0 -equivalent if there exists a homotopy $h_t(p)$, $0 \leq t \leq 1$, $p \in K^n$ and $h_t(p) \in K^n$, such that $h_0 = f_1$, $h_1 = g_1$, and $h_t(\partial \sigma_i^n)$ never contains q_0 .

LEMMA 7.1. For any deformation f_t of K^n we can find a (uniquely determined) deformation \tilde{f}_t of the universal covering complex \tilde{K}^n such that

$$\pi \tilde{f}_t = f_t \pi. \quad 14)$$

It is clear that if f_t is q_0 -regular, then \tilde{f}_t is \tilde{q}_0 -regular, where \tilde{q}_0 is the point over q_0 belonging to σ_{00}^n . Hence the degree n^{ij} of $f_1(x_j \sigma_{i0}^n)$ at \tilde{q}_0 is well defined. We put $t^i = \sum_j n^{ij} x_j$ and define an element $v^n =$

13) $\bar{\kappa}$ is analogously defined as $\underline{\kappa}$.

14) π is the projection of \tilde{K}^n on K^n . For the proof of the LEMMA, see, A. Komatu, Zur Topologie der Abbildungen von Komplexen, Jap. Journ. Math., vol. XVII, 1941, Satz 1.4.

$v^n(f_t) = \sum_i t^i \sigma_i^n \in \mathfrak{C}^n(K^n, I \circ G)$ ¹⁵⁾ $= \mathfrak{B}^n(K^n, I \circ G)$. Evidently if two q_0 -regular deformations f_t, g_t are q_0 -equivalent then $v^n(f_t) = v^n(g_t)$.

LEMMA 7.2. For every q_0 -regular deformation f_t the corresponding $v^n(f_t)$ belongs to a definite cohomology class independent of the special f_t .

Proof : For any $z^n = \sum \rho^{i,j} \sigma_{ji}^n \in Z_n(\tilde{K}^n, \mathfrak{R}_1)$, $KI(z^n, \bar{\kappa}^{-1} v^n)$ is the degree of $\tilde{f}_1(z^n)$ at \tilde{q}_0 . We know that the degree of mapping of an ordinary cycle is invariant under deformation.

Consequently we have

$$KI(z^n, \bar{\kappa}^{-1} v^n(f_t) - \bar{\kappa}^{-1} v^n(g_t)) = 0.$$

By LEMMA 3.1 this implies that

$$\bar{\kappa}^{-1} v^n(f_t) - \bar{\kappa}^{-1} v^n(g_t) \in \mathfrak{B}^n(\tilde{K}^n, I).$$

Hence by LEMMA 4.1

$$v^n(f_t) - v^n(g_t) \in \mathfrak{B}^n(K^n, I \circ G).$$

LEMMA 7.3. If $v^n \infty v^n(f_t)$, there exists a deformation g_t such that $v^n = v^n(g_t)$, ($n > 1$).

Proof: We may assume that f_1 is a simplicial transformation of a subdivision K_1^n of K^n into K^n . Following Hopf ¹⁶⁾ we take a point p on σ_j^{n-1} and join the image point $f_1(p)$ to a point q of σ_0^n different from q_0 by a curve w disjoint from q_0 such that $\overline{q \cdot q} w^{-1} f_1(p) p [p]^{-1} = x_k$, where $\overline{f_1(p) p}$ is the inverse path of $f_t(p)$ ($0 \leq t \leq 1$),

We want to modify f_t within a small neighborhood of p so as to obtain a deformation g_t such that

$$v^n(g_t) - v^n(f_t) = \delta(x_k \sigma_j^{n-1}).$$

Before we carry out this modification, we make several conventions about notations.

Let K_ν^n ($\nu = 2, 3, 4, 5$) be successive subdivisions of K_1^n , and let

¹⁵⁾ Since $f_1^{-1}(p_0)$ is compact there are at most a finite number of $n^{i,j}$ which are different from zero.

¹⁶⁾ H. Hopf, Über wesentliche Abbildungen von Komplexen, Recueil Math., vol. 37, 1930, Beweis von Satz IIIa.

