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A REMARK ON CONJUGACY CLASSES
IN SIMPLE GROUPS

NOBUO NOBUSAWA

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Let $A$ be a union of some conjugacy classes in a group. We define a binary operation on $A$ by $a \circ b = b^{-1}ab$. It satisfies that (1) $a \circ a = a$, (2) $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$ and (3) a mapping $\sigma_x: x \rightarrow x \circ a$ is a permutation on $A$. Generally we call a binary system which satisfies the above three conditions a pseudosymmetric set. It is called a symmetric set if (4) $\sigma_x$ has the order 2 is also satisfied. The set of all nilpotent elements in a Lie algebra is another example of a pseudosymmetric set. Where $\sigma_x = \exp(\text{ad} a)$. The purpose of this note is to generalize the main result on the simplicity of a symmetric set given in [2] to the case of a pseudosymmetric set. As applications, three examples of conjugacy classes in simple groups $\text{A}_n$, $\text{SL}(V)$ and $\text{Sp}(V)$ will be discussed, from which we could derive a new proof of the simplicity of the corresponding groups $\text{A}_n$, $\text{PSL}(V)$ and $\text{PSp}(V)$.

Generally, let $A$ be a pseudosymmetric set and define $G = G(A) = \langle \sigma_a | a \in A \rangle$, a group generated by $\sigma_a$. The above three conditions imply that $G$ is a group of automorphisms of $A$. Note that if $\rho$ is an automorphism of $A$, then $\sigma_x \circ \rho = \rho^{-1} \sigma_x \rho$. $\{\sigma_a | a \in A\}$ is a union of conjugacy classes in $G$ and hence is a pseudosymmetric set, and the mapping $\sigma: a \rightarrow \sigma_a$ is a homomorphism of $A$ to the set. When $\sigma$ is a monomorphism, we say that $A$ is effective. When $A = a^G$ for an element $a$, we say that $A$ is transitive. Let $G'$ be the commutator subgroup of $G$. When $A$ is transitive, $G' = \langle \sigma^{-1}_a \sigma_b | a, b \in A \rangle$, since $b = a^\rho$ with some element $\rho$ in $G$ and $\sigma^{-1}_a \sigma_b = \sigma^{-1}_b \sigma_a \rho \in G'$ and conversely $\sigma^{-1}_a \sigma_b \sigma_a \sigma_b = \sigma^{-1}_a \sigma_a$ with $c = a^\rho$. So, in this case, $G = \langle G', \sigma_a \rangle$ for any $a$. Also note that if $A$ is a union of conjugacy classes in a group $K$ and if $A$ generates $K$, then $G \cong K/Z(K)$, where $Z(K)$ is the center of $K$.

Let $A$ and $B$ be pseudosymmetric sets and suppose that there exists a homomorphism $f$ of $A$ onto $B$. The inverse image $f^{-1}(b)$ for an element $b$ in $B$ is called a coset of $f$. Let $\{C_i \}$ be the set of all cosets of $f$. Then $\{C_i \}$ is a system of blocks of imprimitivity of the permutation group $G$, and if $\sigma$ and $\rho$ belong to the same coset, then $C_\sigma = C_\rho$ for every $i$. When $|B| > 1$ and $f$ is not a monomorphism, we say that $f$ is proper. A pseudosymmetric set $A$ with $|A| > 2$ is called simple if it has no proper homomorphism. Note that if $A$ is simple, then it is transitive. For, consider the canonical homomorphism $a \rightarrow a^G$.
of $A$ onto $B=\{a^g|a\in A\}$. Since $A$ is simple, $|B|=1$ or the mapping is a monomorphism. In the former case, $A=a^g$ is transitive. In the latter case, $a=a^g$ for every $a$, i.e., $G$ is trivial, which is impossible because $|A|>2$ implies that $A$ has a proper homomorphism to the trivial pseudosymmetric set of two elements. The following theorem is established for a symmetric set in [2].

**Theorem.** Let $A$ be a pseudosymmetric set. If $A$ is simple, then $G'$ is the unique minimal normal subgroup of $G$. The converse is also true if $A$ is effective and transitive.

