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Author(s)	Ohmori, Zyozyu
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Osaka University

## ON THE SCHUR INDICES OF THE FINITE UNITARY GROUPS

ZYOZYU OHMORI

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### Introduction

Let  $F_q$  denote the finite group of characteristic  $p$  with  $q$  elements. We consider the finite unitary group  $U(n, q^2)$  of rank  $n$  relative to the quadratic extension  $F_{q^2}/F_q$ . For a complex irreducible character  $\chi$  of a finite group, the Schur index of  $\chi$  with respect to the field  $\mathbb{Q}$  of rational numbers is defined to be the minimal degree among all the extensions  $K/\mathbb{Q}(\chi)$  such that  $\chi$  is realizable in  $K$ . Here  $\mathbb{Q}(\chi)$  is the extension of  $\mathbb{Q}$  generated by the values of  $\chi$ . We denote this index by  $m_{\mathbb{Q}}(\chi)$ . In this paper, we shall determine the Schur indices of all the complex irreducible characters of  $U(n, q^2)$  for sufficiently large  $p$  and  $q$ . Our main result is the following theorem.

**Main Theorem.** *Assume that  $p$  and  $q$  are sufficiently large. Then the Schur index of any complex irreducible character of  $U(n, q^2)$  with respect to the field of rational numbers is one.*

REMARK. If  $n \leq 5$ , it is enough only to assume  $p \neq 2$  (see §2).

The theorem follows from

**Theorem A** (R. Gow [3], p 112). *For any complex irreducible character  $\chi$  of  $U(n, q^2)$ ,  $m_{\mathbb{Q}}(\chi)$  divides 2.*

**Theorem B.** *The values of any complex irreducible character of  $U(n, q^2)$  on unipotent elements are rational integers and its Schur index divides these values.*

This will be proved in Section 4.

**Theorem C.** *Assume that  $p$  and  $q$  are sufficiently large (if  $n \leq 5$ , this assumption can be dropped out). Then for any complex irreducible character  $\chi$  of  $U(n, q^2)$ , there is a unipotent element  $u$  of  $U(n, q^2)$  such that  $\chi(u)$  is equal to the  $p$ -part of the degree of  $\chi$  up to sign.*

This will be proved in Section 2.

I wish to thank Professor R. Hotta for giving me his preprint and for his kind advice.

NOTATION.  $\mathbb{Q}$  is the field of rational numbers. All characters are complex ones. A *character* of a finite group means a non-negative integral linear combination of irreducible characters of the group.

### 1. The Schur index

To determine the Schur indices of irreducible characters of  $U(n, q^2)$ , we shall use the following property of the index.

**Lemma 1.1.** ([1], (70.21)). *Let  $H$  be a finite group and let  $\xi$  be a character of  $H$  which is realizable in  $\mathbb{Q}$ . Then, if  $\chi$  is an irreducible character of  $H$ ,  $m_{\mathbb{Q}}(\chi)$  divides the intertwining number  $(\xi, \chi)$ .*

### 2. Ennola's conjecture implies Theorem C

In [2], V. Ennola stated the following conjecture (cf. [8]).

**Conjecture of Ennola.** *The system of irreducible characters of  $U(n, q^2)$  coincides with that of irreducible  $C$ -functions, which are obtained from the irreducible characters of the general linear group  $GL(n, q)$  by the simple formal change that  $q$  is everywhere replaced by  $-q$ .*

This is checked by himself in [2] for  $n \leq 3$  and by S. Nozawa [8], [9] for  $n=4, 5$ . Recently, G. Lusztig, B. Srinivasan, R. Hotta, D. Kazhdan and T.A. Springer have proved the conjecture for sufficiently large  $p$  and  $q$  (See [5], [7]). Thus Theorem C follows from Theorem C of [10], which is the counterpart for  $GL(n, q)$ .

### 3. Some lemmas on representation theory of algebraic groups

Let  $G$  be a connected, reductive linear algebraic group defined over  $\mathbb{F}_q$  and  $F$  the corresponding Frobenius endomorphism. Then  $G^F$  is the finite group of  $\mathbb{F}_q$ -rational points in  $G$ . Let  $Z$  denote the centre of  $G$ . Throughout this section, we shall assume that  $Z$  is connected and that  $p$  is not a bad prime for  $G$  for all the simple components of  $G$ .

**Lemma 3.1.** *Let  $S$  be a Sylow  $p$ -subgroup of  $G^F$ . Then, if  $\lambda$  is a linear character of  $S$ ,  $\text{Ind}_S^{G^F}(\lambda)$  is a character of  $G^F$  which is realizable in  $\mathbb{Q}$ .*

For a proof, see [11], Cor. 2.3..

**Corollary 3.2.** *If  $\chi$  is an irreducible character of  $U(n, q^2)$  of degree coprime to  $p$ , then  $m_{\mathbb{Q}}(\chi)$  is equal to 1.*

Proof. Since the centre of the unitary group is connected, (3.2) follows from the main theorem of [11].

Recall that an element  $x$  of  $G$  ( $G$  is reductive) is called *regular* if  $Z_G(x)$  (=the centralizer of  $x$ ) has the minimal dimension.

**Lemma 3.3.** (i)  $G^F$  contains a regular unipotent element of  $G$ .  
 (ii) The set of regular unipotent elements in  $G^F$  form a single conjugacy class.

