Minisuperspace in Two- and Four-Dimensional Quantum Cosmology

Atushi Ishikawa

Department of Physics, Osaka University,

Machikaneyama, Toyonaka, 560 Japan

DISSERTATION IN PHYSICS

THE OSAKA UNIVERSITY

GRADUATE SCHOOL OF SCIENCE

TOYONAKA, OSAKA

OSAKA univ.

February, 3 1994

Minisuperspace in Two- and Four-Dimensional Quantum Cosmology

Atushi Ishikawa

Department of Physics, Osaka University,

Machikaneyama, Toyonaka, 560 Japan

A thesis submitted in partial fulfillment

of the requirements for the degree of

Doctor of Science

Osaka University

In this thesis, we investigate the validity of the minisuperspace in the context of quantum cosmology in 2 dimensions and 4 dimensions.

The aim of quantum cosmology is to describe the quantum initial states of the universe where the quantum gravitational effects is important. The quantum states of the universe is described by the wave function of the universe which can be defined individually by canonical quantization or by path integral quantization. In the canonical quantization of general relativity, the four-dimensional diffeomorphisms impose two constrains on the wave function; the Wheeler-DeWitt equation and the momentum constraints. The quantum states of the universe can be determined by solving these constraint equations in principle. These equations, however, are second order hyperbolic functional differential equations and are quite difficult to obtain the general solution. Therefore people have solved these equations in the minisuperspace approximation and/or the WKB approximation.

In 4 dimensions, it is assumed that the minisuperspace approximation is meaningful in the region where the WKB approximation is valid. In order to investigate this conjecture, we consider the system of the Einstein gravity coupled to a scalar field, and show numerically that the Friedmann minisuperspace approximation is valid only after the universe grows bigger than the Planck scale. This result indicates that the validity region for the minisuperspace approximation coincides with the one for the WKB approximation.

On the other hand, the 2-dimensional gravity is fascinating. The Einstein action is, however, trivial in 2 dimensions, and therefore we must consider the Liouville action which originates from the conformal anomaly of the path integral measure. In 2 dimensions, it is assumed that the minisuperspace is exact. In order to investigate this proposal, we consider the Liouville field and conformal matter fields. We concentrate on the case of annulus topology, and obtain the transition amplitude consisting only of the zero modes of the fields when we take the $c = 1$ conformal matter field and impose the Neumann boundary condition on the system. This result indicates that, in the our model, the minisuperspace represents the superspace exactly in 2 dimensions.

Contents

I. INTRODUCTION

The aim of this thesis is to investigate how minisuperspace exists in superspace in the context of quantum cosmology.

What is meant by quantum cosmology? It is the theory which completely explains the presently observed state of the universe by the initial conditions in cosmology. The Big Bang model explains some of the features of the observed universe. There are, however, a number of problems which are not explained; the flatness, absence of horizons and the origin of the density fluctuations required to produce galaxies. The inflationary universe scenario, which started from the appearance of Guth's paper [1] and was constructed by quantized matter fields on a classically fixed gravitational background, has provided some possible explanation to the horizon and the flatness problems. Moreover, the desired density fluctuation spectrum is obtained by assuming that the matter fields start out in some particular quantum state. Recently, the anisotropy of the cosmic background radiation has been found by COBE [2]. It strongly supports the inflationary scenario as well as the Big Bang model. In the inflationary scenario, however, there is no way to incorporate the effects coming from initial states. It is impossible to treat them in the framework of the inflationary universe scenario.

Therefore, the most significant aim of quantum cosmology now may be determination of the initial states of the universe. In order to investigate this problem, it is necessary to go back beyond the inflationary period to the stage near the Planck scale where quantum gravitational effects can not be ignored. The initial states of the universe may be determined through the quantization of the gravitational fields as well as the matter fields. In order to describe the quantum states of the universe, we use the wave function or the transition amplitude of the universe;

$$\Psi[h_{ij}, \Phi], \tag{1.1}$$

which can be defined individually by canonical quantization or by path integral quantization. Here $h_{ij}(\mathrm{x})$ is the 3-metric and $\Phi(\mathrm{x})$ is the matter field configuration on a 3-hypersurface. Historically, the early works in quantum cosmology by the canonical quantization were established by DeWitt [3], by Misner [4] and by Wheeler [5] in 1960's. In 1980's, the modern approaches including the path integral quantization were established by Hartle and Hawking [6] [7] [8], by Vilenkin [9] and by Linde [10]. These authors proposed possible boundary conditions or initial conditions on the wave function of the universe.

The explicit evaluation of the wave function, however, has not been established in the full quantum gravitational field theory. The proposals in the above works have been stated in a general form. The concrete calculations are all evaluated in some approximation; the minisuperspace approximation and/or the WKB approximation. This is unavoidable, because we have not yet obtained the complete gravitational theory beyond the Einstein gravity. One can straightforwardly quantize the classical Einstein theory in the similar way to the quantization of the ordinary matter or gauge fields. The present quantum gravity has a number of defects; the most difficult problem may be its nonrenormalizability. We notice that the defects are caused by the incompleteness of the present quantum gravitational theory, and hence we should regard it as an effective theory for the complete quantum gravitational theory.

From this view point, we assume that our present quantum gravitational theory (cosmology) is meaningful only in the region where the WKB approximation works, and that the validity region of the minisuperspace approximation may coincide with that of the WKB approximation. Briefly speaking, the minisuperspace approximation reduces the field theory to the quantum mechanics by assuming some symmetries; homogeneity, isotropy, etc. In this approximation, we can easily treat the quantum gravitational system. In other words, we assume that the calculations obtained in the minisuperspace are meaningful in a region where the WKB approximation is valid.

In this thesis, we investigate explicitly this proposal in 4 and 2 dimensions by following our discussions in Ref. [11] and Ref. [12]. In 4 dimensions, we consider the system of the Einstein gravity and a scalar field, and show numerically that the Friedmann minisuperspace approximation is valid only when the universe is bigger than the Planck scale. This result indicates that the validity region for the minisuperspace approximation coincides with the one for the WKB approximation. On the other hand, the Einstein action is trivial in 2 dimensions, although 2-dimensional gravity is a fascinating object in studying the nature of quantum gravity. Therefore we must consider the Liouville action which originates from the anomaly of the path integral measure. In 2 dimensions, we consider the Liouville field and conformal matter fields. We concentrate on the case of annulus topology, and obtain the transition amplitude consisting of the zero modes of the fields when we take the $c = 1$ conformal matter field and impose the Neumann boundary condition on the system. This result indicates that the minisuperspace represents the superspace exactly in 2 dimensions.

We should remark the difference between the gravitational theory in 4 dimensions and the one in 2 dimensions*. First of all, the classical theory exists in 4 dimensions (see §II.A), while it does not exist in 2 dimensions (see §IV). In other words, the 4 dimensional quantum gravity (cosmology) starts from the classical theory and always has its classical limit. On the other hand, the 2 dimensional quantum gravity starts as the quantum

*Here we consider the Liouville field theory.

theory from beginning and does not have the classical limit†. Therefore, there are two quantization procedures in 4 dimensions; the canonical quantization and the path integral quantization (see §II.B and §II.C). On the contrary, there is no procedure but the path integral quantization in 2 dimensions, because there are no classical theories. Secondly, the dynamical degrees of freedom for the gravitational field are different. The number of independent components is 2 in 4 dimensions, 0 in 3 dimensions (3-dimensional gravity is out of the scope in this thesis) and -1 in 2 dimensions. The number of elements of the symmetric metric is ten in 4 dimensions, and there are four primary constraints and four secondary constraints related with the 4-dimensional diffeomorphisms. Therefore the number of independent dynamical degrees of freedom is 2 in 4 dimensions, and the same calculations hold in 3 dimensions. In 2 dimensions, however, the situation is different and this fact is related to the lack of the canonical structure (see §IV).

This thesis is organized as follows. In §II, we will briefly review the general formulation of 4-dimensional quantum cosmology (see for example [13]). We will see the canonical structure of the classical Einstein gravity coupled with some matter field, then we will quantize the system by the canonical quantization and by the path integral quantization. In addition, we will discuss equivalence between the wave function of the universe by the canonical quantization and the one defined by the path integral quantization. In this section, we will introduce superspace and minisuperspace where the classical and quantum theory are set up.

In §III, we will show our work that the validity region for the minisuperspace approximation coincides with the one for the WKB approximation in 4 dimensions. We will

---

†When we take account of the dilaton field for instance, the classical theory exists. Thus the classical limit exists.

introduce the Friedmann minisuperspace model and the perturbed Friedmann minisuperspace model by following the work of Halliwell and Hawking [14]. Then we will make the numerical analysis of the stability of the Friedmann minisuperspace by following our work [11].

In §IV, we will briefly review the general formulation of 2-dimensional quantum gravity (see for example [15]), and will rewrite it along the ADM decomposition used in the 4-dimensional quantum cosmology [12] [16].

In §V, we will concentrate on the case of annulus topology and will obtain the transition amplitude consisting only of the zero modes of the fields. The result crucially depends on the case that we take the $c = 1$ conformal matter field and impose the Neumann boundary condition on the system [12].

Finally, we will conclude this thesis in §VI.

II. THE GENERAL FORMULATION OF 4-DIMENSIONAL QUANTUM COSMOLOGY

In this section, we will review the general formalism of 4-dimensional quantum cosmology in the Lorentzian signature. There are two methods to quantize general relativity. The one is the canonical quantization and the other is the path integral quantization (see for example [13]).

In the canonical quantization, time plays a special role, while the idea of relativity is to unify time and space. We consider a 3-hypersurface embedded in a 4-manifold on which the 4-metric is $g_{\mu\nu}$. The embedding is described by the ADM (1+3) decomposition of the 4-metric,

$$\begin{aligned} ds^2 &= -N^2 dt^2 + h_{ij}\left(N^i dt + dx^i\right)\left(N^j dt + dx^j\right) \\ &= -(N^2 - N_i N^i)dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j. \end{aligned} \tag{2.1}$$

Here N and N^i are the lapse and shift functions, respectively, which become Lagrange multipliers related to a 4-diffeomorphism in the Einstein theory, and h_{ij} is the 3-metric on the 3-hypersurface. In 4 dimensions, we use the notation that the 4-index μ runs from 0 to 3, and the spatial index i runs from 1 to 3.

On the other hand, the path integral quantization is a very powerful procedure when we impose a boundary condition on the 4-manifold [6]. However we must also consider the 3-hypersurface in order to obtain the quantum states of the universe. Therefore the ADM (1+3) decomposition is important even in the path integral quantization.

By following the ADM (1+3) decomposition, we will see the canonical structure of general relativity with some matter field in §II.A. In §II.B and §II.C, we will review the canonical quantization and the path integral quantization respectively. After seeing these two quantizations, we will discuss the equivalence between them formally in §II.D and explicitly in §II.E. In §II.E, we will introduce minisuperspace.

A. Classical theory -canonical structure-

We consider a system of the Einstein gravity and a scalar field. The action is the standard Einstein-Hilbert action coupled to a matter field,

$$S = S_{gravity} + S_{matter}, \tag{2.2}$$

where

$$S_{gravity} = \frac{m_p^2}{16\pi}\left\{\int_M d^4x\sqrt{-g}\,(R-2\Lambda) + 2\int_{\partial M} d^3x\sqrt{h}K\right\} \tag{2.3}$$

and

$$S_{matter} = -\frac{1}{2}\int d^4x\left\{g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2\right\}. \tag{2.4}$$

Here m_p is the Planck mass; $m_p^2 = 1/G$, and K is the trace part of the extrinsic curvature K_{ij} given by

$$K_{ij} = \frac{1}{2N}\left\{-\frac{\partial h_{ij}}{\partial t} + 2D_{(i}N_{j)}\right\}. \tag{2.5}$$

The extrinsic curvature describes a way of embedding the 3-hypersurface into the 4-manifold, and D_i is the covariant derivative in the 3-hypersurface. By using the (1+3) variables, the action takes the form

$$\begin{aligned} S_{gravity} &= \frac{m_p^2}{16\pi}\int_M dtd^3x\sqrt{h}N\left\{K_{ij}K^{ij} - K^2 + {}^{(3)}R - 2\Lambda\right\} \\ &\equiv \int L_{gravity}, \\ S_{matter} &\equiv \int L_{matter}. \end{aligned} \tag{2.6}$$

The boundary term in the gravitational action (2.3) is needed for obtaining the gravitational form above. The canonical conjugate momenta to h_{ij} and Φ are respectively given by

$$\pi^{ij} = \frac{\partial L_{gravity}}{\partial \dot{h}_{ij}} = -\frac{\sqrt{h}m_p^2}{16\pi}\left(K^{ij} - h^{ij}K\right) \tag{2.7}$$

and

$$\pi_\Phi = \frac{\partial L_{matter}}{\partial \dot{\Phi}} = N^{-1}\sqrt{h}\left(\dot{\Phi} - N^i \frac{\partial \Phi}{\partial x^i}\right). \tag{2.8}$$

We can derive the Hamiltonian form of the action

$$S = \int dt d^3x \left\{\dot{h}_{ij}\pi^{ij} + \dot{\Phi}\pi_\Phi - NH_0 - N^i H_i\right\}, \tag{2.9}$$

where

$$\begin{aligned} H_0 &= 16\pi m_p^{-2} G_{ijkl}\pi^{ij}\pi^{kl} - \frac{m_p^2\sqrt{h}}{16\pi}\left({}^{(3)}R - 2\Lambda\right) \\ &+ \frac{\sqrt{h}}{2}\left(\frac{\pi_\Phi^2}{h} + h^{ij}\partial_i\Phi\partial_j\Phi + m^2\Phi^2\right), \\ H^i &= -2D_j\pi^{ij} + h^{ij}\pi_\Phi\partial_j\Phi. \end{aligned} \tag{2.10}$$

Here G_{ijkl} is the DeWitt metric given by

$$G_{ijkl} = \frac{1}{2}h^{-1/2}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right). \tag{2.11}$$

The lapse N and shift N^i play the role of Lagrange multipliers as stated before. Thus the solution obeys the Hamiltonian constraint

$$H_0 = 0 \tag{2.12}$$

and the momentum constraints

$$H^i = 0. \tag{2.13}$$

These constraints are equivalent, respectively, to the time-time and time-space components of the classical Einstein equations. The constraints play a central role in the canonical quantization, as we shall see in the next subsection. The arena where the classical dynamics holds is called superspace; the space of all 3-metrics and matter field configurations $(h_{ij}(\mathrm{x}),\ \Psi(\mathrm{x}))$ on a 3-hypersurface. The superspace has a finite number of coordinates $(h_{ij}(\mathrm{x}),\ \Psi(\mathrm{x}))$ at every point x on the 3-hypersurface and is infinite dimensional.

