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EMBEDDING MANIFOLDS IN EUCLIDEAN SPACE

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1. Introduction. We consider here the problem of whether a smooth manifold M (compact, without boundary) embeds in Euclidean space of a given dimension. Our results are of two kinds: first we give sufficient conditions for an orientable n -manifold to embed in R^{2n-2} , and we then give necessary and sufficient conditions for RP^n ($=n$ -dimensional real projective space) to embed in R^{2n-6} . We obtain these results using the embedding theory of A. Haefliger [6].

Recall that by Whitney [37], every n -manifold embeds in R^{2n} . Combining results of Haefliger [6], Haefliger-Hirsch [9] and Massey-Peterson [16] one knows that every orientable n -manifold embeds in R^{2n-1} ($n > 4$), and if n is not a power of two, every n -manifold embeds in R^{2n-1} . Finally, if n is a power of two ($n > 4$), by [9] and [26] one has: a non-orientable n -manifold embeds in R^{2n-1} if and only if $\bar{w}_{n-1} = 0$. Here $\bar{w}_i, i \geq 0$, denotes the (mod 2) normal Stiefel-Whitney class of a manifold M .

We give two sets of sufficient conditions for embedding an n -manifold in R^{2n-2} ; in order to use the theory of Haefliger, we assume $n \geq 7$.

Theorem 1.1. *Let M be an orientable n -manifold, with $\bar{w}_{n-3+i} = 0$, for $i \geq 0$. If either $w_3 \neq 0$, or $w_2 \neq 0$ and $H_1(M; Z)$ has no 2-torsion, then M embeds in R^{2n-2} .*

Here w_i denotes the i^{th} mod 2 (tangent) Stiefel-Whitney class of M . A necessary condition for M^n to embed in R^{2n-2} is that $\bar{w}_{n-2} = 0$. Note, however, that if $n-1$ is a power of two, then RP^n does not embed in R^{2n-2} , even though $\bar{w}_{n-2} = 0$. (In this case $\bar{w}_{n-3} \neq 0$ and $H_1(RP^n; Z) = Z_2$).

By Massey-Peterson [16] one has that $\bar{w}_{n-3+i} = 0, i \geq 0$, for M^n , provided one of the following conditions is satisfied: $n \equiv 3 \pmod{4}$; $n \equiv 0, 2 \pmod{4}$ and $\alpha(n) \geq 3$; $n \equiv 1 \pmod{4}$ and $\alpha(n) \geq 4$. Here $\alpha(n)$ denotes the number of one's in the dyadic expansion of the integer n .

Recall that an orientable manifold is called a spin manifold if $w_2 = 0$. As a complement to Theorem (1.1) we have:

Theorem 1.2. *Let M be an n -dimensional spin manifold with $\bar{w}_{n-5+i} = 0$,*

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$i \geq 0$. Then M^n embeds in R^{2^n-2} , provided that $H_1(M; Z)$ has no 2-torsion when $n \equiv 0 \pmod{4}$.

Again by [16] and [26] we have: let M^n be a spin manifold with $n \equiv i \pmod{8}$, $4 \leq j \leq 7$; then, $\bar{w}_{n-j} = 0$. Thus by (1.2) we obtain: if M^n is a spin manifold with $n \equiv 5, 6, 7 \pmod{8}$, M embeds in R^{2^n-2} .

We consider now the problem of embedding RP^n . Quite good results have been obtained by geometric methods. In particular, the work of Mahowald-Milgram [15], Steer [31], and Rees [25] gives a good picture for large values of $\alpha(n)$. However, we now show that for small values of $\alpha(n)$ the known results are not best possible.

Theorem 1.3. *Let s be a positive integer, not a power of two. Set $n = 8s + t$, $0 \leq t \leq 7$. Then RP^n embeds in R^{2^n-6} , provided $\alpha(n) \geq 4$ when $t = 1$ or 2.*

To my knowledge this is a new result in the following cases (all congruences are mod 8).

$$\begin{array}{ll} n \equiv 0, & 2 \leq \alpha(n) \leq 8, \\ n \equiv 1, & 4 \leq \alpha(n) \leq 6, \\ n \equiv 2, & 4 \leq \alpha(n) \leq 7, \\ n \equiv 3, 5, & 4 \leq \alpha(n) \leq 5, \\ n \equiv 4, & 3 \leq \alpha(n) \leq 7. \end{array}$$

Combining results of [31] and [25] one has (cf., [12, 5.3]): If $n \equiv 7 \pmod{8}$, RP^n embeds in $R^{2^n-\alpha(n)-3}$; thus, if $n \equiv 6 \pmod{8}$, RP^n embeds in $R^{2^n-\alpha(n)-2}$. Consequently by (1.3) we have:

Corollary 1.4. *Let n be an integer such that $n \geq 15$ and $n \equiv 2, 4 \pmod{8}$. Then RP^n embeds in R^{2^n-6} if, and only if,*

$$\begin{array}{ll} \alpha(n) \geq 2, & \text{and } n \equiv 0 \\ \alpha(n) \geq 4, & \text{and } n \equiv 1, 3, 5, 6, 7. \end{array}$$

Of course, by (1.3), if $n \equiv 2$ and $\alpha(n) \geq 4$ or $n \equiv 4$ and $\alpha(n) \geq 3$, then RP^n does embed in R^{2^n-6} . Note [29], [1], [21], [4], that when $n \equiv 2$ and $\alpha(n) = 3$ or $n \equiv 4$ and $\alpha(n) = 2$, RP^n immerses in R^{2^n-6} but not in R^{2^n-7} . Thus the following conjecture seems reasonable.

Conjecture 1.5. *If $n \equiv 2 \pmod{8}$ and $\alpha(n) = 3$, or if $n \equiv 4 \pmod{8}$ and $\alpha(n) = 2$, then RP^n does not embed in R^{2^n-6} .*

The method of proof developed in this paper also gives one new result for complex projective n -space, CP^n .

Theorem 1.6. *Let n be a positive integer with $n \equiv 3 \pmod{4}$ and $\alpha(n) \geq 4$. Then CP^n embeds in R^{4n-6} .*

For $\alpha(n) \geq 5$ this follows by work of Steer [31].

The specific result of Haefliger that we use is the following. For a topological space X let X^2 denote the product $X \times X$ and let Δ denote the diagonal in X^2 . The group of order 2, Z_2 , acts freely on $X^2 - \Delta$ by interchanging factors; we set $X^* = (X^2 - \Delta)/Z_2$. The projection $p: X^2 - \Delta \rightarrow X^*$ is a 2-fold covering map; denote by ξ the associated line bundle and by $S_q(\xi)$ the $(q-1)$ sphere bundle associated to the q -plane bundle $q\xi$. Haefliger proves (see [5] and [6, §1.7]):

Theorem 1.7 (Haefliger). *Let M be a smooth n -manifold and let q be a positive integer such that $2q \geq 3(n+1)$. Then M embeds in R^q if, and only if, the bundle $S_q(\xi)$ has a section.*

REMARK. A similar theorem has been proved by Weber [36] for PL -manifolds (and semi-linear embeddings) and by J.A. Lees [41] for topological manifolds with locally flat embeddings (assuming $2q > 3(n+1)$). Thus Theorems (1.1) and (1.2) can be stated for these categories of manifolds. In connection with Theorems (1.3) and (1.6), note the work of Rigdon [27].

Our method of proof is to use various techniques of obstruction theory to show that the bundle $S_q(\xi)$ has a section. Briefly, the following techniques will occur: (i) indeterminacy, (ii) relations, (iii) naturality, (iv) generating class, (v) Whitney product formulae.

The remainder of the paper is organized as follows: in section 2 we develop some facts about the space M^* . Section 3 is a brief survey of obstruction theory, while in section 4 we give the proofs of Theorems (1.1) and (1.2). In section 5 we prove Theorem (1.3) and in section 6, Theorem (1.6). Finally, sections 7 and 8 contain proofs omitted in previous sections.

2. Properties of M^*

In order to use Theorem 1.7, we need to know the cohomology of M^* , especially mod 2. For the rest of the paper all cohomology will be with mod 2 coefficients unless otherwise indicated.

To compute $H^*(M^*)$ (mod 2 coefficients!) we use another result of Haefliger [7], as reworded by Rigdon [26]. We set (cf., [19]).

$$\Gamma M = S^\infty \times_{Z_2} (M^2),$$

where S^∞ is the unit sphere in R^∞ , and where Z_2 acts by the diagonal action. Also, let P^∞ denote the infinite dimensional real projective space, and for a

manifold M let $P(M)$ denote the projective line bundle associated to the tangent bundle.

