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Osaka University
A DECOMPOSITION OF THE SPACE $\mathcal{M}$ OF RIEMANNIAN METRICS ON A MANIFOLD

NORIHITO KOISO

(Received March 6, 1978)

0. Introduction

Let $M$ be a compact $C^\infty$-manifold. We denote by $\mathcal{M}$, $\mathcal{D}$ and $\mathcal{F}$ the space of all riemannian metrics on $M$, the diffeomorphism group of $M$, and the space of all positive functions on $M$, respectively. Then the group $\mathcal{D}$ and $\mathcal{F}$ acts on $\mathcal{M}$ by pull back and multiplication, respectively. D. Ebin and N. Koiso establish Slice theorem [4, Theorem 2.2] on the action of $\mathcal{D}$.

In this paper, we shall give a decomposition theorem on the action of $\mathcal{F}$ (Theorem 2.5). That is, there is a local diffeomorphism from $\mathcal{F} \times \bar{\mathcal{C}}$ into $\mathcal{M}$ where $\bar{\mathcal{C}}$ is a subspace of $\mathcal{M}$ of riemannian metrics with volume 1 and of constant scalar curvature $\tau_g$ such that $\tau_g=0$ or $\tau_g/(n-1)$ is not an eigenvalue of $\Delta_g$. Combining the above theorems, we get the following decomposition of a deformation (Corollary 2.9). Let $g \in \bar{\mathcal{C}}$ and $g(t)$ be a deformation of $g$. Then there are a curve $f(t)$ in $\mathcal{F}$, a curve $\gamma(t)$ in $\mathcal{D}$ and a curve $g(t)$ in $\mathcal{F}$ such that $\delta g'(0)=0$, which satisfy the equation $g(t) = f(t)\gamma(t)^*g(t)$. (For the operator $\delta$, see 1.)

The author wishes to express his thanks to the referee.

1. Preliminaries

First, we introduce notation and definitions which will be used throughout this paper. Let $M$ be an $n$-dimensional, connected and compact $C^\infty$-manifold, and we always assume $n \geq 2$. For a vector bundle $T$ over $M$, we denote by $H'(T)$ the space of all $H'$-sections, where $H'$ means an object which has derivatives defined almost everywhere up to order $r$ and such that each partial derivative is square integrable. Then $H'(T)$ is isomorphic to a Hilbert space and the space $C^\infty(T)$ of all $C^\infty$-sections becomes an inverse limit of $\{H'(T)\}_{r=1,2,\ldots}$. Therefore such a space is said to be an ILH-space. If a topological space $\mathcal{X}$ is isomorphic to an ILH-space locally, $\mathcal{X}$ is said to be an ILH-manifold. For details, see [5].
Let \( g \) be an \( H^s \)-metric on \( M \). We consider the riemannian connection and use the following notations:

- \( v_g \); the volume element with respect to \( g \),
- \( R \); the curvature tensor,
- \( \rho \); the Ricci tensor,