B. Canonical quantization

We follow Dirac's prescription of quantization for the constrained system. The momenta (2.7) and (2.8) are replaced by the differential operators

$$\pi^{ij}(x) \to -i\frac{\delta}{\delta h_{ij}(x)} \tag{2.14}$$

and

$$\pi_\Phi(x) \to -i\frac{\delta}{\delta\Phi(x)}. \tag{2.15}$$

The Wheeler-DeWitt equation and the momentum constraints are the quantum versions of the Hamiltonian constraint (2.12) and the momentum constraints (2.13) as follows

$$\hat{H}_0\Psi[h_{ij},\Phi] = 0 \tag{2.16}$$

and

$$\hat{H}^i\Psi[h_{ij},\Phi] = 0. \tag{2.17}$$

Here the momenta in H_0 and H^i are replaced by the differential operators;

$$\begin{aligned}\hat{H}_0 &= -16\pi m_p^{-2}G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} - \frac{m_p^2\sqrt{h}}{16\pi}\left({}^{(3)}R - 2\Lambda\right)\\ &\quad+\frac{\sqrt{h}}{2}\left(-\frac{1}{h}\frac{\delta^2}{\delta\Phi^2} + h^{ij}\partial_i\Phi\partial_j\Phi + m^2\Phi^2\right)\\ \hat{H}^i &= 2iD_j\frac{\delta}{\delta h_{ij}} - ih^{ij}\partial_j\Phi\frac{\delta}{\delta\Phi}.\end{aligned} \tag{2.18}$$

The Wheeler-DeWitt equation is related to a reparametrization invariance of the theory and the momentum constraints represent that the wave function of the universe is invariant under 3-dimensional diffeomorphisms. In order to show this, let us restrict our attention to the case of no matter field and consider only the closed universe. We consider the effect of shifting the argument of the wave function by an infinitesimal diffeomorphism in the 3-hypersurface; $x^i \to x^i - \xi^i$. From this transformation, we obtain

$$\Psi[h_{ij} + D_{(i}\xi_{j)}] = \Psi[h_{ij}] + \int d^3 D_{(i}\xi_{j)} \frac{\delta \Psi}{\delta h_{ij}} \tag{2.19}$$

and therefore the change in Ψ is given by

$$\begin{aligned} \delta\Psi[h_{ij}] &= -\int d^3x \xi_j D_i \left(\frac{\delta \Psi}{\delta h_{ij}}\right) \\ &= -\frac{1}{2i}\int d^3x \xi_i \hat{H}^i \Psi = 0. \end{aligned} \tag{2.20}$$

Here we use the momentum constraints (2.17). Consequently the arguments of the wave function should be the invariant quantity under the 3-diffeomorphisms.

The Wheeler-DeWitt equation is a second order hyperbolic functional differential equation which describes the dynamical evolution of the wave function in the superspace. In general, there are many solutions for the Wheeler-DeWitt equation, and we need boundary conditions to pick up a solution. It is more convenient for imposing boundary conditions to take the path integral quantization [6], as will be discussed in the next subsection.

C. Path integral quantization

In the path integral quantization, the wave function is described as a transition amplitude

$$\Psi[h''_{ij}, \Phi''; h'_{ij}, \Phi'] = \int \frac{[\mathcal{D}g_{\mu\nu}]}{vol(\text{Gauge})}[\mathcal{D}\Phi] e^{iS[g_{\mu\nu},\Phi]}, \tag{2.21}$$

where (h''_{ij}, Φ'') and (h'_{ij}, Φ') represent induced values at the boundaries 3-hypersurface. Alternatively, in many cases, we take Euclidean path integral for convenience (see for example [6] [17] [18] [19]), and its formulation is briefly given in Appendix A.

When the 4-manifold has topology $\mathbf{R} \times \Sigma$, the path integral is defined as

$$\Psi[h''_{ij}, \Phi''; h'_{ij}, \Phi'] = \int [\mathcal{D}N^{\mu}][\mathcal{D}h_{ij}][\mathcal{D}\Phi]\delta[\dot{N}^{\mu} - \chi^{\mu}] \triangle_{\chi}\, e^{iS[g_{\mu\nu},\Phi]} \tag{2.22}$$

with the gauge fixing condition $\dot{N}^{\mu} - \chi^{\mu} = 0$ and the associated Faddeev-Popov determinant. Note that the path integral measure $[\mathcal{D}g_{\mu\nu}]$ is firstly defined as $[\mathcal{D}N^{\mu}][\mathcal{D}h_{ij}]$ in the form of equation (2.22). The Jacobian is not considered. This path integral formulation is very powerful to impose the no-boundary boundary condition to a 4-manifold in the Euclidean framework [6]. Although the boundary condition can be formally stated in general cases, we must also follow the ADM $(1+3)$ decomposition in order to obtain the transition amplitude of the universe.

D. Formal equivalence in superspace

In order to show formally the equivalence between the wave function by the canonical quantization and that by the path integral quantization, the boundary conditions for the path integral given in §II.C plays a central role. We set that the lapse and shift (N, N^i) are unrestricted at the boundaries, and the 3-metric and mater field ($h_{ij}(\mathrm{x})$, $\Psi(\mathrm{x})$) match the arguments of the transition amplitude. In other words, the transition amplitude is a functional of the induced values of the 3-metric and matter field at the boundaries 3-hypersurface (h''_{ij}, Φ''), (h'_{ij}, Φ') only and the arguments do not contain the lapse or the shift functions (N^{μ}). Therefore we can show the equivalence as follows

$$\begin{aligned}
0 = \frac{\delta}{\delta N^{\mu}}\Psi[h''_{ij}, \Phi''; h'_{ij}, \Phi'] &= \frac{\delta}{\delta N^{\mu}} \int \frac{[\mathcal{D}g_{\mu\nu}]}{vol(\text{Gauge})}[\mathcal{D}\Phi]e^{iS[g_{\mu\nu},\Phi]} \\
&= \int \frac{[\mathcal{D}g_{\mu\nu}]}{vol(\text{Gauge})}[\mathcal{D}\Phi]i\frac{\delta}{\delta N^{\mu}}S[g_{\mu\nu}, \Phi]e^{iS[g_{\mu\nu},\Phi]} \\
&= i\hat{H}_{\mu} \int \frac{[\mathcal{D}g_{\mu\nu}]}{vol(\text{Gauge})}[\mathcal{D}\Phi]e^{iS[g_{\mu\nu},\Phi]} \\
&= i\hat{H}_{\mu}\Psi[h''_{ij}, \Phi''; h'_{ij}, \Phi'].
\end{aligned} \tag{2.23}$$

Consequently we obtain the Wheeler-DeWitt equation and the momentum constraints;

$$\hat{H}_\mu \Psi[h''_{ij}, \Phi''; h'_{ij}, \Phi'] = 0. \tag{2.24}$$

E. Explicit equivalence in minisuperspace

In the previous subsection, we have shown equivalence between the wave function of the universe by the canonical quantization and the one defined by the path integral quantization formally. The key point is that the arguments of the transition amplitude are only the induced values at the boundaries 3-hypersurface of the 4-manifold. We will show the equivalence more explicitly in the minisuperspace model.

The superspace is infinite dimensional. It is, therefore, difficult to deal with a wave function and a transition amplitude in practice. In other words, we cannot solve the Wheeler-DeWitt equation and the momentum constraints easily without any approximation, because they are second order functional differential equations. In order to make this technical problem tractable, we often reduce the degrees of freedom of the superspace to a finite number by assuming some symmetries. This reduced finite dimensional superspace is called the minisuperspace. The validity of the minisuperspace will be discussed in §III.

In minisuperspace, we restrict the metric and matter field to be homogeneous as follows. The lapse function is taken to be homogeneous; $N = N(t)$, and the shift functions are set to be zero; $N^i = 0$. The 3-metric h_{ij} are restricted to be homogeneous, and therefore they are described by a finite number of functions of t; $q^\alpha(t)$ $(\alpha = 1,\ 2, \cdots, n)$. For example, the Robertson-Walker metric is given by

$$h_{ij} dx^i dx^j = a^2(t) d\Omega^2_{(3)}, \tag{2.25}$$

where $d\Omega^2_{(3)}$ is the metric on the 3-sphere and $a(t)$ is a scale factor of the universe. In this case, we take $q^\alpha(t) = a(t)$. We can also take an anisotropic metric for example

$$h_{ij} dx^i dx^j = a^2(t) dx^2 + b^2(t) dy^2 + c^2(t) dz^2, \tag{2.26}$$

which is known as the Bianchi Type I model. In this case, we take $q^\alpha(t) = (a(t),\ b(t),\ c(t))$.

In general, we obtain the Hamiltonian form of the action

$$S = \int_{t'}^{t''} dt \left\{ p_\alpha \dot{q}^\alpha - NH \right\} \tag{2.27}$$

and the lapse function N is a Lagrange multiplier enforcing the Hamiltonian constraint

$$H(p_\alpha, q^\alpha) = \frac{1}{2} f^{\alpha\beta} p_\alpha p_\beta + U(q). \tag{2.28}$$

Here $f_{\alpha\beta}$ is the reduced version of the DeWitt metric (2.11), and we also include the matter field into q^α in the same way as the 3-metrics. The same setting for the path integral is imposed such that p_α and N are free at the end points and that q^α are fixed and satisfy the boundary conditions;

$$q^\alpha(t') = q^{\alpha\prime}, \quad q^\alpha(t'') = q^{\alpha\prime\prime}. \tag{2.29}$$

The Hamiltonian constraint indicates the presence of a reparametrization invariance under the transformations

$$\begin{aligned} &\delta_\epsilon q^\alpha = \epsilon(t) \left\{ q^\alpha, H \right\}, \quad \delta_\epsilon p_\alpha = \left\{ p_\alpha, H \right\}, \\ &\delta_\epsilon N = \dot{\epsilon}(t), \end{aligned} \tag{2.30}$$

where $\epsilon(t)$ is an arbitrary parameter. Under these transformations, the action changes by an amount

$$\delta S = \left[\epsilon(t) \left\{ p_\alpha \frac{\partial H}{\partial p_\alpha} - H \right\} \right]_{t'}^{t''}. \tag{2.31}$$

Therefore the action is unchanged if and only if $\epsilon(t)$ satisfies the boundary conditions

$$\epsilon(t') = 0 = \epsilon(t''). \tag{2.32}$$

We can write down the path integral in minisuperspace

$$\Psi(q^{\alpha\prime\prime};q^{\alpha\prime}) = \int [\mathcal{D}N][\mathcal{D}p_\alpha][\mathcal{D}q^\alpha]\delta[\dot{N}-\chi]\,\triangle_\chi\, e^{iS[p,q,N]}, \tag{2.33}$$

where we take the $\dot{N}=\chi$ gauge and χ is an arbitrary function of $(p_\alpha,\, q^\alpha,\, N)$. By following the discussion in Ref. [20], we will obtain the gauge fixing action and the ghost action.

The gauge condition $\dot{N}=\chi$ is imposed by using a Lagrange multiplier $\Pi(t)$ as

$$S_{gf} = \int_{t'}^{t''} dt\Pi(t)\left\{\dot{N}(t)-\chi\right\}. \tag{2.34}$$

We can write down the ghost action according to the Batalin, Fradkin and Vilkovisky (BFV) formalism [21]. This formalism is a more systematic fashion. The BRS symmetry involves the replacement of the parameter $\epsilon(t)$ with $\Lambda c(t)$ where Λ is a constant anti-commuting parameter and $c(t)$ is an anti-commuting ghost field. Eventually we wish to generate the BRS transformations using a Poisson bracket, and therefore we must eliminate the time derivatives from the transformations. We write $\dot{c}=\rho$, and this equation is imposed on the action by adding a term $\bar{\rho}(\dot{c}-\rho)$, where ρ and $\bar{\rho}$ are anti-commuting ghost field momenta. In order to make ρ dynamical, we also add a term $\bar{c}\dot{\rho}$. Therefore the ghost action is given by

$$S_{gh} = \int_{t'}^{t''} dt\left\{\bar{\rho}\dot{c}+\bar{c}\dot{\rho}-\bar{\rho}\rho\right\}. \tag{2.35}$$

We obtain the BRS transformations from the transformations (2.30) and from the gauge fixing action (2.34) and the ghost action (2.35);

$$\begin{aligned}
&\delta p_\alpha = -\Lambda c\frac{\partial H}{\partial q^\alpha},\quad \delta q^\alpha = \Lambda c\frac{\partial H}{\partial p_\alpha},\quad \delta N = \Lambda\rho,\\
&\delta\Pi = 0,\quad \delta c = 0,\quad \delta\rho = 0,\\
&\delta\bar{c} = -\Lambda\Pi,\quad \delta\bar{\rho} = -\Lambda H,
\end{aligned} \tag{2.36}$$

with the boundary conditions

$$
\begin{aligned}
&\Pi(t') = 0 = \Pi(t''), \\
&c(t') = 0 = c(t''), \\
&\bar{c}(t') = 0 = \bar{c}(t'').
\end{aligned}
\tag{2.37}
$$

Here we have taken the $\chi = 0$ gauge, because it is guaranteed by the Fradkin-Vilkovisky theorem [21] that the resulting path integral is independent of a choice of a gauge fixing function χ (see for the detail discussion in Ref. [20]). Consequently the path integral is described as

$$
\Psi(q^{\alpha''}; q^{\alpha'}) = \int [\mathcal{D}\Pi][\mathcal{D}N][\mathcal{D}\rho][\mathcal{D}\bar{c}][\mathcal{D}\bar{\rho}][\mathcal{D}c][\mathcal{D}p_\alpha][\mathcal{D}q^\alpha] e^{i\{S+S_{gf}+S_{gh}\}}. \tag{2.38}
$$

By taking $\chi = 0$, the ghost field decouples from the other field and the ghost field integration can be performed. We define this integration by splitting the time interval into $n+1$. Therefore the ghost field integration is defined as

$$
\begin{aligned}
\int [\mathcal{D}\rho][\mathcal{D}\bar{c}][\mathcal{D}\bar{\rho}][\mathcal{D}c] e^{iS_{gh}} &= \int \prod_{j=0}^{n} d\bar{\rho}_j \prod_{j=0}^{n} d\rho_j \prod_{j=1}^{n} dc_j \prod_{j=1}^{n} d\bar{c}_j \\
&\quad \times \exp\left\{ i\epsilon \sum_{j=0}^{n} \left[\bar{\rho}_j \frac{c_{j+1} - c_j}{\epsilon} + \bar{c}_j \frac{\rho_{j+1} - \rho_j}{\epsilon} - \bar{\rho}_j \rho_j \right] \right\} \\
&= (i\epsilon)^{n+1} \int \prod_{j=1}^{n} dc_j \prod_{j=1}^{n} d\bar{c}_j \exp\left\{ -\frac{i}{\epsilon} \sum_{j=0}^{n} (\bar{c}_{j+1} - \bar{c}_j)(c_{j+1} - c_j) \right\} \\
&= t'' - t',
\end{aligned}
\tag{2.39}
$$

where $c_0 = c(t') = 0$, $c_{n+1} = c(t'') = 0$ and $\bar{c}_0 = \bar{c}(t') = 0$, $\bar{c}_{n+1} = \bar{c}(t'') = 0$. Then we can also carry out the integrations over $\Pi(t)$ and $N(t)$ such that

$$
\begin{aligned}
\int [\mathcal{D}\Pi][\mathcal{D}N] e^{iS_{gf}} &= \prod_{j=0}^{n} dN_j \prod_{j=1}^{n} d\Pi_j \exp\left\{ i \sum_{j=0}^{n} \Pi_j (N_{j+1} - N_j) \right\} \\
&= \int \prod_{j=0}^{n} dN_j \prod_{j=1}^{n} \delta(N_{j+1} - N_j) \\
&= dN_0.
\end{aligned}
\tag{2.40}
$$

Note that $\Pi(t)$ plays a role as a coordinate and that $N(t)$ plays a role as a momentum from the boundary conditions (2.37). Finally, we obtain the path integral form in the minisuperspace

$$\Psi(q^{\alpha\prime\prime};q^{\alpha\prime}) = \int dN\,(t''-t')\int[\mathcal{D}p_\alpha][\mathcal{D}q^\alpha]e^{iS}, \tag{2.41}$$

where we drop out the suffix 0 in N_0. From this expression, we can show explicitly the equivalence between the wave function by the canonical quantization and the transition amplitude defined by path integral.