Theorem 2.1 (Haefliger). *Given an n -manifold M , there is a commutative diagram of mod 2 cohomology, as shown below, in which each row is an exact sequence ($i \geq 0$):*

$$\begin{array}{ccccccc}
 0 & \rightarrow & H^{i-n}(M) & \xrightarrow{\varphi_1} & H^i(M \times M) & \xrightarrow{\rho_1} & H^i(M \times M - \Delta) \rightarrow 0 \\
 & & \uparrow r^* & & \uparrow q^* & & \uparrow p^* \\
 0 & \rightarrow & H^{i-n}(P^\infty \times M) & \xrightarrow{\varphi} & H^i(\Gamma M) & \xrightarrow{\rho} & H^i(M^*) \rightarrow 0 \\
 & & \parallel & & \downarrow k^* & & \downarrow j^* \\
 0 & \rightarrow & H^{i-n}(P^\infty \times M) & \xrightarrow{\varphi_2} & H^i(P^\infty \times M) & \xrightarrow{\rho_2} & H^i(P(M)) \rightarrow 0
 \end{array}$$

All the morphisms in the diagram, except the φ 's, are induced by mappings between spaces. φ_1 and φ_2 can be thought of as Gysin maps. Specifically, given $x \in H^{i-n}(M)$, then

$$(2.2) \quad \varphi_1(x) = U \cdot (1 \otimes x), \quad \text{where } U \in H^n(M^2)$$

is the mod 2 "Thom class" of M , as given, e.g., by Milnor [20]. φ_2 is computed as follows. Let $u \in H^1(P^\infty)$ denote the generator. Then, for $x \in H^*(M)$, and $j \geq 0$,

$$(2.3) \quad \varphi_2(u^j \otimes x) = \sum_0^n u^{i+j} \otimes w_{n-i}(M) \cdot x.$$

The key space in 2.1 is ΓM ; Steenrod [30] has computed the cohomology of this as follows.

Let t be the involution of $M \times M$ which transposes the factors, and set $\sigma = 1 + t^*: H^*(M^2) \rightarrow H^*(M^2)$. Let K^* and I^* denote, respectively, the kernel and image of σ . Thus K^* and I^* are graded groups with $I^* \subset K^*$; set $\bar{K}^* = K/I$.

Using the obvious projection $\Gamma M \rightarrow P^\infty$, we regard $H^*(\Gamma M)$ as an $H(P^*)$ -module.

Theorem 2.4 (Steenrod). *There is an isomorphism of $H^\infty(P^*)$ -modules,*

$$H^*(\Gamma M) \approx (H^*(P^\infty) \otimes \bar{K}^*) \oplus I^*,$$

where $H^*(P^\infty)$ acts trivially on I^* .

Note that \bar{K}^* is zero in odd dimensions. For each $n \geq 0$ we have an isomorphism

$$H^n(M) \approx \bar{K}^{2n}, \quad x \rightarrow (x)^2$$

where $(x)^2$ denotes the coset of I containing $x \otimes x$.

We now describe the morphisms q^* and k^* in (2.1).

Proposition 2.5.

- (i) $q^*|(\bar{K}^* \oplus I^*) = \text{identity}$.
- (ii) $q^*(u^m \otimes (x)^2) = 0$, if $m > 0$
- (iii) $k^*(I^*) = 0$,
- (iv) $k^*(u^m \otimes (x)^2) = \sum_0^q u^{m+q-i} \otimes \text{Sq}^i(x)$, if $\text{deg } x = q$.

For the proof, see Haefliger [7] and Steenrod [30].

Note that by 2.5 (iv), $k^*|(H^*(P^\infty) \otimes \bar{K}^*)$ is injective. Thus, for $y \in H^*(\Gamma M)$,

$$(2.5) \quad (v) \quad y = 0 \text{ if and only if } q^*(y) = 0 \text{ and } k^*(y) = 0.$$

Returning to diagram (2.1), the map $r: M \rightarrow P^\infty \times M$ is simply the inclusion; the morphism ρ_2 is computed as follows. As before, let ξ denote the canonical line bundle over M^* ; set $\eta = j^*\xi$, $v = w_1 \eta \in H^1(P(M))$. Recall (e.g. [11]) that $H^*(P(M))$ is a free $H^*(M)$ -module on $1, v \cdots v^{n-1}$, with the relation

$$(2.6) \quad v^n = \sum_{i=1}^n v^{n-i} \cdot w_i(M).$$

Given $x \in H^*(M)$, we have

$$(2.7) \quad \rho_2(u^m \otimes x) = v^m \cdot x, \quad m \geq 0.$$

Our goal is to find ways of showing that the obstructions vanish for a section of the bundle $S_q(\xi)$ over M^* . For this we need ways of showing that a class in $H^*(M^*)$ is zero. The following result is useful for this.

Define B^* to be the subspace of $H^*(\Gamma M)$ generated by all classes of the form

$$(2.8) \quad u^j \otimes (x)^2, \quad \text{with } j + \text{deg } x < \dim M.$$

We set $\lambda = \rho_2 k^* = j^* \rho$, in (2.1), and write $\Lambda^* = \lambda(B^*) \subset H^*(P(M))$. Note that $B^* \cap I^* = 0$.

Proposition 2.9.

- (a) *Kernel* $j^* = \rho(I^*)$,
- (b) $\rho|I^*$ is injective,
- (c) *Image* $j^* = \lambda(B^*) (= \Lambda^*)$,
- (d) ρ maps $B^* \oplus I^*$ isomorphically onto $H^*(M^*)$.

The proof is given in §7.

Set $w = w_1 \xi \in H^1(M^*)$. Since $\rho(u) = w$ and $j^*w = v$, we have (by 2.4),

Corollary 2.10. $w \cdot (\text{kernel } j^*) = 0$.

3. Obstruction theory for sphere bundles

We discuss here the general problem of finding a section to a sphere bundle. At the end of the section we consider the special case posed by Theorem 1.7—the sphere bundle is one associated to a multiple of a line bundle.

Let X be a complex and ω an oriented q -plane bundle over X , $q \geq 8$. We assume that $\dim X \leq q + 5$. Then the (mod 2) obstructions to a section in the associated sphere bundle are the following (see [14], [34]), using the fact that the 4 and 5-stems are zero [35].

$$\begin{aligned} \chi(\omega) &\in H^q(X; Z), \\ (\alpha_1, \alpha_3)(\omega) &\in H^{q+1}(X) \oplus H^{q+3}(X), \\ (\beta_2, \beta_3)(\omega) &\in H^{q+2}(X) \oplus H^{q+3}(X), \\ \gamma_3(\omega) &\in H^{q+3}(X). \end{aligned}$$

In our applications, ω comes from a double cover and so the mod 3 obstruction in $\dim q + 3$ is zero ([3], [28]). Also, for such a bundle, $w_{2i+1}(\omega) = 0$, $i \geq 0$.

These obstructions have the following indeterminacies: for $j \geq 1$, define $\theta_j: H^*(X) \rightarrow H^*(X)$ by

$$x \rightarrow \text{Sq}^j(x) + w_j(\omega) \cdot x.$$

Then, assuming that $w_1(\omega) = w_3(\omega) = 0$,

$$\begin{aligned} (3.1) \quad \text{Indet } (\alpha_1, \alpha_3) &= (\theta_2, \theta_4) H^{q-1}(X; Z). \\ \text{Indet } (\beta_2, \beta_3) &= (\theta_2, \text{Sq}^2 \text{Sq}^1) H^q(X) + \text{Sq}^1 H^{q+2}(X), \\ \text{Indet } (\gamma_3) &= \theta_2 H^{q+1}(X) + \text{Sq}^1 H^{q+2}(X). \end{aligned}$$

In the case of the β 's and γ , this is just the indeterminacy obtained by passing from one stage of the Postnikov resolution to the next—not the “full” indeterminacy in the sense of [18]. At one point we will need the full indeterminacy for (β_2, β_3) . Specifically, one can show (see [17], [18]):

$$\text{Indet } (\beta_2, \beta_3)(\omega) = \Psi_\omega H^{q-1}(X; Z),$$

where Ψ_ω is a “twisted” secondary cohomology operation [17], [32] defined on $\text{Kernel } \theta_2 \cap \text{Kernel } \theta_4 \cap H^{q-1}(X; Z)$, taking values in $H^{q+2}(X) \oplus H^{q+3}(X)$, and with $\text{Indet } \Psi_\omega = (\theta_2, \text{Sq}^2 \text{Sq}^1) H^q(X) + \text{Sq}^1 H^{q+2}(X)$. Note the simple, but important, fact:

$$\text{if } \text{Kernel } \theta_2 \cap \text{Kernel } \theta_4 \cap H^{q-1}(X; Z) = 0,$$

then

$$\text{Indet } (\beta_2, \beta_3) = \text{Indet } \Psi_\omega = (\theta_2, \text{Sq}^2 \text{Sq}^1) H^q(X) + \text{Sq}^1 H^{q+2}(X).$$

