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PSEUDO-RANK FUNCTIONS ON SKEW GROUP RINGS AND ON FIXED SUBRINGS OF AUTOMORPHISMS OF UNIT-REGULAR RINGS

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Let R be a unit-regular ring and G a finite subgroup of $Aut(R)$ with $|G|^{-1} \in R$. This paper is concerned with relationships between the pseudo-rank functions of the skew group ring $R*G$ and ones of the fixed subring R^G. We introduce such relationships by studying certain homomorphisms between $K_0(R*G)$ and $K_0(R^G)$.

In §1, under the assumption that $R*G$ is a unit-regular ring and R is a finitely generated projective left R^G-module, we shall investigate the following two homomorphisms:

$$\bar{\mu}\colon K_0(R^G) \to K_0(R*G)\,, \quad \text{defined by} \quad \bar{\mu}([M]) = [R*Ge \otimes_{R^G} M]$$

$$\bar{\lambda}\colon K_0(R*G) \to K_0(R^G)\,, \quad \text{defined by} \quad \bar{\lambda}([A]) = [\mathrm{Hom}_{R*G}(R*Ge, A)]\,,$$

where $e = |G|^{-1} \sum_{g \in G} g$ in $R*G$. Then we shall show that $\bar{\lambda}\,\bar{\mu}$ is the identity map and $\bar{\mu}$ is an order-embedding map.

The maps $\bar{\mu}$, $\bar{\lambda}$ induce maps μ^*, λ^* between $\mathrm{P}(R*G)$ and $\mathrm{P}(R^G)$, where $\mathrm{P}(T)$ (resp. $\partial_e\mathrm{P}(T)$) is the family of all pseudo-rank functions (resp. extremal pseudo-rank functions) of a regular ring T. For any $N \in \mathrm{P}(R*G)$ with $N(e) > 0$ and any $a \in R^G$, we define

$$\mu^*(N)\,(a) = N(e)^{-1}\, D_N(R*Ge \otimes_{R^G} R^G\, a)\,,$$

where D_N is the dimension function which corresponds to N. For any $Q \in P(R^G)$ and any $x \in R*G$, we define

$$\lambda^*(Q)\,(x) = D_Q({}_{R^G}R)^{-1} D_Q(\mathrm{Hom}_{R*G}(R*Ge,\, R*Gx))\,,$$

where D_Q is the dimension function which corresponds to Q. Then we shall show that $\mu^*(N)$ (resp. $\lambda^*(Q)$) is a pseudo-rank function of R^G (resp. $R*G$) and $\mu^*\lambda^*$=identity and μ^* preserves extremal pseudo-rank functions.

In §2, for a directly finite, left self-injective, regular ring R and an X-

outer group G, we shall determine all extremal pseudo-rank functions of $R*G$ from ones of R. It is shown from above results that $R*G\simeq M_n(R^G)$ as rings, where $n=|G|$, and $R\simeq R^G[G]$ as left $R^G[G]$-modules.

In §3, assuming that R is a left and right self-injective regular ring and R is a finitely generated, projective, left R^G-module, we shall show that there exists a bijection from some subset of $\text{Max}(R*G)$ into $\text{Max}(R^G)$. Using this result, we obtain that for any $Q\in\partial_e P(R^G)$, there exists an unique $N\in\partial_e P(R*G)$ with $N(e)>0$ such that $Q(a)=N(e)^{-1}N(ae)$ for any $a\in R^G$.

1. Relations between $P(R*G)$ and $P(R^G)$

Given regular ring T, we use $\text{FP}(T)$ to denote the set of all finitely generated projective left T-modules. For modules A, B, $A\lesssim B$ means that A is isomorphic to a submodule of B and we use $n.A$ to denote the direct sum of n copies of A.

According to [1, p. 226], we mean by a pseudo-rank function on R is a map $N: R\rightarrow[0, 1]$ such that

(1) $N(1)=1$.

(2) $N(rs)\leqq N(r)$ and $N(rs)\leqq N(s)$ for all $r, s\in R$.

(3) $N(e+f)=N(e)+N(f)$ for all orthogonal idempotent $e, f\in R$.

If, in addition

(4) $N(r)>0$ for all non-zero $r\in R$,

then N is called a rank function. We use $P(R)$ to denote the set of all pseudo-rank functions on R

For a regular ring R, we view $P(R)$ as a subset of the real vector space $\boldsymbol{R}^R$, which we equipped with the product topology [1, Ch. 16 and Appendix]. Then $P(R)$ is a compact convex subset of $\boldsymbol{R}^R$ by [1, Prop. 16, 17]. We use $\partial_e P(R)$ to denote the set of all extreme points of $P(R)$. It is known that $P(R)$ is equal to the closure of the convex hull of $\partial_e P(R)$ by Krein-Milman Theorem.