(For the standard sphere with orthonormal basis, \( R_{1212} = -R_{1221} < 0 \) and \( \rho_{11} < 0 \).)
- \( \tau \); the scalar curvature,
- \( ( , ) \); the inner product in fibres of a tensor bundle defined by \( g \),
- \( \langle , \rangle \); the global inner product for sections of a tensor bundle over \( M \), i.e., \( \langle , \rangle = \int_M ( , ) v_g \),
- \( S^2 \); the symmetric covariant 2-tensor bundle over \( M \),
- \( H'(M) \); the Hilbert space of all \( H' \)-functions,
- \( H'_{\phi}(M) \); the Hilbert space of all \( H' \)-functions \( f \) such that \( \int_M f v_g = 0 \),
- \( H'_\phi(S^2) \); the Hilbert space of all symmetric bilinear \( H' \)-forms \( h \) such that \( \langle h, g \rangle = 0 \),
- \( \nabla \); the covariant derivation,
- \( \delta \); the formal adjoint of \( \nabla \) with respect to \( \langle , \rangle \),
- \( \delta^* \); the formal adjoint of \( \delta|H'(S^2) \),
- \( \Delta = \delta d \); the Laplacian operating on the space \( H'(M) \),
- \( \Delta = \delta \nabla \); the rough Laplacian operating on the space \( H'(T_p^S) \),
- \( \text{Hess} = \nabla d \); the Hessian on the space \( H'(M) \),
- \( \mathcal{F} \); the ILH-manifold of all positive \( C^\infty \)-functions on \( M \),
- \( \mathcal{F}' \); the Hilbert manifold of all positive \( H' \)-functions on \( M \),
- \( \mathcal{M} \); the ILH-manifold of all \( C^\infty \)-metrics on \( M \),
- \( \mathcal{M}' \); the Hilbert manifold of all \( H' \)-metrics on \( M \),
- \( \mathcal{M}'_1 \); the ILH-manifold of all \( C^\infty \)-metrics with volume 1,
- \( \mathcal{M}'_1 \); the Hilbert manifold of all \( H' \)-metrics with volume 1.

When we consider the metric space \( \mathcal{M}' \), the covariant derivation, the curvature tensor and the Ricci tensor with respect to an element \( g \) of \( \mathcal{M}' \) will be denoted by \( \nabla_g, R_g \) or \( \rho_g \). By a deformation of \( g \) we mean a \( C^\infty \)-curve \( g(t) : I \rightarrow \mathcal{M} \) such that \( g(0) = g \), where \( I \) is an open interval. The differential \( g'(0) \) is called an infinitesimal deformation, or simply an \( i \)-deformation. If there is a 1-parameter family \( \gamma(t) \) of diffeomorphisms such that \( g(t) = \gamma(t)^*g \) then the deformation \( g(t) \) is said to be trivial. If there is a 1-form \( \xi \) such that \( h = \delta^*\xi \), then the \( i \)-deformation \( h \) is said to be trivial. On the other hand, an \( i \)-deformation \( h \) is said to be essential if \( \delta h = 0 \).

Now, we give some fundamental propositions.

**Lemma 1.1** [6,11.3]. Let \( E \) and \( F \) be vector bundles over \( M \) and \( f : E \rightarrow F \) be a fiber preserving \( C^\infty \)-map. If \( s > \frac{n}{2} \), then the map \( \phi : H^s(E) \rightarrow H^s(F) \) which is defined by \( \phi(\alpha) = f \circ \alpha \) is \( C^\infty \).
Proposition 1.2. If $s > \frac{n}{2}$, then the map $D: \mathcal{M}^{s+1} \times H^{s+1}(T\xi) \to H^{k}(T\xi_{s+1})$ which is defined by $D(g, \xi) = \nabla_{g, \xi}$ is $C^{\infty}$.

Proof. Let $g_0$ be a fixed $C^\infty$-metric on $M$. We define the tensor field $T(g)$ by $T(g)(X, Y) = (\nabla_{g_0})_g X - (\nabla_{g_0})_g Y$ for an $H^s$-metric $g$ on $M$. Then we get

$$(T(g))_{ij} = \frac{1}{2} g^{il} \{(\nabla_{g_0})_g g_{lj} - (\nabla_{g_0})_g g_{li} \},$$

and

$$(D(g, \xi))_{ij} - (D(g_0, \xi))_{ij} = \sum_{k=1}^{\beta} (T(g))_{ijkl} (\xi_{ijkl} - (\xi_{ijkl}))_{g_{ij}} + \sum_{k=1}^{\beta} (T(g))_{ijkl} (\xi_{ijkl} - (\xi_{ijkl}))_{g_{ij}} - (D(g_0, \xi))_{ij} \xi_{ijkl} (D(g_0, \xi))_{ijkl} - (D(g_0, \xi))_{ij} (D(g_0, \xi))_{ijkl}.
$$