Proof. Suppose that $A$ is simple. Let $K=1$ be a normal subgroup of $G$, and $B$ the set of all $K$-orbits. $B$ is a pseudosymmetric set, and there is the canonical homorphism $f\colon a\rightarrow a^K$. Since $K=1$, $f$ is not a monomorphism. Therefore, $|B|=1$, which implies that $K$ is transitive on $A$. So, for any elements $a$ and $b$, $a^\sigma=b$ with $\sigma\in K$. Then, $\sigma^a=\rho^{-1}\sigma^a\rho=\sigma^a$, and hence $\sigma^a\sigma^b\in K$ as $K$ is normal. Thus $G'=K$, which proves the first part of Theorem. Conversely, suppose that $A$ is effective and transitive and that $A$ is not simple. We want to show that there is a normal subgroup $K$ such that $1\neq K\subseteq G'$. Since $A$ is not simple, there is a proper homomorphism $f$ of $A$ onto $B$ with $|B|\geq 2$. $f$ induces a homomorphism $\bar{f}$ of $G$ to $G(B)$ in a natural way: $f(a\circ b)=f(a)\circ f(b)=f(a)^{f(b)}$, or, more generally $f(a^\sigma)=f(a)^{f(\sigma)}$. Let $\bar{g}$ be the restriction of $\bar{f}$ to $G'$. Let $K$ be the kernel of $\bar{g}$. Since $f$ is not a monomorphism, there exist $a$ and $b$ such that $a\neq b$ and $f(a)=f(b)$. Then, $\bar{f}(\sigma_a)=\bar{f}(\sigma_b)$ and hence $\bar{g}(\sigma_a^\sigma_b)=1$. Thus $K=1$. Note that $\sigma_a^{-1}\sigma_b\neq 1$ and $\in G'$ as $A$ is effective and transitive. On the other hand, let $f(c)$ and $f(d)$ be two elements in $B$. Since $A$ is transitive, $c^\tau=d$ with some $\tau$ in $G$. We may assume that $\tau$ is in $G'$. For, $G=<G', \sigma_a>=\sum \sigma^i G'$ and we can replace $\tau$ by $\sigma_a^i\tau$. Then, $f(c)^{f(\tau)}=f(c')=f(d)\neq f(c)$. Therefore, $g(\tau)\neq 1$ and $\tau$ is not in $K$. $K\subseteq G'$.

**Corollary.** Let $A$ be an effective and transitive pseudosymmetric set. Suppose $G'=G$. Then $A$ is simple if and only if $G$ is a simple group.

In the following, we show some examples of simple pseudosymmetric sets. Although it is well known that the corresponding groups $G$ are simple, we shall show the simplicity of $A$ directly, thus giving a new proof of the simplicity of $G$ (once we show $G'=G$).

**Example 1.** We consider the alternating group $A_n$, $(n\geq 5)$ Let $A$ be the conjugacy class of the 3-cycle $(1, 2, 3)$. $A$ consists of all 3-cycles and generates $A_n$. So, $G=A_n/Z(A_n)=A_n$. We shall show that $A$ is simple. Let $\{C_i\}$ be the set of all cosets of a homomorphism of $A$ to a pseudosymmetric set $B$. Assume that $|C_i|\geq 2$. Note that all $C_i$ have the same cardinality as $A$ is transitive. Let $C$ be one of $C_i$. 


(1) Suppose that \((1, 2, 3)\) and \((1, 2, 4)\) are both contained in \(C\). It is not hard to check that the pseudosymmetric set \(C\) contains all \((i, j, k), 1 \leq i, j, k \leq 4\). Since \((1, 2, 3)^\sigma = (1, 2, 4) \in C\) where \(\sigma = (3, 4, 5)\), we see that \((1, 2, 4)^\sigma = (1, 2, 5)\) is also contained in \(C\) due to the definition of a block of imprimitivity of a permutation group. So, \(C\) contains all \((i, j, k), l \leq i, j, k \leq 5\) by the above argument. Repeating this process, we have \(C = A\).

(2) Suppose that \((1, 2, 3)\) and \((1, 4, 5) \in C\). Then, \((1, 2, 3)^\sigma = (4, 2, 3)\) is contained in \(C\), where \(\sigma = (1, 2, 3)\) and \(\tau = (2, 1, 3)\). Then both \((2, 4, 5)^\sigma = (3, 4, 5)\) and \((2, 4, 5)^\tau = (1, 4, 5)\) are contained in \(C = C_i = C_j\), where \(C_i\) contains \((2, 4, 5)\). Then \(C' = A\) by (1).

(3) Suppose that \((1, 2, 3)\) and \((2, 1, 3) \in C\). Let \(\sigma = (1, 2, 3)\) and \(\tau = (2, 1, 3)\). Then both \((2, 3, 5)^\sigma = (3, 1, 5)\) and \((2, 3, 5)^\tau = (2, 3, 5)\) are contained in a coset \(C' = C\), and \(C' = A\) by (3).

