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SEMI-PERFECT QF-3 AND PP-RINGS

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Let R be a ring with unit and N be the radical of R. An R-module M is a *minimal faithful* R-module if M is faithful and no proper summand of M is faithful. R is *left QF*-3 if R has a unique minimal faithful left R-module (up to isomorphism). R is *semi-perfect* if R/N has minimum condition and idempotents can be lifted modulo N (see [1]). Let $1=\Sigma E_i$ be a decomposition of the identity of a semi-perfect ring R into a sum of mutually orthogonal idempotents such that E_i modulo N is the identity element of a simple component of R/N. Following Harada [4], we call R a *partially PP-ring* if Rx is R-projective for all $x \in E_i R E_j$.

Mochizuki [7] studied the double centralizer of a minimal faithful left module for a hereditary QF-3 algebra of finite rank over a field. In [4] Harada applied his theory of generalized triangular matrix rings to extend Mochizuki's results to left QF-3 and semi-primary partially PP-rings. The purpose of this note is to give a direct proof of Harada's results which extends them to semi-perfect rings.

Theorem. *Let R be a semi-perfect left QF-3 and partially PP-ring.*
1. *R contains an idempotent e such that Re is (isomorphic to) the minimal faithful left R-module. Furthermore, $eN=0$ and if e' is any primitive idempotent of R such that $e'N=0$, then Re' is isomorphic to a summand of Re.*
2. *R is right QF-3.*
3. *If Rf, $f^2=f$, is any faithful projective, injective left ideal of R and $B=Hom_{fRf}(Rf, Rf)$, where Rf is regarded as a right fRf module, then both eRe and B are semi-simple rings with minimum condition. Furthermore, B is the left and right injective hull of R regarded as an R-module and B is R-projective.*
4. *If R is left hereditary, then R is a generalized uniserial ring.*

Proof. Let $S_1, \cdots, S_n$ be one of each isomorphism type of simple left R-modules and $e_1, \cdots, e_n$ be a complete set of non-isomorphic primitive idempotents of R. Then $Re_1+\cdots+Re_n$ and the injective hull of $S_1+\cdots+S_n$, $E(S_1+\cdots+S_n)=E(S_1)+\cdots+E(S_n)$, are easily seen to be faithful R-modules. Since each $E(S_i)$ is an indecomposable injective it has a local endomorphism ring. By renumbering we may assume $S_1, \cdots, S_k$ is a subset of $S_1, \cdots, S_n$ minimal with respect to $E(S_1)+\cdots+E(S_k)$ being faithful and by the Krull-Schmidt theorem this module is a minimal faithful module. Since each e_iRe_i is a local ring we may apply similar reasoning to obtain a minimal faithful module of the form $Re_1+\cdots+Re_t$. Since R is left QF-3, $E(S_1)+\cdots+E(S_k)\cong Re_1+\cdots+Re_t$ and again by the Krull-Schmidt theorem we have $k=t$ and a permutation π such that $Re_i\cong E(S_{\pi(i)})$ for $i=1, \cdots, k$. Thus we may take $e=e_1+\cdots+e_k$.

We observe that if g and h are primitive idempotents of R and if $gsh\neq 0$ with $s\in R$ then the map of Rg into Rh given by $rg\rightarrow rg^2sh$ is a monomorphsm. If not the kernel would be a proper summand of Rg since the image $Rgsh$ is R-projective as R is partially PP. But this contradicts the fact that g is primitive. For each $i=1, \cdots, k$, $e_iN=0$ since otherwise $e_iNe_j\neq 0$ for some $j=1, \cdots, k$, which implies Re_i is isomorphic to a submodule of $Ne_j\subsetneqq Re_j$. This is a contradiction since Re_i is injective and Re_j is indecomposable. Thus $eN=0$. Also if e' is any primitive idempotent of R, $e'Re_i\neq 0$ for some $i=1, \cdots, k$, and so Re' is isomorphic to a submodule of Re_i and hence Re' contains a unique minimal left ideal which is essential in Re'. Furthermore, if $e'N=0$ the above map is an isomorphism since Re_i/Ne_i is simple. Since R is a finite sum of primitive left ideals, R has an essential left socle E which is a finite sum of simple modules. Moreover, the right annihilator of E is zero, since if $Ex=0$ with $x\neq 0$ there exist primitive idempotents g, h such that $gxh\neq 0$. Then the left annihilator of gxh contains E and is a proper summand of R since it is the kernel of the map of R onto the R-projective module $Rgxh$ given by $r\rightarrow rgxh$. This is a contradiction as E is essential in R.

