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## SEMI-PERFECT QF-3 AND PP-RINGS

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Let  $R$  be a ring with unit and  $N$  be the radical of  $R$ . An  $R$ -module  $M$  is a *minimal faithful*  $R$ -module if  $M$  is faithful and no proper summand of  $M$  is faithful.  $R$  is *left QF-3* if  $R$  has a unique minimal faithful left  $R$ -module (up to isomorphism).  $R$  is *semi-perfect* if  $R/N$  has minimum condition and idempotents can be lifted modulo  $N$  (see [1]). Let  $1 = \sum E_i$  be a decomposition of the identity of a semi-perfect ring  $R$  into a sum of mutually orthogonal idempotents such that  $E_i$  modulo  $N$  is the identity element of a simple component of  $R/N$ . Following Harada [4], we call  $R$  a *partially PP-ring* if  $Rx$  is  $R$ -projective for all  $x \in E_i R E_j$ .

Mochizuki [7] studied the double centralizer of a minimal faithful left module for a hereditary QF-3 algebra of finite rank over a field. In [4] Harada applied his theory of generalized triangular matrix rings to extend Mochizuki's results to left QF-3 and semi-primary partially PP-rings. The purpose of this note is to give a direct proof of Harada's results which extends them to semi-perfect rings.

**Theorem.** *Let  $R$  be a semi-perfect left QF-3 and partially PP-ring.*

1.  *$R$  contains an idempotent  $e$  such that  $Re$  is (isomorphic to) the minimal faithful left  $R$ -module. Furthermore,  $eN=0$  and if  $e'$  is any primitive idempotent of  $R$  such that  $e'N=0$ , then  $Re'$  is isomorphic to a summand of  $Re$ .*
2.  *$R$  is right QF-3.*
3. *If  $Rf, f^2=f$ , is any faithful projective, injective left ideal of  $R$  and  $B = \text{Hom}_{fRf}(Rf, Rf)$ , where  $Rf$  is regarded as a right  $fRf$  module, then both  $eRe$  and  $B$  are semi-simple rings with minimum condition. Furthermore,  $B$  is the left and right injective hull of  $R$  regarded as an  $R$ -module and  $B$  is  $R$ -projective.*
4. *If  $R$  is left hereditary, then  $R$  is a generalized uniserial ring.*

**Proof.** Let  $S_1, \dots, S_n$  be one of each isomorphism type of simple left  $R$ -modules and  $e_1, \dots, e_n$  be a complete set of non-isomorphic primitive idempotents of  $R$ . Then  $Re_1 + \dots + Re_n$  and the injective hull of  $S_1 + \dots + S_n$ ,  $E(S_1 + \dots + S_n) = E(S_1) + \dots + E(S_n)$ , are easily seen to be faithful  $R$ -modules. Since each  $E(S_i)$  is an indecomposable injective it has a local endomorphism ring. By renumbering we may assume  $S_1, \dots, S_k$  is a subset of  $S_1, \dots, S_n$  minimal with

respect to  $E(S_1) + \cdots + E(S_k)$  being faithful and by the Krull-Schmidt theorem this module is a minimal faithful module. Since each  $e_i Re_i$  is a local ring we may apply similar reasoning to obtain a minimal faithful module of the form  $Re_1 + \cdots + Re_t$ . Since  $R$  is left QF-3,  $E(S_1) + \cdots + E(S_k) \cong Re_1 + \cdots + Re_t$  and again by the Krull-Schmidt theorem we have  $k=t$  and a permutation  $\pi$  such that  $Re_i \cong E(S_{\pi(i)})$  for  $i=1, \dots, k$ . Thus we may take  $e=e_1 + \cdots + e_k$ .