By the definition of the $H^s$-topology, we know that the map $g \to (\nabla_{g_0})_g$ is a $C^\infty$-map from $\mathcal{M}^{s+1}$ to $H^{k}(T\xi_0)$. Hence Lemma 1.1 implies that the map $g \to T(g)$ is a $C^\infty$-map from $\mathcal{M}^{s+1}$ to $H^{k}(T\xi_0)$. Applying Lemma 1.1 to the above formula, we see that the map $(T(g), \xi) \to g_0 \nabla_{g_0} \xi$ is a $C^\infty$-map from $H^{k}(T\xi_0)^{(s+1)}$ to $H^{k}(T\xi_{s+1})$. But the map $\xi \to D(g_0, \xi)$ is a continuous linear map from $H^{k}(T\xi_0)$ to $H^{k}(T\xi_{s+1})$, hence the map $(T(g), \xi) \to D(g, \xi)$ is $C^\infty$. Thus we see that the map $D$ is a composition of $C^\infty$-maps, and so is $C^\infty$.

Corollary 1.3. If $s > \frac{n}{2}$, then the map $(g, f) \to \nabla_{g, f}$ is a $C^\infty$-map from $\mathcal{M}^{s+1} \times H^{s+2}(M)$ to $H^{s}(M)$.

Proof. We apply Proposition 1.2 to the formula $\Delta_{g, f} = -g^{ij} \nabla_{g, f} d_{ij, f}$.

Corollary 1.4. If $s > \frac{n}{2}$, then the maps $g \to R$, $\rho$, $\tau$ are $C^\infty$-maps from $\mathcal{M}^{s+2}$ to $H^{k}(T\xi_{s+1})$, $H^{k}(S^2)$ and $H^{k}(M)$, respectively.

Proof. The smoothness of the map $g \to R$ completes the proof. By easy computation, we get the next formula :

$$R(g)_{ijkl} - R(g_0)_{ijkl} = (\nabla_{g_0})_g (T(g))_{ijkl} - (\nabla_{g_0})_g (T(g))_{ijkl} + (T(g))_{ikm} (T(g))_{m, jk} - (T(g))_{ikm} (T(g))_{ikm}.$$ 

Thus, applying Proposition 1.2, we see that the map $g \to R$ is $C^\infty$.

Lemma 1.5 [9,(19.5); 1,(2.11) (2.12)]. Let $g(t)$ be a deformation of $g$. If we set $h = g'(0)$, then we have the following formulae; 

$$\frac{d}{dt} | g(t) = \Delta h \pm 2k h \pm 2k h \pm 2k h \pm 2k h \pm 2k h \pm 2k h \pm 2k h \pm 2k h.$$ 

where $2(2k)_{ij} = \rho_i h_{ij} + \rho_j h_{ik}$ and $(Lh)_{ij} = R_{ikl} h^{kl}$.
2. A decomposition of the space $\mathcal{M}$

We denote by $C^r$ the space of all $H^r$-metrics with constant scalar curvature and with volume 1. Fix a $C^\infty$-metric $g_0 \in \mathcal{M}_1$. For an integer $r > \frac{n}{2}$ and $g \in \mathcal{M}_1$, we define a $C^\infty$-map

$$\sigma^*_g : H^r_\epsilon(M) \rightarrow H^{r-1}_\epsilon(M)$$

by $\sigma^*_g(f) = (n-1)(\Delta_g)^2 f - \tau_g \Delta_g f - \left\{(n-1)(\Delta_g)^2 f - \tau_g \Delta_g f\right\} \nu_\epsilon$.

In fact the map $(g, f) \mapsto \sigma^*_g(f)$ is a $C^\infty$-map from $\mathcal{M}_1 \times H^r_\epsilon(M)$ to $H^{r-1}_\epsilon(M)$ owing to Corollary 1.3 and Corollary 1.4. First we show some lemmas.