(4) Suppose that \((1, 2, 3)\) and \((2, 1, 4) \in C\). Let \(\sigma = (1, 2, 3)\) and \(\tau = (2, 1, 4)\). Then both \((2, 3, 4)^\sigma = (3, 2, 1)\) and \((2, 3, 4)^\tau = (2, 3, 4)\) are contained in a coset \(C' = C\), and \(C' = A\) by (3).

(5) Suppose that \(n \geq 6\) and that \((1, 2, 3)\) and \((4, 5, 6) \in C\). Let \(\sigma = (1, 2, 3)\) and \(\tau = (4, 5, 6)\). Then both \((2, 3, 4)^\sigma = (3, 1, 4)\) and \((2, 3, 4)^\tau = (2, 3, 5)\) are contained in a coset \(C' = C\), and \(C' = A\) by (3).

From the above, we can conclude that \(A\) is simple.

Example 2 (For Examples 2 & 3, see [1]). Let \(V\) be a vector space over a field \(K\). Let \(\tau_{a,f}\) be a transvection: \(x \mapsto x - f(x)a\), where \(a \neq 0\) and \(f\) is a non-zero linear function such that \(f(a) = 0\). A pseudosymmetric set \(A\) is defined as follows. When \(\dim V \geq 3\), let \(A\) be the set of all transvections. It is known in this case that \(A\) is a conjugacy class in \(SL(V)\) and generates \(SL(V)\). When \(\dim V = 2\), let \(\tau\) be a transvection represented by a matrix \(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\) with respect to some basis of \(V\), and let \(A\) be the conjugacy class of \(\tau\) in \(SL(V)\). We show that \(A\) generates \(SL(V)\) in this case. Then \(A\) is seen to be transitive. For \(\lambda \neq 0\), we have

\[
\begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 & \lambda^2 \\ 0 & 1 \end{bmatrix} \in A.
\]

If \(\text{char}(K) \neq 2\) or if \(K\) is finite, then \(\mu = \alpha^2 - \beta^2 - \gamma^2\) has solutions \(\alpha\), \(\beta\) and \(\gamma\) in \(K\) for any given \(\mu\) as we see easily. Then,

\[
\begin{bmatrix} 1 & \alpha^2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \beta^2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \gamma^2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \in \langle A \rangle.
\]

Then, also,

\[
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\mu & 1 \end{bmatrix} \in \langle A \rangle.
\]

We see that \(\langle A \rangle = SL(V)\) in this case. Next, assume that \(\text{char}(K) = 2\) and \(K\)
is infinite. Then,\[
\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \langle A \rangle .
\]
Hence,\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \langle A \rangle .
\]
For any non-zero \(\mu\),\[
\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \mu^{-2} \\ 0 & 0 \end{pmatrix} \in \langle A \rangle .
\]
Hence,\[
\begin{pmatrix} 1 & \mu^{-2} \\ \mu^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \mu^{-2} & 1 \\ 0 & \mu^2 \end{pmatrix} \in \langle A \rangle .
\]
Therefore,\[
\begin{pmatrix} \mu^{-2} & 1 \\ 0 & \mu^2 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mu^{-2} & 1 \\ 0 & \mu^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \lambda (\mu^{-4} - 1) \\ 0 & 1 \end{pmatrix} \in \langle A \rangle .
\]
Since \(K\) is infinite, \(\lambda (\mu^{-4} - 1)\) can be any non-zero element in \(K\). As in the first case, we can show \(\langle A \rangle = SL(V)\). So, we can also conclude that for any \(\langle a \rangle\) there exists \(c \in \langle a \rangle\) and \(f\) such that \(\tau_{a,f} \in A\) if \(\dim V = 2\).