Again according to [1, p. 232], we mean by a dimension function on $\text{FP}(T)$ is a map D: $\text{FP}(T)\rightarrow\boldsymbol{R}^+$ such that

(1) $D(T)=1$

(2) If $A, B\in\text{FP}(T)$ and $A\lesssim B$, then $D(A)\leqq D(B)$.

(3) $D(A\oplus B)=D(A)+D(B)$ for all $A, B\in\text{FP}(T)$.

Let $D(T)$ denote the set of all dimension functions on $\text{FP}(T)$. There is a bijection Γ_T: $P(T)\rightarrow D(T)$ such that $\Gamma_T(P)(Tt)=P(t)$ for all $P\in P(T)$ and $t\in T$ by [1, Prop. 16.8]. For $P\in P(T)$, we use D_P to denote the dimension function $\Gamma_T(P)$.

Let T be a ring with identity element 1 and let G be a finite group of automorphisms of T with $|G|^{-1}\in T$. The skew group ring, $T*G$, is defined to be a free left T-module with basis $\{g: g\in G\}$ and multiplication given as follows: if $r, s\in T$ and $g, h\in G$, then $(rg)(sh)=rs^{g^{-1}}gh$ ([9]).

Throughout this paper, put $e=|G|^{-1}\sum_{g\in G}g$ and denote by θ the map $e(T*G)e\to T^G$ which is given by $\theta[e(\sum_{g\in G}r_g g)e]=\sum_{g\in G}t(r_g)$, where $t(r)=|G|^{-1}\sum_{g\in G}r^g$ for $r\in T$. Then e is an idempotent and θ is an isomorphism by [9, Lemma 0.1].

Let R be a unit-regular ring and G a finite subgroup of $Aut(R)$ with $|G|^{-1}\in R$. In [8], we have studied relationships between $P(R*G)$ and $P(R)$ (resp. $\partial_e P(R*G)$ and $\partial_e P(R)$). Especially we have shown that all G-invariant $P\in P(R)$ can be extended to pseudo-rank functions of $R*G$. In this paper, we shall study the relation between $P(R*G)$ and $P(R^G)$ (resp. $\partial_e P(R*G)$ and $\partial_e P(R^G)$). If $R*G$ and R^G are Morita equivalent, then K.R. Goodearl has shown under a general situation that there is a bijection between $P(R*G)$ and $P(R^G)$ in [1, Cor. 16.9]. We shall define maps between $P(R*G)$ and $P(R^G)$, which are more concrete than the Goodearl's bijection, without the assumption of Morita Equivalence.

A partially ordered abelian group is an abelian group K equipped with a partial order $\leqq$ which is translation invariant ([1, p. 202]). The positive cone of K is the set $K^+=\{x\in K;\ x\geqq 0\}$. If the partial order on K is directed (upward or downward), then K is called a directed abelian group. An order unit in K is an element $u>0$ such that for any $x\in K$, there exists a positive integer n for which $x\leqq nu$. We denote by a pair (G, u) a partially ordered abelian group with order-unit u.

For a unit-regular ring T, the Grothendieck group $K_0(T)$ is an abelian group with generators $[A]$, where $[A]$ is the isomorphism class for $A\in \mathrm{FP}(T)$ and with relation $[A\oplus B]=[A]+[B]$ ([1, §15]). Every element of $K_0(T)$ has the form $[A]-[B]$ for some $A, B\in \mathrm{FP}(T)$. $K_0(T)$ is a partially ordered abelian group with order-unit $[T]$ and positive cone $K_0(T)^+$ coincides with $\{[A]: A\in \mathrm{FP}(T)\}$ by [1, Prop. 15.2].

Let R be a unit-regular ring and let G be a finite subgroup of $Aut(R)$ with $|G|^{-1}\in R$. The skew group ring $R*G$ is a regular ring by [5]. Unfortunately we don't know whether $R*G$ is unit-regular or not. Therefore, from now on, we assume that $R*G$ is unit-regular in many cases. We regard $R*Ge$ as a (left $R*G$, right R^G)-bimodule, where $e=|G|^{-1}\sum_{g\in G}g$.

There exists a natural functor μ; $\mathrm{FP}(R^G)\to \mathrm{FP}(R*G)$ given by the rule $\mu(M)=R*Ge\otimes_{R^G}M$. Then we have a positive homomorphism $\bar{\mu}$: $K_0(R^G)\to K_0(R*G)$, defined by $\bar{\mu}([M])=[\mu(M)]$. Set $F=\{N\in P(R*G):\ N(e)=0\}$. Then μ also induces a map μ^*: $P(R*G)\backslash F\to P(R^G)$ given by the rule $\mu^*(N)(a)=N(e)^{-1}D_N(\mu(R^Ga))$ for any $N\in P(R*G)\backslash F$ and any $a\in R^G$, where D_N is the dimension function which corresponds to N. In fact, since $\mu(R^Ga)=R*Ge\otimes R^Ga\cong R*Gea$, we have $D_N(\mu(R^Ga))=N(ea)$. Then $\mu^*(N)(a)=N(e)^{-1}N(ea)$ for all $a\in R^G$. Thus $\mu^*(N)$ is a pseudo-rank function by the isomorphism θ: $eR*Ge\to R^G$ and [1, Lemma 16.2].