A second useful fact about these obstructions is that they satisfy certain universal relations, see [14], [33], [34, 4.2]. Namely

$$(3.2) \quad \begin{aligned} \theta_2\alpha_1(\omega) &= 0 \\ \text{Sq}^2\text{Sq}^1\alpha_1(\omega) + \text{Sq}^1\alpha_3(\omega) &= 0 \\ \theta_2\beta_2(\omega) + \text{Sq}^1\beta_3(\omega) &= 0, \end{aligned}$$

assuming, as above, that $w_1(\omega) = w_3(\omega) = 0$. Moreover, if $w_i(\omega) = 0$ for $1 \leq i \leq 7$, we then have

$$(3.3) \quad \text{Sq}^6\alpha_1(\omega) + \text{Sq}^4\alpha_3(\omega) = 0.$$

Suppose now that Y is a second complex and $f: Y \rightarrow X$ a map. One then has naturality relations for the obstructions: e.g.,

- (3.4) (i) *If $\chi(\omega) = 0$, then $(\alpha_1, \alpha_3)f^*\omega$ is defined and $f^*(\alpha_1, \alpha_3)(\omega) \subset (\alpha_1, \alpha_3)f^*\omega$.*
 (ii) *If $\chi(\omega) = 0$ and $(\alpha_1, \alpha_3)(\omega) \equiv 0$, then $(\beta_2, \beta_3)(f^*\omega)$ is defined and $f^*(\beta_2, \beta_3)\omega \subset (\beta_2, \beta_3)f^*\omega$.*

We consider now the special case $\omega = q\xi$, ξ a line bundle over X . We take q even, say $q = 2s$, so that ω is orientable. Let $v = w_1\xi \in H^1(X)$. Also, denote by δ_2 the Bockstein coboundary associated with the exact sequence $Z \xrightarrow{\times 2} Z \rightarrow Z_2$. Since $\chi(2\xi) = \delta_2 v$, one has

$$(3.5) \quad \chi(q\xi) = \delta_2(v^{q-1}).$$

To compute $\alpha_1(q\xi)$ (assuming $\chi(q\xi) = 0$), we use the theory of “twisted” cohomology operations, as developed in [17] and [32]. Write θ_2 for $\theta_2(q\xi)$. One then has a secondary operation $\Phi_{3,,}$, of degree 3, associated with the following relation (see p. 206 in [32]):

$$(3.6) \quad \Phi_3: \theta_2 \circ \theta_2 = 0, \text{ on integral classes.}$$

Our result is:

Proposition 3.7. *Let ξ be a line bundle over X , with $v = w_1\xi$. Suppose that $\chi(q\xi) = 0$, for some $q = 2s$, $s \geq 2$. If $\theta_2 H^{q-1}(X; Z) = \theta_2 H^{q-1}(X)$, then $\alpha_1(q\xi) \equiv \Phi_3(\delta_2(v^{q-3}) \cdot sv^2)$.*

This is proved at the end of the section, using the “generating class” theorem of [32].

One final technique we will need is the Whitney product formula for higher order obstructions: see [22] and [34, 4.3]. We keep the notation of 3.7.

Proposition 3.8.

- (i) *Suppose that $\chi(q\xi) = 0$. Then*

$$\alpha_1(q+2)\xi \equiv \alpha_1(q\xi) \cdot v^2,$$

$$\alpha_3(q+2)\xi \equiv \alpha_3(q\xi) \cdot v^2 + \alpha_1(q\xi) \cdot v^4.$$
- (ii) *Suppose that $\chi(q\xi) = 0$ and that $(\alpha_1, \alpha_3)(q\xi) \equiv 0$.*
Then,

$$\beta_j(q+2)\xi \equiv \beta_j(q\xi) \cdot v^2, \quad j=2, 3.$$

Proof of 3.7. Let η denote the canonical oriented 2-plane bundle over CP^∞ . The sphere bundle associated to $s\eta$ is

$$S^{2s-1} \xrightarrow{i} CP^{s-1} \xrightarrow{\pi} CP^\infty,$$

where π is homotopic to the inclusion. Let $x = \chi(\eta) \in H^2(CP^\infty; Z)$ denote the Euler class of η . Thus, $\chi(s\eta) = x^s \in H^{2s}(CP^\infty; Z)$. Consider now the first stage in a Postnikov resolution of π .

$$\begin{array}{ccccc}
 & & E & & \\
 & q \nearrow & \downarrow p & & \\
 CP^{s-1} & \xrightarrow{\pi} & CP^\infty & \xrightarrow{x^s} & K(Z, 2s)
 \end{array}$$

Let $\alpha \in H^{2s+1}(E)$ denote the second obstruction. Then α arises because of the relation

$$\theta_2(x^s) = 0.$$

(Note §§3–5 of [32]). But

$$x^s \text{ mod } 2 = \theta_2(x^{s-1}).$$

Thus, in the language of §5 of [32], x^{s-1} is a “generating class” for α ; and hence, by Theorem 5.9 of [32].,

$$(*) \quad \alpha \in \Phi_3(p^*x^{s-1}, sx \text{ mod } 2).$$

To prove 4.4, let $f: X \rightarrow CP^\infty$ be a map such that $f^*(x) = \delta_2 v$. Then, $q\xi = f^*(s\eta)$ —since $q = 2s$. By hypothesis, $\chi(q\xi) = 0$ and so f lifts to a map $g: X \rightarrow E$. Moreover, $g^*\alpha \in \alpha_1(q\xi)$. But by (*), $g^*\alpha \in \Phi_3((\delta_2 v)^{s-1}, sv^2)$. By the hypotheses of 4.4, α_1 and Φ_3 have the same indeterminacy, and so the theorem is proved.

REMARK. A similar result has been obtained independently by Rigdon [26].

4. Embedding n -manifolds in R^{2n-2}

If M is an n -manifold, then M^* has the homotopy type of a $(2n-1)$ -complex, and so to see whether $(2n-2)\xi$ has a section (i.e., by 1.7, whether M embeds in R^{2n-2}) we need only consider $\chi(2n-2)\xi$ and $\alpha_1(2n-2)\xi$. To compute χ we use the following important result of Haefliger [7].

Theorem 4.1 (Haefliger). *If M is an n -manifold, then $v^{n+k}=0$ if, and only if, $\bar{w}_{k+i}=0, i \geq 0$.*

The following result implies Theorem (1.1).

Proposition 4.2. *Let M be an orientable n -manifold. If $Sq^2H^{n-2}(M; Z)=H^n(M)$, then $\theta_2H^{2n-3}(M^*; Z)=H^{2n-1}(M^*)$.*

We give the proof at the end of the section.

Proof of Theorem 1.1. Since $\bar{w}_{n-3+i}=0$, for $i \geq 0$, it follows from (4.1) and (3.5) that $\chi(q\xi)=0$, where $q=2n-2$. We will show that $\alpha_1(q\xi) \equiv 0$ by showing that $Sq^2H^{n-2}(M; Z)=H^n(M)$. For then by (4.2), $H^{2n-1}(M^*)=Indet \alpha_1(q\xi)$ and hence $\alpha_1(q\xi) \equiv 0$.

Let $\mu \in H^n(M)$ denote the generator. Suppose first that $w_3 \neq 0$. Then there is a class $y \in H^{n-3}(M)$ such that $y \cdot w_3 = \mu$. But by Wu [38], $\mu = y \cdot w_3 = Sq^2Sq^1y$, and so $\mu \in Sq^2H^{n-2}(M; Z)$. On the other hand, suppose that $H_1(M; Z)$ has no 2-torsion. Then by Poincare duality, $H^{n-1}(M; Z)$ has no 2-torsion and so $H^{n-2}(M) = H^{n-2}(M; Z) \pmod 2$. Assume that $w_2 \neq 0$, and let $z \in H^{n-2}(M)$ be a class such that $\mu = z \cdot w_2 = Sq^2z$. But $z = \hat{z} \pmod 2$, for some $\hat{z} \in H^{n-2}(M; Z)$, and so again $\mu \in Sq^2H^{n-2}(M; Z)$, which completes the proof of the Theorem.

We turn now to the proof of Theorem (1.2). Since M is a spin manifold, $Sq^2H^{n-2}(M)=0$, and so we cannot use Proposition (4.2); instead we have the following:

Proposition 4.3. *Let M be an n -dimensional spin manifold. If $n \equiv 0 \pmod 4$ or if $H_1(M; Z)$ has no 2-torsion, then $\theta_2H^{2n-3}(M^*; Z) = \theta_2H^{2n-3}(M^*)$.*

Here $\theta_2 = \theta_2(2n-2)\xi$. We give the proof at the end of the section.