Now we are in a position to show that \(A\) is simple. Let \(\{C_i\}\) be the set of all cosets of a homomorphism where \(|C_i| \geq 2\). First, we prove that there is a coset \(C\) which contains two elements \(\tau_{a,f}\) and \(\tau_{b,g}\) such that \(f(b) \neq 0\). For it, let \(\sigma\) and \(\rho\) be two elements in some coset. There is a hyperplane \(H\) such that \(H^\sigma \neq H^\rho\), since otherwise \(\sigma \rho^{-1}\) fixes every line and hence \(\sigma = \rho\) as both \(\sigma\) and \(\rho\) are transvections. So, we can choose an element \(c\) in \(H\) such that \(c^\sigma \in H^\rho\). Let \(h\) be a linear function defining \(H\); \(H = H_k = \{x | h(x) = 0\}\). Let \(a = c^\sigma, b = c^\rho, f = h^\rho\) and \(g = h^\sigma\). Let \(C = C_i^\sigma\), where \(C_i\) is a coset containing \(\tau_{c,k}\). Note that we can make \(\tau_{c,k} \in A\) if \(\dim V = 2\) by the above remark. Note also \(C_i^\sigma = C_i^\rho\) as \(\sigma\) and \(\rho\) belong to the same coset. \(C\) satisfies the above condition. For, \(f(b) = h^\rho(b) = h(b^{\rho^-1}) \neq 0\) as \(b \in H^\rho\). \(C\) contains \(\tau_{c,k}^\rho = \tau_{a,f}^\sigma\) and \(\tau_{c,k}^\rho = \tau_{b,g}^\sigma\). Next, we prove that, for every line \(\langle d \rangle\), \(C\) contains an element \(\tau_{d',*}\) such that \(d' \in \langle d \rangle\). For it, we may assume that \(d \in \langle a \rangle \cup \langle b \rangle\). If \(d \notin H_f\), we can choose \(\varphi\) is \(SL(V)\) such that \(\varphi\) is the identity on \(H_f\) and that \(b^\varphi \in \langle d \rangle\). Note that \(f(b) \neq 0\) implies \(b \in H_f\). \(\varphi\) fixes \(\tau_{a,f}\) as it is a unimodular linear transformation identically on \(H_f\). Therefore, \(C^\varphi = C\). Since \(\tau_{b,g}^\varphi \in C\), we can let \(d' = b^\varphi\). If \(d \in H_f\) and \(d \notin H_g\), we can choose \(\xi\) in \(SL(V)\) such that \(\xi\) is the identity on \(H_g\) and that \(d^\xi \in H_f\). Since \(\xi\) fixes \(\tau_{b,g}\) this time, \(C^\xi = C\). From the above, we can find \(d_0\) such that \(\tau_{a_0,*} \in C\) and that \(d_0 \in \langle d \rangle\). So, in
this case, let $d'=d_{z}^{-1}$. Finally, suppose that $d\in H_f \cap H_g$. In this case, we can choose $z$ in $SL(V)$ such that $z$ induces a unimodular linear transformation on $H_f$, $a^z=a$ and $d^z\in H_f$. It follows that $\tau_{d,f}=\tau_{z,f}$ since $\tau\in SL(V)$ and its restriction on $H_f$ is a unimodular linear transformation of $H_f$. Hence, $C'=C$. Then, as above, we can show the existence of a required element $d'$. It is now easy to conclude that $C=A$. For, let $\tau_{d',z}$ be given as above. $\tau_{d',z}$ and $\tau_{d',z}$ are commutative as $d'\in \langle d \rangle$. For every $d$, $\tau_{d,f}$ leaves $C$ fixed. Since $A$ is transitive, this implies $C=A$. We have proven that $A$ is simple.

**Example 3.** Suppose that $V$ has a non-singular symplectic metric $(x, y)$. Let $\sigma_{\alpha,\lambda}$ be a symplectic transvection: $x\mapsto x+\lambda(x, a)a$, where $a$ is a non-zero element in $V$ and $\lambda$ is a non-zero element in $K$. We define a pseudosymmetric set $A$ by $A=\{\sigma_{\alpha,\lambda}| a, \lambda \in V^*\}$. We want to show that $A$ generates $Sp(V)$ and that $A$ is simple. In order to show that $A$ generates $Sp(V)$, first suppose that char $(K)\neq 2$ or that $K$ is finite. Since $\sigma_{\alpha,\lambda}=\sigma_{z,\lambda}$ and $\sigma_{z,-1}=\sigma_{\alpha,-1}$, we can show that $\langle A \rangle$ contains all $\sigma_{\alpha,\mu}$ as in Example 2. Thus, $\langle A \rangle=Sp(V)$ in this case, since $\sigma_{z,\mu}$ generate $Sp(V)$. Next, suppose that char $(K)=2$ and that $K$ is infinite. We reduce our problem to the case of dim 2 and solve it. To show $\sigma_{\alpha,\lambda}\in \langle A \rangle$, consider $V'=\langle a, a'\rangle$, a hyperbolic plane. Let $V=V'\bigoplus V''$ be an orthogonal decomposition. Then $\sigma_{\alpha,\lambda}=\sigma_{\alpha,\lambda}'+1_{V''}$, where $\sigma_{\alpha,\lambda}'$ is a symplectic transvection on $V'$. Now, $Sp(V')=SL(V')=PSL(V')$ because $K$ is infinite and char $(K)=2$. (See [1], p. 174.) If we let $A'=\{\sigma_{\alpha,\lambda}'| a, \lambda \in V^*\}$, then $\langle A' \rangle$ is a normal subgroup of $SL(V')$ and hence $\langle A' \rangle=Sp(V')$, since the latter is a simple group by the above. This implies that $\sigma_{\alpha,\lambda}\in \langle A' \rangle+1_{V''}\subset \langle A \rangle$. Thus, $A$ generates $Sp(V)$.