Proposition 1. *Let μ^*: $P(R*G)\backslash F \to P(R^G)$ be the map given above. If $N \in P(R*G)\backslash F$ is extremal in $P(R*G)$, then $\mu^*(N)$ is also extremal.*

Proof. It is sufficient to prove that

$$\mu^*(N)(a) \wedge \mu^*(N)(b) = sup\{\mu^*(N)(arb) : r \in R^G\}$$

for all a, $b \in R^G$ by [1, Prop. 19.16]. We compute as follows;

$$\begin{aligned} sup\{\mu^*(N)(arb) : r \in R^G\} &= sup\{N(earb). N(e)^{-1} : r \in R^G\} \\ &= sup\{N(ea.er.eb) : r \in R^G\}. N(e)^{-1}. \end{aligned}$$

If r runs over all element of R^G, $ea.er.eb$ runs over all generators of $aeR*Gbe$ by θ. Then, since N is extremal, we have

$$sup\{N(ea.er.eb); r \in R^G\} = N(ea) \wedge N(eb)$$

by [1, Th. 19.16]. Consequently we see that

$$sup\{\mu^*(N)(arb) : r \in R^G\} = \mu^*(N)(a) \wedge \mu^*(N)(b).$$

In general, there may not exist any map from $P(R^G) \to P(R*G)$. Under the assumption that R is a finitely generated, projective, left R^G-module, there exists such a map ([8]). For the sake of completeness, we shall again define it. We assume that R is a finitely generated, projective, left R^G-module. For any $A \in \mathrm{FP}(R*G)$, define $\lambda(A) = \mathrm{Hom}_{R*G}(R*Ge, A)$. Since $\mathrm{Hom}_{R*G}(R*Ge, R*G) \cong eR*G \cong R$ as left R^G-modules, $\lambda(A)$ is a finitely generated, projective, left R^G-module. The functor λ induces a positive homomorphism

$$\bar{\lambda} : K_0(R*G) \to K_0(R^G) \quad \text{defined by the rule;} \quad \bar{\lambda}([A]) = [\lambda(A)].$$

Since $\mathrm{Hom}_{R*G}(R*Ge, R*G) \cong eR*G \cong R$ as left R^G-modules, we have $\bar{\lambda}([R*G]) = [{}_{R^G}R]$. We define

$$\lambda^*(Q)(x) = D_Q(R)^{-1} D_Q(\lambda(R*Gx))$$

for any $Q \in P(R^G)$ and for all $x \in R*G$, where D_Q is the dimension function which corresponds to Q. By [8, §3], $\lambda^*(Q)$ is a pseudo-rank function on $R*G$.

REMARK 1. Since $\lambda(R*Ge) \cong eR*Ge \cong R^G$, we have the relation that $\lambda^*(Q)(e) = D_Q({}_{R^G}R)^{-1}$ for all $Q \in P(R^G)$.

Now we shall determine pseudo-rank functions on R^G from ones on $R*G$.

Theorem 2. *Let R be a unit-regular ring, G a finite subgroup of $Aut(R)$ with $|G|^{-1} \in R$ and $R*G$ a skew group ring of G over R. Put $e = |G|^{-1} \sum_{g \in G} g$ and set $F = \{N \in P(R*G) : N(e) = 0\}$. We assume that $R*G$ is a unit-regular* *ring and that R is a finitely generated, projective, left R^G-module. Then the following hold;*

(1) $\bar{\mu}$: $K_0(R^G)\to K_0(R*G)$ *is an order-embedding map and* $\bar{\lambda}\,\bar{\mu}$=*identity.*

(2) *For any* $Q\in P(R^G)$, *there exists some* $N\in P(R*G)\backslash F$ *such that* $Q(a)=N(e)^{-1}N(ae)$ *for any* $a\in R^G$.

Proof. (1) First we shall show that for any idempotent $a\in R^G$, $\lambda\ \mu(R^Ga)\cong R^Ga$. In fact, we see that

$$\begin{aligned}\lambda\ \mu(R^Ga) &= \mathrm{Hom}_{R*G}(R*Ge,\ R*Ge\otimes_{R^G}R^Ga)\\ &\cong \mathrm{Hom}_{R*G}(R*Ge,\ R*Gea)\\ &\cong eR*Gea\\ &\cong R^Ga\,,\end{aligned}$$

using the isomorphism $eR*Ge\to R^G$.

