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A REMARK ON CONJUGACY CLASSES
IN SIMPLE GROUPS

NOBUO NOBUSAWA

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Let $A$ be a union of some conjugacy classes in a group. We define a binary operation on $A$ by $a \circ b = b^{-1}ab$. It satisfies that (1) $a \circ a = a$, (2) $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$ and (3) a mapping $\sigma_a : x \mapsto x \circ a$ is a permutation on $A$. Generally we call a binary system which satisfies the above three conditions a pseudosymmetric set. It is called a symmetric set if (4) $\sigma_a$ has the order 2 is also satisfied. The set of all nilpotent elements in a Lie algebra is another example of a pseudosymmetric set. We.e.e $\sigma_a = \exp(\text{ad } a)$. The purpose of this note is to generalize the main result on the simplicity of a symmetric set given in [2] to the case of a pseudosymmetric set. As applications, three examples of conjugacy classes in simple groups $A_n$, $SL(V)$ and $Sp(V)$ will be discussed, from which we could derive a new proof of the simplicity of the corresponding groups $A_n$, $PSL(V)$ and $PSp(V)$.

Generally, let $A$ be a pseudosymmetric set and define $G = G(A) = \langle \sigma_a | a \in A \rangle$, a group generated by $\sigma_a$. The above three conditions imply that $G$ is a group of automorphisms of $A$. Note that if $\rho$ is an automorphism of $A$, then $\sigma_a \circ \rho = \rho^{-1} \sigma_a \rho$. $\{ \sigma_a | a \in A \}$ is a union of conjugacy classes in $G$ and hence is a pseudosymmetric set, and the mapping $\sigma : a \mapsto \sigma_a$ is a homomorphism of $A$ to the set. When $\sigma$ is a monomorphism, we say that $A$ is effective. When $A = a^G$ for an element $a$, we say that $A$ is transitive. Let $G'$ be the commutator subgroup of $G$. When $A$ is transitive, $G' = \langle \sigma_a^{-1} \sigma_b | a, b \in A \rangle$, since $b = a^\rho$ with some element $\rho$ in $G$ and $\sigma_a^{-1} \sigma_b = \rho^{-1} \sigma_a \rho \in G'$ and conversely $\sigma_a^{-1} \sigma_b \sigma_a \sigma_b = \sigma_a^{-1} \sigma_a$ with $c = a^\rho$. So, in this case, $G = \langle G', \sigma_a \rangle$ for any $a$. Also note that if $A$ is a union of conjugacy classes in a group $K$ and if $A$ generates $K$, then $G \approx K/Z(K)$, where $Z(K)$ is the center of $K$.

Let $A$ and $B$ be pseudosymmetric sets and suppose that there exists a homomorphism $f$ of $A$ onto $B$. The inverse image $f^{-1}(b)$ for an element $b$ in $B$ is called a coset of $f$. Let $\{ C_i \}$ be the set of all cosets of $f$. Then $\{ C_i \}$ is a system of blocks of imprimitivity of the permutation group $G$, and if $\sigma$ and $\rho$ belong to the same coset, then $C_\sigma = C_\rho$ for every $i$. When $|B| > 1$ and $f$ is not a monomorphism, we say that $f$ is proper. A pseudosymmetric set $A$ with $|A| > 2$ is called simple if it has no proper homomorphism. Note that if $A$ is simple, then it is transitive. For, consider the canonical homomorphism $a \mapsto a^G$.
of $A$ onto $B=\{a^g \mid a \in A\}$. Since $A$ is simple, $|B|=1$ or the mapping is a monomorphism. In the former case, $A=a^G$ is transitive. In the latter case, $a=a^g$ for every $a$, i.e., $G$ is trivial, which is impossible because $|A|>2$ implies that $A$ has a proper homomorphism to the trivial pseudosymmetric set of two elements. The following theorem is established for a symmetric set in [2].

**Theorem.** Let $A$ be a pseudosymmetric set. If $A$ is simple, then $G'$ is the unique minimal normal subgroup of $G$. The converse is also true if $A$ is effective and transitive.