In the minisuperspace, we can obtain the explicit path integral form (2.41) where the Faddeev-Popov determinant; $\triangle_{\chi=0} = t''-t'$, and the functional integral over the lapse function N reduces to a single ordinary integration over the constant N. The transition amplitude (2.41) can be viewed as the integration over all times $T \equiv N\cdot(t''-t')$;

$$\Psi(q^{\alpha\prime\prime};q^{\alpha\prime}) = \int dT\psi(q^{\alpha\prime\prime},T;q^{\alpha\prime},0), \tag{2.42}$$

of an ordinary quantum mechanical propagator;

$$\psi(q^{\alpha\prime\prime},T;q^{\alpha\prime},0) = \int[\mathcal{D}p_\alpha][\mathcal{D}q^\alpha]e^{iS}. \tag{2.43}$$

Here $\psi(q^{\alpha\prime\prime},T;q^{\alpha\prime},0)$ satisfies the Schrödinger equations at the initial and final state with time coordinate T;

$$\begin{aligned}\hat{H}'\psi(q^{\alpha\prime\prime},T;q^{\alpha\prime},0) &= i\frac{\partial}{\partial T}\psi(q^{\alpha\prime\prime},T;q^{\alpha\prime},0),\\ \hat{H}''\psi(q^{\alpha\prime\prime},T;q^{\alpha\prime},0) &= i\frac{\partial}{\partial T}\psi(q^{\alpha\prime\prime},T;q^{\alpha\prime},0).\end{aligned} \tag{2.44}$$

The Hamiltonian operators are obtained by replacing the momenta in the Hamiltonian (2.28) by the operators at the end points;

$$\begin{aligned}\hat{H}' &= -\frac{1}{2}f^{\alpha\beta}\frac{\partial}{\partial q^{\alpha\prime}}\frac{\partial}{\partial q^{\beta\prime}} + U(q'),\\ \hat{H}'' &= -\frac{1}{2}f^{\alpha\beta}\frac{\partial}{\partial q^{\alpha\prime\prime}}\frac{\partial}{\partial q^{\beta\prime\prime}} + U(q'').\end{aligned} \tag{2.45}$$

By using these Schrödinger equations, we can easily show that the transition amplitude (2.41) satisfies the Wheeler-DeWitt equation as follows

$$\begin{aligned}\hat{H}''\Psi(q^{\alpha\prime\prime};q^{\alpha\prime}) &= \int dT i\frac{\partial}{\partial T}\psi(q^{\alpha\prime\prime},T;q^{\alpha\prime},0)\\ &= i\left[\psi(q^{\alpha\prime\prime},T;q^{\alpha\prime},0)\right]_{T_1}^{T_2}\\ &= 0.\end{aligned}\tag{2.46}$$

Likewise,

$$\hat{H}'\Psi(q^{\alpha\prime\prime};q^{\alpha\prime}) = 0.\tag{2.47}$$

Here T_1 and T_2 are the end points of the T integration, and we choose them so that the right-hand side of equation (2.46), (2.47) vanishes. In general, the T is integrated along a contour in the complex plain in the Euclidean path integral framework [6] [17] [18] [19]. This contour is usually taken to be infinite, with $\psi(q^{\alpha\prime\prime},T;q^{\alpha\prime},0)$ going to zero at the ends $(T_1,\ T_2)$, or closed; $T_1 = T_2$.

In the minisuperspace, we can treat the wave function or the transition amplitude of the universe explicitly, while the validity of the minisuperspace approximation is unclear. We believe that the classical solutions in the minisuperspace may be solutions to the full field equations in the superspace, and therefore the classical action in the minisuperspace will be the action of a solution to the full Einstein equations. In other words, the lowest order semi-classical approximation to the minisuperspace wave function coincides with the one of the full theory. Therefore the validity region of the minisuperspace approximation should coincide with one of the WKB approximation. We will investigate this conjecture in the next section.

III. THE VALIDITY REGION OF THE MINISUPERSPACE IN 4 DIMENSIONS

When we discuss the classical dynamics of the universe, it is very useful and physically reasonable to assume that the universe has certain symmetries. A well-known example is the Friedmann (Robertson-Walker) model which has the spatial homogeneity and isotropy.

The quantization of the universe was first introduced by DeWitt [3] using the Friedmann minisuperspace approximation. The homogeneous anisotropic cases were then studied by Misner [4] and others [22]. Moreover the minisuperspace model has received the renewed attention since the appearance of the paper by Hartle and Hawking [6] on the proposal for no-boundary boundary condition of the universe. Although the boundary condition can be formally stated in general cases, analytic calculations are performed only in the minisuperspace model (see for example [17] [18] [19]) and all inhomogeneous modes are omitted before quantization. Some attempts to introduce the inhomogeneous modes along the path integral method using the WKB approximation are given in Ref. [14] and Ref. [23].

The above naive assumption that some degrees of freedom are fixed to zero, say, before quantization apparently violates the uncertainty principle and therefore a question arises; whether or not the minisuperspace approximation is a physically reasonable approximation to the true quantum gravity. Although this problem has been well known to many people working in the field of quantum cosmology for a long time, only few people [24] [25] have checked the validity of the minisuperspace model. In Ref. [25], Kuchar and Ryan discussed the quantum cosmology of a Taub universe embedded in a mixmaster universe to assess the validity of the minisuperspace model. They analyzed the lower-dimensional homogeneous superspace model (Taub universe) and the higher(but finite)-dimensional homogeneous superspace model (mixmaster universe) in a region where exact solutions for both models exist. In their work, the inhomogeneous modes are not considered. While in Ref. [24] Sinha and Hu analyzed an effect of discarding an infinite number of inhomogeneous modes using the WKB approximation. They found that the WKB approximation has the limited validity and that it is important to go beyond the WKB when classical background is not available and full quantum behavior of the minisuperspace sector is important.

In this section, we will investigate the validity region of the minisuperspace beyond the WKB approximation by examining numerically the stability of the Friedmann minisuperspace model in the superspace [11]. We consider a system of the Einstein gravity with a scalar field as a model of the quantum cosmology introduced in the previous section. In order to check the stability of the minisuperspace model, we have only to consider a configuration space of the metric and the scalar field near the minisuperspace. In other words, we can regard all inhomogeneous modes as very small. In this approximation, the inhomogeneous modes are decoupled to one another. When we solve the Wheeler-DeWitt equation, we pick up some inhomogeneous modes out of the infinite number of the other modes; the degrees of freedom of the superspace of this system are restricted to a finite number. Our method is therefore similar to that of Kuchar and Ryan. We consider some higher-dimensional (but finite) superspace which includes the lower-dimensional Friedmann minisuperspace. It should be noted that, as we compute the wave function of the universe numerically, our analysis is not necessarily limited within the validity region in which the WKB approximation is applicable, however our result shows that the validity region of the minisuperspace coincides with the region of the WKB approximation.

In §III.A, we will review the analysis of a homogeneous and isotropic minisuperspace model, which is known as the Friedmann (Robertson-Walker) model, with a scalar field. The Friedmann model will be extended to all order the matter and gravitational degrees of freedom in §III.B, following the Halliwell and Hawking work [14]. In §III.C, we will pick up some particular inhomogeneous modes out of all possible infinite perturbative modes to the minisuperspace, and will solve the Wheeler-DeWitt equation on the newly defined superspace numerically.

A. The Friedmann minisuperspace model

Here we will briefly review the Friedmann (Robertson-Walker) minisuperspace model. It is assumed that a metric of this model has the spatial homogeneity and isotropy, which has the advantage of reducing the infinite dimensional superspace to a finite dimensional minisuperspace as stated in II.E. The metric of this model is given by

$$ds^2 = \sigma^2 \left(-N^2 dt^2 + a^2 d\Omega_{(3)}^2\right), \tag{3.1}$$

where $d\Omega_{(3)}^2$ is the metric of the unit 3-sphere and σ is a normalization factor defined by $\sigma^2 = 2/3\pi m_p^2$. The action of this model is described by

$$S_0 = -\frac{1}{2}\int dt N a^3 \left(\frac{\dot{a}^2}{N^2 a^2} - \frac{1}{a^2} - \frac{\dot{\phi}^2}{N^2} + m^2\phi^2\right). \tag{3.2}$$

The suffix 0 represents that this action is the minisuperspace action. It should be noted that the action consists of rescaled scalar field $\phi/\sqrt{2}\pi\sigma$ and mass m/σ in the action (2.4), and that the scalar field is a function of t only and independent of the spatial coordinate. The classical Hamiltonian is

$$H_{|0} = \frac{1}{2}N\left(-a^{-1}\pi_a^2 + a^{-3}\pi_\phi^2 - a + a^3 m^2\phi^2\right), \tag{3.3}$$

where the conjugate momenta to a and ϕ are respectively

$$\pi_a = -\frac{a\dot{a}}{N} \tag{3.4}$$

and

$$\pi_\phi = \frac{a^3\dot{\phi}}{N}. \tag{3.5}$$

Because the action has a time reparametrization invariance, the classical field equations are constrained by the Hamiltonian constraint equation $H_{|0} = 0$. By replacing the momenta (3.4) (3.5) by the operators

$$\pi_a \to -i\frac{\partial}{\partial a}, \quad \pi_\phi \to -i\frac{\partial}{\partial \phi}, \tag{3.6}$$

we obtain the Wheeler-DeWitt equation

$$\frac{N}{2}e^{-3\alpha}\left(\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} + e^{6\alpha}m^2\phi^2 - e^{-4\alpha}\right)\Psi[\alpha,\phi] = 0, \tag{3.7}$$

where we have introduced $\alpha = \ln(a)$. One can regard equation (3.7) as the hyperbolic differential equation on the two dimensional minisuperspace with coordinates (α, ϕ). We can therefore regard α as a time coordinate.

B. The perturbed Friedmann minisuperspace model

We first describe the metric near the Friedmann metric as

$$ds^2 = \sigma^2\left\{-\left(N^2 - N_iN^i\right)dt^2 + 2N_idx^idt + h_{ij}dx^idx^j\right\}. \tag{3.8}$$

The metric of the spatial hypersurface h_{ij} has the form

$$h_{ij} = a^2\left(\Omega_{ij} + \epsilon_{ij}\right), \tag{3.9}$$

where ϵ_{ij} is a small quantity around the metric of the unit 3-sphere Ω_{ij}. Following Halliwell and Hawking [14], the quantities ϵ_{ij} is expanded in harmonics;

$$\begin{aligned}\epsilon_{ij} = \sum_{n,l,m}\Big[&\sqrt{6}a_{nlm}\Omega_{ij}Q^n_{lm}/3 + \sqrt{6}b_{nlm}(P_{ij})^n_{lm} + \sqrt{2}c^o_{lmn}(S^o_{ij})^n_{lm} \\ &+\sqrt{2}c^e_{lmn}(S^e_{ij})^n_{lm} + \sqrt{2}d^o_{lmn}(G^o_{ij})^n_{lm} + \sqrt{2}d^e_{lmn}(G^e_{ij})^n_{lm}\Big].\end{aligned} \tag{3.10}$$

The lapse, shift and the scalar field can also be expanded by harmonics;

$$N = N_0 \left[1 + 6^{-1/2} \sum_{n,l,m} g_{nlm} Q^n_{lm} \right],$$
$$N_i = a \sum_{n,l,m} \left[6^{-1/2} k_{nlm} (P_i)^n_{lm} + 2^{1/2} j_{nlm} (S_i)^n_{lm} \right],$$
$$\Phi = \sigma^{-1} \left[2^{-1/2} \pi \phi(t) + \sum_{n,l,m} f_{nlm} Q^n_{lm} \right]. \tag{3.11}$$

The definition and the normalization of these harmonics are given in the Appendix B. The coefficients a_{nlm}, b_{nlm}, c^o_{nlm}, c^e_{nlm}, d^o_{nlm}, d^e_{nlm}, g_{nlm}, j_{nlm}, k_{nlm} and f_{nlm} are only functions of time. We expand the action to all orders in the zero mode of the fields (a, ϕ and N_0) but only to the second order in the non-zero modes (a_{nlm}, b_{nlm}, $\cdots$, f_{nlm});

$$S = S_0(a, \phi, N_0) + \sum_n S_n, \tag{3.12}$$

where S_0 is the action of the Friedmann minisuperspace model given by equation (3.2) and S_n are terms quadratic in perturbations. Hereafter, the labels n, l, m, o and e will be denoted simply by n. The exact forms of S_n are given by

$$S_n = \int dt \left(S^n_{gravity} + S^n_{matter} \right), \tag{3.13}$$

where

$$\begin{aligned}
S^n_{gravity} = \frac{e^\alpha}{2} N_0 \Bigg[& (n^2 - \frac{5}{2}) \frac{a_n^2}{3} + \frac{(n^2-7)(n^2-4)}{3(n^2-1)} b_n^2 - 2(n^2-4) c_n^2 \\
& -(n^2+1) d_n^2 + 2(n^2-4) \frac{a_n b_n}{3} + \frac{2}{3} g_n \left\{ (n^2-4) b_n + (n^2 + \frac{1}{2}) a_n \right\} \\
& + \frac{1}{N_0^2} \left\{ -\frac{k_n^2}{3(n^2-1)} + (n^2-4) j_n^2 \right\} \Bigg] \\
& + \frac{e^{3\alpha}}{2N_0} \Bigg[-\dot{a}_n^2 + \frac{(n^2-4)}{(n^2-1)} \dot{b}_n^2 + (n^2-4) \dot{c}_n^2 + \dot{d}_n^2 \\
& + g_n \left\{ 2\dot{\alpha}\dot{a}_n + \dot{\alpha}^2 (3a_n - g_n) \right\} \\
& + \dot{\alpha} \left\{ -2a_n \dot{a}_n + 8 \frac{(n^2-4)}{(n^2-1)} b_n \dot{b}_n + 8(n^2-4) c_n \dot{c}_n + 8 d_n \dot{d}_n \right\}
\end{aligned}$$

$$
\begin{aligned}
&+\dot{\alpha}^2\left\{-\frac{3}{2}a_n^2+6\frac{(n^2-4)}{(n^2-1)}b_n^2+6(n^2-4)c_n^2+6d_n^2\right\}\\
&+e^{-\alpha}\left\{k_n\left(-\frac{2}{3}\dot{a}_n+\frac{2(n^2-4)}{3(n^2-1)}\dot{b}_n+\frac{2}{3}\dot{\alpha}g_n\right)-2(n^2-4)j_n\dot{c}_n\right\}\Bigg]
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
S^n_{matter}=\frac{N_0e^{3\alpha}}{2}&\left[\frac{\left(\dot{f}_n^2+6a_n\dot{f}_n\dot{\phi}\right)}{N_0^2}-m^2\left(f_n^2+6a_nf_n\phi\right)-e^{-2\alpha}(n^2-1)f_n^2\right.\\
&+\frac{3}{2}\left(\frac{\dot{\phi}^2}{N_0^2}-m^2\phi^2\right)\left\{a_n^2-\frac{4(n^2-4)}{(n^2-1)}b_n^2-4(n^2-4)c_n^2-4d_n^2\right\}\\
&\left.-g_n\left(2m^2f_n\phi+3m^2a_n\phi^2+\frac{2\dot{f}_n\dot{\phi}}{N_0^2}+\frac{3a_n\dot{\phi}^2}{N_0^2}\right)-\frac{2k_nf_n\dot{\phi}}{e^\alpha N_0^2}\right].
\end{aligned}
\tag{3.15}
$$