We observe that if  $g$  and  $h$  are primitive idempotents of  $R$  and if  $gsh \neq 0$  with  $s \in R$  then the map of  $Rg$  into  $Rh$  given by  $rg \rightarrow rg^2sh$  is a monomorphism. If not the kernel would be a proper summand of  $Rg$  since the image  $Rgsh$  is  $R$ -projective as  $R$  is partially PP. But this contradicts the fact that  $g$  is primitive. For each  $i=1, \dots, k$ ,  $e_i N = 0$  since otherwise  $e_i Ne_j \neq 0$  for some  $j=1, \dots, k$ , which implies  $Re_i$  is isomorphic to a submodule of  $Ne_j \subsetneq Re_j$ . This is a contradiction since  $Re_i$  is injective and  $Re_j$  is indecomposable. Thus  $eN=0$ . Also if  $e'$  is any primitive idempotent of  $R$ ,  $e' Re_i \neq 0$  for some  $i=1, \dots, k$ , and so  $Re'$  is isomorphic to a submodule of  $Re_i$  and hence  $Re'$  contains a unique minimal left ideal which is essential in  $Re'$ . Furthermore, if  $e'N=0$  the above map is an isomorphism since  $Re_i/Ne_i$  is simple. Since  $R$  is a finite sum of primitive left ideals,  $R$  has an essential left socle  $E$  which is a finite sum of simple modules. Moreover, the right annihilator of  $E$  is zero, since if  $Ex=0$  with  $x \neq 0$  there exist primitive idempotents  $g, h$  such that  $gxh \neq 0$ . Then the left annihilator of  $gxh$  contains  $E$  and is a proper summand of  $R$  since it is the kernel of the map of  $R$  onto the  $R$ -projective module  $Rgxh$  given by  $r \rightarrow rgxh$ . This is a contradiction as  $E$  is essential in  $R$ .

Let  $Q = \text{Hom}_R({}_R E, {}_R E)$ . Then  $Q$  is a semi-simple ring with minimum condition since  ${}_R E$  is a finite sum of simple  $R$ -modules. Now note that  $\lambda: R \rightarrow Q$  by  $(s)(r)\lambda = sr$ ,  $r \in R$  and  $s \in E$  is a unital ring monomorphism. Furthermore, if  $q \in Q$  and  $s, s' \in E$

$$(s')[(s)\lambda q] = (s's)q = s'(sq) = (s')[(sq)\lambda].$$

Hence  $\lambda$  restricted to  $E$  is a right  $Q$ -monomorphism and we have  $(E)\lambda \subseteq (E)\lambda Q \subseteq (EQ)\lambda \subseteq E$ . Thus we may regard  $R$  as a unital subring of  $Q$  containing  $E$  which is a faithful right ideal of  $Q$ . Thus  ${}_R Q$  is an essential extension of  ${}_R R$ . Now let  $Rf, f^2=f$ , be any faithful injective left ideal of  $R$ . Since  ${}_R Qf$  is essential over  ${}_R Rf$ , we have  $Qf = Rf$  and so  $Rf$  is a faithful left ideal of  $Q$ . Thus  $Q_R$  is essential over  $R_R$ . Also  $fRf = fQf$  and so is semi-simple with minimum condition. Moreover,  $B = \text{Hom}_{fRf}(Rf, Rf) = \text{Hom}_{fQf}(Qf, Qf) = Q$  since  $Qf$  is a faithful left ideal of  $Q$ .

We now show that  $E_R$  is  $R$ -injective. Let  $J$  be any right ideal of  $R$  and  $\alpha: J \rightarrow E_R$  be an  $R$ -homomorphism. If  $q_i \in Q$  and  $a_i \in J$ , let  $(\sum a_i q_i)\alpha = \sum (a_i)\alpha q_i$ . Suppose  $\sum a_i q_i = 0$ . Then for any  $rf \in Rf$ ,  $q_i rf \in Qf = Rf$  and so

$$0 = (0)\alpha = ((\sum a_i q_i)rf)\alpha = \sum (a_i)\alpha q_i rf = (\sum (a_i)\alpha q_i)rf.$$