Since $K_0(R^G)$ (resp. $K_0(R*G)$) is generated by the set $\{[I]$: I is a principal left ideal$\}$ by [1, Prop. 2.6], we see that $\bar{\lambda}\,\bar{\mu}$=identity. For any M, $M'\in \mathrm{FP}(R^G)$, we assume that $\bar{\mu}([M])\leqq\bar{\mu}([M'])$. By definitions and [1, Prop. 15.2], we see that $\mu(M)\lesssim\mu(M')$ and$M\cong\lambda\ \mu(M)\lesssim\lambda\ \mu(M')\cong M'$. Hence we conclude that $[M]\leqq[M']$.

(2) For maps μ^*: $P(R*G)\backslash F\to P(R^G)$ and λ^*: $P(R^G)\to P(R*G)\backslash F$, we may show that $\mu^*\lambda^*$=identity. For any $Q\in P(R^G)$ and any $a\in R^G$,

$$\begin{aligned}\mu^*\ \lambda^*(Q)\,(a) &= \lambda^*(Q)\,(e)^{-1}\cdot D_{\lambda^*(Q)}\,(\lambda\,(R^Ga))\\ &= D_Q(R)\cdot D_Q(R)^{-1}\,D_Q(\mu\ \lambda\,(R^Ga))\\ &= D_Q(R^Ga)\\ &= Q(a)\,.\end{aligned}$$

REMARK 2. By Proposition 1, the restriction map of μ^* on $\partial_eP(R*G)\backslash F$ is a map into $\partial_eP(R^G)$. Unfortunately we can't prove that it is also an epimorphism. We shall prove in §3 that it is an epimorphism for self-injective regular rings.

Next we shall determine a condition that $R*G$ and R^G are Morita equivalent.

Proposition 3. *Let R be a unit-regular ring and let G be a finite subgroup of Aut(R) with* $|G|^{-1}\in R$. *We assume that R*G is also a unit-regular ring. The following conditions are equivalent.*

(1) *R*Ge (resp. eR*G) is a generator as a R*G-module.*

(2) $N(e)>0$ *for all* $N\in\partial_eP(R*G)$.

Proof. (1)⇒(2). By the assumption of (1), there exists some natural number k such that $R*G\lesssim k\cdot(R*Ge)$. Then, for any $N\in P(R*G)$, we have

$kN(e) \geqq 1$ and so $N(e) > 0$.

(2)⇒(1). We shall show that $R*GeR*G = R*G$. Put $H = R*GeR*G$. Assume that $H \neq R*G$. Let f; $R*G \to R*G/H$ be a natural epimorphism. Since $R*G/H$ is also unit-regular, we have that $P(R*G/H)$ is not empty by [1, Cor 18.5]. By [1, Th. A.6], there exist $N' \in \partial_e P(R*G/H)$. We consider the function $N'f$. Then $N = N'f$ is an extreme pseudo-rank function on $R*G$ by [1, Prop. 16.19]. Since $H \subset ker(N)$, $N(e) = 0$. This is a contradiction. Hence $R*GeR*G = R*G$ and we see that $R*Ge$ is a generator.

REMARK 3. In above case, since $\mathrm{End}_{R*G}(R*Ge) \cong R^G$, $R*G$ and R^G are Morita equivalent. So, $\lambda^* \mu^* =$ identity and hence μ^* induces a bijection from $\partial_e P(R*G)$ into $\partial_e P(R^G)$.

2. *X*-outer automorphisms

In this section, let R be a directly finite, left self-injective, regular ring and G a finite group of automorphisms of R with $|G|^{-1} \in R$. It is known that both $R*G$ and R^G are directly finite, left self-injective, regular rings ([12]) and that such rings are unit-regular rings ([1, Th. 9.17]). K.R. Goodearl has shown that there exists a bijection $\partial_e P(R) \to \mathrm{Max}(R)$ which is defined by the rule; $P \to ker(P)$ and that $R/ker(P)$ is a simple self-injective regular ring with the unique rank function [4, II. 14.5]. We use repeatedly that fact.

An automorphism g of R is called an X-inner if there exists a non-zero element $x \in R$ such that $rx = xr^g$ for all $r \in R$ ([10]). If g is not X-inner, we call g X-outer. For a subgroup G of $Aut(R)$, we call G X-outer if all $g \neq 1 \in G$ are X-outer. Let $Z(R)$ be the center of R.

First we shall determine the structure of $\mathrm{Max}(R*G)$ for an X-outer group G. The following Lemma has been essentially proved in [5], but we shall prove it in this note for the sake of completeness. We denote the set of all central idempotents of a ring T by $B(T)$.

Lemma 4. *Let R be a directly finite, left self-injective, regular ring and G a finite group of automorphisms of R with $|G|^{-1} \in R$. We assume that G is X-outer. Then* $\mathrm{Max}(R*G) = \{(\cap_{g \in G} M^g)*G : M \in \mathrm{Max}(R)\}$.

