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ON THE SCHUR INDICES OF THE FINITE UNITARY GROUPS

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Introduction

Let $\boldsymbol{F}_q$ denote the finite group of characteristic p with q elements. We consider the finite unitary group $U(n, q^2)$ of rank n relative to the quadratic extension $\boldsymbol{F}_{q^2}/\boldsymbol{F}_q$. For a complex irreducible character χ of a finite group, the Schur index of χ with respect to the field $\boldsymbol{Q}$ of rational numbers is defined to be the minimal degree among all the extensions $K/\boldsymbol{Q}(\chi)$ such that χ is realizable in K. Here $\boldsymbol{Q}(\chi)$ is the extension of $\boldsymbol{Q}$ generated by the values of χ. We denote this index by $m_{\boldsymbol{Q}}(\chi)$. In this paper, we shall determine the Schur indices of all the complex irreducible characters of $U(n, q^2)$ for sufficiently large p and q. Our main result is the following theorem.

Main Theorem. *Assume that p and q are sufficiently large. Then the Schur index of any complex irreducible character of $U(n, q^2)$ with respect to the field of rational numbers is one.*

REMARK. If $n \leq 5$, it is enough only to assume $p \neq 2$ (see §2).

The theorem follows from

Theorem A (R. Gow [3], p 112). *For any complex irreducible character χ of $U(n, q^2)$, $m_{\boldsymbol{Q}}(\chi)$ divides 2.*

Theorem B. *The values of any complex irreducible character of $U(n, q^2)$ on unipotent elements are rational integers and its Schur index divides these values.*

This will be proved in Section 4.

Theorem C. *Assume that p and q are sufficiently large (if $n \leq 5$, this assumption can be dropped out). Then for any complex irreducible character χ of $U(n, q^2)$, there is a unipotent element u of $U(n, q^2)$ such that $\chi(u)$ is equal to the p-part of the degree of χ up to sign.*

This will be proved in Section 2.

I wish to thank Professor R. Hotta for giving me his preprint and for his kind advice.

NOTATION. $\boldsymbol{Q}$ is the field of rational numbers. All characters are complex ones. A *character* of a finite group means a non-negative integral linear combination of irreducible characters of the group.

1. The Schur index

To determine the Schur indices of irreducible characters of $U(n, q^2)$, we shall use the following property of the index.

Lemma 1.1. ([1], (70.21)). *Let H be a finite group and let ξ be a character of H which is realizable in $\boldsymbol{Q}$. Then, if χ is an irreducible character of H, $m_{\boldsymbol{Q}}(\chi)$ divides the intertwining number (ζ, χ).*

2. Ennola's conjecture implies Theorem C

In [2], V. Ennola stated the following conjecture (cf. [8]).

Conjecture of Ennola. *The system of irreducible characters of $U(n, q^2)$ coincides with that of irreducible C-functions, which are obtained from the irreducible characters of the general linear group $GL(n, q)$ by the simple formal change that q is everywhere replaced by $-q$.*

This is checked by himself in [2] for $n\leq 3$ and by S. Nozawa [8], [9] for $n=4$, 5. Recently, G. Lusztig, B. Srinivasan, R. Hotta, D. Kazhdan and T.A. Springer have proved the conjecture for sufficiently large p and q (See [5], [7]). Thus Theorem C follows from Theorem C of [10], which is the counterpart for $GL(n, q)$.

3. Some lemmas on representation theory of algebraic groups

Let G be a connected, reductive linear algebraic group defined over $\boldsymbol{F}_q$ and F the corresponding Frobenius endomorphism. Then G^F is the finite group of $\boldsymbol{F}_q$-rational points in G. Let Z denote the centre of G. Throughout this section, we shall assume that Z is connected and that p is not a bad prime for G for all the simple components of G.

Lemma 3.1. *Let S be a Sylow p-subgroup of G^F. Then, if λ is a linear character of S, $Ind_S^{G^F}(\lambda)$ is a character of G^F which is realizable in $\boldsymbol{Q}$.*

For a proof, see [11], Cor. 2.3..

Corollary 3.2. *If χ is an irreducible character of $U(n, q^2)$ of degree coprime to p, then $m_{\boldsymbol{Q}}(\chi)$ is equal to* 1.

Proof. Since the centre of the unitary group is connected, (3.2) follows from the main theorem of [11].

Recall that an element x of G (G is reductive) is called *regular* if $Z_G(x)$ (=the centralizer of x) has the minimal dimension.

Lemma 3.3. (i) *G^F contains a regular unipotent element of G.*
(ii) *The set of regular unipotent elements in G^F form a single conjugacy class.*

This is well known.

Corollary 3.4. *Any character of G^F takes rational integral values on regular unipotent elements.*

Proof. If u is a regular unipotent element in G^F, for any integer k coprime to p, u^k is also regular; u and u^k are conjugate in G^F (by (3.3)). Then (3.4) follows from [10], (1.1) lemma or from [4], Lemma 2.

Lemma 3.5. *If U is an F-stable maximal unipotent subgroup of G and ρ is an irreducible character of U^F of degree greater than one, we have that $\rho(u)=0$ for any regular unipotent element u in U^F.*

Proof. Since the image of U in G/Z is isomorphic to U, we may assume that Z is trivial. Then (3.5) follows from Theorems A, A′ of [6].

REMARK. In (3.5), the assumption that Z is connected is not needed.