Proof of Theorem 1.2. Since $\bar{w}_{n-5+i}=0$ ($i \geq 0$), $\chi(2n-2)\xi=0$, using (3.5) and (4.1), and so $\alpha_1(2n-2)\xi$ is defined. By (4.3) and (3.7), $\alpha_1(q\xi) \equiv \Phi_3(\delta_r(v^{q-3}, sv^2))$, where $q=2n-2, s=n-1$. Since $\bar{w}_{n-5+i}=0$ ($i \geq 0$), then by 4.1 $v^{2n-5} = v^{q-3} = 0$, and so $\alpha_1(q\xi) \equiv 0$, which gives an embedding of M^n in R^{2n-2} , by Theorem (1.7).

Proof of Proposition 4.2. Note that by (2.8), $B^{2n-1}=0$, and hence by (2.9), $H^{2n-1}(M^*) = \rho I^{2n-1}$.

Let $\hat{\xi}$ denote the line bundle over ΓM with $w_1 \hat{\xi} = u$, and let $\hat{\theta}_2 = \theta_2(q\hat{\xi})$. Then,

$$\rho \hat{\theta}_2 = \theta_2 \rho, \hat{\theta}_2(I^*) = \text{Sq}^2(I^*).$$

To prove (4.2), let $y \in H^{2n-1}(M^*)$ and let $z \in I^{2n-1}$ with $\rho(z) = y$. Then, $q^*z = \sigma(\mu \otimes b)$, for some $b \in H^{n-1}(M)$. By hypothesis, there is a class $\hat{a} \in H^{n-2}(M; Z)$ with $\text{Sq}^2 \hat{a} = \mu$. Also, since M is orientable, $b = \hat{b} \bmod 2$ for some $\hat{b} \in H^{n-1}(M; Z)$. By analyzing $H^*(\Gamma M; Z)$ (e.g. [2]), one sees that there is a class $\hat{x} \in H^{2n-3}(\Gamma M; Z)$ such that $q^*(\hat{x}) = \hat{a} \otimes \hat{b} + \hat{b} \otimes \hat{a} \in H^{2n-3}(M^2; Z)$, and $\hat{x} \bmod 2 \in I^{2n-3}$. Thus,

$$q^*(\text{Sq}^2 \hat{x}) = \text{Sq}^2(\hat{a} \otimes \hat{b} + \hat{b} \otimes \hat{a}) = \sigma(\mu \otimes b) = q^*(z).$$

Since $z, \text{Sq}^2 \hat{x} \in I^*$ this means that $z = \text{Sq}^2 \hat{x}$ and so

$$y = \rho(z) = \rho \text{Sq}^2 \hat{x} = \rho \hat{\theta}_2 \hat{x} = \theta_2 \rho(\hat{x}),$$

as desired.

Proof of Proposition 4.3. Let $\theta_2 = \theta_2(2n-2)\xi$, $\hat{\theta}_2 = \theta_2(2n-2)\hat{\xi}$, as above. Let $x \in H^{2n-3}(M^*)$. Then (see (2.9)), one may choose $b \in H^{2n-3}(\Gamma M)$ so that $\rho(b) = x$ and

$$k^*(b) = u^{n-1} \otimes h + u^{n-2} \otimes \text{Sq}^1 h,$$

for some $h \in H^{n-2}(M)$. Since M is spin, $\text{Sq}^2 h = 0$ and $q^* \hat{\theta}_2(b) = 0$. Moreover,

$$k^* \hat{\theta}_2(b) = \hat{\theta}_2 k^*(b) = \binom{n}{2} u^{n+1} \otimes h + \binom{n-2}{2} u^n \otimes \text{Sq}^1 h.$$

Note that

$$u^{n+1} \otimes h = \varphi_2(u \otimes h) = k^* \varphi(u \otimes h),$$

and $q^*(u \otimes h) = 0$. Set

$$\beta = \hat{\theta}_2(b) - \varphi\left(\binom{n}{2}(u \otimes h)\right).$$

Then,

$$(*) \quad \rho(\beta) = \theta_2(x), \quad q^*(\beta) = 0, \quad k^*(\beta) = \binom{n-2}{2} u^n \otimes \text{Sq}^1 h.$$

Case I, $n \not\equiv 0 \pmod{4}$. If $n \equiv 2, 3 \pmod{4}$, then $\binom{n-2}{2} = 0 \pmod{2}$ and so $\beta = 0$, which means that $\theta_2(x) = 0$. If $n \equiv 1 \pmod{4}$, then $u^n \otimes \text{Sq}^1 h = \hat{\theta}_2 \text{Sq}^1(u^{n-2} \otimes h)$, and $k^* \text{Sq}^1(1 \otimes (h)^2) = \text{Sq}^1(u^{n-2} \otimes h)$. Set

$$\beta' = \beta - \hat{\theta}_2 \delta_2(1 \otimes (h)^2).$$

Then, $q^*\beta' = q^*\beta = 0$, $k^*\beta' = 0$, so $\beta' = 0$. Thus,

$$\theta_2(x) = \rho(\beta) = \theta\delta_2\rho(1 \otimes (h)^2) \in \theta_2 H^{2n-3}(M^*; Z);$$

this completes the proof in this case.

Case II, $H_1(M; Z)$ has no 2-torsion. By Poincare duality, $H^{n-1}(M; Z)$ has no 2-torsion and so $Sq^1 H^{n-2}(M) = 0$. Thus, in equation (*), $Sq^1 h = 0$ and so $\beta = 0$. This means that $\theta_2(x) = 0$, which completes the proof.

One can deduce other embedding results from (4.2), such as:

Theorem 4.4. *Let M be an n -dimensional, non-orientable manifold, such that $\bar{w}_{n-3+i} = 0$, $i \geq 0$. If $Sq^2 H^{n-3}(M) = H^{n-1}(M)$ and if $w_3 = w_1^3$, then M embeds in R^{2n-2} .*

Note that this gives as a special case the result of Handel [10]: if $n = 4k + 2$, $k \geq 2$ then RP^n embeds in R^{2n-2} .*)

5. Embedding real projective space

Before proving Theorem (1.3) we develop some preliminary material. For convenience we write P^n for RP^n , $n \geq 1$. In order to use Theorem (1.7), we need some rather detailed information about $H^*(P^{n*})$. We obtain this mainly by studying Λ^* and I^* —see (2.4) and (2.8).

We begin with some notation. In $H^*(P^\infty \times P^n)$, we set

$$(5.1) \quad [d, e] = \sum_{i=0}^e u^{d-i} \otimes Sq^i x^e,$$

where d, e are positive integers and x generates $H^1(P^n)$. By an abuse of notation, we use the same symbol to denote the image of $[d, e]$ by ρ_2 ; thus, in $H^*(P(P^n))$,

$$[d, e] = \sum_{i=0}^e v^{d-i} \cdot Sq^i x^e.$$

Note that by (2.5) (iv),

$$(5.2) \quad k^*(u^d \otimes (x^e)^2) = [d+e, e].$$

Also, from (2.8), (2.9) we have

$$(5.3) \quad \text{In } H^*(P(P^n)), \Lambda^* \text{ is spanned by the classes } [d, e], \text{ where } e \leq d < n.$$

In §8 we prove:

Proposition 5.4. *In $H^*(P^\infty \times P^n)$ and $H^*(P(P^n))$,*

$$Sq^1 [d, e] = d[d+1, e],$$

$$Sq^2 [d, e] = \binom{d}{2} [d+2, e] + e[d+1, e+1].$$

*) Remark (added in proof). These results overlap some with recent work of D. Bausam (Trans. A.M.S. 213 (1975), 263–303).

Similarly, in $I^* \subset H^*(\Gamma M)$, we write $\sigma(d, e)$ for $\sigma(x^d \otimes x^e)$. By the Cartan formula we have:

Proposition 5.5. $Sq^1\sigma(d, e) = d\sigma(d+1, e) + e\sigma(d, e+1)$,

$$Sq^2\sigma(d, e) = \binom{d}{2}\sigma(d+2, e) + de\sigma(d+1, e+1) + \binom{e}{2}\sigma(d, e+2).$$

Combining (5.4) and (5.5) we prove in §8:

Proposition 5.6.

$$Sq^2H^{4k-1}(P^{n*}) = Sq^2Sq^1H^{4k-2}(P^{n*}).$$

At one point we will need to know something about the integral cohomology of $P(P^n)$. The following result (proved in §8) suffices.

Proposition 5.7. *If n is even and $k \equiv 1 \pmod{4}$, then*

$$H^k(P^{n*}; Z) = \delta_2 H^{k-1}(P^{n*}).$$

We now can give the proof of Theorem 1.3. We do this by a series of lemmas that fit together to prove all parts of the Theorem.

Lemma 5.8. *Let q be a power of two, $q \geq 8$. Then,*

$$\alpha_1(q+4)\xi \equiv 0 \text{ in } H^{q+5}(M^*), M = P^{q-1}.$$

Proof. Since q is a power of two, $\bar{w}_i(P^{q-1}) = 0, i > 0$, and so by 4.1, $v^q = 0$ in $H^q(M^*), M = P^{q-1}$. Thus by (3.5) $\chi(q+4)\xi = 0$, and so $\alpha_1(q+4)\xi$ is defined. But by (3.7) and (5.6),

$$\alpha_1(q+4)\xi \equiv \Phi_3(\delta_2 v^{q+1}, 0) \equiv 0,$$

since $v^{q+1} = 0$. This completes the proof.