$s_\nu(p)$ ($\nu = 2, 3, 4, 5$) be the stars of p corresponding to these subdivisions. We suppose that the subdivision K_1^n be sufficiently fine, so that every point of the image $f_1(s_2(p))$ can be joined to $f_1(p)$ by a segment and that q does not lie on any of these segments. We suppose further that the stars $s_\nu(p)$ are all similar and similarly placed with p as center of similarity.

Given a point p' in $s_\nu(p) - \overline{s_{\nu+1}(p)}$ ($\nu = 2, 3, 4$) we denote by $\lambda_\nu(p')$ the ratio $\overline{p_\nu p'} : \overline{p_\nu p_{\nu+1}}$ where p_ν is the point of intersection of $\overline{pp'}$ with $\partial s_\nu(p)$.

Let us now return to the original question. Our modification shall be achieved after the following three steps.

FIRST STEP: Let $\phi_t(p')$ ($1 \leq t \leq 2$) be a deformation of K^n under which each point p' of K^n moves along a segment at a constant velocity to $\phi_2(p')$, where $\phi_2(p')$ is given by:

$$\phi_2(p') = p' \quad \text{for} \quad p' \in K^n - s_2(p),$$

$$\phi_2(p') = p \quad \text{for} \quad p' \in \overline{s_3(p)},$$

$$\phi_2(p') = \text{the point } p'' \text{ on } \overline{pp'} \text{ such that}$$

$$\overline{p_2 p''} = \overline{p_2 p} \lambda_2(p').$$

We define

$$f_t(p') \quad \text{for} \quad (1 \leq t \leq 2) \text{ by}$$

$$f_t(p') = f_1(\phi_t(p')) \quad (1 \leq t \leq 2).$$

Then we have

$$f_2(s_3(p)) = f_1(p).$$

SECOND STEP: Let the curve w be given with respect to the parameter t ($2 \leq t \leq 3$) by $w(t)$.

We define $f_t(p')$ ($2 \leq t \leq 3$) as follows:

$$f_t(p') = f_2(p') \quad \text{for} \quad p' \in K^n - s_3(p),$$

$$f_t(p') = w(t) \quad \text{for} \quad p' \in \overline{s_4(p)},$$

$$f_t(p') = w(2 + \lambda_3(p')(t-2)) \quad \text{for} \quad p' \in s_3(p) - \overline{s_4(p)}.$$

Then we have

$$f_3(s_4(p)) = w(3) = q.$$

THIRD STEP : Let τ^n be a subsimplex of σ_0^n containing q_0 and having q as a vertex, and let the boundary simplex of τ^n opposite to q be τ^{n-1} . We suppose that τ^n is given the orientation induced from the one of σ_0^n , and that τ^{n-1} is oriented in such a way that

$$\partial \tau^n = + \tau^{n-1} + \dots \dots \dots$$

Now we consider $s_5(p)$. Let t^{n-1} be the $(n-1)$ -simplex which is the common part of σ_j^{n-1} with $s_5(p)$, and t_i^n the n -simplex which is the common part of σ_i^n and $s_5(p)$. Let further e^i be the vertex of t_i^n in σ_i^n .

We map the simplex t_i^n affinely on τ^n in such a way that the point e^i is mapped into q and the simplex t^{n-1} is mapped positively on τ^{n-1} , where the positive orientation of t^{n-1} is the one induced from σ_j^{n-1} . We name this mapping $f_4(p')$, and extend this outside of $s_5(p)$ requiring :

$$\begin{aligned} f_4(p') &= f_3(p') && \text{for } p' \in K^n - s_4(p), \\ f_4(p') &= \text{the point } p'' \text{ on } \overline{f_4(p_4) f_4(p_5)} \text{ such that} \\ \overline{f_4(p_4) p''} (= \overline{q p''}) &= \lambda_4(p') \overline{f_4(p_4) f_4(p_5)}, && \text{for } p' \in s_4(p) - \overline{s_5(p)}. \end{aligned}$$

We then define $f_t(p')$ for $(3 \leq t \leq 4)$ by

$$f_t(p') = f_3(p') = f_4(p') \text{ for } p' \in K^n - s_4(p), \quad f_t(p') = \text{the point } p'' \text{ on } \overline{q f_4(p')} \text{ such that}$$

$$\overline{p q''} = t \overline{q f_4(p')} \text{ for } p' \in s_4(p).$$

Thus we obtain a deformation $f_t(p)$ ($0 \leq t \leq 4$) which is a modification of $f_t(p)$ ($0 \leq t \leq 4$) within a small neighborhood of p .

