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A DECOMPOSITION OF THE SPACE $\mathcal{M}$ OF RIEMANNIAN METRICS ON A MANIFOLD

NORIHITO KOISO

(Received March 6, 1978)

0. Introduction

Let $M$ be a compact $C^\infty$-manifold. We denote by $\mathcal{M}$, $\mathcal{D}$ and $\mathcal{F}$ the space of all riemannian metrics on $M$, the diffeomorphism group of $M$, and the space of all positive functions on $M$, respectively. Then the group $\mathcal{D}$ and $\mathcal{F}$ acts on $\mathcal{M}$ by pull back and multiplication, respectively. D. Ebin and N. Koiso establish Slice theorem [4, Theorem 2.2] on the action of $\mathcal{D}$.

In this paper, we shall give a decomposition theorem on the action of $\mathcal{F}$ (Theorem 2.5). That is, there is a local diffeomorphism from $\mathcal{F} \times \bar{\mathcal{C}}$ into $\mathcal{M}$ where $\bar{\mathcal{C}}$ is a subspace of $\mathcal{M}$ of riemannian metrics with volume 1 and of constant scalar curvature $\tau_g$ such that $\tau_g=0$ or $\tau_g/(n-1)$ is not an eigenvalue of $\Delta_g$. Combining the above theorems, we get the following decomposition of a deformation (Corollary 2.9). Let $g \in \bar{\mathcal{C}}$ and $g(t)$ be a deformation of $g$. Then there are a curve $f(t)$ in $\mathcal{F}$, a curve $\gamma(t)$ in $\mathcal{D}$ and a curve $g(t)$ in $\mathcal{C}$ such that $\delta g'(0)=0$, which satisfy the equation $g(t)=f(t) \gamma(t) \ast g(t)$. (For the operator $\delta$, see 1.)

The author wishes to express his thanks to the referee.

1. Preliminaries

First, we introduce notation and definitions which will be used throughout this paper. Let $M$ be an $n$-dimensional, connected and compact $C^\infty$-manifold, and we always assume $n \geq 2$. For a vector bundle $T$ over $M$, we denote by $H'(T)$ the space of all $H'$-sections, where $H'$ means an object which has derivatives defined almost everywhere up to order $r$ and such that each partial derivative is square integrable. Then $H'(T)$ is isomorphic to a Hilbert space and the space $C^\infty(T)$ of all $C^\infty$-sections becomes an inverse limit of $\{H'(T)\}_{r=1,2,\ldots}$. Therefore such a space is said to be an ILH-space. If a topological space $\mathcal{X}$ is isomorphic to an ILH-space locally, $\mathcal{X}$ is said to be an ILH-manifold. For details, see [5].
Let $g$ be an $H^t$-metric on $M$. We consider the riemannian connection and use the following notations:

- $v_g$: the volume element with respect to $g$,
- $R$: the curvature tensor,
- $\rho$: the Ricci tensor,

(For the standard sphere with orthonormal basis, $R_{212}^2=R_{121}<0$ and $\rho_{11}<0$.)

- $\tau$: the scalar curvature,
- $(\ ,\ )$: the inner product in fibres of a tensor bundle defined by $g$,
- $\langle\ ,\ \rangle$: the global inner product for sections of a tensor bundle over $M$,

i.e., $\langle\ ,\ \rangle=\int_M (\ ,\ )v_g$,

- $S^2$: the symmetric covariant 2-tensor bundle over $M$,
- $H'(M)$: the Hilbert space of all $H'$-functions,
- $H'_g(M)$: the Hilbert space of all $H'$-functions $f$ such that $\int_M fv_g=0$,
- $H'_i(S^2)$: the Hilbert space of all symmetric bilinear $H'$-forms $h$ such that $\langle h, g \rangle=0$,