One should note that we consider the configuration space near the Friedmann minisuperspace and that the zero-mode of the fields (a, ϕ and N_0) are not classical solutions. The Hamiltonian can be expressed as

$$
\begin{aligned}
H=&N_0\left(H_{|0}+\sum_n H^n_{|2}+\sum_n g_nH^n_{|1}\right)\\
&+\sum_n\left(k_nH^{(S)n}_{-1}+j_nH^{(V)n}_{-1}\right),
\end{aligned}
\tag{3.16}
$$

where $H_{|0}$ is the Hamiltonian of the Friedmann minisuperspace model given by equation (3.3). The second-order Hamiltonian $H_{|2}$ is given by

$$
H_{|2}=\sum_n H^n_{|2}=\sum_n\left(H^{(S)n}_{|2}+H^{(V)n}_{|2}+H^{(T)n}_{|2}\right),
\tag{3.17}
$$

where

$$
\begin{aligned}
H^{(S)n}_{|2}=\frac{1}{2e^{3\alpha}}&\left[\left\{\frac{a_n^2}{2}+\frac{10(n^2-4)}{(n^2-1)}b_n^2\right\}\pi_\alpha^2+\left\{\frac{15a_n^2}{2}+\frac{6(n^2-4)}{(n^2-1)}b_n^2\right\}\pi_\phi^2\right.\\
&-\pi_{a_n}^2+\frac{(n^2-1)}{(n^2-4)}\pi_{b_n}^2+\pi_{f_n}^2+2a_n\pi_{a_n}\pi_\alpha+8b_n\pi_{b_n}\pi_\alpha\\
&-6a_n\pi_{f_n}\pi_\phi-e^{4\alpha}\left\{\frac{(n^2-5/2)}{3}a_n^2+\frac{(n^2-7)(n^2-4)}{3(n^2-1)}b_n^2\right.\\
&\left.+\frac{2(n^2-4)}{3}a_nb_n-(n^2-1)f_n^2\right\}
\end{aligned}
$$

$$+e^{6\alpha}m^2\left(f_n^2+6a_nf_n\phi\right)+e^{6\alpha}m^2\phi^2\left\{\frac{3a_n^2}{2}-\frac{6(n^2-4)}{(n^2-1)}b_n^2\right\}\Bigg], \tag{3.18}$$

$$H_{|2}^{(V)n}=\frac{1}{2e^{3\alpha}}\Bigg[(n^2-4)c_n^2\left(10\pi_\alpha^2+6\pi_\phi^2\right)+\frac{1}{(n^2-4)}\pi_{c_n}^2$$
$$+8c_n\pi_{c_n}\pi_\alpha+(n^2-4)c_n^2\left(2e^{4\alpha}-6e^{6\alpha}m^2\phi^2\right)\Bigg], \tag{3.19}$$

$$H_{|2}^{(T)n}=\frac{1}{2e^{3\alpha}}\Bigg[d_n^2\left(10\pi_\alpha^2+6\pi_\phi^2\right)+\pi_{d_n}^2+8d_n\pi_{d_n}\pi_\alpha$$
$$+d_n^2\left\{(n^2+1)e^{4\alpha}-6e^{6\alpha}m^2\phi^2\right\}\Bigg]. \tag{3.20}$$

The first order Hamiltonian is

$$H_{|1}^{n}=\frac{1}{2e^{3\alpha}}\Bigg[-a_n\left(\pi_\alpha^2+3\pi_\phi^2\right)+2\left(\pi_{f_n}\pi_\phi-\pi_{a_n}\pi_\alpha\right)$$
$$+e^{6\alpha}m^2\left(3a_n\phi^2+2f_n\phi\right)-\frac{2e^{4\alpha}}{3}\left\{(n^2-4)b_n+(n^2+\frac{1}{2})a_n\right\}\Bigg]. \tag{3.21}$$

The shift parts of the Hamiltonian are

$$H_{\_1}^{(S)n}=\frac{1}{3e^{3\alpha}}\left[-\pi_{a_n}+\pi_{b_n}+\left\{a_n+\frac{4(n^2-4)}{(n^2-1)}b_n\right\}\pi_\alpha+3f_n\pi_\phi\right], \tag{3.22}$$
$$H_{\_1}^{(V)n}=e^{-\alpha}\left[\pi_{c_n}+4(n^2-4)c_n\pi_\alpha\right]. \tag{3.23}$$

The conjugate momenta have been defined in the usual manner (see Appendix C).

Because the Lagrange multipliers N_0, g_n, k_n and j_n are all independent, the Wheeler-DeWitt equation is decomposed into two parts,

$$\left\{\hat{H}_{|0}+\sum_n\left(\hat{H}_{|2}^{(S)n}+\hat{H}_{|2}^{(V)n}+\hat{H}_{|2}^{(T)n}\right)\right\}\Psi[\alpha,\phi,a_n,b_n,c_n,d_n,f_n]=0 \tag{3.24}$$

and

$$\hat{H}_{|1}^{n}\Psi[\alpha,\phi,a_n,b_n,c_n,d_n,f_n]=0, \tag{3.25}$$

where we have replaced the momenta by the operators;

$$\pi_{a_n}\to-i\frac{\partial}{\partial a_n},\quad \pi_{b_n}\to-i\frac{\partial}{\partial b_n},\quad \pi_{c_n}\to-i\frac{\partial}{\partial c_n},$$
$$\pi_{d_n}\to-i\frac{\partial}{\partial d_n},\quad \pi_{f_n}\to-i\frac{\partial}{\partial f_n}, \tag{3.26}$$

with the replacements of the zero mode (3.6). The Hamiltonian operators are given by

$$
\begin{aligned}
\hat{H}_{|0} &= \frac{1}{2}e^{-3\alpha}\left(\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\phi^2}-e^{4\alpha}+e^{6\alpha}m^2\phi^2\right),\\
\hat{H}^{(S)n}_{|2} &= \frac{1}{2e^{3\alpha}}\Bigg[-\left\{\frac{a_n^2}{2}+\frac{10(n^2-4)}{(n^2-1)}b_n^2\right\}\frac{\partial^2}{\partial\alpha^2}-\left\{\frac{15a_n^2}{2}+\frac{6(n^2-4)}{(n^2-1)}b_n^2\right\}\frac{\partial^2}{\partial\phi^2}\\
&+\frac{\partial^2}{\partial{a_n}^2}-\frac{(n^2-1)}{(n^2-4)}\frac{\partial^2}{\partial{b_n}^2}-\frac{\partial^2}{\partial{f_n}^2}-2a_n\frac{\partial^2}{\partial a_n\partial\alpha}-8b_n\frac{\partial^2}{\partial b_n\partial\alpha}\\
&+6a_n\frac{\partial^2}{\partial f_n\partial\phi}-e^{4\alpha}\Bigg\{\frac{(n^2-5/2)}{3}a_n^2+\frac{(n^2-7)(n^2-4)}{3(n^2-1)}b_n^2\\
&+\frac{2(n^2-4)}{3}a_nb_n-(n^2-1)f_n^2\Bigg\}\\
&+e^{6\alpha}m^2\left(f_n^2+6a_nf_n\phi\right)+e^{6\alpha}m^2\phi^2\left\{\frac{3a_n^2}{2}-\frac{6(n^2-4)}{(n^2-1)}b_n^2\right\}\Bigg],\\
\hat{H}^{(V)n}_{|2} &= \frac{1}{2e^{3\alpha}}\Bigg[-(n^2-4)c_n^2\left(10\frac{\partial^2}{\partial\alpha^2}+6\frac{\partial^2}{\partial\phi^2}\right)-\frac{1}{(n^2-4)}\frac{\partial^2}{\partial{c_n}^2}\\
&-8c_n\frac{\partial^2}{\partial c_n\partial\alpha}+(n^2-4)c_n^2\left(2e^{4\alpha}-6e^{6\alpha}m^2\phi^2\right)\Bigg],\\
\hat{H}^{(T)n}_{|2} &= \frac{1}{2e^{3\alpha}}\Bigg[-d_n^2\left(10\frac{\partial^2}{\partial\alpha^2}+6\frac{\partial^2}{\partial\phi^2}\right)-\frac{\partial^2}{\partial{d_n}^2}-8d_n\frac{\partial^2}{\partial d_n\partial\alpha}\\
&+d_n^2\left\{(n^2+1)e^{4\alpha}-6e^{6\alpha}m^2\phi^2\right\}\Bigg]
\end{aligned}
\tag{3.27}
$$

and

$$
\begin{aligned}
\hat{H}^{n}_{|1} &= \frac{1}{2e^{3\alpha}}\Bigg[a_n\left(\frac{\partial^2}{\partial\alpha^2}+3\frac{\partial^2}{\partial\phi^2}\right)-2\left(\frac{\partial^2}{\partial f_n\partial\phi}-\frac{\partial^2}{\partial a_n\partial\alpha}\right)\\
&+e^{6\alpha}m^2\left(3a_n\phi^2+2f_n\phi\right)-\frac{2e^{4\alpha}}{3}\left\{(n^2-4)b_n+(n^2+\frac{1}{2})a_n\right\}\Bigg].
\end{aligned}
\tag{3.28}
$$

Following Halliwell and Hawking [14], we call equation (3.24) the master equation and equation (3.25) the linear Hamiltonian constraint. The momentum constraints are

$$
\hat{H}^{(S)n}_{\_1}\Psi[\alpha,\phi,a_n,b_n,c_n,d_n,f_n]=0 \tag{3.29}
$$

and

$$
\hat{H}^{(V)n}_{\_1}\Psi[\alpha,\phi,a_n,b_n,c_n,d_n,f_n]=0, \tag{3.30}
$$

where

$$\hat{H}_{-1}^{(S)n} = \frac{1}{3e^{3\alpha}} \left[-\frac{\partial}{\partial a_n} + \frac{\partial}{\partial b_n} + \left\{ a_n + \frac{4(n^2-4)}{(n^2-1)} b_n \right\} \frac{\partial}{\partial \alpha} + 3 f_n \frac{\partial}{\partial \phi} \right],$$
$$\hat{H}_{-1}^{(V)n} = e^{-\alpha} \left[\frac{\partial}{\partial c_n} + 4(n^2-4) c_n \frac{\partial}{\partial \alpha} \right]. \qquad (3.31)$$

C. The analysis of the stability of the Friedmann minisuperspace

While we consider only the second order in the non-zero modes, the master equation (3.24) is difficult to solve analytically. We solve it numerically by picking up some inhomogeneous mode from the infinite number of modes adequately. In other words, we restrict the degrees of freedom of the superspace of this system to a finite number. The reason why we can pick up some particular inhomogeneous mode is that the perturbation modes are not coupled directly and that the master equation (3.24) is expressed as a sum of the each mode.

There are many ways of picking up the inhomogeneous modes. For example, we can fix the mode number n for some value. Furthermore we can pick up only the scalar harmonics part of the gravity (a_n, b_n, g_n and k_n), the vector harmonics part (c_n, j_n) or the tensor part (d_n). The easiest non-trivial one is described in what follows. For the metric, we set the scalar type perturbations (a_n, b_n, g_n and k_n) equal to zero and consider only the vector and the tensor type perturbations (c_n, d_n and j_n) for some fixed n. For the matter field, we also fix the mode number to n. We consider the higher-dimensional (but finite) space W which includes the lower-dimensional Friedmann minisuperspace and then we solve a reduced Wheeler-DeWitt equation on the space W. We are not bothered by the problem of solving the Wheeler-DeWitt equation and the linear Hamiltonian constraint equation simultaneously, because we have omitted the scalar type perturbations (a_n, b_n, g_n and k_n). We do not discuss the effect of the scalar harmonics part.

Following the above restriction, we obtain a master equation on the restricted space W:

$$\left(\hat{H}_{|0} + \hat{H}_{|2}^{(V)n} + \hat{H}_{|2}^{(T)n}\right) \Psi[\alpha, \phi, c_n, d_n, f_n] = 0. \qquad (3.32)$$

Hereafter we call the equation (3.32) the fixed master equation. There is neither the scalar momentum constraint (3.29) nor the linear Hamiltonian constraint (3.25). Following the discussion in the paper of Halliwell and Hawking [14], we can eliminate c_n from the fixed master equation (3.32) by using the vector momentum constraint (3.30). We can use it to substitute for the partial derivatives with respect to c_n and then solve the resultant differential equation on $c_n = 0$. After we know the wave function on $c_n = 0$, we can use the vector momentum constraint (3.30) to calculate the wave function at other values of c_n.

Consequently we obtain the modified fixed master equation;

$$\begin{aligned}
&\frac{N}{2}\frac{1}{a^3}\left[\left(1 - 10d_n^2\right)\left(a^2\frac{\partial^2}{\partial a^2} + a\frac{\partial}{\partial a}\right) - \left(1 + 6d_n^2\right)\frac{\partial^2}{\partial \phi^2} - \frac{\partial^2}{\partial d_n^2}\right. \\
&-8d_n a\frac{\partial^2}{\partial d_n \partial a} - \frac{\partial^2}{\partial f_n^2} + m^2 a^6 \phi^2 - a^4 \\
&\left.+d_n^2\left\{(n^2+1)a^4 - 6m^2a^6\phi^2\right\} + (n^2-1)a^4 f_n^2 + m^2 a^6 f_n^2\right]\Psi[a, \phi, d_n, f_n] = 0. \qquad (3.33)
\end{aligned}$$

This is the hyperbolic partial differential equation. We can therefore regard the scale factor a as time.