We shall show that this deformation $f_t(p)$ ($0 \leq t \leq 4$) (rewritten as $g_t(p)$ ($0 \leq t \leq 1$)) is the one with the required property.

Consider the covering deformation \tilde{g}_t of g_t . Then \tilde{g}_t is the modification of \tilde{f}_t within a neighborhood of the inverse image $\pi^{-1}(p)$, but, as can be easily seen, it has no influence upon the values of $n^{th}(f_t)$ except at the point $x_k \tilde{p}$. As for the influence at $x_k \tilde{p}$ Hopf's calculation

shows that the difference $n^{in}(g) - n^{in}(f) =$ the incidence number $[\sigma_{kj}^n : \sigma_{kj}^{n-1}]$. From this, after a simple calculation, we obtain

$$v^n(g) - v^n(f) = \delta(x_k \sigma_j^{n-1}).$$

Now let $v^n \infty v^n(f_t)$, then there exists an $(n-1)$ -cochain

$$\sum m^{jk} x_k \sigma_j^{n-1} \text{ such that } v^n - v^n(f_t) = \delta(\sum m^{jk} x_k \sigma_j^{n-1}).$$

Repeating the above process for $x_k \sigma_j^{n-1}$'s successively we can conclude that there exists a deformation g_t such that

$$v^n(g_t) - v^n(f_t) = \delta(\sum m^{jk} x_k \sigma_j^{n-1}).$$

Comparing the last two identities we obtain

$$v^n(g_t) = v^n.$$

III. STABILITY

8. Let K^n be as in § 2-7. K^n is called $*$ -cyclic if each n -simplex σ_i^n is contained in an n -cycle (finite or infinite) with suitable local coefficients.

THEOREM B'. A $*$ -cyclic complex K^n is stable for any dimension n .

Proof: If K^n were not stable, then there exists a deformation f_t such that a point of K^n (for example q_0) is not contained in $f_2(K^n)$. f_t is then q_0 -regular and $v^n(f_t)$ is well defined and $= 0$. On the other hand by assumption there exists an n -cycle

$$z^n = \sum s^i \sigma_i^n \in Z_n(K^n, J^q) \text{ with } s_0 \neq 0.$$

Now by LEMMA 7.2 $v^n(f_t) \infty v^n(I)$, where I is the identity deformation. By LEMMA 5.2 we have

$$0 = KI(z^n, v^n(f_t)) = KI(z^n, v^n(I)) = s_0.$$

This is a contradiction.

THEOREM C'. For $n \geq 3$ a stable complex is $*$ -cyclic.

Proof: Let K^n be not $*$ -cyclic, then at least one n -simplex (for example σ_0^n) is never contained in an n -cycle $z^n \in Z_n(K^n, \mathfrak{B}_1 * G)$. Con-

sequently $KI(z^n, v^n) = 0$. Hence by LEMMA 5.3 $v^n(\mathbf{I}) \infty 0$, and by LEMMA 7.3 there exists a deformation f_t such that $0 = v^n(f_t)$.

We may assume that f_1 (and hence \tilde{f}_1) is simplicial, and the inverse image $\tilde{f}_1^{-1}(\tilde{q}_0)$ in $\sigma_{j_i}^n$ consists of a finite number of points $\tilde{p}_{j_1}, \tilde{p}_{j_2}, \dots, \tilde{p}_{j_{s_j}}$ with projections $p_{j_1}, p_{j_2}, \dots, p_{j_{s_j}}$ in σ_i^n .