- $\nabla$: the covariant derivation,
- $\delta$: the formal adjoint of $\nabla$ with respect to $\langle\ ,\ \rangle$,
- $\delta^*$: the formal adjoint of $\delta|H'(S^2)$,
- $\Delta=\delta d$; the Laplacian operating on the space $H'(M)$,
- $\Delta' =\delta \nabla$: the rough Laplacian operating on the space $H'(T_e)$,
- $\text{Hess}=\nabla d$: the Hessian on the space $H'(M)$,

- $\mathcal{F}$: the ILH-manifold of all positive $C^\infty$-functions on $M$,
- $\mathcal{F}'$: the Hilbert manifold of all positive $H'$-functions on $M$,
- $\mathcal{M}$: the ILH-manifold of all $C^\infty$-metrics on $M$,
- $\mathcal{M}'$: the Hilbert manifold of all $H'$-metrics on $M$,
- $\mathcal{M}_1$: the ILH-manifold of all $C^\infty$-metrics with volume 1,
- $\mathcal{M}'_1$: the Hilbert manifold of all $H'$-metrics with volume 1.

When we consider the metric space $\mathcal{M}'$, the covariant derivation, the curvature tensor and the Ricci tensor with respect to an element $g$ of $\mathcal{M}'$ will be denoted by $\nabla_g$, $R_g$ or $\rho_g$. By a deformation of $g$ we mean a $C^\infty$-curve $g(t): I \to \mathcal{M}$ such that $g(0)=g$, where $I$ is an open interval. The differential $g'(0)$ is called an infinitesimal deformation, or simply an $i$-deformation. If there is a 1-parameter family $\gamma(t)$ of diffeomorphisms such that $g(t)=\gamma(t)^*g$ then the deformation $g(t)$ is said to be trivial. If there is a 1-form $\xi$ such that $h=\delta^*\xi$, then the $i$-deformation $h$ is said to be trivial. On the other hand, an $i$-deformation $h$ is said to be essential if $\delta h=0$.

Now, we give some fundamental propositions.

**Lemma 1.1** [6,11.3]. Let $E$ and $F$ be vector bundles over $M$ and $f: E \to F$ be a fiber preserving $C^\infty$-map. If $s>\frac{n}{2}$, then the map $\phi: H'(E) \to H'(F)$ which is defined by $\phi(\alpha)=f\circ\alpha$ is $C^\infty$. 
Proposition 1.2. If \( s > \frac{n}{2} \), then the map \( D: M^{s+1} \times H^{s+1}(T^p) \to H^s(T^p) \) which is defined by \( D(g, \xi) = \nabla_{g^*} \xi \) is \( C^\infty \).

Proof. Let \( g_0 \) be a fixed \( C^\infty \)-metric on \( M \). We define the tensor field \( T(g) \) by \( T(g)(X, Y) = (\nabla_{g^*})_g Y - (\nabla_{g^*})_0 Y \) for an \( H^s \)-metric \( g \) on \( M \). Then we get
\[
(T(g))_{ij} = \frac{1}{2} g^{kl} \{(\nabla_{g^*})_g g_{ij} - (\nabla_{g^*})_0 g_{ij}\},
\]
and
\[
(D(g, \xi))_{ij} = (D(g_0, \xi))_{ij} - (D(g_0, \xi))_{ij}.
\]
By the definition of the \( H^s \)-topology, we know that the map \( g \to (\nabla_{g^*})_g \) is a \( C^\infty \)-map from \( M^{s+1} \) to \( H^s(T^p) \). Hence Lemma 1.1 implies that the map \( g \to T(g) \) is a \( C^\infty \)-map from \( M^{s+1} \) to \( H^s(T^p) \). Applying Lemma 1.1 to the above formula, we see that the map \( (T(g), \xi) \to D(g, \xi) \) is a \( C^\infty \)-map from \( H^s(T^p) \) to \( H^s(T^p) \). But the map \( \xi \to D(g_0, \xi) \) is a continuous linear map from \( H^s(T^p) \) to \( H^s(T^p) \), hence the map \( (T(g), \xi) \to D(g, \xi) \) is \( C^\infty \).