To analyze the stability of the minisuperspace, we first set a wave packet which is centered around the minisuperspace sector as the initial condition;

$$\Psi(a = a_0) \propto \exp\left(-u_0 d_n^2\right), \qquad (3.34)$$

where u_0 is an arbitrary constant (see Figs.(1, 2)). We then investigate the development of the wave packet prepared. We impose the boundary condition of the wave function; $\Psi \to 0$

as the inhomogeneous perturbation d_n or f_n goes to infinity. The free parameters of this system are the mass of the scalar field (m), the mode number (n) and the initial value of the scale factor (a_0). We compute the development of the wave packet with respect to the scale factor (a) in the various cases of these parameters. And the results are summarized in Tables.(1, 2). For the minisuperspace case, we can see that the wave function of the universe grows exponentially (see Fig.(3)). For the perturbed minisuperspace case, if the wave packet develops exponentially without much spreading into the surrounding minisuperspace, we can consider the minisuperspace stable (for example, in a case $a_0 = 20$, $m = 0$ and $n = 2$, see Fig.(1)). On the other hand, if the wave packet dose not grow exponentially or much spreading into the surrounding minisuperspace, we can say that the minisuperspace is unstable (for example, in a case $a_0 = 2$, $m = 0$ and $n = 2$, see Fig.(2)). Furthermore we compute the development of a standard deviation (Δd_n) of the wave packet to check whether the minisuperspace is really stable. The standard deviation Δd_n is used as an index which represents the width of the wave function in the direction of d_n. The results are in Fig.(4). From this figure, we can conclude that the minisuperspace is stable when the initial value of the scale factor (a_0) of the universe is larger than a few times of the Planck length. In this case, we can regard the wave packet (which grows exponentially) approximately as the wave function of the universe on the minisuperspace. On the other hand, the minisuperspace becomes unstable when a_0 is near the Planck length. The results do not change even if the mass of the scalar field or the mode number n is set large (see Table.(1, 2)).

Our result agrees with the validity region in which the WKB approximation is applicable.

IV. THE LAPSE, SHIFT AND THE LIOUVILLE FIELD IN 2-DIMENSIONAL GRAVITY

It is well-known that string theory can be viewed as 2-dimensional gravity. Here we want to consider it as a toy model for 4-dimensional quantum cosmology.

In 4 dimensions, we have seen, in §II, that a quantum state of the universe is described by a wave function of the universe on superspace (see for example Ref. [13]). Superspace has a finite number of coordinates at every point on the 3-hypersurface and is infinite-dimensional. In order to discuss quantum mechanical properties of the universe, many people want to solve zero-energy Schrödinger equations that are decomposed into the Wheeler-DeWitt equation and the momentum constraints by the (3+1) decomposition consisting of the lapse and shift functions (N, N^i) and the 3-metric on the hypersurface (h_{ij}). It is very hard, however, to solve such infinite-dimensional differential equations without any approximation apart from many difficult conceptual problems. In order to make this problem tractable, we often reduce the degrees of freedom of the superspace to a finite number by assuming certain symmetries (see §II.E). This reduced finite dimensional superspace is called minisuperspace. In the minisuperspace, the transition amplitude of the universe defined by path integral can be easily shown to obey the Wheeler-DeWitt equation in the $\dot{N} = 0$ gauge [20].

In contrast to the above 4-dimensional case, the Einstein action becomes a topological number in 2 dimensions. In order to obtain a gravitational theory, we have to treat an anomaly of the path integral measure exactly (see for example Ref. [15]). It has been proposed that the minisuperspace represents the superspace exactly in 2 dimensions [26], [27]. The reason may be that an argument of a wave function of a 1-dimensional universe may be the length of the universe itself (see §II.B). This conjecture is partially supported by the calculations of the $c = 0$ matrix model [26].

In this section and next section, in the context of quantum cosmology, we want to investigate this proposal in the framework of the continuum Liouville theory. In §IV.A, in order to investigate the lapse, shift and the Liouville field in 2-dimensional gravity, we will start from the Polyakov action and will briefly review the known formulation to rewrite it by using the (1+1) decomposition [28]. Some of this discussion in what follows is developed in Ref. [16]. We will show that the path integral measure $[\mathcal{D}g_{ab}]$ can be decomposed into $[\mathcal{D}N][\mathcal{D}M][\mathcal{D}\phi]$ by using the lapse and shift functions (N, M) and the Liouville field (ϕ). In §IV.B, however, we will find that there is an undesirable term consisting only of N and M in the Liouville action.

A. The ADM decomposition in 2-dimensional gravity

In this subsection, we will briefly review the formulation of 2-dimensional gravity (see for example Ref. [15]). Starting from the Polyakov action, we will rewrite it by the (1+1) decomposition. We will obtain the transition amplitude of the sting universe, which will be used in §V.

We write the coordinates on the worldsheet as

$$\xi^a = (\xi^0, \xi^1). \tag{4.1}$$

In this coordinate, the transition amplitude is represented as

$$Z[X_F^\mu, \phi_F; X_I^\mu, \phi_I] = \int \frac{[\mathcal{D}g_{ab}][\mathcal{D}X^\mu]}{vol(\text{Gauge})} \exp\left\{-\frac{1}{2}\int d^2\xi\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu\right\}, \tag{4.2}$$

where the metric on the worldsheet is given by

$$ds^2 = g_{ab}(\xi)d\xi^a d\xi^b = e^{\phi(\xi)}\hat{g}_{ab}(\xi)d\xi^a d\xi^b. \tag{4.3}$$

Here we take a conformal time on the worldsheet, and can take out the dynamical variable as an overall conformal mode. We also parametrize the fiducial metric $\hat{g}_{ab}(\xi)$ by using the lapse and shift functions $N(\xi)$, $M(\xi)$, following the ADM decomposition in 2 dimensions [28];

$$\hat{g}_{ab}(\xi) = \begin{pmatrix} N(\xi)^{-2} + M(\xi)^2 & M(\xi) \\ M(\xi) & 1 \end{pmatrix}. \tag{4.4}$$

The reason why we take an inverse of the lapse function will be discussed later. The most general local metric on deformations δg_{ab} of the metric is given by

$$||\delta g||^2 = \int d^2\xi \sqrt{g} \left\{ G^{abcd} + u g^{ab} g^{cd} \right\} \delta g_{ab} \delta g_{cd}, \tag{4.5}$$

where u is an arbitrary positive real number and G^{abcd} is the identity operator in the space of symmetric traceless tensors;

$$G^{abcd} = \frac{1}{2} \left(\delta_a^c \delta_b^d + \delta_a^d \delta_b^c - g_{ab} g^{cd} \right). \tag{4.6}$$

The decomposition of the measure $[\mathcal{D} g_{ab}]$ in the transition amplitude (4.2) is given by the orthogonal decomposition;

$$\delta g_{ab} = \delta h_{ab} + (\delta\rho) g_{ab}, \tag{4.7}$$

where

$$\delta\rho = -\frac{\delta N}{N} + \delta\phi \tag{4.8}$$

and

$$\delta h_{ab} = e^{\phi} \begin{pmatrix} (M^2/N - 1/N^3)\,\delta N + 2M\delta M & M\delta N/N + \delta M \\ M\delta N/N + \delta M & \delta N/N \end{pmatrix}. \tag{4.9}$$

Here $\delta\rho$ is the trace part of the metric deformations δg_{ab}, and δh_{ab} is the symmetric traceless part. Then the metric on deformations of δg_{ab} is decomposed as

$$||\delta g||^2 = \int d^2\xi \sqrt{g} G^{abcd} \delta h_{ab} \delta h_{cd} + 4u \int d^2\xi \sqrt{g} (\delta\rho)^2. \tag{4.10}$$

From this decomposition, we can separate the measure $[\mathcal{D}g_{ab}]$ in the form of $[\mathcal{D}\rho][\mathcal{D}h_{ab}]$. Next we change variables from ρ, h_{ab} to ϕ, v_a;

$$\delta\rho = \delta\phi + g^{ab}\nabla_a(\delta v_b),$$
$$\delta h_{ab} = 2G^{cd}_{ab}\nabla_c(\delta v_d) = (P_1\delta v)_{ab}, \tag{4.11}$$

where δv_a are infinitesimal generators with respect to a 2-dimensional diffeomorphism. The operator P_1 maps vectors into symmetric traceless tensors. We obtain another decomposition with the Jacobian $\left\{det'(P_1^\dagger P_1)\right\}_g^{1/2}$;

$$[\mathcal{D}g_{ab}] = \prod_i d\tau_i \frac{1}{vol(\mathrm{CK})} \frac{det < \psi^{(j)}|\frac{\partial g}{\partial\tau_k} >_g}{det < \psi^{(j)}|\psi^{(k)} >_g^{1/2}} [\mathcal{D}\phi][\mathcal{D}v_a]\left\{det'(P_1^\dagger P_1)\right\}_g^{1/2}, \tag{4.12}$$

where τ_i are moduli parameters and the factor $\frac{det<\psi^{(j)}|\frac{\partial g}{det\partial\tau_k}>_g}{det<\psi^{(j)}|\psi^{(k)}>_g^{1/2}}$ is the Weil-Peterson measure which represents the angle between the moduli space and the gauge orbit of a diffeomorphism. The *prime* in equation (4.12) denotes the omission of zero mode with respect to conformal Killing vectors dV; $P_1(dV) = 0$. We must divide by the volume of conformal Killing vectors $vol(\mathrm{CK})$. The Liouville action S_ϕ arises in the formula;

$$\frac{det < \psi^{(j)}|\frac{\partial g}{\partial\tau_k} >_g}{det < \psi^{(j)}|\psi^{(k)} >_g^{1/2}}\left\{det'(P_1^\dagger P_1)\right\}_g^{1/2} = \frac{det < \psi^{(j)}|\frac{\partial\hat{g}}{\partial\tau_k} >_{\hat{g}}}{det < \psi^{(j)}|\psi^{(k)} >_{\hat{g}}^{1/2}}\left\{det'(P_1^\dagger P_1)\right\}_{\hat{g}}^{1/2} e^{-\frac{26}{48\pi}S_\phi[\phi,\hat{g}]}. \tag{4.13}$$

Accordingly the measure is decomposed as

$$\begin{aligned}[\mathcal{D}g_{ab}] &= \prod_i d\tau_i \frac{det < \psi^{(j)}|\frac{\partial\hat{g}}{\partial\tau_k} >_{\hat{g}}}{det < \psi^{(j)}|\psi^{(k)} >_{\hat{g}}^{1/2}} [\mathcal{D}\phi][\mathcal{D}v_a]\left\{det'(P_1^\dagger P_1)\right\}_{\hat{g}}^{1/2} e^{-\frac{26}{48\pi}S_\phi[\phi,\hat{g}]} \\ &\equiv [\mathcal{D}\phi][\mathcal{D}\hat{h}_{ab}]e^{-\frac{26}{48\pi}S_\phi[\phi,\hat{g}]},\end{aligned} \tag{4.14}$$

where $\hat{h}_{ab}$ is defined by above equation. The most general local metric on deformations $\delta\hat{h}_{ab}$ is given by

$$\begin{aligned}||\delta\hat{h}||^2 &= \int d^2\xi\sqrt{\hat{g}}\left\{\hat{G}^{abcd} + u\hat{g}^{ab}\hat{g}^{cd}\right\}\delta\hat{h}_{ab}\delta\hat{h}_{cd} \\ &= \int d^2\xi\left\{2\frac{(\delta N)^2}{N^2} + 2N^2(\delta M)^2\right\}.\end{aligned} \tag{4.15}$$

From this decomposition, we obtain another separation of the measure $[\mathcal{D}g_{ab}]$;

$$[\mathcal{D}g_{ab}] = [\mathcal{D}\phi][\mathcal{D}N][\mathcal{D}M]e^{-\frac{26}{48\pi}S_\phi[\phi,\hat{g}]}. \tag{4.16}$$

In obtaining this form, we have taken the parametrization of the fiducial metric given by equation (4.4). Here we find the same separation form of the measure that we take in 4 dimensions (see §II.C). This discussion is developed in Ref. [16]. We should be careful about treating the measure of the Liouville field $[\mathcal{D}\phi]$, because it is not translationally invariant. We will discuss it in §V.

The difference of the path integral for conformal matter fields (central charge c) evaluated on g_{ab} and that on $\hat{g}_{ab}$ is also represented in terms of the Liouville action;

$$\begin{aligned}&\int [\mathcal{D}X^\mu]_g \exp\left\{-\frac{1}{2}\int d^2\xi\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu\right\} \\ &= \int [\mathcal{D}X^\mu]_{\hat{g}} \exp\left\{-\frac{1}{2}\int d^2\xi\sqrt{\hat{g}}\hat{g}^{ab}\partial_a X^\mu \partial_b X_\mu\right\} e^{\frac{c}{48\pi}S_\phi[\phi,\hat{g}]}. \end{aligned} \tag{4.17}$$

Consequently the transition amplitude (4.2) can be expressed as

$$\begin{aligned}Z[X_F^\mu, \phi_F; X_I^\mu, \phi_I] = &\int \frac{[\mathcal{D}N][\mathcal{D}M]}{vol(\text{Gauge})} \int [\mathcal{D}X^\mu] \exp\left\{-\frac{1}{2}\int d^2\xi\sqrt{\hat{g}}\hat{g}^{ab}\partial_a X^\mu \partial_b X_\mu\right\} \\ &\times \int [\mathcal{D}\phi] e^{-\frac{26-c}{48\pi}S_\phi[\phi,\hat{g}]}. \end{aligned} \tag{4.18}$$

B. The action for conformal matter fields and that for the Liouville field

By using the parametrization of the fiducial metric (4.4), we can write the action of conformal matter fields as

$$\begin{aligned}S_{X^\mu} &= \frac{1}{2}\int d^2\xi\sqrt{\hat{g}}\hat{g}^{ab}\partial_a X^\mu \partial_b X_\mu \\ &= \int d^2\xi \left\{P_{X\mu}\dot{X}^\mu - N^{-1}H_0^{X^\mu} - MH_1^{X^\mu}\right\}, \end{aligned} \tag{4.19}$$

where

$$P_{X^\mu} = N\left(\dot{X}^\mu - MX^{\mu\prime}\right),$$
$$H_0^{X^\mu} = \frac{1}{2}\left(P_{X^\mu}P_{X_\mu} - X^{\mu\prime}X_\mu{}'\right), \quad H_1^{X^\mu} = P_{X_\mu}X^{\mu\prime}. \tag{4.20}$$

Likewise,

$$\begin{aligned} S_\phi &= \kappa\int d^2\xi\sqrt{\hat{g}}\left\{\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi + \hat{R}\phi + \mu e^\phi\right\} \\ &= \int d^2\xi\left\{P_\phi\dot{\phi} - N^{-1}H_0^\phi - MH_1^\phi + 2\kappa NM'^2\right\}, \end{aligned} \tag{4.21}$$

where

$$P_\phi = \kappa N\left(\dot{\phi} - M\phi' - 2M'\right),$$
$$H_0^\phi = \frac{1}{2\kappa}P_\phi^2 - \frac{\kappa}{2}\phi'^2 + 2\kappa\phi'' + \mu e^\phi, \quad H_1^\phi = P_\phi\phi' - 2\phi'. \tag{4.22}$$

The *dot* and the *prime* represent the derivative with respect to ξ^0 and ξ^1 respectively, and we take $\kappa = \frac{26-c}{48\pi}$.