Proof. Suppose that $A$ is simple. Let $K \trianglelefteq 1$ be a normal subgroup of $G$, and $B$ the set of all $K$-orbits. $B$ is a pseudosymmetric set, and there is the canonical homomorphism $f: a \rightarrow a^K$. Since $K \trianglelefteq 1$, $f$ is not a monomorphism. Therefore, $|B|=1$, which implies that $K$ is transitive on $A$. So, for any elements $a$ and $b$, $a^\rho=b$ with $\rho$ in $K$. Then, $\sigma_\rho^a=\rho^{-1}\sigma_\rho^a=\sigma_\rho$, and hence $\sigma_\rho^a\sigma_\rho \in K$ as $K$ is normal. Thus $G' \subseteq K$, which proves the first part of Theorem. Conversely, suppose that $A$ is effective and transitive and that $A$ is not simple. We want to show that there is a normal subgroup $K$ such that $1 \nmid K \subseteq G'$. Since $A$ is not simple, there is a proper homomorphism $f$ of $A$ onto $B$ with $|B| \geq 2$. $f$ induces a homomorphism $\bar{f}$ of $G$ to $G(B)$ in a natural way: $f(ab)=f(a)f(b)=f(a)^{\bar{f}(\sigma_\rho)}$, or, more generally $f(a^\rho)=f(a)^{\bar{f}(\sigma_\rho)}$. Let $g$ be the restriction of $\bar{f}$ to $G'$. Let $K$ be the kernel of $g$. Since $f$ is not a monomorphism, there exist $a$ and $b$ such that $a\neq b$ and $f(a)=f(b)$. Then, $\bar{f}(\sigma_\rho)=\bar{f}(\sigma_a)$ and hence $g(\sigma_\rho^{-1}\sigma_a)=1$. Thus $K \trianglelefteq 1$. Note that $\sigma_\rho^{-1}\sigma_a \neq 1$ and $\in G'$ as $A$ is effective and transitive. On the other hand, let $f(c)$ and $f(d)$ be two elements in $B$. Since $A$ is transitive, $c^\tau=d$ with some $\tau$ in $G$. We may assume that $\tau$ is in $G'$. For, $G=\langle G', \sigma_\rho \rangle = \sum \sigma_\rho^i G'$ and we can replace $\tau$ by $\sigma_\rho^i \tau$. Then, $f(c)^{\bar{f}(\tau)}=f(c')=f(d) \neq f(c)$. Therefore, $g(\tau) \neq 1$ and $\tau$ is not in $K$. $K \subseteq G'$.

**Corollary.** Let $A$ be an effective and transitive pseudosymmetric set. Suppose $G'=G$. Then $A$ is simple if and only if $G$ is a simple group.

In the following, we show some examples of simple pseudosymmetric sets. Although it is well known that the corresponding groups $G$ are simple, we shall show the simplicity of $A$ directly, thus giving a new proof of the simplicity of $G$ (once we show $G'=G$).

**Example 1.** We consider the alternating group $A_n$. $(n \geq 5)$ Let $A$ be the conjugacy class of the 3-cycle $(1, 2, 3)$. $A$ consists of all 3-cycles and generates $A_n$. So, $G \approx A_n/Z(A_n)=A_n$. We shall show that $A$ is simple. Let $\{C_i\}$ be the set of all cosets of a homomorphism of $A$ to a pseudosymmetric set $B$. Assume that $|C_i| \geq 2$. Note that all $C_i$ have the same cardinality as $A$ is transitive. Let $C$ be one of $C_i$. 


(1) Suppose that (1, 2, 3) and (1, 2, 4) are both contained in $C$. It is not hard to check that the pseudosymmetric set $C$ contains all $(i, j, k)$, $1 \leq i, j, k \leq 4$. Since $(1, 2, 3)^\sigma = (1, 2, 4) \in C$ where $\sigma = (3, 4, 5)$, we see that $(1, 2, 4)^\sigma = (1, 2, 5)$ is also contained in $C$ due to the definition of a block of imprimitivity of a permutation group. So, $C$ contains all $(i, j, k)$, $l \leq i, j, k \leq 5$ by the above argument. Repeating this process, we have $C = A$.

(2) Suppose that $(1, 2, 3)$ and $(1, 4, 5) \in C$. Then, $(1, 2, 3)^\sigma = (4, 2, 3)$ is contained in $C$, where $\sigma = (1, 2, 3)$ and $\tau = (2, 1, 3)$. Then both $(2, 4, 5)^\sigma = (3, 4, 5)$ and $(2, 4, 5)^\tau = (3, 4, 5)$ are contained in $C = C_1 = C_2$, where $C_1$ contains $(2, 4, 5)$. Then $C' = A$ by (1).

(3) Suppose that $(1, 2, 3)$ and $(2, 1, 3) \in C$. Then both $(2, 3, 5)^\sigma = (3, 4, 5)$ and $(2, 3, 5)^\tau = (2, 3, 5)$ are contained in a coset $C'$, and $C' = A$ by (3).