Corollary 1.3. If \( s > \frac{n}{2} \), then the map \( (g, f) \to \nabla_g f \) is a \( C^\infty \)-map from \( M^{s+1} \times H^{s+2}(M) \) to \( H^s(M) \).

Proof. We apply Proposition 1.2 to the formula \( \Delta f = -g^{ij} \nabla_i d_j f \).

Corollary 1.4. If \( s > \frac{n}{2} \), then the maps \( g \to R, \rho, \tau \) are \( C^\infty \)-maps from \( M^{s+2} \) to \( H^s(T^p), H^s(S^2) \) and \( H^s(M) \), respectively.

Proof. The smoothness of the map \( g \to R \) completes the proof. By easy computation, we get the next formula:
\[
R(g)_{ijk}^l - R(g_0)_{ijk}^l = (\nabla_{g^*})_g (T(g))_{ijk}^l - (\nabla_{g_0^*})_0 (T(g))_{ijk}^l + (T(g))_{ijm}^l (T(g))_{km}^j - (T(g))_{ijm}^l (T(g))_{km}^j.
\]
Thus, applying Proposition 1.2, we see that the map \( g \to R \) is \( C^\infty \).

Lemma 1.5 [9,(19.5); 1,(2.11) (2.12)]. Let \( g(t) \) be a deformation of \( g \). If we set \( h = g'(0) \), then we have the following formulae:
\[
d \frac{d}{dt} |_0 g'(t) = \Delta tr h + \delta \delta h - (h, \rho), \tag{1.5.1}
\]
\[
d \frac{d}{dt} |_0 \rho g'(t) = \frac{1}{2} \{\Delta h + 2Qh + 2Lh - 2\delta^* \delta h - Hess tr h, \} \tag{1.5.2}
\]
where \( 2(Qh)_{ij} = \rho^i h_{kj} + \rho^j h_{ik} \) and \( \rho h = R_{ikj} h^{kl} \).
2. A decomposition of the space $\mathcal{M}$

We denote by $\mathcal{C}'$ the space of all $H'$-metrics with constant scalar curvature and with volume 1. Fix a $\mathcal{C}^\infty$-metric $g_0 \in \mathcal{M}_1$. For an integer $r > \frac{n}{2} + 4$ and $g \in \mathcal{M}_1$, we define a $\mathcal{C}^\infty$-map

$$\sigma^*_g : H'_{g_0}(M) \rightarrow H'_{g_0}^{-r}(M)$$

by $\sigma^*_g(f) = (n - 1)(\Delta_g^2)f - \tau_g \Delta_g f - \{(n - 1)(\Delta_g^2)f - \tau_0 \Delta_g f\} \psi_{g_0}$.

In fact the map $\mathcal{g}, f) \rightarrow \sigma^*_g(f)$ is a $\mathcal{C}^\infty$-map from $\mathcal{M}_1 \times H'_{g_0}(M)$ to $H'_{g_0}^{-r}(M)$ owing to Corollary 1.3 and Corollary 1.4. First we show some lemmas.

**Lemma 2.1.** If we denote by $K'$ the subset of $\mathcal{M}_1$ of all metrics $g \in \mathcal{M}_1$ such that $\sigma^*_g$ is an isomorphism, then $K'$ is open in $\mathcal{M}_1$.

**Proof.** The map $g \rightarrow \sigma^*_g$ is a $\mathcal{C}^\infty$-map from $\mathcal{M}_1$ to the space $L(H'_{g_0}(M), H'_{g_0}^{-r}(M))$ of all continuous linear maps from $H'_{g_0}(M)$ to $H'_{g_0}^{-r}(M)$. On the other hand the set of all isomorphisms is open in $L(H'_{g_0}(M), H'_{g_0}^{-r}(M))$, hence $K'$ is open in $\mathcal{M}_1$.