The extra term $2\kappa NM'^2$ in the Liouville action (4.21) prevents us from interpreting the lapse and shift functions as the Lagrange multipliers which enforce the Hamiltonian constraint and the momentum constraints [16] [28] [29]. This is related to the fact that there is no invariance under the diffeomorphisms of the fiducial metric $\hat{g}_{ab}$ where the Liouville mode ϕ is taken out. Therefore we cannot construct the canonical formulation of this system; the degrees of freedom are insufficient and there is no canonical structure (classically). In order to avoid this problem, Teitelboim added an extra field to the system in order to recover the canonical structure [28] [29] [30]. The meaning of this extra field is unclear however. In the next section, we will bypass this problem for the case of annulus topology.

V. THE EXACTNESS OF THE MINISUPERSPACE IN 2 DIMENSIONS

In this section, we take a conformal gauge in order to eliminate the problematic term mentioned in the previous section. We restrict ourselves to the case of annulus topology and calculate the transition amplitude of the 1-dimensional loop universe. We will show that the non-zero modes of the fields are all cancelled out in the transition amplitude only when we take the $c = 1$ conformal matter field and impose the Neumann boundary condition on the system. The reason is as follows. When the cosmological constant is ignored, the Liouville field acts as an extra conformal matter field. Therefore the target space is 2-dimensional but a string cannot vibrate in 2 dimensions. The rational for ignoring the cosmological constant will be discussed later in this section. This cancellation of the non-zero modes is similar to the case of torus topology which was investigated by Bershadsky and Klebanov [31]. Our guiding principle is that ghost fields should not appear on the boundary in the context of quantum cosmology, and this requirement is satisfied only by the Neumann boundary condition. In open string theory, the Neumann boundary condition is taken, because the end points of a string are free. In our case, the Neumann boundary condition means that we must sum over all allowed values of the non-zero modes in the initial and final states. As a result of these settings, we obtain the transition amplitude constructed only by the zero modes. It can be said that the Wheeler-DeWitt equation, that this transition amplitude obeys, exists only on the minisuperspace, and therefore the minisuperspace represents the superspace exactly. Our result crucially depends on the fact that we consider the case with the $c = 1$ conformal matter field.

A. The transition amplitude for the case of annulus

Here we concentrate on the case of annulus topology and fix the diffeomorphisms completely by taking the easiest gauge, namely the conformal gauge, in order to eliminate the problematic term mentioned in the previous section. This gauge fixing is quite different from the $\dot{N} = 0$ gauge used in the minisuperspace treatment of a 4-dimensional gauge fixing (see for example Ref. [13] [20]). If we fix the diffeomorphisms completely, this problematic term disappears and the degrees of freedom in the fiducial metric $\hat{g}_{ab}$ become finite, and is described by a modular parameter t in the annulus case. In other words, we can take $N^{-1} = t$ and $M = 0$ in the parametrization (4.4). In this gauge, from the decomposition of the measure $[\mathcal{D}g_{ab}]$ given by the first line of equation (4.14), we obtain the transition amplitude of the 1-dimensional loop universe;

$$
\begin{aligned}
Z[X^{\mu}_F, \phi_F; X^{\mu}_I, \phi_I] &= \int_0^{\infty} dt \frac{1}{\Omega(CK)} \frac{< \psi | \frac{\partial \hat{g}}{\partial t} >_{\hat{g}}}{< \psi | \psi >_{\hat{g}}^{1/2}} \left\{ det'(P_1^{\dagger} P_1) \right\}_{\hat{g}}^{1/2} \\
&\quad \times \int [\mathcal{D}X^{\mu}] e^{-S_{X^{\mu}}[X^{\mu}, t]} \int [\mathcal{D}\phi] e^{-S_{\phi}[\phi, t]} \\
&= \int_0^{\infty} \frac{dt}{t} \left\{ det'(P_1^{\dagger} P_1) \right\}_{\hat{g}}^{1/2} \int [\mathcal{D}X^{\mu}] e^{-S_{X^{\mu}}[X^{\mu}, t]} \int [\mathcal{D}\phi] e^{-S_{\phi}[\phi, t]}, \qquad (5.1)
\end{aligned}
$$

where we have used the following results of the calculations of the Weil-Peterson measure and the volume of a conformal Killing vector;

$$
\frac{< \psi | \frac{\partial \hat{g}}{\partial t} >_{\hat{g}}}{< \psi | \psi >_{\hat{g}}^{1/2}} = \left(\frac{2}{t} \right)^{1/2} \qquad (5.2)
$$

and

$$
\Omega(CK) = t^{1/2}. \qquad (5.3)
$$

Note that the volume of v_a is divided by $vol(\text{Gauge})$. Here the action of the conformal matter fields and that of the Liouville field can be rewritten respectively as

$$
S_{X^{\mu}}[X^{\mu}, t] = \frac{1}{2} \int_M d^2\sigma \left\{ \dot{X}^{\mu} \dot{X}_{\mu} + X^{\mu\prime} X_{\mu}{}' \right\} \qquad (5.4)
$$

and

$$S_\phi[\phi,t]=\frac{\kappa}{2}\int_M d^2\sigma\left\{\dot{\phi}^2+\phi'^2-4\phi''\right\}. \tag{5.5}$$

Here we have simply discarded the cosmological constant (see the subsequent discussion for the case of $c=1$).

In order to obtain the actions (5.4) and (5.5), we have made a coordinate transformation from ξ^a to σ^a ($\sigma^0=t\xi^0$, $\sigma^1=\xi^1$) and the region M of the new coordinates σ^a is given by

$$M: \quad 0\le\sigma^0\le t, \quad 0\le\sigma^1\le 1. \tag{5.6}$$

Here the space coordinate σ^1 is periodic and the boundaries exist at $\sigma^0=0,t$. From this transformation, we can interpret the modular parameter t as a time variable of the system consisting of conformal matter fields and the Liouville field.

For the computation of the transition amplitude (5.1), we will make the mode expansion. For conformal matter fields, we expand them as follows

$$X^\mu(\sigma^0,\sigma^1)=X_0^\mu(\sigma^0)+\sum_{n\neq 0}a_n^\mu(\sigma^0)e^{-2\pi i\sigma^1}. \tag{5.7}$$

The first term is the zero mode and the second term represents the vibrations. We can separate the partition function of matter fields into the zero mode part and the non-zero modes parts;

$$\begin{aligned}\int[\mathcal{D}X^\mu]e^{-S_{X^\mu}[X^\mu,t]}&=\int[\mathcal{D}X_0^\mu]e^{-S_{X_0^\mu}[X_0^\mu,t]}\prod_{n\neq 0}\int[\mathcal{D}a_n^\mu]e^{-S_{a_n^\mu}[a_n^\mu,t]}\\&=K(X_{0F}^\mu,t;X_{0I}^\mu,0)\prod_{n\neq 0}\left\{\frac{1}{n\sinh(2\pi nt)}\right\}^{c/2},\end{aligned} \tag{5.8}$$

where

$$
\begin{aligned}
S_{X^\mu}[X^\mu,t] &= S_{X_0^\mu}[X_0^\mu,t] + \sum_{n\neq 0} S_{a_n^\mu}[a_n^\mu,t],\\
S_{X_0^\mu}[X_0^\mu,t] &= \frac{1}{2}\int_0^t d\sigma^0 \dot{X}_0^\mu \dot{X}_{0\mu},\\
S_{a_n^\mu}[a_n^\mu,t] &= \frac{1}{2}\int_0^t d\sigma^0 \left\{\dot{a}_n^\mu \dot{a}_{n\mu} + (2\pi n)^2 a_n^\mu a_{n\mu}\right\},
\end{aligned}
\tag{5.9}
$$

and

$$
\begin{aligned}
K(X_{0F}^\mu,t;X_{0I}^\mu,0) &\equiv \int [\mathcal{D}X_0^\mu] e^{-S_{X_0^\mu}[X_0^\mu,t]}\\
&= \left(\frac{1}{2\pi t}\right)^{c/2} \exp\left\{-\frac{(X_{0F}^\mu - X_{0I}^\mu)(X_{0F\mu} - X_{0I\mu})}{2t}\right\},\\
\int [\mathcal{D}a_n^\mu] e^{-S_{a_n^\mu}[a_n^\mu,t]} &= \left\{\frac{1}{n\sinh(2\pi n t)}\right\}^{c/2}.
\end{aligned}
\tag{5.10}
$$

We have used the knowledge of the path integral in quantum mechanics for a free particle and a harmonic oscillator. Here we have taken the Neumann boundary condition as stated before. Note that the Neumann boundary condition puts no restriction on the zero mode. For the Liouville field, we also make the same mode expansion;

$$
\phi(\sigma^0,\sigma^1) = \phi_0(\sigma^0) + \sum_{n\neq 0} b_n(\sigma^0) e^{-2\pi i \sigma^1}.
\tag{5.11}
$$

We also separate the partition function of the Liouville field into two parts;

$$
\int [\mathcal{D}\phi] e^{-S_\phi[\phi,t]} = K(\phi_{0F},t;\phi_{0I},0) \prod_{n\neq 0} \left\{\frac{1}{n\sinh(2\pi n t)}\right\}^{1/2},
\tag{5.12}
$$

where

$$
\begin{aligned}
S_\phi[\phi,t] &= S_{\phi_0}[\phi_0,t] + \sum_{n\neq 0} S_{b_n}[b_n,t],\\
S_{\phi_0}[\phi_0,t] &= \frac{\kappa}{2}\int_0^t d\sigma^0 \dot{\phi}_0^2,\\
S_{b_n}[b_n,t] &= \frac{\kappa}{2}\int_0^t d\sigma^0 \left\{\dot{b}_n^2 + (2\pi n)^2 b_n^2\right\}.
\end{aligned}
\tag{5.13}
$$

The second derivative term in the Liouville action (5.5) disappears in the mode expansion. For the ghost fields, we also take the Neumann boundary condition. We obtain the Faddeev-Popov determinant

$$\left\{det'(P_1^\dagger P_1)\right\}_{\hat{g}}^{1/2} = 2\pi t \prod_{n\neq 0} n\sinh(2\pi n t). \tag{5.14}$$

There is no zero mode, because it is absent from beginning.

Inserting the partition functions of conformal matter fields and the Liouville field (5.8), (5.12) and the Faddeev-Popov determinant (5.14) into equation (5.1), we obtain finally the transition amplitude for the case of annulus topology explicitly;

$$\begin{aligned} Z[X_F^\mu, \phi_F; X_I^\mu, \phi_I] = 2\pi \int_0^\infty dt K(X_{0F}^\mu, t; X_{0I}^\mu, 0) K(\phi_{0F}, t; \phi_{0I}, 0) \\ \times \left\{ \prod_{n\neq 0} n\sinh(2\pi n t) \right\}^{(1-c)/2} . \end{aligned} \tag{5.15}$$

B. The minisuperspace Wheeler-DeWitt equation

We can easily show that the partition function for the zero mode and the one for the non-zero modes satisfy the Euclidean Schrödinger equation individually;

$$\begin{aligned} \hat{H}_0^{X_0^\mu} K(X_{0F}^\mu, t; X_{0I}^\mu, 0) &= -\frac{\partial}{\partial t} K(X_{0F}^\mu, t; X_{0I}^\mu, 0), \\ \hat{H}_0^{a_n^\mu} K(a_{nF}^\mu, t; a_{nI}^\mu, 0) &= -\frac{\partial}{\partial t} K(a_{nF}^\mu, t; a_{nI}^\mu, 0) \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} \hat{H}_0^{\phi_0} K(\phi_{0F}, t; \phi_{0I}, 0) &= -\frac{\partial}{\partial t} K(\phi_{0F}, t; \phi_{0I}, 0), \\ \hat{H}_0^{b_n} K(b_{nF}, t; b_{nI}, 0) &= -\frac{\partial}{\partial t} K(b_{nF}, t; b_{nI}, 0). \end{aligned} \tag{5.17}$$

The Hamiltonian operators $\hat{H}_0^{X_0^\mu}$, $\hat{H}_0^{a_n^\mu}$, $\hat{H}_0^{\phi_0}$ and $\hat{H}_0^{b_n}$ are defined as follows. From the mode-expanded actions (5.9) and (5.13), we can rewrite the actions and Hamiltonians;

$$S_{X^\mu}[X^\mu, t] = \int_0^t d\sigma^0 \left\{ P_{X_0\,\mu} \dot{X}_0^\mu + \sum_{n\neq 0} P_{a_n\,\mu} \dot{a}_n^\mu - H_0^{X^\mu} \right\} \tag{5.18}$$

and

$$S_\phi[\phi, t] = \int_0^t d\sigma^0 \left\{ P_{\phi_0}\dot{\phi}_0 + \sum_{n\neq 0} P_{b_n}\dot{b}_n - H_0^\phi \right\}, \tag{5.19}$$

where

$$H_0^{X^\mu} = \frac{1}{2}\left[P_{X_0}{}^\mu P_{X_0\,\mu} + \sum_{n\neq 0}\left\{ P_{a_n}{}^\mu P_{a_n\,\mu} - (2\pi n)^2 a_n^\mu a_{n\mu}\right\}\right] \equiv H_0^{X_0^\mu} + \sum_{n\neq 0} H_0^{a_n^\mu},$$

$$H_0^\phi = \frac{\kappa}{2}\left[P_{\phi_0}^2 + \sum_{n\neq 0}\left\{ P_{b_n}{}^2 - (2\pi n)^2 b_n^2\right\}\right] \equiv H_0^{\phi_0} + \sum_{n\neq 0} H_0^{b_n}. \tag{5.20}$$

In order to obtain the Hamiltonian operators, we replace the momentum by the differential operator.