4. Proof of Theorem B

Let $G=U_n$ $(=GL_n)$ be the unitary group of rank n defined over $\boldsymbol{F}_q$. The Frobenius F is given by $F((x_{ij}))={}^t(x_{ij}^q)^{-1}$. We also introduce an endomorphism F_0 of U_n defined by $F_0((x_{ij}))=(x_{ij}^{q^2})$. Then $U_n^{F_0}=GL(n, q^2)$.

Lemma 4.1. *Two elements of $U(n, q^2)$ are conjugate in $U(n, q^2)$ if and only if they are conjugate in $GL(n, q^2)$.*

For a proof, see [12], I.3.6.; also see [2].

Now we prove Theorem B. Let u be a unipotent element of U_n^F, $\mu=(\mu_1, \mu_2, \cdots, \mu_s)$ the corresponding partition of n and P_μ the standard parabolic subgroup of U_n corresponding to μ i.e.

$$P_\mu=\left\{\begin{pmatrix} A_{11} & & * & \\ & A_{22} & & \\ 0 & & \ddots & \\ & & & A_{ss}\end{pmatrix}; \ A_{ii}\in U_{\mu_i}\right\}.$$

Then $L=U_{\mu_1}\times\cdots\times U_{\mu_s}$, which is embedded diagonally into P_μ, is an F-stable

Levi-subgroup of P; in particular, L is reductive and connected. Let u_0 be an F-stable regular unipotent element of L (such element exists by (3.3)) and let u_μ be the Jordan canonical form of u. We may assume that u_μ belongs to L. Since the centre of L is connected and since u_μ is a regular unipotent element of L, u_μ and u_0 are conjugate in $L^{F_0}=GL(\mu_1, q^2)\times\cdots\times GL(\mu_s, q^2)$. (Note that u_0 and u_μ are rational over $\boldsymbol{F}_{q^2}$). Then u and u_0 are conjugate in $GL(n, q^2)$; by (4.1), they are conjugate in U_n^F. We may assume u to be u_0. Let U be the unipotent radical of the F-stable Borel subgroup of L containing u. Then U is F-stable and contains u. Now let χ be an irreducible character of U_n^F. Then we have that

$$\chi\,|\,U^F=\sum_\lambda a_\lambda\cdot\lambda+\sum_\rho b_\rho\cdot\rho\,,$$

where the first summation is over all the linear characters of U^F and the second summation is over all the non-linear irreducible characters of U^F. By (3.5), we have

$$\chi(u)=\sum_\lambda a_\lambda\cdot\lambda(u)\,.$$

Since $a_\lambda=(\chi, \operatorname{Ind}_{U^F}^{U_n^F}(\lambda))$ and $\operatorname{Ind}_{U^F}^{U_n^F}(\lambda)=\operatorname{Ind}_{L^F}^{U_n^F}(\operatorname{Ind}_{U^F}^{L^F}(\lambda))$ is realizable in $\boldsymbol{Q}$ by (3.1), we see by (1.1) that $m_{\boldsymbol{Q}}(\chi)$ divides a_λ. We can rewrite the above expression as

$$\chi(u)/m_{\boldsymbol{Q}}(\chi)=\sum_\lambda (a_\lambda/m_{\boldsymbol{Q}}(\chi))\cdot\lambda(u)\,.$$

In this expression, the right hand side is an algebraic integer. Then, to prove Theorem B, it suffices to show that $\chi(u)$ is rational. But any character of L^F takes a rational integral value on u(by (3.4)) and the restriction of χ to L^F is a non-negative integral linear combination of irreducible characters of L^F; hence $\chi(u)$ is an integer. This completes the proof of Theorem B.

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References

[1] C.W. Curtis and I. Reiner: Representation theory of finite groups and associative algebras, John Wiley and Sons, Interscience, New York, 1962.

[2] V. Ennola: *On the characters of the finite unitary groups*, Ann. Acad. Sci. Fenn. **323** (1963), 1–34.

[3] R. Gow: *Schur indices of some groups of Lie type*, J. Algebra **42** (1976), 102–120.

[4] J.A. Green, G.I. Lehrer and G. Lusztig: *On the degrees of certain group characters*, Quart. J. Math. Oxford (2) **27** (1976), 1–4.

[5] R. Hotta and T.A. Springer: *A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups*, to appear.

[6] G.I. Lehrer: *Adjoint groups, regular unipotent elements and discrete series characters*, Trans. Amer. Math. Soc. **214** (1975), 249–260.

[7] G. Lusztig and B. Srinivasan: *The characters of the finite unitary groups*, to appear.

[8] S. Nozawa: *On the characters of the finite general unitary group* $U(4, q^2)$, J. Fac. Sci. Univ. Tokyo **19** (1972), 257–293.

[9] ————: *On the characters of the finite general unitary group* $U(5, q^2)$, J. Fac. Sci. Univ. Tokyo **23** (1976), 23–74.

[10] Z. Ohmori: *On the Schur indices of* $GL(n, q)$ *and* $SL(2n+1, q)$, J. Math. Soc. Japan **29** (1977), 693–707.

[11] ————: *On the Schur indices of reductive groups*, Quart. J. Math. Oxford (2) **28** (1977), 357–361.

[12] T.A. Springer and R. Steinberg: *Conjugacy classes* (Springer Lecture Notes Series 131), Springer-Verlag, New York/Berlin, 1970, 167–266.