Now let s be an integer that is not a power of two, as in (1.3), and set $k = s - 1$, so that

$$8s + t = 8k + 8 + t, \quad 0 \leq t \leq 7.$$

Let q be the largest power of two such that $q/2 < 8s$. Then, $q + 4 \leq 16k + 4$, and so using the embedding $P^{8k+15} \subset P^{q-1}$, together with (3.8), we have:

Corollary 5.9.

$$\alpha_2(16k+4)\xi \equiv 0 \text{ in } H^{16k+5}(M^*), M = P^{8k+15}.$$

Recall the map $j: P(M) \rightarrow M^*$, given in diagram (2.1).

Lemma 5.10. *Taking $M = P^{8k+15}$, we have: There is a class a_3 such that*

$(0, a_3) \in (\alpha_1, \alpha_3)(16k+8)\xi$ and $j^*a_3 = r[8k+10, 8k+1]$, in $H^{16k+11}(P(M))$, $r \in Z_2$.

Proof. By (5.9), $\alpha_1(16k+8)\xi \equiv 0$; let $a_3 \in H^{16k+11}(M^*)$ be such that $(0, a_3) \in (\alpha_1, \alpha_3)(16k+8)\xi$. To prove (5.10) we show that a_3 can be chosen so that $j^*a_3 = r[8k+10, 8k+1]$, $r \in Z_2$.

Note that $w_i(16k+8)\xi = 0$, $1 \leq i \leq 7$, and so by (3.2), and (3.3),

$$\text{Sq}^1 a_3 = 0, \quad \text{Sq}^4 a_3 = 0.$$

Using (5.3) and (5.4), we have (since $\text{Sq}^1 j^*a_3 = 0$), $j^*a_3 = \sum_{i=0}^4 c_i [8k+6+2i, 8k+5-2i]$, where $c_i \in Z_2$. Now $\text{Sq}^4(j^*a_3) = 0$, and so the proof of (5.10) is complete when we show:

- A) $\text{Sq}^4 [8k+10, 8k+1] = 0$
- B) Sq^4 is injective on the subspace spanned by $[8k+14, 8k-3]$, $[8k+12, 8k-1]$, $[8k+8, 8k+3]$, $[8k+6, 8k+5]$.

We will use one more piece of notation: we set $(i, j) = v^i \cdot x^j$ in $H^{i+j}(P(P^n))$. Thus,

$$\begin{aligned} [8k+10, 8k+1] &= (8k+10, 8k+1) + (8k+9, 8k+2) + \\ &\quad k(8k+2, 8k+9) + k(8k+1, 8k+10), \end{aligned}$$

and so $\text{Sq}^4 [8k+10, 8k+1] = 0$, as claimed.

To prove (B), note that

$$\begin{aligned} \text{Sq}^4 [8k+14, 8k-3] &= [8k+14, 8k+1] + \dots \\ \text{Sq}^4 [8k+12, 8k-1] &= [8k+12, 8k+3] + \dots \\ \text{Sq}^4 [8k+8, 8k+3] &= [8k+11, 8k+4] + \dots \\ \text{Sq}^4 [8k+6, 8k+5] &= [8k+10, 8k+5] + \dots, \end{aligned}$$

where in each case the terms omitted have a left-hand coordinate smaller than that of the term shown. Thus Sq^4 is injective as claimed, which completes the proof of (5.10).

REMARK. In doing calculations such as above, we continually use the fact that, by (2.6),

$$(5.11) \quad (n, 0) = \sum_{i=1}^n \binom{n+1}{i} (n-i, i), \quad \text{in } P(P^n).$$

Also, note that $(i, 0) \cdot [d, e] = [d+i, e]$.

Lemma (5.10) will suffice to calculate the obstructions (α_1, α_3) in all the cases of Theorem (1.3).

We now jump ahead to compute the obstruction γ_3 .

Lemma 5.12. For $n \geq 15$, if $\gamma_3(2n-6)\xi$ is defined on P^{**} , then $\gamma_3(2n-6)\xi \equiv 0$.

This follows at once from (3.1), using the following fact, which we prove in §8.

$$(5.13) \quad H^{2n-3}(P^{**}) = \theta_2 H^{2n-5}(P^{**}) + \text{Sq}^1 H^{2n-4}(P^{**}), \quad \text{where } \theta_2 = \theta_2(2n-6)\xi.$$

We now come to the proof of Theorem (1.3): we divide the proof into three cases. As before, set $n=8s+t=8k+8+t$, $0 \leq t \leq 7$, s not a power of two.

Case I. $n \equiv 3, 4, 5 \pmod 8$.

Let $q=8k+15$, we do all our calculation on P^{q*} .

$$(5.14) \quad \text{On } P^{q*}, (\alpha_1, \alpha_3)(16k+16)\xi \equiv 0.$$

By (5.10) and the Whitney formula, (3.8), there is a class $a_3 \in H^{16k+17}(P^{q*})$ such that $(0, a_3) \in (\alpha_1, \alpha_3)(16k+14)\xi$ and $j^*(0, a_3) = (0, r[8k+16, 8k+1])$. But by (5.11),

$$\begin{aligned} [8k+16, 8k+1] &= (8k+16, 8k+1) + (8k+15, 8k+2) + \\ &k(8k+8, 8k+9) + k(8k+7, 8k+10) = 0, \end{aligned}$$

and so $j^*a_3 = 0$. Thus, by (3.8) and (2.10), $(\alpha_1, \alpha_3)(16k+16)\xi \equiv 0$, as desired.

We now show

$$(5.15) \quad (\beta_2, \beta_3)(16k+16)\xi \equiv 0, \quad \text{on } P^{q*}, q = 8k+15.$$

Note first that

$$(C) \quad \text{On } P(P^q), q=8k+15,$$

$$\Lambda^{16k+19} = \text{Sq}^1 \Lambda^{16k+18} + \text{Sq}^2 \text{Sq}^1 \Lambda^{16k+16}.$$

This is a simple calculation using (5.4) and (5.3)-e.g., $[8k+14, 8k+5] = \text{Sq}^1[8k+13, 8k+5]$, $[8k+13, 8k+6] = \text{Sq}^2 \text{Sq}^1[8k+11, 8k+5]$. Thus, by (3.1), $j^*\beta_3(16k+16)\xi \equiv 0$, on $P(P^q)$. Choose classes $(b_2, b_3) \in (\beta_2, \beta_3)(16k+16)\xi$ such that $j^*b_3 = 0$. Since $\theta_2(16k+16)\xi = \text{Sq}^2$, we have by (3.2), $j^*\text{Sq}^2 b_2 = 0$. By (5.3), $j^*b_2 = \sum_{i=0}^5 c_i[8k+9+i, 8k+9-i]$, $c_i \in Z_2$. Using (5.4), one finds that Kernel Sq^2 on Λ^{16k+18} is generated by $[8k+12, 8k+6]$. Since,

$$\begin{aligned} \text{Sq}^2[8k+10, 8k+6] &= [8k+12, 8k+6], \\ \text{Sq}^2 \text{Sq}^1[8k+10, 8k+6] &= 0, \end{aligned}$$

this means that one can alter b_2 to a class b_2' (without changing b_3), so that $(b_2', b_3) \in (\beta_2, \beta_3)(16k+16)\xi$ and $j^*b_2' = j^*b_3 = 0$. Hence, by (2.9) there are classes $(c_2, c_3) \in I^*$ with $\rho(c_2) = b_2'$, $\rho(c_3) = b_3$. Using (5.5) one easily shows:

$$(5.16) \quad \text{On } \Gamma P^q, I^{16k+19} = \text{Sq}^1 I^{16k+18} + \text{Sq}^2 \text{Sq}^1 I^{16k+16}.$$

This shows that $\rho(c_3) \in \text{Indet } \beta_3$, and so we may choose c_3 to be zero-i.e., $\rho(c_2, 0) \in (\beta_2, \beta_3)(16k+16)\xi$. By (3.2), $\rho(\text{Sq}^2 c_2) = 0$, and so by (2.9), $\text{Sq}^2 c_2 = 0$. Using (5.5) one finds that on I^{16k+18} , Kernel Sq^2 is generated by $\sigma(8k+14, 8k+4)$ and $\sigma(8k+12, 8k+6) + \sigma(8k+10, 8k+8)$. Since $\text{Sq}^2 \sigma(8k+14, 8k+2) = \sigma(8k+14, 8k+4)$ and $\text{Sq}^2 \sigma(8k+10, 8k+6) = \sigma(8k+12, 8k+6) + \sigma(8k+10, 8k+8)$, we see that $(b_2, b_3) \in \text{Indet } (\beta_2, \beta_3)$, and so $(\beta_2, \beta_3)(16k+16)\xi \equiv 0$, as claimed.