Proof. Since G is X-outer, $Z(R*G)$ is contained in $Z(R) \cap R^G$. Hence $B(R*G) \subset B(R) \cap R^G$. First we choose any $P \in \mathrm{Max}(R*G)$. Put $m = P \cap B(R*G)$, then $m \in \mathrm{Max}(B(R*G))$ and P is the unique maximal ideal containing m by [1, Th. 8.25]. Let m_0 be a maximal ideal of $B(R)$ containing m. Then there exists a unique maximal ideal M of R containing m_0 by [1, Th. 8.25]. Put $\bar{M} = \cap_{g \in G} M^g$. We note that $m \subset \bar{M}$. By [11, Lemma 4.1], $\bar{M}*G$ is a finite intersection of maximal ideals of $R*G$ and P is the unique maximal ideal of

$R*G$ containing m by [1, Th. 8.25]. Therefore we have $P=\bar{M}*G$. Conversely for any $M\in\mathrm{Max}(R)$, put $m=M\cap B(R*G)$. Then we see that $m\in\mathrm{Max}(B(R*G))$. Since $(\cap_{g\in G}M^g)*G$ is a finite intersection of maximal ideals of $R*G$ by [11, Lemma 4.1] and containing m, it is a maximal ideal by [1, Th. 8.25].

In [8], we have studied the relation between $P(R*G)$ and $P(R)$. Especially we can extend a G-invariant pseudo-rank function P on R to one, P^G, on $R*G$ defined by the rule; $P^G(x)=|G|^{-1}D_P({}_R(R*Gx))$ for all $x\in R*G$ ([8, Cor. 4]). If P is not G-invariant, then we consider the trace $t(P)=|G|^{-1}\sum_{g\in G}P^g$, where $P^g(r)=P(r^{g^{-1}})$. Now we shall determine all elements in $\partial_e P(R*G)$, using Lemma 4 and [8, Cor. 4].

Proposition 5. *Let R be a directly finite, left self-injective, regular ring and G a finite group of automorphisms of R with $|G|^{-1}\in R$. We assume that G is X-outer. Then $\partial_e P(R*G)=\{t(Q)^G\colon Q\in\partial_e P(R)\}$.*

Proof. For any $N\in\partial_e P(R*G)$, we see that $ker(N)\in\mathrm{Max}(R*G)$ by [4, II. 14.5]. By Lemma 4, we have that $ker(N)=(\cap_{g\in G}M^g)*G$, where $M\in\mathrm{Max}(R)$. We choose $Q\in\partial_e P(R)$ such that $ker(Q)=M$. Since $ker\ t(Q)=\cap_{g\in G}M^g/ker\ t(Q)^G\supset(\cap_{g\in G}M^g)*G$. Hence we have $\ker\ (t(Q)^G)=ker\ (N)$ and hence $t(Q)^G=N$. Conversely for any $Q\in\partial_e P(R)$, we proved above that $ker\ (t(Q)^G)$ is a maximal ideal of $R*G$. Thus $t(Q)^G$ is extremal by [4. II. 14.5].

Lemma 6. *Let R be a directly finite, left self-injective, regular ring and G a finite group of automorphisms of R with $|G|^{-1}\in R$. We assume that G is X-outer. Then the following hold:*

(1) *$N(e)=n^{-1}$ for all $N\in\partial_e P(R*G)$, where $n=|G|$.*

(2) *$R*G\cong M_n(R^G)$.*

Proof. By Proposition 5, we have $N=t(Q)^G$ for some $Q\in\partial_e P(R)$. Since $R*Ge\cong R$ as a left R-module, $N(e)=t(Q)^G(e)=n^{-1}$ by [8, Corollary 4]. Consequently we have $R*G\cong n(R*Ge)$ as a left $R*G$-module by [2, Cor. 2.7]. Hence $R*G\cong M_n(R^G)$, because $eR*Re\cong G^G$.

Now, using Lemma 6, we shall prove an interesting result concerning with "a normal basis" of R over R^G.

Proposition 7. *Let R be a directly finite, left self-injective, regular ring and G a finite group of automorphisms of R with $|G|^{-1}\in R$. We assume that G is X-outer. Then $R\cong R^G[G]$ as $R^G[G]$-modules.*

Proof. We can easily see that $R*GeR*G=R*G$ by Lemma 4. Then $R*Ge$ is a generator as a $R*G$-module and R is a finitely generated, projective, left R^G-module. We know that there exist maps $\mu^*\colon P(R*G)\to P(R^G)$ and

λ^*: $P(R^G)\to P(R*G)$ such that $\lambda^*\mu^*$=identity, $\mu^*\lambda^*$=identity and both maps are also bijection on the extremal boundary by §1. Especially we have an important relation that $\lambda^*(Q)(e)=D_Q(R)^{-1}$ for all $Q\in P(R^G)$. Therefore any $Q\in\partial_e P(R^G)$, we have $\lambda^*(Q)\in\partial_e P(R*G)$ and $\lambda^*(Q)(e)=D_Q(R)^{-1}$ by the above remark. Put $n=|G|$. By Lemma 6, we have $D_Q(R)=n$ for all $Q\in\partial_e P(R^G)$. Then by [2, Cor. 2.7], we see that ${}_{R^G}R\cong n.R^G$.