There are at most a finite number of i 's such that the system $\{p_{j_1}, p_{j_2}, \dots, p_{j_{s_j}}\}$ is not vacuous, and any two systems $\{p_{j_1}, p_{j_2}, \dots, p_{j_{s_j}}\}$ and $\{p_{k_1}, p_{k_2}, \dots, p_{k_{s_k}}\}$ are disjoint when $j \neq k$. Therefore, since $n \geq 3$, we can choose a finite number of disjoint (topological) subsimplexes τ_j^n of σ_i^n such that non-vacuous system $\{p_{j_1}, p_{j_2}, \dots, p_{j_{s_j}}\}$ is contained in τ_j^n .

Now since the closed paths $f_1(\overline{p_{j_{\nu'}} p_{j_{\nu''}}})$ ($\nu', \nu'' = 1, 2, \dots, s_k$) are easily seen to be contractible to a point, and (the degree of $f_1(\tau_j^n)$ at $q_0 = n^{i,j}(f_t) = 0$, and since $n \geq 3$, we can apply Hopf's lemmas ¹⁷⁾ to obtain a deformation g_t such that:

$$g_t \text{ is a modification of } f_t \text{ within } \tau_j^n, \text{ and } g_t^{-1}(q_0) \cap \tau_j^n = 0.$$

Since $\{\tau_j^n\}$'s are disjoint and g_t coincides with f_t outside of τ_j^n , we can repeat the above process for the remaining τ_k^n 's and arrive at a deformation g_t' such that:

$$g_t' \text{ is a modification of } f_t \text{ within } \sigma_i^n, \text{ and } g_t'^{-1}(q_0) \cap \sigma_i^n = 0.$$

We then repeat the above total process for the remaining σ_h^n 's and finally obtain a deformation g_t'' such that:

$$g_t''^{-1}(q_0) = 0.$$

COROLLARY TO THEOREM C'. If an n -simplex σ_0^n of K^n is never contained in an n -cycle from $Z_n(K^n, \mathfrak{R}_1 * G)$, then there exists a deformation g_t such that $\sigma_0^n \cap g_t(K^n) = 0$.

THEOREM A'. A linear graph K^1 is stable if and only if it is $*$ -cyclic.

Proof: We have only to show that a stable K^1 is $*$ -cyclic. By THEOREM A K^1 has no free side. Let $\sigma_0^1 = \overline{q_1 q_2}$ be any 1-simplex of

¹⁷⁾ H. Hopf and E. Pannwitz, l. c., p. 453.

K^1 . Then there are two possibilities :

1°. $K^1 - \sigma_0^1$ is connected.

2°. $K^1 - \sigma_0^1$ is not connected and consists of two disjoint linear graphs K_1^1 and K_2^1 . In the case 1°, it is evident that σ_0^1 is contained in a 1-cycle. In the case 2°, K_i^1 ($i=1, 2$) contains a subcomplex c_i^1 which is homeomorphic with either a circle or a ray.

Let w_i be a broken line joining q_1 to a point of c_i^1 in K_i^1 , and let p_i be the first point on w_i which is contained in c_i^1 .

For simplicity we suppose that p_i is the end point of w_i and, when c_i^1 is homeomorphic to a ray, p_i is the origin of the ray.

Let $L^1 = c_1^1 + w_1 + \sigma_0^1 + w_2 + c_2^1$, then we can easily construct a 1-cycle z^1 with local coefficients having L^1 as its carrier.

In this z^1 σ_0^1 appears with a non-vanishing coefficient.

Combining THEOREM A', THEOREM B' and THEOREM C' we have

THEOREM D. For a locally finite homogeneous n -complex K^n ($n \neq 2$) stability is equivalent to $*$ -cyclicity.

Finally we state the following theorem which can be easily proved using LEMMA 4.1 and COROLLARY to THEOREM C' :

THEOREM E. For $n \neq 2$ a locally finite homogeneous n -complex K^n is stable if and only if its universal covering complex \tilde{K}^n is stable.

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