Combining (5.14), (5.15) and (5.12), we see (cf. §2) that on $(P^{8k+11})^*$ the sphere bundle associated to $(16k+16)\xi$ has a section and hence, by (1.7), P^{8k+11} embeds in R^{16k+16} . Similarly, using (3.8), we see that P^{8k+12} embeds in R^{6k+18} and P^{8k+13} in R^{16k+20} , thus proving Theorem (1.3) in Case I.

Case II. $n \equiv 0 \pmod 8$.

We first prove:

$$(5.17) \quad (\alpha_1, \alpha_3)(16k+8)\xi \equiv 0, \text{ on } P^{8k+8*}.$$

By (5.10) there is a class a_3 such that $(0, a_3) \in (\alpha_1, \alpha_3)(16k+8)\xi$ (on P^{8k+8*}) and $j^* a_3 = r[8k+10, 8k+1]$, $r \in \mathbb{Z}_2$. But by (5.11),

$$[8k+10, 8k+1] = (8k+10, 8k+1) + (8k+9, 8k+2) = 0,$$

which shows that $j^* a_3 = 0$.

We now use the fact [15], [25] that P^{8k+7} embeds in R^{16k+8} ; thus, if $i: P^{8k+7*} \rightarrow P^{8k+8*}$ is induced by the inclusion $P^{8k+7} \subset P^{8k+8}$, we have (by (1.7)), $i^*(\alpha_1, \alpha_3)(16k+8)\xi \equiv 0$ and hence, by (3.1), there is a class $y \in H^{16k+7}(P^{8k+7*}; \mathbb{Z})$ such that

$$(0, i^* a_3) \in (\text{Sq}^2, \text{Sq}^4)(y).$$

Let z be a (mod 2) class in $B^* \oplus I^*$ such that $\rho(z) = y \pmod 2$, and let

$$z = b + e, \quad b \in B^*, \quad e \in I^*.$$

We have

$$b = s(u^5 \otimes (x^{8k+1})^2) + t(u^3 \otimes (x^{8k+2})^2) + q(u \otimes (x^{8k+3})^2),$$

and so $k^*(b) = \sum_{i=1}^6 r_i(8k+7-i, 8k+i)$, $r_i \in \mathbb{Z}_2$. Consequently,

$$\begin{aligned} k^*(\text{Sq}^4 b) &= s_1(8k+10, 8k+1) + s_2(8k+9, 8k+2) + \\ &\quad s_3(8k+8, 8k+3), \quad s_i \in \mathbb{Z}_2 \\ &= k^* \varphi(s_1(u^3 \otimes x^{8k+1}) + s_2(u^2 \otimes x^{8k+2}) + s_3(u \otimes x^{8k+3})). \end{aligned}$$

Thus, by (2.5) (ii) and (iv),

$$\text{Sq}^4 b = 0, \text{ mod Image } \varphi.$$

Since $I^* = \text{kernel } k^*$, we have $\text{Sq}^4 I^* \subset I^*$. A simple calculation shows that $\text{Sq}^4 I^{16k+7} = 0$, in $H^*(\Gamma P^{8k+7})$. Consequently, $\text{Sq}^4 z = 0$, mod image, φ and so

$$i^* a_3 = \text{Sq}^4 y = \rho(\text{Sq}^4 z) = 0.$$

Recall that back on P^{8k+8^*} , $j^* a_3 = 0$. Hence there is a class $d \in I^{16k+7}$ (on ΓP^{8k+8}), with $\rho(d) = a_3$; since $i^* d$ also is in I^* , and since $\rho(i^* d) = i^* a_3 = 0$, it follows that $i^* d = 0$. Thus, $d = r\sigma(8k+8, 8k+3)$, $r \in Z_2$. But by (3.2), since $\alpha_1 \equiv 0$,

$$\rho(\text{Sq}^1 d) = \text{Sq}^1 a_3 = 0,$$

and hence, $\text{Sq}^1 d = 0$. Since $\text{Sq}^1 \sigma(8k+8, 8k+3) = \sigma(8k+8, 8k+4) \neq 0$, this shows that $r = 0$, and so $a_3 = 0$, completing the proof of (5.17).

By (5.17), $(\beta_2, \beta_3)(16k+8)\xi$ is defined, on P^{8k+8^*} . We now show:

$$(5.18) \quad j^*(\beta_2, \beta_3)(16k+8)\xi \equiv 0.$$

This follows easily from (3.1), (3.2) and (5.3). We leave the details to the reader.

Finally, by (2.10) and (3.8), (5.18) implies that $(\beta_2, \beta_3)(16k+10)\xi \equiv 0$, and so, by (5.12) and (1.7), P^{8k+8} embeds in R^{16k+10} , as desired. This completes the proof for Case II.

Case III. $n \equiv 1, 2 \pmod{8}$, $\alpha(n) \geq 4$.

We do all the argument on P^{8k+10} ; set $m = 8k+10$. The first result is:

$$(5.19) \quad (\alpha_1, \alpha_3)(16k+10)\xi \equiv 0, \text{ on } P^{m^*}.$$

As before, by (5.10), there is a class a_3 such that $(0, a_3) \in (\alpha_1, \alpha_3)(16k+8)\xi$ (on P^{m^*}) and $j^* a_3 = r[8k+10, 8k+1]$ in $H^{16k+11}(P(P^m))$, $r \in Z_2$. Thus by (3.8), $j^*(\alpha_1, \alpha_3)(16k+10)\xi \equiv (0, r[8k+12, 8k+1])$. Let $l: P^{m-1} \rightarrow P^{m^*}$, $\hat{l}: P(P^{m-1}) \rightarrow P(P^m)$ denote the maps induced by the inclusion $P^{m-1} \subset P^m$. We now use the fact that P^{8k+9} immerses in R^{16k+10} , see [23]. (It is at this point that we require $\alpha(n) \geq 4$.) Thus the bundle $\hat{l}^* i^*(16k+10)\xi$ has a (nowhere zero) section on $P(P^{m-1})$, by Haefliger-Hirsch [8]. Consequently, by (3.1),

$$(*) \quad (0, r[8k+12, 8k+1]) \in (\theta_2, \text{Sq}^4) H^{16k+9}(P(P^{8k+9}); Z).$$

Using (5.11) one sees that $[8k+12, 8k+1] = [8k+8, 8k+5]$ on $P(P^{8k+9})$. Moreover, by (5.4), θ_2 is an injection on Kernel $\text{Sq}^1 \cap H^{16k+9}(P(P^{8k+9}))$, $\theta_2 = \theta_2(16k+10)\xi$. Since $[8k+12, 8k+1] \neq 0$, it follows from (*) that $r = 0$. Thus, $j^* a_3 = 0$ and so $j^*(\alpha_1, \alpha_3)(16k+8)\xi \equiv 0$; consequently, by (3.8) and (2.10), $(\alpha_1, \alpha_3)(16k+10)\xi \equiv 0$, on P^{m^*} , which proves (5.19).

The obstruction $(\beta_2, \beta_3)(16k+10)\xi$ is consequently defined, and we now show:

$$(5.20) \quad j^*(\beta_2, \beta_3)(16k+10)\xi \equiv 0, \quad \text{on } P^{m*}, m=8k+10.$$

The first step is to show:

A) We may choose classes $(b_2, b_3) \in (\beta_2, \beta_3)(16k+10)\xi$ so that

$$j^*(b_2, b_3) = (r[8k+6, 8k+6], 0), r \in Z_2.$$

Note that

$$\begin{aligned} j^*b_2 &= r[8k+6, 8k+6] + s[8k+7, 8k+5] + \\ &\quad t[8k+8, 8k+4] + q[8k+9, 8k+3], \\ j^*b_3 &= c[8k+7, 8k+6] + d[8k+8, 8k+5] + e[8k+9, 8k+4], \end{aligned}$$

where the coefficients all lie in Z_2 . Since

$$\begin{aligned} \text{Sq}^1[8k+7, 8k+5] &= [8k+8, 8k+5], \\ \text{Sq}^2\text{Sq}^1[8k+7, 8k+3] &= [8k+9, 8k+4], \quad \text{and} \\ \text{Sq}^2\text{Sq}^1[8k+5, 8k+5] &= [8k+8, 8k+5] + [8k+7, 8k+6], \end{aligned}$$

we see that b_3 can be chosen so that $j^*b_3=0$. Thus, by (3.2), $j^*(\theta_2 b_2)=0$, where $\theta_2=\theta_2(16k+10)\xi$. Using (5.4) and (5.11) one finds that this implies: $s=0, t=q$. But $\theta_2[8k+8, 8k+2]=[8k+9, 8k+3]+[8k+8, 8k+4]+[8k+6, 8k+6]$. Hence, b_2 can be altered (without changing b_3) so that $j^*b_2=r[8k+6, 8k+6]$, as claimed.