**Lemma 2.2.** Let $\mathcal{C}$ be the subset of $\mathcal{M}$ of all metrics $g$ with constant scalar curvature $\tau_g$ such that $\tau_g = 0$ or $\tau_g(n - 1)$ is not an eigenvalue of $\Delta_g$. Then $\mathcal{C}' \cap K' \cap \mathcal{M} = \mathcal{C}$.

**Proof.** Let $g \in \mathcal{C}$. Then $g \in \mathcal{C}' \cap \mathcal{M}$, and so it is sufficient to prove that $g \in K'$. If $f \in \ker \sigma^*_g$ then $(n - 1)(\Delta_g^2)f - \tau_g \Delta_g f$ is a constant. By integration we see

$$(n - 1)(\Delta_g^2)f - \tau_g \Delta_g f = 0.$$ 

But here $\tau_g = 0$ or $\tau_g$ is not an eigenvalue of $\Delta_g$. Hence $\Delta_g f$ is a constant, and so the assumption that $f \in H'_{g_0}(M)$ implies $f = 0$. Thus we see $\sigma^*_g$ is injective. On the other hand $\text{Im} \{(n - 1)(\Delta_g^2)f - \tau_g \Delta_g f\} = H'_{g_0}^{-r}(M)$ implies $\sigma^*_g$ is surjective. Therefore $\mathcal{C} \subset \mathcal{C}' \cap K' \cap \mathcal{M}$, and by the definition of $\mathcal{C}$ and $K'$ we see $\mathcal{C} \subset \mathcal{C}' \cap K' \cap \mathcal{M}$.

**Lemma 2.3.** $\mathcal{C}' \cap K'$ is an submanifold of $\mathcal{M}_1$.

**Proof.** We define a $\mathcal{C}^\infty$-map $\widetilde{\Delta} : \mathcal{M}_1 \rightarrow H'_{g_0}^{-r}(M)$ by

$$\widetilde{\Delta}(g) = \Delta_g \tau_g - \int \Delta_g \tau_g \psi_{g_0}.$$ 

Then $\mathcal{C}' = (\widetilde{\Delta}^{-1})(0)$. By differentiation we get

(1) A.E. Fischer and J.E. Marsden [8, Theorem 3] show that the space $\mathcal{R} \cdot \mathcal{C}$ becomes a submanifold of $\mathcal{M}$. 

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\[
T_g(\Delta \tau)(h) = \Delta'_{(g, h)} \tau_g + \Delta \tau'_{(g, h)} - \int \{(\Delta'_{(g, h)} + \Delta \tau'_{(g, h)}) v_{\delta_0} \cdot
\]

Let \( g \in C' \). Then we get

\[
\Delta'_{(g, h)} \tau_g = \frac{d}{dt} g \Delta + \int \tau_g = 0.
\]

If \( h \) is conformal, i.e., there is \( f \in H'_g(M) \) such that \( h = f g \), by substituting to the formula (1.5.1) we get

\[
\tau'_{(g, f g)} = (n-1) \Delta g f - \tau g f.
\]

Thus we get \( T_g(\Delta \tau) (f g) = \sigma_g'(f) \), and \( T_g(\Delta \tau) \) is surjective. This implies, by implicit function theorem, \( C' \cap K' \) is a submanifold of \( \mathcal{M}_g \), and so of \( \mathcal{M} \).

**Lemma 2.4.** Define a \( C^\infty \)-map \( \chi' : T' \times (C \cap K') \to \mathcal{M} \) by \( \chi'(f, g) = f g \). If \( g \in C \) then \( T_{(f, g)} \chi' \) is an isomorphism.