Next, we consider R as a left $R*G$-module by the rule: $(\sum_{g\in G}r_g g)r=\sum_{g\in G}r_g r^g$. Since it is known that $R\cong R*Ge$ as $R*G$-modules, we have that $R*G\cong n.R$ as $R*G$-modules by Lemma 6. Let $S=R^G[G]$ be an ordinary group ring of G over R^G, which is a left self-injective, regular, subring of $R*G$. Since $R\cong n.R^G$ as left R^G-modules, we have that $R*G\cong n.S$ as left S-modules. On the other hand, since $R*G\cong n.R$ as left $R*G$-modules, we have that $n.R\cong n.S$ as left S-modules. By [1, Th. 10.34], we can conclude that $R\cong S$ as left S-modules.

3. N^*-metric

K.R. Goodearl and D. Handelman have introduced the N^*-metric which is induced by $P(R)$ for a regular ring R. In this section, we shall study the bijectiveness of the map μ^*: $\partial_e P(R*G)\to\partial_e P(R^G)$ for a self-injective regular ring R, using the N^*-metrics of $R*G$, R and R^G.

Let T be a unit-regular ring. We assume that for a given non-zero $x\in T$, there exists $P\in P(T)$ such that $P(x)>0$. For each $x\in T$, according to [7], we define

$$N^*_T(x)=sup\{P(x):\ \ P\in P(T)\}\ .$$

Thus $N^*_T(x)$ is a real number, and $0\leqq N^*_T(x)\leqq 1$. N^*_T induces a metric d^* on T given by the rule $d^*(x,y)=N^*_T(x-y)$, which we call the N^*_T-metric and T becomes a topological ring with respect to N^*_T-metric. If T is complete with respect to N^*_T-metric, T is called N^*_T-complete. It is known that regular rings with bounded index of nilpotence and $\aleph_0$-continuous regular rings are N^*_T-complete [3, Th. 1.3 and Th. 1.8]. We define $ker\ (P(T))=\bigcap_{P\in P(T)} ker\ (P)$.

Lemma 8. *Let R be a unit-regular ring with $ker\ (P(R))=0$ and G a finite subgroup of $Aut(R)$ with $|G|^{-1}\in R$ and let $R*G$ be a skew group ring of G over R. We assume that $R*G$ is a unit-regular ring and R is a finitely generated projective left R^G-module. Then the following hold.*

(1) *$ker\ (P(R*G))=0$ and*

$$N^*_{R*G}(r)\leqq N^*_R(r)\leqq|G|N^*_{R*G}(r)$$

for all $r\in R$.

(2) *There exists a natural number t such that*

$$N_R^*(a) \leqq N_{R^G}^*(a) \leqq tN_R^*(a)$$

for all $a \in R^G$. Consequently, the topology defined by $N_{R^G}^*$-metric are coincide the topology induced by N_R^*-metric on R^G.

Proof. (1) For any $P \in P(R)$, let $t(P) = |G|^{-1} \sum_{g \in G} P^g$, which is a G-invariant pseudo-rank function. By [8, Cor. 4], the extension $t(P)^G$ is a pseudo-rank function on $R*G$ and $t(P)^G|_R = t(P)$. For $x \in ker\ (P(R*G))$, we assume that $R*Gx \cong \oplus_i Rr_i$ as R-modules. Then $t(P)^G(x) = |G|^{-1} \sum_i t(P)(r_i)$ by [8, Cor. 4] and so $t(P)(r_i) = 0$ for all i. Since $P(r_i) \leqq |G| t(P)(r_i)$ by definition, we see that $P(r_i) = 0$ for all i and so that $r_i = 0$ for all i by assumption. Next, for any $r \in R$, we see that

$$P(r) \leqq |G| t(P)(r) = |G| t(P)^G(r) \leqq |G| N_{R*G}^*(r)$$

for any $r \in R$. Therefore $N_R^*(r) \leqq |G| N_{R*G}^*(r)$.

(2) Since R is also a finitely generated, projective, left R^G-module by assumption, let $R \lesssim t.(R^G)$ for some $t > 0$. Then $D_Q(R) \leqq t$ for all $Q \in P(R^G)$. Using Theorem 2, we see that for $Q \in P(R^G)$ and any $a \in R^G$,

$$\begin{aligned} Q(a) &= \mu^*(\lambda^*(Q))\,(a) \\ &= \lambda^*(Q)\,(e)^{-1} \cdot \lambda^*(Q)\,(ea) \\ &\leqq D_Q(R) \cdot \lambda^*(Q)\,(a) \\ &\leqq tN_R^*(a)\,. \end{aligned}$$

Thus we see that $N_{R^G}^*(a) \leqq tN_R^*(a)$ for all $a \in R^G$.