To complete the proof of (5.20), we use the map $i: P^{m-2*} \rightarrow P^{m*}$. In Case II we proved that $i^*(\beta_2, \beta_3)(16k+10)\xi \equiv 0$ on P^{m-2*} , and so $i^*(b_2, b_3) \in \Psi_\omega H^{16k+9}(P^{m-2*}; Z)$, (see discussion following (3.1)), where $\omega=(16k+10)\xi$. Now by (5.7), a class in $H^{16k+9}(P^{m-2*}; Z)$ is determined by its mod 2 reduction. Suppose then that y is in domain Ψ_ω , and let $\bar{y}=y \text{ mod } 2$. Then (see §3), $\text{Sq}^1\bar{y}=0, \theta_2\bar{y}=0, \text{Sq}^4\bar{y}=0$. But a calculation shows that

$$H^{16k+9}(P^{m-2*}) \cap \text{Kernel Sq}^1 \cap \text{Ker } \theta_2 \cap \text{Ker Sq}^4 = 0,$$

and so $\bar{y} \text{ mod } 2=0$. Thus, $y=0$, and so by §3,

$$i^*(b_2, b_3) \in \text{indet } \Psi_\omega = (\theta_2, \text{Sq}^1\text{Sq}^1)H^{16k+10}(P^{m-2*}) + \text{Sq}^1H^{6k+12}(P^{m-2*}).$$

Also, by what we have already proved, $j^*i^*(b_2, b_3)=(r[8k+6, 8k+6], 0)$, in $P(P^{m-2})$. Thus, there is a class $y \in H^{16k+10}(P^{m-2*})$ with $\theta_2(j^*y)=r[8k+6, 8k+6]$. A simple calculation using (5.4) shows that this is possible only if $j^*y=0$ and $r=0$. Hence, back on P^{m*} , $j^*(\beta_2, \beta_3)(16k+10)\xi \equiv 0$, as claimed.

Therefore, by (3.8) and (5.11), $(\beta_2, \beta_3)(16k+12)\xi \equiv 0$ on P^{m*} , and hence on P^{m-1*} . Thus by (5.12) and (1.7), P^n embeds in R^{2n-6} , for $n=8k+9$ and $8k+10$, completing the proof of Theorem (1.3).

6. Embedding complex projective space

Our goal is to show that if $n=4s+3$, s not a power of two, then CP^n embeds in R^{4n-6} . We do this by showing that the sphere bundle over CP^n , associated to $(4n-6)\xi$, has a section. Since the methods here are very similar to those used in §5, we only sketch the proof.

We use the following notation: $y \in H^2(CP^n)$ denotes the generator, and in $H^*(P(CP^n))$ we set

$$[d, 2j] = \sum_{i=0}^j v^{d-2i} \cdot \text{Sq}^{2i}(y^j).$$

As before, $\Lambda^* \subset H^*(P(CP^n))$ denotes $j^*H^*(CP^n)$, and as in (5.3) we have:

$$(6.1) \quad \text{The classes } [d, 2i] \text{ generate } \Lambda^*, \text{ where } 2i \leq d \leq 2n-1.$$

Finally, we set $s=k+1$, so that $n=4k+7$. The first step in the proof of (1.5) is to show:

$$(6.2) \quad \text{the obstruction } (\alpha_1, \alpha_3)(16k+12)\xi \text{ is defined and there are classes } (a_1, a_3) \in (\alpha_1, \alpha_3)(16k+12)\xi \text{ such that}$$

$$j^*a_1 = r[8k+13, 8k] + s[8k+9, 8k+4]$$

$$j^*a_3 = s[8k+11, 8k+4], \quad r, s \in \mathbb{Z}.$$

This is proved using (3.1) and (3.2). Now $w_i(16k+12)\xi \neq 0$, while $w_i(16k+12)\xi = 0$, for $1 \leq i \leq 7$, $i \neq 4$. Thus, one has a relation analogous to (3.3):

$$\text{Sq}^6 a_1 + \theta_4 a_3 = 0.$$

Using this on (6.2) one finds that $s=0$ (in (6.2)). But by a formula analogous to (5.11), $[8k+15, 8k]=0$, and hence $j^*(\alpha_1, \alpha_3)(16k+14)\xi \equiv 0$, using (3.8). Therefore, by (2.10) and (3.8), $(\alpha_1, \alpha_3)(16k+16)\xi \equiv 0$.

The next step is to show:

$$(6.3) \quad (\beta_2, \beta_3)(16k+22)\xi \equiv 0.$$

Starting with classes $(b_2, b_3) \in (\beta_2, \beta_3)(16k+18)\xi$, one finds that by using the indeterminacy of β_2 (i.e., θ_2), b_2 can be chosen so that $j^*b_2 = r[8k+12, 8k+8]$. And by (3.2), one has $j^*b_3 = s[8k+13, 8k+8]$. But

$$[8k+14, 8k+8] = [8k+15, 8k+8] = 0,$$

and so $j^*(\beta_2, \beta_3)(16k+20)\xi \equiv 0$. Consequently, by (2.10) and (3.8), (β_2, β_3)

$(16k+22)\xi \equiv 0$, as desired.

By an indeterminacy argument (use θ_2) one shows that $j^*\gamma_3(16k+22)\xi \equiv 0$. But $I^{16k+25} = 0$, and so by 2.9, $\gamma_3(16k+22)\xi \equiv 0$, which means by (1.7) that CP^n embeds in R^{2n-6} .

7. The cohomology of M^*

This section contains the proofs that were omitted in sections 2 and 4. We begin with the proof of Proposition (2.9.)

(a) Kernel $j^* = \rho(I^*)$.

This follows at once from the exactness of (2.1), given that kernel $k^* = I^*$ (see (2.5)).

(b) $\rho|I^*$ is injective.

Set $D^* = H^*(P^\infty) \otimes \bar{K}^*$, see (2.4). Note that $D^* \cap I^* = 0$ and that $\varphi(u^i \otimes x) \in D^*$, if $i > 0$ and $x \in H^*(M)$. Suppose that $e \in I^*$ with $\rho(e) = 0$. Then, by (2.1) and the above remarks, $e = \varphi(1 \otimes y)$, for some $y \in H^*(M)$. By (2.5) and (2.1), since $e \in I^*$,

$$0 = k^*(e) = k^*\varphi(1 \otimes y) = \varphi_2(1 \otimes y).$$

But φ_2 is injective, and so $y = 0$, which proves $e = 0$, as claimed.

(c) Image $j^* = \lambda(B^*) = \Lambda^*$.

Note that by (2.3), Image $\varphi_2 = u^n \otimes H^*(M) \oplus u^{n+1} \otimes H^*(M) \oplus \dots$, where $n = \dim M$. Thus, if we set $C = \sum_{i=0}^n u^i \otimes H^*(M)$, we have that ρ_2 maps C isomorphically onto $H^*(P(M))$. Set $\tilde{C} = C \cap \rho_2^{-1}(\text{Image } j^*)$. Note that $k^*(B^*) \subset C$ and hence $k^*(B^*) \subset \tilde{C}$, we show:

$$(*) \quad k^*(B^*) = \tilde{C},$$

which proves (c). Moreover, by (*), λ maps B^* isomorphically onto Image j^* and hence $\rho|B^*$ is an inverse to j^* , which proves (d), in (2.9).

To prove (*) all we need show is that k^* maps B^* onto \tilde{C} . This is a consequence of the following:

Proposition 7.1. *Given $y \in H^*(\Gamma M)$, there is a class $b \in B^*$ such that $\lambda(b) = \lambda(y)$.*

Before proving this we develop some preliminary material. Given a class y in $H^*(\Gamma M)$ we associate with it a unique class in $H^*(P^\infty) \otimes \bar{K}$, called the leading term of y . Suppose that degree $y = d$, and set $s = [d/2]$. Then we can write $y = \sum_0^s u^{d-2i} \otimes (x_i)^2 + l$, where $l \in I$ and where degree $x_i = i$, $0 \leq i \leq s$. Let j be

the integer such that $x_j \neq 0$ and $x_i = 0$ for $i < j$. We define *leading term* $y = u^{d-2j} \otimes (x_j)^2$. If $y = l$, we set *leading term* $y = 0$.

We will need the following key fact.