**Proof.** Injectivity. We see

\[(T_{(f, g)} \chi')(\phi, h) = fh + \phi g.\]

If \( fh + \phi g = 0 \), then \( \phi g \in \text{Ker} \ T_g(\Delta \tau) \), where \( \phi = -\phi f / f \). Hence

\[
\Delta_g \text{tr}_g(\phi g) + \delta_g \delta(h_g) - (\phi g, \rho_g)_g = 0,
\]

therefore \( (n-1) \Delta_g \phi - \tau_g \phi = 0 \).

But here \( g \in C \), which implies \( \phi = 0 \), and so \( h = 0 \), \( \phi = 0 \).

Surjectivity. The equation \( \text{Im} \ T_{(f, g)} \chi' = f T_g(C') + H'(M)g \) shows that \( \text{Im} \ T_{(f, g)} \chi' \) is closed in \( H'(S^2) \). Hence, if \( T_{(f, g)} \chi' \) is not surjective then there exists a non-zero element \( \bar{h} \) in \( H'(S^2) \) orthogonal to \( f T_g(C') \) and \( H'(M)g \). We set

\[
K_g(h) = \Delta_g(\Delta_g \text{tr}_g h + \delta_g \delta_g h - (h, \rho_g)_g).
\]

Then we get \( T_g(C') = \text{Ker} \ T_g(\Delta \tau) = \text{Ker} \ T_g(\Delta \tau) = \text{Ker} \ K_g \). On the other hand \( K_g \) has surjective symbol. Hence [2, Corollary 6.9] implies that \( H'(S^2) \) has the decomposition

\[
H'(S^2) = Rg \oplus T_g(C') \oplus \text{Im} K_g^*,
\]

where \( K_g^* \) is the formal adjoint of \( K_g \). \( f \bar{h} \) is orthogonal to \( T_g(C') \) and \( H'(M)g \), hence \( f \bar{h} \in \text{Im} K_g^* \). If we set \( f \bar{h} = K_g^*(\psi) \), then we see

\[
f \bar{h} = (\Delta_g)^2 \psi + \nabla_n \Delta_g \psi - \Delta_g \psi \rho_g.
\]

Since \( f \bar{h} \) is orthogonal to \( H'(M)g \), we see
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0 = \text{tr}_g(f \bar{h}) = (n-1)(\Delta g)^2 \phi - \tau_g \Delta g \phi.

By the assumption that \( g \in \mathcal{C} \), we see \( \Delta g \phi = 0 \) and so \( f \bar{h} = 0 \), which contradicts the assumption that \( \bar{h} \neq 0 \).

**Theorem 2.5.** (2) The space \( \mathcal{C} \) is an ILH-submanifold of \( \mathcal{M} \) and the map \( \chi : \mathcal{F} \times \mathcal{C} \rightarrow \mathcal{M} \) is a local ILH-diffeomorphism into \( \mathcal{M} \), where \( \chi \) is defined by \( \chi(f,g) = fg \).

(For the notation ILH, see [5, pp. 168–169].)

**Remark 2.6.** J.L. Kazdan and F.W. Warner [3, Theorem 1.1] show that \( \mathcal{C} \) is not empty.

**Remark 2.7.** When \( n = 2 \), this result is classical. That is, any metric \( g \) is conformal to some metric with constant scalar curvature.

**Proof.** We fix a sufficiently large integer \( r \). By Lemma 2.2, Lemma 2.4 and the inverse function theorem there is an open neighbourhood \( W' \) of \( \mathcal{C} \) in \( \mathcal{F} \times (C' \cap K') \) such that \( \chi' | W' \) is a local diffeomorphism. We denote by \( \mathcal{C}' \) the set of all metrics \( g \in C' \cap K' \) such that there is an \( H' \)-function \( f \) such that \( (f,g) \in W' \). For an integer \( s \geq r \) we set \( \mathcal{C}' = \mathcal{C}' \cap \bigcap_{s=r}^{s=1}(C' \cap K') \). We easily see that \( \mathcal{C}' \supset \mathcal{C}'+1 \) and, by Lemma 2.1, that \( \mathcal{C}' \) is open in \( C' \cap K' \). Moreover we see \( \lim_{s \to r} \mathcal{C}' = \mathcal{C} \) by Lemma 2.2, and thus we can define an ILH-structure on \( \mathcal{C} \) as \( \mathcal{C} = \lim \mathcal{C}' \).