Since  ${}_Q Rf$  is faithful we see that  $\bar{\alpha}$  is a well defined  $Q$ -homomorphism of  $JQ$  into  $E$ . Since  $JQ$  is a summand of  $Q_Q$  there exists  $s \in E$  such that  $(t)\bar{\alpha} = st$  for all  $t \in JQ$  and so  $(j)\alpha = (j)\bar{\alpha} = sj$  for all  $j \in J$ . Thus  $E_R$  is  $R$ -injective and hence also  $R$ -projective. Now  ${}_Q Rf$  (resp.  $E_Q$ ) is a faithful left (resp. right) ideal of  $Q$ . Thus  ${}_Q Q$  (resp.  $Q_Q$ ) is a  $Q$ -direct summand of a direct sum of copies of  ${}_Q Rf$  (resp.  $E_Q$ ) and so is certainly an  $R$ -summand. Thus  ${}_R Q$  (resp.  $Q_R$ ) is  $R$ -projective and injective and being essential over  ${}_R R$  (resp.  $R_R$ ) is the injective hull.

Since  $ReR$  has zero left annihilator, where  $e$  is as in statement 1 of the theorem, it is essential in  $R_R$  and so since  $(ReR)N = 0$ , it is the right socle of  $R$ . Now since  $R$  contains an essential right socle and a faithful projective injective right ideal one can use a standard argument to conclude that the injective hull of the direct sum of one copy of each isomorphism class of simple right ideals of  $R$  is a unique minimal faithful right  $R$ -module (see [6]). Thus  $R$  is right QF-3.

Now suppose  $R$  is left hereditary. Bass [1] has shown that if  $P$  is a non-zero projective module over any ring with radical  $N$ ,  $NP \neq P$ . We have  $Re \supseteq Ne \supseteq \dots \supseteq N^t e \supseteq \dots$  which is a decreasing sequence of  $eRe$  submodules of  $Re$  and since  $Re$  is finitely generated over  $eRe$  it must eventually be constant, say  $N^t e = N^{t+1} e = \dots$ . But since  $N^t e$  is  $R$ -projective Bass' result implies that  $N^t e = 0$  and  $Re$  being faithful, we have  $N^t = 0$ . Thus  $R$  is semi-primary and hence also right hereditary. Thus by what has already been established the conditions on  $R$  are symmetric. We will, therefore, show only that  $R$  is left generalized uniserial. For this it suffices to show that if  $Rg$ ,  $g^2 = g$ , is any primitive left ideal of  $R$  and  $i$  is any positive integer for which  $N^i g \neq 0$ , then  $N^i g / N^{i+1} g$  is simple. However,  $Rg$  contains a unique minimal left ideal which is essential and so  $N^i g$  is an indecomposable projective  $R$ -module. Since  $R$  is semi-primary this implies that  $N^i g$  is isomorphic to a primitive left ideal of  $R$  (see [2]) and so  $N(N^i g) = N^{i+1} g$  is the unique maximal left ideal of  $N^i g$ . This completes the proof of the theorem.

REMARK 1. If  $R$  satisfies the hypothesis of the theorem one can easily show that  $R = R_1 + \dots + R_n$ , where the  $R_i$  are indecomposable ideals of  $R$  which are semi-perfect QF-3 and partially PP-rings. Each  $R_i$  contains a unique primitive idempotent  $e_i$  (up to isomorphism) such that  $e_i N_i = 0$  where  $N_i$  is the radical of  $R_i$ . Furthermore  $e_i R e_i$  is a division ring and  $B_i = \text{Hom}_{e_i R e_i} (R_i e_i, R_i e_i) = (e_i R e_i)_{n_i}$ . When  $R$  is hereditary each  $R_i$  is a complete blocked triangular matrix ring over a division ring (see Goldie [3] or Harada [4]).

REMARK 2. With minor modifications the above proof serves to establish the conclusions of the theorem for semi-perfect left QF-3 rings with zero left singular ideal which contain no infinite direct sum of left (right) ideals. Furthermore, these conditions are easily seen to be necessary as well as sufficient. In this connection Harada [5, p. 23] has given an interesting example of a semi-

primary left QF-3 ring with zero left and right singular ideals for which the conclusions of the theorem fail almost entirely.

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