**Lemma 2.1.** If we denote by $K'$ the subset of $\mathcal{M}_1$ of all metrics $g \in \mathcal{M}_1$ such that $\sigma^*_g$ is an isomorphism, then $K'$ is open in $\mathcal{M}_1$.

**Proof.** The map $g \rightarrow \sigma^*_g$ is a $C^\infty$-map from $\mathcal{M}_1$ to the space $L(H^r_\epsilon(M), H^{r-1}_\epsilon(M))$ of all continuous linear maps from $H^r_\epsilon(M)$ to $H^{r-1}_\epsilon(M)$. On the other hand the set of all isomorphisms is open in $L(H^r_\epsilon(M), H^{r-1}_\epsilon(M))$, hence $K'$ is open in $\mathcal{M}_1$.

**Lemma 2.2.** Let $\tilde{C}$ be the subset of $\mathcal{M}$ of all metrics $g$ with constant scalar curvature $\tau_g$ such that $\tau_g = 0$ or $\tau_g(n-1)$ is not an eigenvalue of $\Delta_g$. Then $C' \cap K' \cap \mathcal{M} = \tilde{C}$.

**Proof.** Let $g \in \tilde{C}$. Then $g \in C' \cap \mathcal{M}$, and so it is sufficient to prove that $g \in K'$. If $f \in \text{Ker} \sigma^*_g$ then $(n-1)(\Delta_g)^2 f - \tau_g \Delta_g f$ is a constant. By integration we see

$$(n-1)(\Delta_g)^2 f - \tau_g \Delta_g f = 0.$$ 

But here $\tau_g = 0$ or $\tau_g$ is not an eigenvalue of $\Delta_g$. Hence $\Delta_g f$ is a constant, and so the assumption that $f \in H^r_\epsilon(M)$ implies $f = 0$. Thus we see $\sigma^*_g$ is injective. On the other hand $\text{Im} \{(n-1)(\Delta_g)^2 - \tau_g \Delta_g\} = H^{r-1}_\epsilon(M)$ implies $\sigma^*_g$ is surjective. Therefore $\tilde{C} \subseteq C' \cap K' \cap \mathcal{M}$, and by the definition of $\tilde{C}$ and $K'$ we see $\tilde{C} = C' \cap K' \cap \mathcal{M}$.

**Lemma 2.3.** $C' \cap K'$ is an submanifold of $\mathcal{M}_1$.

**Proof.** We define a $C^\infty$-map $\tilde{\Delta} : \mathcal{M}_1 \rightarrow H^{r-1}_\epsilon(M)$ by

$$\tilde{\Delta}(g) = \Delta_g \tau_g - \int \Delta_g \tau_g \nu_\epsilon.$$ 

Then $C' = (\tilde{\Delta})^{-1}(0)$. By differentiation we get

(1) A.E. Fischer and J.E. Marsden [8, Theorem 3] show that the space $R \cdot \tilde{C}$ becomes a submanifold of $\mathcal{M}$. 

Let $g \in C'$. Then we get
\[
\Delta'_{(g, h)} \tau_g = \frac{d}{dt} \Delta_{g + th} \tau_g = 0.
\]
If $h$ is conformal, i.e., there is $f \in H'_g(M)$ such that $h = fg$, by substituting to the formula (1.5.1) we get
\[
\tau'_{(g, f)} = (n-1) \Delta g - \tau_f g.
\]
Thus we get $T'_{g}(\Delta \tau)(fg) = \sigma g^*(f)$, and $T'_{g}(\Delta \tau)$ is surjective. This implies, by implicit function theorem, $C' \cap K'$ is a submanifold of $M'$, and so of $M$.

**Lemma 2.4.** Define a $C^\infty$-map $\mathcal{X}' : T' \times (C' \cap K') \to M'$ by $\mathcal{X}'(f, g) = fg$. If $g \in C$ then $T'_{(f, g)} \mathcal{X}'$ is an isomorphism.