(7.2) *Let $x \in \bar{H}^q(M)$, $x \neq 0$, and let j be a non-negative integer. Then, leading term $\phi(u^j \otimes x) = u^{n-q+j} \otimes (x)^2$.*

Proof. Write $d = n + q + j$, and set $s = [d/2]$. There are classes $l \in I$ and $y_i \in \bar{K}^{2i}$ such that

$$\phi(u^j \otimes x) = \sum_0^s u^{d-2i} \otimes y_i + l.$$

Also, by 2.3,

$$\phi_2(u^j \otimes x) = \sum_0^m u^{n+j-i} \otimes w_i M \cdot x = \sum_0^m u^{d-q-i} \otimes w_i M \cdot x.$$

Thus the term in $\phi_2(u^j \otimes x)$ with highest power of u is $u^{d-q} \otimes x$. But $\phi_2 = k^* \phi$, and so $y_i = 0$ for $i < q$ and

$$k^*(u^{d-2q} \otimes y_q) = u^{d-q} \otimes x + \text{terms with lower degree in } u.$$

Using 2.5 (iv), and recalling that $\text{Sq}^0(x) = x$, we have $y_q = (x)^2 \pmod{I}$, which implies that

$$\begin{aligned} \text{leading term } \phi(u^j \otimes x) &= u^{d-2q} \otimes (x)^2 \\ &= u^{n-q+j} \otimes (x)^2, \end{aligned}$$

as claimed. This completes the proof of (7.2).

Proof of 7.1. Let $y \in H^*(\Gamma M)$. Since $\lambda(I) = 0$, we may suppose that $y \in H^*(P^\infty) \otimes \bar{K}$. Let leading term $y = u^k \otimes (x)^2$, where $k \geq 0$ and $\text{deg } x = q$, say. If $k + q < n$, then $y \in B^*$ and there is nothing to prove, so suppose that $k + q \geq n$. Let $y_1 = \phi(u^{k+q-n} \otimes x)$. Then, by (2.1)

$$\lambda(y) = \lambda(y - y_1).$$

But by (7.2), y and y_1 have the same leading term, and so

$$\text{leading term } (y - y_1) = u^{k_1} \otimes (x^{(1)})^2,$$

where $k_1 \leq k - 2$ and $k_1 + 2 \text{ deg } x^{(1)} = \text{deg } y$. Thus,

$$k_1 + \text{deg } x^{(1)} < k + \text{deg } x.$$

Continuing in this way we obtain classes y_1, y_2, \dots, y_r , say, such that

$$\lambda(y) = \lambda(y - (y_1 + \dots + y_r)), \text{ and } y - (y_1 + \dots + y_r) \in B^*,$$

thus completing the proof of (7.1) and hence of (2.9).

8. The cohomology of P^{n*}

This section contains the proofs that were omitted in §5. We begin with the following useful fact.

Lemma 8.1. *Let $d \in H^q(\Gamma P^n)$, $q > 0$, and let $k^*(d) = \sum_{i=0}^q a_i(u^{q-i} \otimes x^i)$, where $a_i \in Z_2$. If $a_i = 0$ for $2i \leq q$, then $k^*(d) = 0$.*

This is an immediate consequence of (5.2). Using this we have:

Proof of 5.4. We do the proof in $H^*(P^\infty \times P^n)$, giving the details only for Sq^2 . Suppose then that d and e are positive integers with $d \geq e$. Note that

$$Sq^2[d+4, e] = Sq^2(u^4 \cdot [d, e]) = u^4 \cdot Sq^2[d, e],$$

and so to prove (5.4) we may assume $e \leq d \leq e+3$, since $\binom{d}{2} \equiv \binom{d+4}{2} \pmod{2}$. Now for $j \geq 0$, $[e+j, e] \in \text{Image } k^*$, by (5.2). Also, if $d \leq e+3$, we find that

$$Sq^2[d, e] + \binom{d}{2}[d+2, e] + e[d+1, e+1] = \sum a_i(u^i \otimes x^j),$$

where the sum is over all, $i+j=d+e+2$ and where $a_i=0$ for $i \geq i$. Thus by (8.1), $Sq^2[d, e] + \binom{d}{2}[d+2, e] + e[d+1, e+1] = 0$ as claimed.

The proof for Sq^1 is similar, using the fact that

$$Sq^1[d+2, e] = Sq^1(u^2 \cdot [d, e]) = u^2 \cdot Sq^1[d, e].$$

Hence, we need only take $d=e, e+1$.

REMARK. A proof for Sq^1 is given in [40], and [2, section 7]; note also [13].

Proof of 5.6. Since $H^*(P^{n*}) = \rho H^*(\Gamma P^n)$, and since $H^*(\Gamma P^n)$ is determined by k^* and q^* , (5.6) will follow when we show:

$$(8.2) \quad \begin{aligned} (i) \quad & Sq^2 k^* H^{4k-1}(\Gamma P^n) = Sq^2 Sq^1 k^* H^{4k-2}(\Gamma P^n) \\ (ii) \quad & Sq^2 q^* I^{4k-1} = Sq^2 Sq^1 q^* I^{4k-2}. \end{aligned}$$

Now (i) follows at once from (5.4), while (ii) may be proved by an inductive argument using (5.5). We omit the details.

To prove (5.6), let $y \in H^{4k-1}(\Gamma P^n)$. By (8.2) (i), we may choose $d \in H^{4k-2}(\Gamma P^n)$ so that $k^*(Sq^2 Sq^1 d) = k^*(Sq^2 y)$. Set $\hat{y} = y - Sq^1 d$. By (8.2)(ii), there is a class $e \in I^*$ such that $q^*(Sq^2 Sq^1 e) = q^* Sq^2 \hat{y}$. Since $Sq^2 Sq^1 I^* \subset I^*$ and $k^* I^* = 0$, we see that $k^* Sq^2 Sq^1(d+e) = k^* Sq^2 y$, $q^* Sq^2 Sq^1(d+e) = q^* Sq^2 y$, and hence $Sq^2 y = Sq^2 Sq^1(d+e)$, completing the proof of (5.6).

We will need the following well-known fact in the proof of (5.7).

Lemma 8.3. *Let X be a space and k a positive integer such that $H^k(X; Z)$ is finitely generated and has no odd torsion. Then, $H^k(X; Z) = \delta_2 H^{k-1}(X; Z_2)$ if, and only if,*

$$\text{Kernel Sq}^1 = \text{Image Sq}^1 \text{ on } H^k(X; Z_2).$$

Proof of 5.7. Note that $\text{Sq}^1 I^* \subset I^*$, and since k is odd, $\text{Sq}^1 B^k \subset B^{k+1}$. Thus by (8.3), (5.7) is proved when we show:

$$\text{Sq}^1 B^{k-1} = \ker \text{Sq}^1 \cap B^k, \quad \text{Sq}^1 I^{k-1} = \text{Ker Sq}^1 \cap I^k.$$

Since $\lambda : B^* \approx \Lambda^*$ (see 2.9), we do the argument for B^* in Λ^* . Define $V \subset \Lambda^*$ to be the subspace spanned by generators $[d, e]$, with $e \leq d \leq n-2$. Since n is even (in 5.7), $\text{Sq}^1 V \subset V$, by (5.4). Let k (in 5.7) be written, $k=4s+1$. We assume $k > n$, since this is the only case of interest to us. Then,

$$\begin{aligned} \Lambda^k &= \{[n-1, k-n+1]\} \oplus V. \text{ But} \\ \text{Sq}^1[n-1, k-n+1] &= [n, k-n+1] = \\ &[n-1, k-n+2] + v, \text{ where } v \in V. \end{aligned}$$

(We use here 5.11 and the fact that $k-n+1$ is even.) Thus $\text{Sq}^1[n-1, k-n+1] \notin V$ and so $\text{Ker Sq}^1 \cap \Lambda^k = \text{Ker Sq}^1 \cap V$. An easy calculation shows that $\text{Ker Sq}^1 \cap V \subset \text{Sq}^1 \Lambda^{k-1}$. Finally, since

$$k^* \text{Sq}^1(1 \otimes (x^r)^2) = q^* \text{Sq}^1(1 \otimes (x^r)^2) = 0,$$

where $r=(k-1)/2$, we see that $\text{Sq}^1 B^{k-1} \subset B^k$ and hence $\text{Ker Sq}^1 \cap B^k = \text{Sq}^1 B^{k-1}$, as claimed. Similarly, one shows that $\text{Sq}^1 I^{k-1} = \text{Ker Sq}^1 \cap I^k$, thus proving (5.7).

Proof of (5.13). For this it suffices to show:

$$\begin{aligned} \text{Sq}^2 I^{2n-5} + \text{Sq}^1 I^{2n-4} &= I^{2n-3}, \\ \theta_2 \Lambda^{2n-5} + \text{Sq}^1 \Lambda^{2n-4} &= \Lambda^{2n-3}, \end{aligned}$$

recalling that $\theta_2 = \text{Sq}^2$ on I^* . Now the first equation follows by a straightforward calculation (consider the cases, n odd and n even); for the second equation, note that Λ^{2n-3} is generated by $[n-1, n-2]$. But if n is odd, then $\text{Sq}^1[n-2, n-2] = [n-1, n-2]$, while if n is even, one shows that $\theta_2[n-2, n-3] = [n-1, n-2]$. This completes the proof of (5.13).

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