(4) Suppose that $(1, 2, 3)$ and $(2, 1, 4) \in C$. Then both $(2, 3, 4)^\sigma = (3, 1, 5)$ and $(2, 3, 4)^\tau = (2, 3, 5)$ are contained in a coset $C''$, and $C'' = A$ by (2).

(5) Suppose that $n \geq 6$ and that $(1, 2, 3)$ and $(4, 5, 6) \in C$. Then both $(2, 3, 4)^\sigma = (3, 4, 5)$ and $(2, 3, 4)^\tau = (2, 3, 5)$ are contained in a coset $C''$, and $C'' = A$ by (3).

From the above, we can conclude that $A$ is simple.

**Example 2** (For Examples 2 & 3, see [1]). Let $V$ be a vector space over a field $K$. Let $\tau_{a, f}$ be a transvection: $x \rightarrow x - f(x)a$, where $a \neq 0$ and $f$ is a non-zero linear function such that $f(a) = 0$. A pseudosymmetric set $A$ is defined as follows. When $\dim V \geq 3$, let $A$ be the set of all transvections. It is known in this case that $A$ is a conjugacy class in $SL(V)$ and generates $SL(V)$. When $\dim V = 2$, let $\tau$ be a transvection represented by a matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ with respect to some basis of $V$, and let $A$ be the conjugacy class of $\tau$ in $SL(V)$. We show that $A$ generates $SL(V)$ in this case. Then $A$ is seen to be transitive. For $\lambda \neq 0$, we have

$$\begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 & \lambda^2 \\ 0 & 1 \end{bmatrix} \in A.$$  

If $\text{char}(K) \neq 2$ or if $K$ is finite, then $\mu = \alpha^2 - \beta^2 - \gamma^2$ has solutions $\alpha$, $\beta$, and $\gamma$ in $K$ for any given $\mu$ as we see easily. Then,

$$\begin{bmatrix} 1 & \alpha^2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \beta^2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \gamma^2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \in \langle A \rangle.$$  

Then, also,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\mu & 1 \end{bmatrix} \in \langle A \rangle.$$  

We see that $\langle A \rangle = SL(V)$ in this case. Next, assume that $\text{char}(K) = 2$ and $K$
is infinite. Then,
\[
\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in A.
\]
Hence,
\[
\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \langle A \rangle.
\]
For any non-zero \( \mu \),
\[
\begin{bmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{bmatrix} = \begin{bmatrix} 1 & \mu^{-2} \\ \mu^2 & 0 \end{bmatrix} \in \langle A \rangle.
\]
Hence,
\[
\begin{bmatrix} 1 & \mu^{-2} \\ \mu^2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mu^{-2} & 1 \\ 0 & \mu^2 \end{bmatrix} \in \langle A \rangle.
\]
Therefore,
\[
\begin{bmatrix} \mu^{-2} & 1 \\ 0 & \mu^2 \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu^{-2} & 1 \\ 0 & \mu^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda(\mu^{-4} - 1) \\ 0 & 1 \end{bmatrix} \in \langle A \rangle.
\]
Since \( K \) is infinite, \( \lambda(\mu^{-4} - 1) \) can be any non-zero element in \( K \). As in the first case, we can show \( \langle A \rangle = SL(V) \). So, we can also conclude that for any \( \langle a \rangle \) there exists \( c \in \langle a \rangle \) and \( f \) such that \( \tau_{e,f} A \) if \( \dim V = 2 \).

Now we are in a position to show that \( A \) is simple. Let \( \{ C_i \} \) be the set of all cosets of a homomorphism where \(| C_i | \geq 2 \). First, we prove that there is a coset \( C \) which contains two elements \( \tau_{e,f} \) and \( \tau_{e,g} \) such that \( f(b) \neq 0 \). For it, let \( \sigma \) and \( \rho \) be two elements in some coset. There is a hyperplane \( H \) such that \( H^\sigma = H^\rho \), since otherwise \( \sigma \rho^{-1} \) fixes every line and hence \( \sigma = \rho \) as both \( \sigma \) and \( \rho \) are transvections. So, we can choose an element \( c \) in \( H \) such that \( c^\sigma \in H^\rho \). Let \( h \) be a linear function defining \( H \); \( H = H_k = \{ x | h(x) = 0 \} \). Let \( a = e^\sigma, b = e^\rho, f = h^\rho \) and \( g = h^\sigma \). Let \( C = C_i^\sigma \), where \( C_i \) is a coset containing \( \tau_{e,k}. \)

Note that we can make \( \tau_{e,k} A \) if \( \dim V = 2 \) by the above remark. Note also \( C_i^\sigma = C_i^\rho \) as \( \sigma \) and \( \rho \) belong to the same coset. \( C \) satisfies the above condition. For, \( f(b) = h^\rho(b) = h(b^{-1}) \neq 0 \) as \( b \in H^\rho \). \( C \) contains \( \tau_{e,k} = \tau_{e,f} \) and \( \tau_{e,k} = \tau_{e,g} \).