We observe that the last term of the transition amplitude (5.15) becomes a number only when we take $c = 1$. This means that there is no vibration of the string as stated before. Therefore only in the case of $c = 1$, we can show that the transition amplitude of the string universe obeys the minisuperspace Wheeler-DeWitt equation, by using the Euclidean Schrödinger equations (5.16) and (5.17);

$$\begin{aligned}
\hat{H}_0 Z[X_F, \phi_F; X_I, \phi_I] &= \left(\hat{H}_0^X + \hat{H}_0^\phi\right) Z[X_F, \phi_F; X_I, \phi_I] \\
&= \left(\hat{H}_0^{X_0} + \hat{H}_0^{\phi_0}\right) Z[X_F, \phi_F; X_I, \phi_I] \\
&= -2\pi \int_0^\infty dt \frac{\partial}{\partial t}\left\{ K(X_{0F}, t; X_{0I}, 0) K(\phi_{0F}, t; \phi_{0I}, 0)\right\} \\
&= -2\pi \left[K(X_{0F}, t; X_{0I}, 0) K(\phi_{0F}, t; \phi_{0I}, 0)\right]_0^\infty \\
&= \delta(X_{0F} - X_{0I})\delta(\phi_{0F} - \phi_{0I}).
\end{aligned} \tag{5.21}$$

Here, in the case of $c = 1$, the reason why we could ignore the cosmological constant is clear by following the discussions in Ref. [31] [32]; the theory cut-off by the exponential interaction which originates from the cosmological constant may be identified with the free field theory with an appropriate renormalization. We do not have to take the modification of the Liouville field dynamics proposed by David and by Distler and Kawai [33] [34], when we only consider the annulus topology without cosmological constant.

As is well known, the quantity $\prod_{n\neq 0} n\sinh(2\pi nt)$ in the Faddeev-Popov determinant diverges. If we regularize this quantity by using ζ function, we obtain the well-known form

$$\begin{aligned}\left\{det'(P_1^\dagger P_1)\right\}_{\hat{g}}^{1/2} &= 2\pi t \prod_{n\neq 0} n\sinh(2\pi nt) \\ &= 2\pi t e^{-\frac{\pi t}{3}} \prod_{n=1}^{\infty} \left(1 - e^{-4\pi nt}\right)^2 .\end{aligned} \tag{5.22}$$

This regularization is not important, because these divergent quantities are all cancelled out in the transition amplitude (5.15) in the case of $c = 1$.

On the other hand, if we take the Dirichlet boundary condition for the conformal matter fields and the Liouville field, we obtain the transition amplitude;

$$\begin{aligned}&Z[X_F^\mu, \phi_F; X_I^\mu, \phi_I] \\ &= 2\pi \int_0^\infty dt K(X_{0F}^\mu, t; X_{0I}^\mu, 0) K(\phi_{0F}, t; \phi_{0I}, 0) \prod_{n\neq 0} n^{(3+c)/2} \left\{\sinh(2\pi nt)\right\}^{(1-c)/2} \\ &\times \exp\left[- \sum_{n\neq 0} \frac{\pi n}{\sinh(2\pi n)} \left\{ \cosh(2\pi nt) \left(a_{nF}^\mu a_{nF\mu} + a_{nI}^\mu a_{nI\mu} + b_{nF}^\mu b_{nF\mu} + b_{nI}^\mu b_{nI\mu} \right) \right.\right. \\ &\qquad\qquad \left.\left. - \left(2a_{nF}^\mu a_{nI\mu} + 2b_{nF}^\mu b_{nI\mu} \right) \right\} \right].\end{aligned} \tag{5.23}$$

The Dirichlet boundary condition is not suited even for the $c = 1$ case. One reason is that if we regularize the quantity $\prod_{n\neq 0} n^2$, it becomes zero and the transition amplitude given by equation (5.23) becomes meaningless. The other reason is as follows. In the Dirichlet case, we are left with the non-zero modes of the conformal matter field and those of the Liouville field on the boundary. In addition, the factor $\sinh^{-1}(2\pi n)$ is cancelled out by the Faddeev-Popov determinant, and therefore the non-zero modes on the boundary cannot satisfy the Euclidean Schrödinger equations (5.16), (5.17) by using the parts which do not cancel out. In this case, the transition amplitude (5.23) cannot obey the Wheeler-DeWitt equation. These are the reasons why we have chosen the Neumann boundary condition.

Our result depends crucially on the fact that we have considered the case with the $c = 1$ conformal matter field. The situation for $c \neq 1$ is beyond the scope of this discussion, because it is impossible to cancel out the vibrations of the non-zero modes.

VI. CONCLUSIONS AND REMARKS

In this thesis, we have examined the validity of the minisuperspace in the context of quantum cosmology in 4 and 2 dimensions.

In 4 dimensions, we have investigated the stability of the Friedmann minisuperspace model by numerically solving the Wheeler-DeWitt equation [11]. In our analysis, a fixed mode number n is picked up and we considered only the scalar harmonics part. We can conclude that the minisuperspace is stable when the initial scale factor a_0 of the universe is lager than several times the Planck length. The minisuperspace becomes unstable when a_0 is about the Planck length. This result indicates that the minisuperspace approximation is meaningful in the region where the WKB approximation works within the limitation of our analysis.

In 2 dimensions, we have pointed out that the canonical structure is absent [12] [16]. In order to eliminate the problematic term, we have taken a conformal gauge by concentrating on the case of annulus topology. In this gauge fixing, the modular parameter plays a role of the lapse function used in the minisuperspace $\dot{N} = 0$ gauge. We have obtained the transition amplitude of the string universe which obeys the minisuperspace Wheeler-DeWitt equation. We have restricted ourselves to the case of the $c = 1$ conformal matter field and have imposed the Neumann boundary condition [12]. There may be other methods to avoid this problem, and it is worth investigating different gauge fixing (for example Ref. [35] [36] [37]).

Especially in Ref. [36] or in Ref. [37], they obtain the reduced quantum mechanical action for the length of a loop universe in the pure gravity. In Ref. [36], the proper-time gauge

$$g_{ab}(t,x) = \begin{pmatrix} 1 & 0 \\ 0 & \gamma(t,x) \end{pmatrix} \tag{6.1}$$

is taken and the reduced action is derived by solving the momentum constrain $T_{01} = 0$. In Ref. [36], the temporal gauge

$$g_{ab}(t,x) = \begin{pmatrix} 1 + l(t)^2 k(t,x)^2 & l(t)^2 k(t,x) \\ l(t)^2 k(t,x) & l(t)^2 \end{pmatrix}. \tag{6.2}$$

is taken and the reduced action is derived as the difference of the Faddeev-Popov determinant in the similar way to obtain the Liouville action. In these gauge fixings, the transition amplitude depends on the length of the boundary, and therefore it can be said that the minisuperspace is exact in the pure gravity. Their results is natural, because the above situation is expected only in the case of the pure gravity. In general, when we take into account of the conformal matter field coupled to the two-dimensional gravity, it is possible to consider another arguments of the transition amplitude except the length of the boundary. Our result may depend on the speciality of the case of $c = 1$. We should remark that the Wheeler-DeWitt equation derived in this thesis may be related to the moduli invariance.

ACKNOWLEDGMENTS

I would like to express my gratitude to Prof. H. Itoyama, Prof. T. Kubota, Prof. H. Kunitomo and Dr. T. Isse for their helpful discussions. Thanks are also due to Prof. K. Kikkawa, Prof. H. Ohtsubo, Prof. E. Takasugi Prof. H. Itoyama, Prof. T. Kubota and Prof. H. Kunitomo for a careful reading of the manuscript and the other members of High Energy Physics group in Osaka University for their warm encouragements.

APPENDIX A: EUCLIDEAN PATH INTEGRAL IN 4 DIMENSIONS

In the Euclidean signature, the ADM $(1+3)$ decomposition form of 4-metric is

$$\begin{aligned} ds^2 &= N_E^2 d\tau^2 + h_{ij}\left(N_E^i d\tau + dx^i\right)\left(N_E^j d\tau + dx^j\right) \\ &= (N_E^2 + N_{Ei}N_E^i)d\tau^2 + 2N_{Ei}dx^i d\tau + h_{ij}dx^i dx^j. \end{aligned} \tag{A1}$$

Here we use the relations

$$\begin{aligned} &\tau = it, \quad N_E = N, \quad iN_E^i = N^i, \\ &iK_E^{ij} = K^{ij}, \quad iI = S. \end{aligned} \tag{A2}$$

From these relations, we obtain the Euclidean action;

$$I = I_g + I_m, \tag{A3}$$

where

$$\begin{aligned} I_g &= -\frac{m_p^2}{16\pi}\left\{\int_M d^4x\sqrt{g}\,(R-2\Lambda) + 2\int_{\partial M} d^3x\sqrt{h}K_E\right\} \\ &= \frac{m_p^2}{16\pi}\int_M d\tau d^3x\sqrt{h}N\left\{K_{Eij}K_E^{ij} - K_E^2 - {}^{(3)}R + 2\Lambda\right\}, \\ I_m &= \frac{1}{2}\int d^4x\left\{g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2\right\}. \end{aligned} \tag{A4}$$

By using this action, we define the Euclidean path integral

$$\begin{aligned} \Psi[h''_{ij}, \Phi''; h'_{ij}, \Phi'] &= \int \frac{[\mathcal{D}g_{\mu\nu}]}{vol(\text{Gauge})}[\mathcal{D}\Phi]e^{-I[g_{\mu\nu},\Phi]} \\ &\equiv \int [\mathcal{D}N^\mu][\mathcal{D}h_{ij}][\mathcal{D}\Phi]\delta[\dot{N}^\mu - \chi^\mu]\,\triangle_\chi\, e^{-I[g_{\mu\nu},\Phi]}. \end{aligned} \tag{A5}$$

APPENDIX B: THE DEFINITION AND THE NORMALIZATION OF HARMONICS

The scalar spherical harmonics $Q^n_{lm}(\chi, \theta, \varphi)$ are defined as

$$Q^n_{lm}(\chi,\theta,\varphi) = \Pi^n_l(\chi) Y_{lm}(\theta,\varphi), \tag{B1}$$

where $\Pi^n_l(\chi)$ are the Fock harmonics [38] [39] and $Y_{lm}(\theta,\varphi)$ are the spherical harmonics on S^2. The scalar harmonics Q^n_{lm} are the eigenfunctions of the Laplacian operator on S^3,

$$Q^{(n)\,|k}_{\;\;|k} = -(n^2-1)Q^{(n)}, \quad n = 1, 2, \cdots. \tag{B2}$$

The spherical harmonics Q^n_{lm} constitute a complete orthogonal set for the expansion of any scalar function on S^3.

The transverse vector harmonics $S^{(n)}_i(\chi,\theta,\varphi)$ are the eigenfunctions of the Laplacian operator on S^3,

$$\begin{aligned} S^{(n)\,|k}_{i\;\;|k} &= -(n^2-2)S^{(n)}_i, \quad n = 2, 3, \cdots, \\ S^{(n)\,|i}_i &= 0. \end{aligned} \tag{B3}$$

The exact expressions of $S^{(n)}_i$ are given in Ref. [39] and $S^{(n)}_i$ are classified by as odd (o) or even (e) parts using a parity transformation. The third vector harmonics P_i are defined by the scalar harmonics Q^n_{lm},

$$P_i = \frac{1}{(n^2-1)} Q_{|i}, \quad n = 2, 3, \cdots \tag{B4}$$

and satisfy

$$\begin{aligned} P^{(n)\,|k}_{i\;\;|k} &= -(n^2-3)P^{(n)}_i, \\ P^{(n)\,|i}_i &= -Q. \end{aligned} \tag{B5}$$

The three types of vector harmonics $S^{(o)}_i$, $S^{(e)}_i$ and P_i constitute a complete orthogonal set for any vector function on S^3.

The transverse traceless tensor harmonics $G^{(n)}_{ij}$ are the eigenfunctions of the Laplacian operators on S^3 which are transverse and traceless,

$$
\begin{aligned}
G^{(n)}_{ij}{}^{|k}{}_{|k} &= -(n^2-3)G^{(n)}_{ij}, \quad n = 3,4,\cdots, \\
G^{(n)}_{ij}{}^{|i} &= 0, \\
G^{(n)i}_{i} &= 0.
\end{aligned} \tag{B6}
$$

They are also classified as odd $G^{(o)}_{ij}$ or even $G^{(e)}_{ij}$ parts. Explicit expressions for $G^{(o)}_{ij}$ and $G^{(e)}_{ij}$ are given in Ref. [39]. The traceless tensor harmonics $S^{(o)}_{ij}$ and $S^{(e)}_{ij}$ are defined, both for odd and even, by

$$
S_{ij} = S_{i|j} + S_{j|i} \tag{B7}
$$

and they satisfy

$$
\begin{aligned}
S_{ij}{}^{|j} &= -(n^2-4)S_i, \\
S_{ij}{}^{|ij} &= 0, \\
S_{ij}{}^{|k}{}_{|k} &= -(n^2-6)S_{ij}.
\end{aligned} \tag{B8}
$$

The two tensors Q_{ij} and P_{ij} are defined by the scalar harmonics,

$$
\begin{aligned}
Q_{ij} &= \frac{1}{3}\Omega_{ij}Q, \quad n = 1,2,\cdots, \\
P_{ij} &= \frac{1}{(n^2-1)}Q_{|ij} + \frac{1}{3}\Omega_{ij}Q, \quad n = 2,3,\cdots
\end{aligned} \tag{B9}
$$

and satisfy the following equations,

$$
\begin{aligned}
P_{ij}{}^{|j} &= -\frac{2}{3}(n^2-4)P_i, \\
P_{ij}{}^{|k}{}_{|k} &= -(n^2-7)P_{ij}, \\
P_{ij}{}^{|ij} &= -\frac{2}{3}(n^2-4)Q.
\end{aligned} \tag{B10}
$$

The six tensor harmonics $G^{(o)}_{ij}$, $G^{(e)}_{ij}$, $S^{(o)}_{ij}$, $S^{(o)}_{ij}$, Q_{ij} and P_{ij} constitute a complete orthogonal set for any symmetric second rank tensor function on S^3.

The normalization and the orthogonality relations are described by the following equations,

$$
\begin{aligned}
\int d\mu Q^n_{lm} Q^{n'}_{l'm'} &= \delta^{nn'}\delta_{mm'}\delta_{ll'}, \\
\int d\mu (P_i)^n_{lm} (P^i)^{n'}_{l'm'} &= \frac{1}{n^2-1}\delta^{nn'}\delta_{mm'}\delta_{ll'}, \\
\int d\mu (P_{ij})^n_{lm} (P^{ij})^{n'}_{l'm'} &= \frac{2(n^2-4)}{3(n^2-1)}\delta^{nn'}\delta_{mm'}\delta_{ll'}, \\
\int d\mu (S_{ij})^n_{lm} (S^{ij})^{n'}_{l'm'} &= 2(n^2-4)\delta^{nn'}\delta_{mm'}\delta_{ll'}, \\
\int d\mu (G_{ij})^n_{lm} (G^{ij})^{n'}_{l'm'} &= \delta^{nn'}\delta_{mm'}\delta_{ll'}, \qquad \text{(B11)}
\end{aligned}
$$

where $d\mu$ is a integration measure on S^3,

$$
\begin{aligned}
d\mu &= dx^3(\det \Omega_{ij})^{1/2} \\
&= \sin\theta d\chi d\theta d\varphi. \qquad \text{(B12)}
\end{aligned}
$$

The reader may find further details on the harmonics in Ref. [38] and Ref. [39].