**Proof.** Injectivity. We see
\[
(T'_{(f, g)} \mathcal{X}')(\phi, h) = fh + \phi g.
\]
If $fh + \phi g = 0$, then $\phi g \in \text{Ker} \ T'_{g}(\Delta \tau)$, where $\phi = -\phi f$. Hence
\[
\Delta_g \text{tr}_g(\phi g) + \delta_g \delta_g(\phi g) - (\phi g, \rho_g)_g = 0,
\]
therefore $(n-1)\Delta g \phi - \tau_g \phi = 0$.

But here $g \in \bar{C}$, which implies $\phi = 0$, and so $h = 0$, $\phi = 0$.

Surjectivity. The equation $\text{Im} T'_{(f, g)} \mathcal{X}' = fT'_{g}(C') + H'(M)g$ shows that $\text{Im} T'_{(f, g)} \mathcal{X}'$ is closed in $H'(S^2)$. Hence, if $T'_{(f, g)} \mathcal{X}'$ is not surjective then there exists a non-zero element $h$ in $H'(S^2)$ orthogonal to $fT'_{g}(C')$ and $H'(M)g$. We set
\[
K_g(h) = \Delta_g(\text{tr}_g h + \delta_g \delta_g h - (h, \rho_g)_g).
\]
Then we get $T'_{g}(C') = \text{Ker} \ T'_{g}(\Delta \tau) = \text{Ker} \ T'_{g}(\Delta \tau) = \text{Ker} \ K_g$. On the other hand $K_g$ has surjective symbol. Hence [2, Corollary 6.9] implies that $H'(S^2)$ has the decomposition
\[
H'(S^2) = Rg \oplus T'_{g}(C') \oplus \text{Im} K_g^*,
\]
where $K_g^*$ is the formal adjoint of $K_g$. If $h$ is orthogonal to $T'_{g}(C')$ and $H'(M)g$, hence $h \in \text{Im} K_g^*$. If we set $f h = K_g^*(\psi)$, then we see
\[
f h = (\Delta g)^2 \psi + \nabla_g \nabla_g \Delta g \psi - \Delta_g \psi \rho_g.
\]
Since $f h$ is orthogonal to $H'(M)g$, we see

\[
T'_{g}(\Delta \tau)(h) = \Delta'_{(g, h)} \tau_g + \Delta g \tau'_g - \int ((\Delta'_{(g, h)} + \Delta g \tau'_g) v_g).
\]
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\[ 0 = \text{tr}_g(f\tilde{h}) = (n-1)(\Delta_g)^2\psi - \tau_g\Delta_g\psi. \]

By the assumption that \( g \in \mathcal{C} \), we see \( \Delta_g\psi = 0 \) and so \( f\tilde{h} = 0 \), which contradicts the assumption that \( \tilde{h} \neq 0 \).

**Theorem 2.5.** (2) The space \( \mathcal{C} \) is an ILH-submanifold of \( \mathcal{M} \) and the map \( \chi : \mathcal{F} \times \mathcal{C} \rightarrow \mathcal{M} \) is a local ILH-diffeomorphism into \( \mathcal{M} \), where \( \chi \) is defined by \( \chi(f, g) = fg \).

(For the notation ILH, see [5, pp. 168–169].)

**Remark 2.6.** J.L. Kazdan and F.W. Warner [3, Theorem 1.1] show that \( \mathcal{C} \) is not empty.

**Remark 2.7.** When \( n = 2 \), this result is classical. That is, any metric \( g \) is conformal to some metric with constant scalar curvature.