Next, we prove that, for every line \( \langle d \rangle \), \( C \) contains an element \( \tau_{d',*} \) such that \( d' \in \langle d \rangle \). For it, we may assume that \( d \in \langle a \rangle \cup \langle b \rangle \). If \( d \in H_f \), we can choose \( \varphi \) is \( SL(V) \) such that \( \varphi \) is the identity on \( H_f \) and that \( b^\varphi \in \langle d \rangle \). Note that \( f(b) = 0 \) implies \( b \in H_f \). \( \varphi \) fixes \( \tau_{e,f} \) as it is a unimodular linear transformation identically on \( H_f \). Therefore, \( C^\varphi = C \). Since \( \tau_{e,k} \in C \), we can let \( d' = b^\varphi \). If \( d \in H_f \) and \( d \in H_g \), we can choose \( \xi \) in \( SL(V) \) such that \( \xi \) is the identity on \( H_g \) and that \( d^\xi \in H_f \). Since \( \xi \) fixes \( \tau_{e,g} \) this time, \( C^\xi = C \). From the above, we can find \( d_0 \) such that \( \tau_{d_0,*} \in C \) and that \( d_0 \in \langle d \rangle \). So, in
this case, let $d' = d^{-1}$. Finally, suppose that $d \in H_f \cap H_g$. In this case, we can choose $\zeta$ in $SL(V)$ such that $\zeta$ induces a unimodular linear transformation on $H_f$, $d^\zeta = a$ and $d^\zeta \in H_f$. It follows that $\tau_{d,f} = \tau_{d',f}$ since $\zeta \in SL(V)$ and its restriction on $H_f$ is a unimodular linear transformation of $H_f$. Hence, $C^\zeta = C$. Then, as above, we can show the existence of a required element $d'$. It is now easy to conclude that $C = A$. For, let $\tau_{d',f}$ be given as above. $\tau_{d,f}$ and $\tau_{d',f}$ are commutative as $d' \in \langle d \rangle$. For every $d$, $\tau_{d,f}$ leaves $C$ fixed. Since $A$ is transitive, this implies $C = A$. We have proven that $A$ is simple.

**Example 3.** Suppose that $V$ has a non-singular symplectic metric $(x, y)$. Let $\sigma_\alpha$ be a symplectic transvection: $x \to x + \lambda(x, a)a$, where $a$ is a non-zero element in $V$ and $\lambda$ is a non-zero element in $K$. We define a pseudosymmetric set $A$ by $A = \{\sigma_\alpha | a \in V^* = V - \{0\}\}$. We want to show that $A$ generates $Sp(V)$ and that $A$ is simple. In order to show that $A$ generates $Sp(V)$, first suppose that $\text{char}(K) \neq 2$ or that $K$ is finite. Since $\sigma_\alpha = \sigma_\beta \sigma_\alpha$ and $\sigma_\alpha^{-1} = \sigma_{-\alpha}$, we can show that $\langle A \rangle$ contains all $\sigma_{\alpha, \lambda}$ as in Example 2. Thus, $\langle A \rangle = Sp(V)$ in this case, since $\sigma_{\alpha, \lambda}$ generate $Sp(V)$. Next, suppose that $\text{char}(K) = 2$ and that $K$ is infinite. We reduce our problem to the case of dim 2 and solve it. To show $\sigma_\alpha \in \langle A \rangle$, consider $V' = \langle a, a' \rangle$, a hyperbolic plane. Let $V = V' \oplus V''$ be an orthogonal decomposition. Then $\sigma_{\alpha, \lambda} = \sigma_{\alpha', \lambda} \oplus 1_{V''}$, where $\sigma_{\alpha', \lambda}$ is a symplectic transvection on $V'$. Now, $Sp(V') = SL(V') = PSL(V')$ because $K$ is infinite and $\text{char}(K) = 2$. (See [1], p. 174.) If we let $A' = \{\sigma_{\alpha, \lambda} | c \in V^*\}$, then $\langle A' \rangle$ is a normal subgroup of $SL(V')$ and hence $\langle A' \rangle = Sp(V')$, since the latter is a simple group by the above. This implies that $\sigma_{\alpha, \lambda} \in \langle A' \rangle \oplus 1_{V''} \subset \langle A \rangle$. Thus, $A$ generates $Sp(V)$.