Let $Q=\mathrm{Hom}_R({}_RE, {}_RE)$. Then Q is a semi-simple ring with minimum condition since ${}_RE$ is a finite sum of simple R-modules. Now note that λ: $R\rightarrow Q$ by $(s)(r)\lambda=sr$, $r\in R$ and $s\in E$ is a unital ring monomorphism. Furthermore, if $q\in Q$ and $s, s'\in E$

$$(s')[(s)\lambda q]=(s's)q=s'(sq)=(s')[(sq)\lambda]\,.$$

Hence λ restricted to E is a right Q-monomorphism and we have $(E)\lambda\subseteq(E)\lambda Q\subseteq(EQ)\lambda\subseteq E$. Thus we may regard R as a unital subring of Q containing E which is a faithful right ideal of Q. Thus ${}_RQ$ is an essential extension of ${}_RR$. Now let Rf, $f^2=f$, be any faithful injective left ideal of R. Since ${}_RQf$ is essential over ${}_RRf$, we have $Qf=Rf$ and so Rf is a faithful left ideal of Q. Thus Q_R is essential over R_R. Also $fRf=fQf$ and so is semi-simple with minimum condition. Moreover, $B=\mathrm{Hom}_{fRf}(Rf, Rf)=\mathrm{Hom}_{fQf}(Qf, Qf)=Q$ since Qf is a faithful left ideal of Q.

We now show that E_R is R-injective. Let J be any right ideal of R and $\alpha\colon J\rightarrow E_R$ be an R-homomorphism. If $q_i\in Q$ and $a_i\in J$, let $(\Sigma a_iq_i)\bar{\alpha}=\Sigma(a_i)\alpha q_i$. Suppose $\Sigma a_iq_i=0$. Then for any $rf\in Rf$, $q_irf\in Qf=Rf$ and so

$$0=(0)\alpha=((\Sigma a_iq_i)rf)\alpha=\Sigma(a_i)\alpha q_irf=(\Sigma(a_i)\alpha q_i)rf\,.$$

Since ${}_QRf$ is faithful we see that $\bar{\alpha}$ is a well defined Q-homomorphism of JQ into E. Since JQ is a summand of Q_Q there exists $s \in E$ such that $(t)\bar{\alpha}=st$ for all $t \in JQ$ and so $(j)\alpha=(j)\bar{\alpha}=sj$ for all $j \in J$. Thus E_R is R-injective and hence also R-projective. Now ${}_QRf$ (resp. E_Q) is a faithful left (resp. right) ideal of Q. Thus ${}_QQ$(resp. Q_Q) is a Q-direct summand of a direct sum of copies of ${}_QRf$ (resp. E_Q) and so is certainly an R-summand. Thus ${}_RQ$ (resp. Q_R) is R-projective and injective and being essential over ${}_RR$ (resp. R_R) is the injective hull.

Since ReR has zero left annihilator, where e is as in statement 1 of the theorem, it is essential in R_R and so since $(ReR)N=0$, it is the right socle of R. Now since R contains an essential right socle and a faithful projective injective right ideal one can use a standard argument to conclude that the injective hull of the direct sum of one copy of each isomorphism class of simple right ideals of R is a unique minimal faithful right R-module (see [6]). Thus R is right QF-3.

Now suppose R is left hereditary. Bass [1] has shown that if P is a non-zero projective module over any ring with radical N, $NP \neq P$. We have $Re \supseteq Ne \supseteq \cdots \supseteq N^ie \supseteq \cdots$ which is a decreasing sequence of eRe submodules of Re and since Re is finitely generated over eRe it must eventually be constant, say $N^le=N^{l+1}e=\cdots$. But since N^le is R-projective Bass' result implies that $N^le=0$ and Re being faithful, we have $N^l=0$. Thus R is semi-primary and hence also right hereditary. Thus by what has already been established the conditions on R are symmetric. We will, therefore, show only that R is left generalized uniserial. For this it suffices to show that if Rg, $g^2=g$, is any primitive left ideal of R and i is any positive integer for which $N^ig \neq 0$, then $N^ig/N^{i+1}g$ is simple. However, Rg contains a unique minimal left ideal which is essential and so N^ig is an indecomposable projective R-module. Since R is semi-primary this implies that N^ig is isomorphic to a primitive left ideal of R (see [2]) and so $N(N^ig)=N^{i+1}g$ is the unique maximal left ideal of N^ig. This completes the proof of the theorem.

REMARK 1. If R satisfies the hypothesis of the theorem one can easily show that $R=R_1+\cdots+R_n$, where the R_i are indecomposable ideals of R which are semi-perfect QF-3 and partially PP-rings. Each R_i contains a unique primitive idempotent e_i (up to isomorphism) such that $e_iN_i=0$ where N_i is the radical of R_i. Furthermore e_iRe_i is a division ring and $B_i=\mathrm{Hom}_{e_iRe_i}(R_ie_i, R_ie_i)=(e_iRe_i)_{n_i}$. When R is hereditary each R_i is a complete blocked triangular matrix ring over a division ring (see Goldie [3] or Harada [4]).

REMARK 2. With minor modifications the above proof serves to establish the conclusions of the theorem for semi-perfect left QF-3 rings with zero left singular ideal which contain no infinite direct sum of left (right) ideals. Furthermore, these conditions are easily seen to be necessary as well as sufficient. In this connection Harada [5, p. 23] has given an interesting example of a semi-primary left QF-3 ring with zero left and right singular ideals for which the conclusions of the theorem fail almost entirely.

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