This is well known.

**Corollary 3.4.** Any character of  $G^F$  takes rational integral values on regular unipotent elements.

Proof. If  $u$  is a regular unipotent element in  $G^F$ , for any integer  $k$  coprime to  $p$ ,  $u^k$  is also regular;  $u$  and  $u^k$  are conjugate in  $G^F$  (by (3.3)). Then (3.4) follows from [10], (1.1) lemma or from [4], Lemma 2.

**Lemma 3.5.** If  $U$  is an  $F$ -stable maximal unipotent subgroup of  $G$  and  $\rho$  is an irreducible character of  $U^F$  of degree greater than one, we have that  $\rho(u)=0$  for any regular unipotent element  $u$  in  $U^F$ .

Proof. Since the image of  $U$  in  $G/Z$  is isomorphic to  $U$ , we may assume that  $Z$  is trivial. Then (3.5) follows from Theorems A, A' of [6].

REMARK. In (3.5), the assumption that  $Z$  is connected is not needed.

#### 4. Proof of Theorem B

Let  $G=U_n$  ( $=GL_n$ ) be the unitary group of rank  $n$  defined over  $F_q$ . The Frobenius  $F$  is given by  $F((x_{ij}))=(x_{ij}^q)^{-1}$ . We also introduce an endomorphism  $F_0$  of  $U_n$  defined by  $F_0((x_{ij}))=(x_{ij}^{q^2})$ . Then  $U_n^{F_0}=GL(n, q^2)$ .

**Lemma 4.1.** Two elements of  $U(n, q^2)$  are conjugate in  $U(n, q^2)$  if and only if they are conjugate in  $GL(n, q^2)$ .

For a proof, see [12], I.3.6.; also see [2].

Now we prove Theorem B. Let  $u$  be a unipotent element of  $U_n^F$ ,  $\mu=(\mu_1, \mu_2, \dots, \mu_s)$  the corresponding partition of  $n$  and  $P_\mu$  the standard parabolic subgroup of  $U_n$  corresponding to  $\mu$  i.e.

$$P_\mu = \left\{ \begin{pmatrix} A_{11} & & & \\ & A_{22} & & * \\ & & \ddots & \\ 0 & & & \ddots \\ & & & & A_{ss} \end{pmatrix} ; A_{ii} \in U_{\mu_i} \right\}.$$

Then  $L=U_{\mu_1} \times \dots \times U_{\mu_s}$ , which is embedded diagonally into  $P_\mu$ , is an  $F$ -stable

Levi-subgroup of  $P$ ; in particular,  $L$  is reductive and connected. Let  $u_0$  be an  $F$ -stable regular unipotent element of  $L$  (such element exists by (3.3)) and let  $u_\mu$  be the Jordan canonical form of  $u$ . We may assume that  $u_\mu$  belongs to  $L$ . Since the centre of  $L$  is connected and since  $u_\mu$  is a regular unipotent element of  $L$ ,  $u_\mu$  and  $u_0$  are conjugate in  $L^{F_0} = GL(\mu_1, q^2) \times \cdots \times GL(\mu_s, q^2)$ . (Note that  $u_0$  and  $u_\mu$  are rational over  $F_{q^2}$ ). Then  $u$  and  $u_0$  are conjugate in  $GL(n, q^2)$ ; by (4.1), they are conjugate in  $U_n^F$ . We may assume  $u$  to be  $u_0$ . Let  $U$  be the unipotent radical of the  $F$ -stable Borel subgroup of  $L$  containing  $u$ . Then  $U$  is  $F$ -stable and contains  $u$ . Now let  $\chi$  be an irreducible character of  $U_n^F$ . Then we have that

$$\chi|U^F = \sum_{\lambda} a_{\lambda} \cdot \lambda + \sum_{\rho} b_{\rho} \cdot \rho,$$

where the first summation is over all the linear characters of  $U^F$  and the second summation is over all the non-linear irreducible characters of  $U^F$ . By (3.5), we have

$$\chi(u) = \sum_{\lambda} a_{\lambda} \cdot \lambda(u).$$

Since  $a_{\lambda} = (\chi, \text{Ind}_{U^F}^{U_n^F}(\lambda))$  and  $\text{Ind}_{U^F}^{U_n^F}(\lambda) = \text{Ind}_{L^F}^{U_n^F}(\text{Ind}_{U^F}^{L^F}(\lambda))$  is realizable in  $\mathcal{Q}$  by (3.1), we see by (1.1) that  $m_{\mathcal{Q}}(\chi)$  divides  $a_{\lambda}$ . We can rewrite the above expression as

$$\chi(u)/m_{\mathcal{Q}}(\chi) = \sum_{\lambda} (a_{\lambda}/m_{\mathcal{Q}}(\chi)) \cdot \lambda(u).$$

In this expression, the right hand side is an algebraic integer. Then, to prove Theorem B, it suffices to show that  $\chi(u)$  is rational. But any character of  $L^F$  takes a rational integral value on  $u$  (by (3.4)) and the restriction of  $\chi$  to  $L^F$  is a non-negative integral linear combination of irreducible characters of  $L^F$ ; hence  $\chi(u)$  is an integer. This completes the proof of Theorem B.

TOKYO METROPOLITAN UNIVERSITY

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