Next we shall prove that the map \( \chi' | \mathcal{F} \times \mathcal{C}' : \mathcal{F} \times \mathcal{C}' \rightarrow \mathcal{M} \) is a local diffeomorphism. Lemma 1.1 implies the smoothness of this map. To prove the smoothness of the inverse map, we choose an open covering \( \{W'_a\} \) of \( W' \) such that \( \chi' | W'_a \) is a diffeomorphism. We apply the following lemma to \( (\chi' | W'_a)^{-1} \).

**Lemma 2.8** [4, Lemma 2.8]. Let \( E \) and \( F \) be vector bundles over \( M \) associated to the frame bundle of \( M \). Then there exists a canonical linear map \( \eta^* : H^0(E) \rightarrow H^0(E) \) for a diffeomorphism \( \eta \) of \( M \). Let \( A \) be an open set of \( H'(E) \) and \( \phi : A \rightarrow H'(F) \) be a \( C^\infty \)-map which commutes with any \( \eta^* \). If we set \( A' = A \cap H'(E) \) for \( s \geq r \), then \( \phi(A') \subset H'(F) \) and the map \( \phi | A' : A' \rightarrow H'(F) \) is \( C^\infty \).

If we set \( \text{Im}(\chi' | W'_a) = A \) and \( (\chi' | W'_a)^{-1} = \phi \), then \( \phi \) is a \( C^\infty \)-map from \( A \) into \( H'(M) \times H'(S^2) \) which commutes with the action of the diffeomorphism group \( \mathcal{D} \) of \( M \). Hence Lemma 2.8 implies that the map

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(2) J.P. Bourguignon [7, VIII. 8. Proposition] shows that \( \tau : \mathcal{M} \rightarrow \mathcal{F} \) is a submersion around a metric \( g \in \mathcal{M} \) such that \( \tau_g \) is not non-negative constant.
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$$(\chi' | W_s)'^{-1}| A^t : A^t \to H'(M) \times H'(\mathbb{S}^2)$$

is $C^\infty$. But here $\mathcal{F} \times C^a$ is a submanifold of $H'(M) \times H'(\mathbb{S}^2)$, hence the map $(\chi' | W')^{-1}| A^t : A^t \to \mathcal{F} \times C^a$ is $C^\infty$. Thus $\chi^t$ is a local diffeomorphism and $\chi=\lim \chi^t$ is an ILH-diffeomorphism, which implies that $\bar{C}$ is an ILH-submanifold of $\mathcal{M}$.

**Corollary 2.9.** Let $g=fg$, where $f \in \mathcal{F}$ and $g \in \bar{C}$. If $g(t)$ is a deformation of $g$ with sufficiently small domain of $t$, then there exist a 1-parameter family of positive functions $f(t)$ on $M$, a 1-parameter family of diffeomorphisms $\gamma(t)$ of $M$ and a deformation $g(t)$ in $\bar{C}$ such that $f(0)=f$, $\delta g'(0)=0$ and $g(t)=f(t)\gamma(t)*g(t)$.

**Proof.** By Theorem 2.5, $g(t)$ is decomposed into $f(t)\bar{g}(t)$, where $\bar{g}(t)$ is a deformation in $\bar{C}$. Applying Slice theorem [4, Theorem 2.2] to $g(t)$, we get $\bar{g}(t)=\gamma(t)*g(t)$, where $g(t)$ is a deformation such that $\delta g'(0)=0$. Also we easily see that $g(t) \in \bar{C}$ for each $t$.

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**References**