Let $T*G$ be a skew group ring of a finite group G over a ring T such that $|G|^{-1} \in T$ and put $e = |G|^{-1} \sum_{g \in G} g$. M. Lorenz and D.S. Passmann [11] and S. Montgomery [9] have studied the relation between prime ideals of $T*G$, T and T^G. Now we shall study maximal ideals of $T*G$ and T^G, using the manners of [9].

We denote by $\mathrm{Spec}_e(T*G)$ the set of all prime ideals of $T*G$ not containing e and let $\mathrm{I}_e(T*G)$=the set of all ideals of $T*G$ not containing e. There exists a natural map ϕ: $\mathrm{I}_e(T*G) \to$the set of all ideals of T^G, defined by the rule $\phi(M) = \theta(eMe)$, where θ: $eT*Ge \to T^G$ is the isomorphism introduced in §1. In [9], it is shown that ϕ induces a bijection from $\mathrm{Spec}_e(T*G)$ to $\mathrm{Spec}(T^G)$. Therefore ϕ also induces a bijection ϕ': $\overline{\mathrm{Spec}_e}(T*G) \to \mathrm{Max}(T^G)$, where $\overline{\mathrm{Spec}_e}(T*G)$ is the set of $\{M \in \mathrm{Spec}_e(T*G)$: M is maximum in $\mathrm{Spec}_e(T*G)\}$. The following lemma is needed in later propositions.

Lemma 9. *Let T be a ring and G a finite subgroup of $Aut(T)$ with $|G|^{-1} \in T$. The following conditions are equivalent.*

(1) *All $\mathfrak{p} \in \overline{\mathrm{Spec}_e}(T*G)$ are maximal ideals.*

(2) *For any* $m\in \mathrm{Max}(T^G)$, *there exists some* $M\in \mathrm{Max}(T)$ *such that* $M\cap T^G \subset m$.

Proof. (1)$\Rightarrow$(2). For any $m\in \mathrm{Max}(T^G)$, we choose $\mathfrak{p}\in \overline{\mathrm{Spec}_e}(T*G)$ such that $\phi'(\mathfrak{p})=m$. By the assumption of (1) and [11, Lemma 4.2], $\mathfrak{p}\cap T=\cap_{g\in G} M^g$ for some $M\in \mathrm{Max}(T)$. Since $(\cap_{g\in G} M^g)*G\subset \mathfrak{p}$, we see that $M\cap T^G= \phi'((\cap_{g\in G} M^g)*G)\subset \phi'(\mathfrak{p})=m$.

(2)$\Rightarrow$(1). For any $\mathfrak{p}\in \overline{\mathrm{Spec}_e}(T*G)$, put $m=\phi'(\mathfrak{p})$ and choose $M\in \mathrm{Max}(T)$ such that $M\cap T^G\subset m$. Since $\bar{M}=\cap_{g\in G} M^g$ is G-invariant, we see that $\bar{M}*G =\cap_i \mathfrak{g}_i$ for some maximal ideals $\mathfrak{g}_i (i=1, \cdots, t)$ of $T*G$ by [11, Lemma 4.1]. Let $\mathfrak{g}_i (i=1, \cdots, s)$ be the set of all primes in $\{\mathfrak{g}_i\ (i=1, \cdots, t)\}$ not containing e.

Since $\phi(\cap_1^s \mathfrak{g}_i)=\phi(\bar{M}*G)=M\cap T^G\subset m=\phi'(\mathfrak{p})$, we see that $\cap_1^s \mathfrak{g}_i\subset \mathfrak{p}$ by [9, (3) of Lemma 0.2]. By primeness of $\mathfrak{p}$, $\mathfrak{g}_i\subset \mathfrak{p}$ for some i and so $\mathfrak{g}_i=\mathfrak{p}$ by the maximality of $\mathfrak{g}_i$.

Next, for a self-injective regular ring R, we shall consider a condition satisfying (2) of Lemma 9. We note that $R*G$ and R^G are also self-injective regular rings by [12].

Proposition 10. *Let* R *be a left and right self-injective, regular ring and* G *a finite subgroup of* $Aut(R)$ *with* $|G|^{-1}\in R$. *If* R *is a finitely generated projective left* R^G*-module, then, for any* $m\in \mathrm{Max}(R^G)$, *there exists* $M\in \mathrm{Max}(R)$ *such that* $M\cap R^G\subset m$.

Proof. By [5, §II], there exist subgroups $H_1, \cdots, H_s$ of G and orthogonal central idempotents $e_1, \cdots, e_s$ of R such that

(1) for any $f\in B(R)$ such that $fe_i=f$, the stabilizer of f is equal to H_i and the distinct conjugates of f are mutually orthogonal,

(2) $e_1^G+\cdots+e_s^G=1$, where e_i^G is the sum of all distinct conjugates of e_i,

(3) $(Re_i)^{H_i}=(Re_i^G)^G$.