Before we show the simplicity of $A$, we show that $A$ is transitive. $V^*$ is clearly a pseudosymmetric set by $aob=a^z\lambda$. A mapping $f: a\mapsto \sigma_{\alpha,1}$ is a homomorphism of $V^*$ onto $A$, and $f^{-1}(a)=\{\pm a\}$. It suffices to show that $V^*$ is transitive. Fix $a$, and let $x$ be an arbitrary element in $V^*$. If $(a, x)\neq 0$, then $a+x=a^z\lambda$, where $\lambda=(a, x)^{-1}$. Therefore, $a+x$ belongs to the $G^*$-orbit of $a$ where $G^*=G(V^*)$. Then $x$ belongs to the $G^*$-orbit of $a+x$, which is equal to the $G^*$-orbit of $a$, since $(a+x, -a)\neq 0$ and $(a+x)+(-a)=x$. If $(a, x)=0$, we can choose $y$ such that $(a, y)\neq 0$ and $(y, x)\neq 0$. For, let $V'=\langle a, a'\rangle$ as before. If $(a', x)\neq 0$, let $y=a'$. If $(a', x)=0$, let $\langle x, x'\rangle$ be a hyperbolic plane which is orthogonal to $V'$. Let $y=a'+x'$. Thus, $x$ is in the $G^*$-orbit of $y$, which is equal to the $G^*$-orbit of $a$. We have shown that $A$ is transitive. Now we are in a position to prove that $A$ is simple. Let $\{C_i\}$ be the set of cosets as before, where $|C_i|\geq 2$. Let $C_i^*=f^{-1}(C_i)$. Let $C^*$ be one of $C_i^*$.

(1) Suppose that $C^*$ contains $a$ and $b$ such that $(a, b)\neq 0$. Since $C^*\sigma_{b,\lambda}=C^*$ for any $\lambda$ as $\sigma_{b,\lambda}$ fixes $b$, $C^*$ contains all $a+\mu b$. So, more generally, $C^*$
contains $\alpha a + \beta b$ for any $\alpha$ and $\beta$. For any $c$ in $V^*$, $(\alpha a + \beta b, c) = 0$ for some $\alpha a + \beta b$ in $V^*$, which implies that $\sigma_{x, \lambda}$ leaves $\alpha a + \beta b$ fixed. Therefore, $C^*$ is left fixed by any $\sigma_{x, \lambda}$. Since $V^*$ is transitive, this implies $C^* = V^*$, or $C = A$. $A$ is simple in this case.

(2) Suppose that $C^*$ contains $a$ and $b$ such that $(a, b) = 0$ and $a \notin \langle b \rangle$. Then, we can express $b = \alpha a + d$ with a non-zero element $d$ in $V''$, where $V = V' \oplus V''$ (orthogonal), since $(a, b) = 0$ and $a \notin \langle b \rangle$. Now, let $c$ be an element in $V''$ such that $(d, c) \neq 0$. Since $\sigma_{x, \lambda}$ fixes $a$, $C^*$ is left fixed by $\sigma_{x, \lambda}$. Then, $b^x \in C^*$, which implies that $b + c \in C^*$. Since $(b, b + c) = 0$, we have $C = A$ by (1).

(3) Suppose that $C^*$ contains $a$ and $\alpha a$, where $\alpha \neq \pm 1$. Let $b$ be an element such that $(a, b) = 0$. Let $C_f$ be a coset which contains $b$. Then, $C_f = C^* \cap \{a\}$, which contains $d = b^x = b + (b, a)a$ and $e = b^x = b + \alpha^2 (b, a)a$. Since $d \notin \langle e \rangle$, we can apply (1) or (2) and get that $\{C_f\} = \{A\}$, or $A$ is simple.

Remark. If we consider $PSL(V)$ and $PSp(V)$, the "effective" condition is satisfied.

References