Before we show the simplicity of $A$, we show that $A$ is transitive. $V^*$ is clearly a pseudosymmetric set by $aob = a^s \sigma_\lambda$. A mapping $f: a \to a_{\sigma_1}$ is a homomorphism of $V^*$ onto $A$, and $f^{-1}(\sigma_{\alpha,1}) = \{\pm a\}$. It suffices to show that $V^*$ is transitive. Fix $a$, and let $x$ be an arbitrary element in $V^*$. If $(a, x) = 0$, then $a + x = a^x \lambda$, where $\lambda = (a, x)^{-1}$. Therefore, $a + x$ belongs to the $G^*$-orbit of $a$ where $G^* = G(V^*)$. Then $x$ belongs to the $G^*$-orbit of $a + x$, which is equal to the $G^*$-orbit of $a$, since $(a + x, -a) = 0$ and $(a + x) + (-a) = x$. If $(a, x) = 0$, we can choose $y$ such that $(a, y) = 0$ and $(y, x) = 0$. For, let $V' = \langle a, a' \rangle$ as before. If $(a', x) = 0$, let $y = a'$. If $(a', x) = 0$, let $\langle x, x' \rangle$ be a hyperbolic plane which is orthogonal to $V'$. Let $y = a' + x'$. Thus, $x$ is in the $G^*$-orbit of $y$, which is equal to the $G^*$-orbit of $a$. We have shown that $A$ is transitive. Now we are in a position to prove that $A$ is simple. Let $\{C_i\}$ be the set of cosets as before, where $|C_i| \geq 2$. Let $C_i^* = f^{-1}(C_i^*)$. Let $C^*$ be one of $C_i^*$.

(1) Suppose that $C^*$ contains $a$ and $b$ such that $(a, b) = 0$. Since $C^*_{\sigma_{b, \lambda}} = C^*$ for any $\lambda$ as $\sigma_{b, \lambda}$ fixes $b$, $C^*$ contains all $a + \mu b$. So, more generally, $C^*$
contains $\alpha a + \beta b$ for any $\alpha$ and $\beta$. For any $c$ in $V^*$, $(\alpha a + \beta b, c) = 0$ for some $\alpha a + \beta b$ in $V^*$, which implies that $\sigma_{\epsilon, \lambda}$ leaves $\alpha a + \beta b$ fixed. Therefore, $C^*$ is left fixed by any $\sigma_{\epsilon, \lambda}$. Since $V^*$ is transitive, this implies $C^* = V^*$, or $C = A$. $A$ is simple in this case.

(2) Suppose that $C^*$ contains $a$ and $b$ such that $(a, b) = 0$ and $a \in \langle b \rangle$. Then, we can express $b = \alpha a + d$ with a non-zero element $d$ in $V''$, where $V = V' \oplus V''$ (orthogonal), since $(a, b) = 0$ and $a \in \langle b \rangle$. Now, let $c$ be an element in $V''$ such that $(d, c) \neq 0$. Since $\sigma_{\epsilon, \lambda}$ fixes $a$, $C^*$ is left fixed by $\sigma_{\epsilon, \lambda}$. Then, $b^{\sigma_{\epsilon, \lambda}} \in C^*$, which implies that $b + c \in C^*$. Since $(b, b + c) \neq 0$, we have $C = A$ by (1).

(3) Suppose that $C^*$ contains $a$ and $\alpha a$, where $\alpha \neq \pm 1$. Let $b$ be an element such that $(a, b) \neq 0$. Let $C^*_{\epsilon}$ be a coset which contains $b$. Then, $C^*_{\epsilon} = C^*_{\epsilon} = C^*_{\epsilon} = C^*_{\epsilon}$, which contains $d = b^{\epsilon} a = b + (b, a)a$ and $e = b^{\epsilon} a = b + \alpha^2 (b, a)a$. Since $d \in \langle e \rangle$, we can apply (1) or (2) and get that $\{C^*_{\epsilon} = C\}$, or $A$ is simple.

Remark. If we consider $PSL(V)$ and $PSp(V)$, the "effective" condition is satisfied.

References