It follows from the assumption that the pair (R, R^G) satisfies (2) of Lemma 8. Then each $(Re_i, (Re_i)^{H_i})$ also satisfies the same one. Therefore it needs only to prove the assertion in the case that any $f\in B(R)$ is G-invariant.

First we consider the topology τ_1, induced by N_R^*-metric on R^G as a subspace. Put $X=\partial_e P(R)$.

(1) $\mathfrak{a}$ is dense in $\cap_{P\in X}(\mathfrak{a}+(ker(P)\cap R^G))$ with respect to τ_1 for any proper ideal $\mathfrak{a}$.

In fact, for $a\in R$, we define a function $\pi(a)\colon X\to[0, 1]$ by the rule: $\pi(a)(N)=N(a)$. Then $\pi(a)$ is a continuous map by the definitions on the topology of X (See, [7]). We choose any $x\in \cap_{P\in X}(\mathfrak{a}+(ker(P)\cap R^G))$ and for each $P\in X$, we put $x=a_P+y_P$, where $a_P\in\mathfrak{a}$ and $y_P\in ker(P)\cap R^G$. For any real number $\varepsilon>0$, $U(y_P)=\pi(y_P)^{-1}([0, \varepsilon\cdot 2^{-1}])$ is a open set for each y_P and it contains P. Then we have

$$X = \bigcup_{P \in X} U(y_P)\,.$$

We note that X is Boolean space by [1]. By compactness and the partition property, there exist finitely many $U(y_{P_i})$ $i=1, \cdots, t$ corresponding to y_{P_i} and mutually disjoint clopon sets $W_i \subset U(y_{P_i})$ such that $X = \bigcup_1^t W_i$. For the set $\{W_i\colon i=1, \cdots, t\}$, there exists mutually orthogonal central idempotents $\{e_i\colon i=1, \cdots, t\}$ of R such that $W_i = \{N \in X\colon N(e_i)=1\}$ by [1]. Since $e_i \in R^G$ $(i=1, \cdots, t)$, $a = \sum_i e_i \cdot a_{P_i}$ is contained in $\mathfrak{a}$. For any $P \in X$, there exists only one W_i such that $P \in W_i$. Then we see that $P(e_i)=1$ and $P(e_j)=0$, for all $j \neq i$ and so we see that

$$\begin{aligned} P(a-x) &= \left(\textstyle\sum_{j\neq i} P(e_j\, a_{P_j})\right) + P((e_i-1)\, a_{P_i}) + P(y_{P_i}) \\ &< P(y_{P_i}) \\ &< \varepsilon \cdot 2^{-1}\,. \end{aligned}$$

As a result, $N_R^*(a-x) < \varepsilon\, 2^{-1} < \varepsilon$.

(2) For any $m \in \mathrm{Max}(R^G)$, $m = \bigcap_{P \in X}(m + (ker P \cap R^G))$.

In fact, since R^G is complete with respect to the topology τ_2 defined by $N_{R^G}^*$-metric by [3, Th. 1.8], m is closed with respect to τ_2 by [3, Th. 1.13 and Cor. 1.14]. Since $\tau_1 = \tau_2$ by Lemma 8, m is closed with respect to τ_1. Then we can conclude that $m = \bigcap_{P \in X}(m + (ker(P) \cap R^G))$ by (1).

(3) For any $m \in \mathrm{Max}(R^G)$, there exists some $P \in X$ such that $m + (ker(P) \cap R^G) \neq R^G$ by (2) and so $m = m + (ker(P) \cap R^G) \supset ker(P) \cap R^G$. By [4, II. 14.5], $ker(P)$ is a maximal ideal of R.

Theorem 11. *Let R be a left and right self-injective regular ring and G a finite subgroup of $Aut(R)$ with $|G|^{-1} \in R$. Assume that R is a finitely generated projective left R^G-module. Let μ^*, λ^* be the maps defined in* §1. *Then $\mu^*\colon \partial_e P(R*G)\backslash F \to \partial_e P(R^G)$ is a bijection and $(\mu^*)^{-1} = \lambda^*$.*

Proof. We shall consider the following diagram:

$$\begin{array}{ccc} \partial_e P(R*G)/F & \xrightarrow{\ \mu^*\ } & \partial_e P(R^G) \\ \Big\downarrow \pi_1 & & \Big\downarrow \pi_1 \\ \overline{\mathrm{Spec}}_e(R*G) & \xrightarrow{\ \phi^*\ } & \mathrm{Max}(R^G) \end{array}$$

where $\pi_i (i=1, 2)$ is the map defined by $\pi_i(N) = ker\,(N)$. By Lemma 9 and Proposition 10, any $\mathfrak{p} \in \overline{\mathrm{Spec}}_e(R*G)$ is a maximal ideal and so π_i $(i=1, 2)$ is a bijection by [4, II. 14.5]. It is easy to prove that the above diagram is commutative. Then we have that μ^* is a bijection and $(\mu^*)^{-1} = \lambda^*$.

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