APPENDIX C: THE CONJUGATE MOMENTA IN THE PERTURBED FRIEDMANN MINISUPERSPACE MODEL

Here we give the full expression for the conjugate momenta to α, ϕ, a_n, b_n, c_n, d_n and f_n.

$$
\begin{aligned}
\pi_\alpha = \frac{e^{3\alpha}}{N_0}\Bigg[&-\dot\alpha + \sum_n \left\{ -a_n\dot a_n + \frac{4(n^2-4)}{(n^2-1)} b_n\dot b_n + 4(n^2-4)c_n\dot c_n + 4d_n\dot d_n \right\} \\
&+\dot\alpha\sum_n\left\{ -\frac{3}{2}a_n^2 + \frac{6(n^2-4)}{(n^2-1)}b_n^2 + 6(n^2-4)c_n^2 + 6d_n^2 \right\} \\
&+\sum_n g_n\left\{ \dot a_n + \dot\alpha(3a_n - g_n) + \frac{1}{3}e^{-\alpha}k_n \right\}\Bigg], \\
\pi_\phi = \frac{e^{3\alpha}}{N_0}\Bigg[&\dot\phi + \sum_n\left\{ 3a_n\dot f_n + \dot\phi\left(\frac{3}{2}a_n^2 - \frac{4(n^2-4)}{(n^2-1)}b_n^2 - 4(n^2-4)c_n^2 - 4d_n^2 \right) \right\} \\
&+\sum_n\left\{ \dot\phi g_n^2 - g_n\left(\dot f_n + 3a_n\dot\phi\right) - e^{-\alpha}k_n f_n \right\}\Bigg] \qquad \text{(C1)}
\end{aligned}
$$

and

$$\begin{aligned}
\pi_{a_n} &= -\frac{e^{3\alpha}}{N_0}\left\{\dot{a}_n + \dot{\alpha}\left(a_n - g_n\right) + \frac{1}{3}e^{-\alpha}k_n\right\}, \\
\pi_{b_n} &= \frac{e^{3\alpha}}{N_0}\frac{\left(n^2-4\right)}{\left(n^2-1\right)}\left\{\dot{b}_n + 4\dot{\alpha}b_n - \frac{1}{3}e^{-\alpha}k_n\right\}, \\
\pi_{c_n} &= \frac{e^{3\alpha}}{N_0}\left(n^2-4\right)\left\{\dot{c}_n + 4\dot{\alpha}c_n - e^{-\alpha}j_n\right\}, \\
\pi_{d_n} &= \frac{e^{3\alpha}}{N_0}\left\{\dot{d}_n + 4\dot{\alpha}d_n\right\}, \\
\pi_{f_n} &= \frac{e^{3\alpha}}{N_0}\left\{\dot{f}_n + \dot{\phi}\left(3a_n - g_n\right)\right\}.
\end{aligned} \tag{C2}$$

REFERENCES

[1] A.H. Guth, Phys. Rev. **D28** (1981) 347.

[2] G.F. Smoot et al, Astrophys. J. Lett. **396** (1992) L1.

[3] B.S. DeWitt, Phys. Rev. **160** (1967) 1113.

[4] C.W. Misner, Phys. Rev. **186** (1969) 1319;
Phys. Rev. **186** (1969) 1328;
Phys. Rev. Lett. **22** (1969) 1071;
in Relativity, edited by M. Carmeli, S. Fickler and L. Witten (Plenum, New York, 1970). Classical and quantum dynamics of a closed universe;
in Magic Without Magic: John Archibald Wheeler, a Collection of Essays in Honor of his 60th Birthday, edited by J. Klauder (Freeman, San Francisco, 1972). Minisuperspace;
Phys. Rev. **D8** (1973) 3271.

[5] J.A. Wheeler, in Relativity, Groups and Topology, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1963);
in Batelles Rencontres, edited by C. DeWitt and J.A. Wheeler (Benjamin, New York, 1968). Superspace and the nature of quantum geometrodynamics.

[6] J.B. Hartle and S.W. Hawking, Phys. Rev. **D28** (1983) 2960.

[7] S.W. Hawking, in Astrophysical Cosmology, edited by H.A. Brück, G.V. Coyne and M.S. Longair (Pontifica Academia Scientarium, Vatican City, 1982). The boundary conditions of the universe.

[8] S.W. Hawking, Nucl. Phys. **B239** (1984) 257.

[9] A. Vilenkin, Phys. Lett. **B177** (1982) 25;

Phys. Rev. **D27** (1983) 2848;

Phys. Rev. **D30** (1984) 509;

Phys. Rev. **D32** (1985) 2511;

Nucl. Phys. **B252** (1985) 141;

Phys. Rev. **D33** (1986) 3560;

Phys. Rev. **D37** (1988) 888.

[10] A.D. Linde, Zh. Eksp. Teor. Fiz. **87** (1984) 369;

Nuovo Cimento **39** (1984) 401;

Rep. Prog. Phys. **47** (1984) 925.

[11] A. Ishikawa and T. Isse, Mod. Phys. Lett. **A8** (1993) 3413.

[12] A. Ishikawa, preprint OU-HET 184.

[13] J.J. Halliwell, lecture delivered at the Jerusalem Winter School on Quantum Cosmology and Baby Universes (1990).

[14] J.J. Halliwell and S.W. Hawking, Phys. Rev. **D31** (1985) 1777.

[15] E. D'Hoker and D.H. Phong, Rev. Mod. Phys. **60** (1988) 917;

T. Kubota, lecture delivered at Osaka University in 1986-1987, unpublished;

H. Kawai, lecture delivered at the 37-th Summer School in Akakura Myoukou 1991 July;

H. Itoyama, lecture delivered at Osaka University in 1993-1994, unpublished.

[16] A. Ishikawa, H. Itoyama and T. Kubota, unpublished.

[17] J.J. Halliwell and J. Louko, Phys. Rev. **D39** (1989) 2206.

[18] J.J. Halliwell and J. Louko, Phys. Rev. **D42** (1990) 3997.

[19] L.J. Garay, J.J. Halliwell and G.A.M. Marugan, Phys. Rev. **D43** (1991) 2572.

[20] J.J. Halliwell, Phys. Rev. **D38** (1988) 2468.

[21] E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. **B55** (1975) 2;
I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **B69** (1977) 309;
M. Henneaux, Phys. Rep. **126** (1985) 1.

[22] M. Ryan, Lecture Notes in Physics No.13, Hamiltonian Cosmology (Springer, New York, 1972).

[23] S. Wada, Phys. Rev. **D34** (1986) 2272.

[24] S. Sinha and B.L. Hu, Phys. Rev. **D44** (1991) 1028.

[25] K.V. Kuchar and M.P. Ryan, Jr., Phys. Rev **D40** (1990) 3982.

[26] G. Moore, N. Seiberg and M. Staudacher, Nucl. Phys. **B362** (1991) 665.

[27] N. Seiberg, Prog. Theor. Phys. Suppl. **102** (1990) 319.

[28] C. Teitelboim, Phys. Lett. **B126** (1983) 41.

[29] T. Yoneya, Prog. Theor. Phys. Suppl. **85** (1985) 256.

[30] E. Martinec, Phys. Rev. **D30** (1984) 1198.

[31] M. Bershadsky and I.R. Klebanov, Phys. Rev. Lett. **65** (1990) 3088.

[32] I.R. Klebanov, lectures given at Spring School on String Theory and Quantum Gravity, Trieste, Italy (1991).

[33] F. David, Mod. Phys. Lett. **A3** (1988) 1651.

[34] J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 509.

[35] D.Z. Freedman, J.I. Latorre and K. Pilch, Nucl. Phys. **B306** (1988) 77.

[36] R. Nakayama, EPHOU-93004, hep-th/9312158 (December 1993).

[37] M. Fukuma, N. Ishibashi, H. Kawai and M. Ninomiya, YITP/K-1045, KEK-TH-381, KEK-Preprint 93-172, YITP/U-93-37, hep-th/9312175 (December 1993).

[38] E.M. Lifschiz and I.M. Khalatnikov, Adv. Phys. **12** (1963) 185.

[39] U.H. Gerlach and U.K. Sengupta, Phys. Rev. **D18** (1978) 1773.

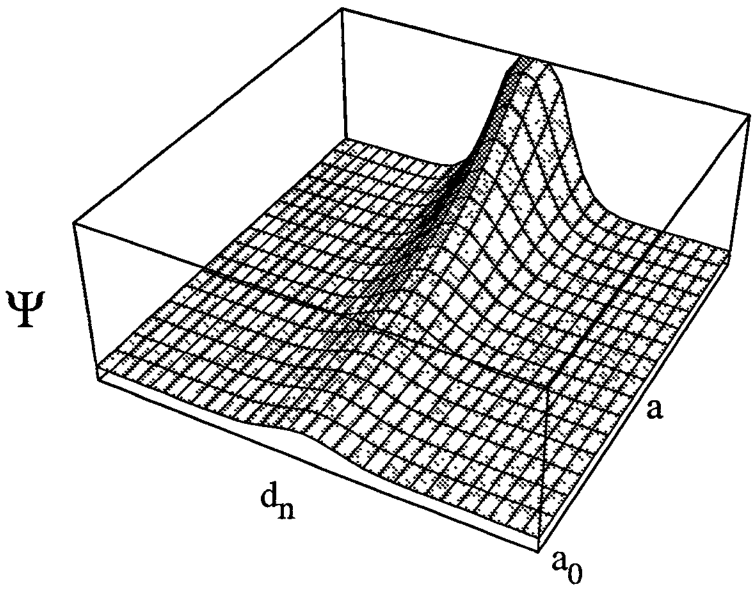

Fig.(1); The wave function Ψ of the universe in a case n=2, a_0=20, mass=0. The figure shows the d_n and a (scale factor) dependence of Ψ.

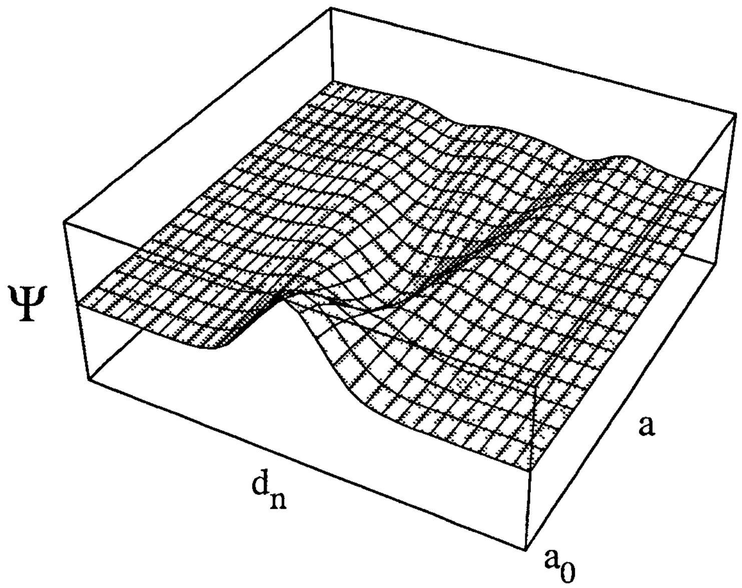

Fig.(2); The wave function Ψ of the universe in a case n=2, a_0=2, mass=0.

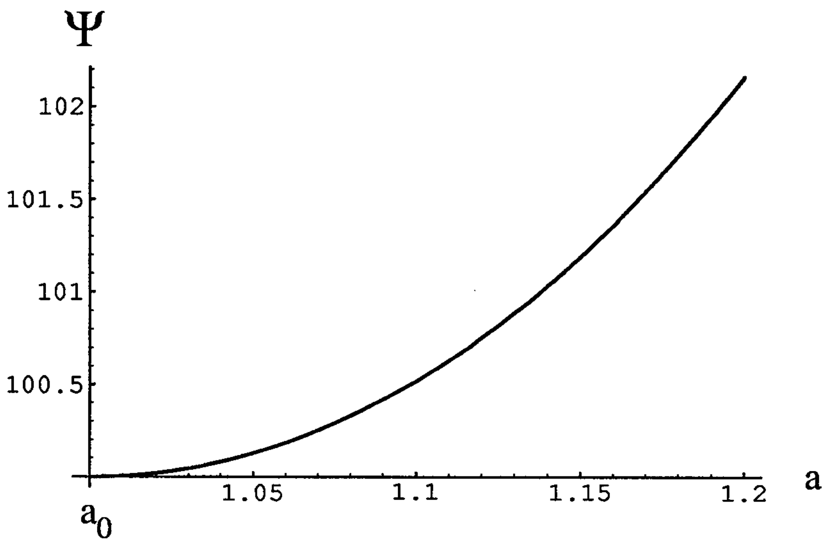

Fig.(3); The exponential growth of the wave function Ψ of the universe of the minisuperspace model without matter fields (where a is a scale factor of the universe and $a_0=1$).

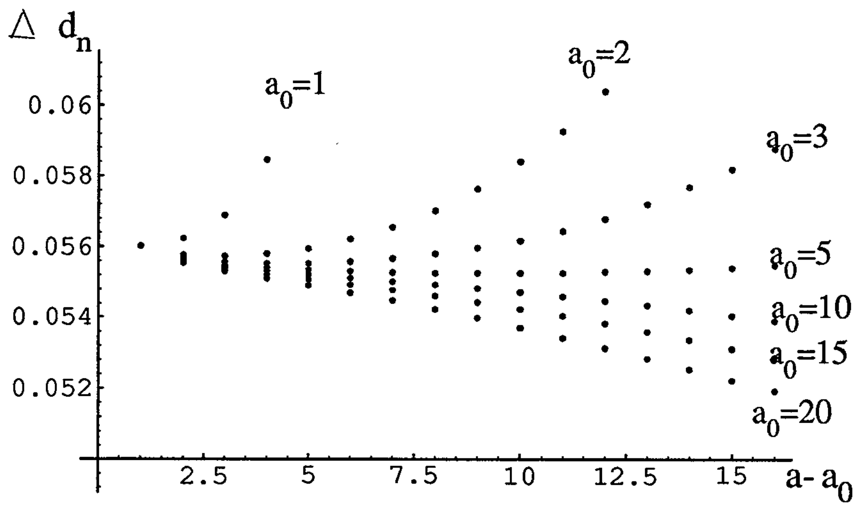

Fig.(4); The evolution of Δd_n with respect to the scale factor ($a-a_0$) in a case n=5, mass=0.

a_0 \ n	5	10	50	100	200
1	×	×	×	×	×
2	×	×	×	×	×
3	×	×	×	×	×
5	△	△	△	△	△
10	○	○	○	○	
15	○	○	○	○	
20	○	○	○	○	

Table.(1); The above table classifies whether minisuperspace is stabel or not (with respect to the intial value of the scale factor and the mode number n). The notation ○ , × or △ means that minisuperspace is stable, unstable or critical respectively.

a_0 \ m	1	5	10
1	×	×	×
2	×	×	×
3	×	×	×
5	△	△	△
10	○	○	
15	○	○	
20	○	○	

Table.(2); The above table classifies whether minisuperspace is stabel or not (with respect to the intial value of the scale factor and the mass parameter m). The notation ○ , × or △ means that minisuperspace is stable, unstable or critical respectively.