Proof. We fix a sufficiently large integer \( r \). By Lemma 2.2, Lemma 2.4 and the inverse function theorem there is an open neighbourhood \( W' \) of \( \mathcal{F} \times \mathcal{C} \) in \( \mathcal{F} \times (\mathcal{C} \cap \mathcal{K}') \) such that \( \chi' | W' \) is a local diffeomorphism. We denote by \( \bar{\mathcal{C}}' \) the set of all metrics \( g \in \mathcal{C} \cap \mathcal{K}' \) such that there is an \( H' \)-function \( f \) such that \( (f, g) \in W' \). For an integer \( s \geq r \) we set \( \bar{\mathcal{C}}^s = \bar{\mathcal{C}}' \cap \bigcap_{s \geq r} (\mathcal{C}^s \cap \mathcal{K}^s) \). We easily see that \( \bar{\mathcal{C}}^s \supset \bar{\mathcal{C}}^{s+1} \) and, by Lemma 2.1, that \( \bar{\mathcal{C}}^s \) is open in \( \mathcal{C}^s \cap \mathcal{K}^s \). Moreover we see \( \bigcup_{s \geq r} \bar{\mathcal{C}}^s = \bar{\mathcal{C}} \) by Lemma 2.2, and thus we can define an ILH-structure on \( \bar{\mathcal{C}} \) as \( \bar{\mathcal{C}} := \lim \bar{\mathcal{C}}^s \).

Next we shall prove that the map \( \chi' | \mathcal{F} \times \bar{\mathcal{C}}^s : \mathcal{F} \times \bar{\mathcal{C}}^s \rightarrow \mathcal{M}^s \) is a local diffeomorphism. Lemma 1.1 implies the smoothness of this map. To prove the smoothness of the inverse map, we choose an open covering \( \{W^s_a\} \) of \( W' \) such that \( \chi' | W^s_a \) is a diffeomorphism. We apply the following lemma to \( (\chi' | W^s_a)^{-1} \).

**Lemma 2.8** [4, Lemma 2.8]. Let \( E \) and \( F \) be vector bundles over \( M \) associated to the frame bundle of \( M \). Then there exists a canonical linear map \( \gamma^* : H^q(E) \rightarrow H^q(F) \) for a diffeomorphism \( \gamma \) of \( M \). Let \( A \) be an open set of \( H'(E) \) and \( \phi : A \rightarrow H'(F) \) be a \( C^\infty \)-map which commutes with any \( \gamma^* \). If we set \( A^s = A \cap H^s(E) \) for \( s \geq r \), then \( \phi(A^s) \subset H^s(F) \) and the map \( \phi | A^s : A^s \rightarrow H^s(F) \) is \( C^s \).

If we set \( \operatorname{Im}(\chi' | W^s_a) = A \) and \( (\chi' | W^s_a)^{-1} = \phi \), then \( \phi \) is a \( C^\infty \)-map from \( A \) into \( H'(M) \times H'(S^3) \) which commutes with the action of the diffeomorphism group \( \mathcal{D} \) of \( M \). Hence Lemma 2.8 implies that the map

(2) J.P. Bourguignon [7, VIII. 8. Proposition] shows that \( \tau : \mathcal{M} \rightarrow \mathcal{F} \) is a submersion around a metric \( g \in \mathcal{M} \) such that \( \tau_g \) is not non-negative constant.
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$$(\mathcal{X}'|W')^{-1}|A^t : A^t \to H'(M) \times H'(S^2)$$

is $C^\infty$. But here $\mathcal{F}' \times \tilde{\mathcal{C}}'$ is a submanifold of $H'(M) \times H'(S^2)$, hence the map $(\mathcal{X}'|W')^{-1}|A^t : A^t \to \mathcal{F}' \times \tilde{\mathcal{C}}'$ is $C^\infty$. Thus $\mathcal{X}'$ is a local diffeomorphism and $\chi = \lim \chi^t$ is an ILH-diffeomorphism, which implies that $\tilde{C}$ is an ILH-submanifold of $\mathcal{M}$.

**Corollary 2.9.** Let $g = fg$, where $f \in \mathcal{F}$ and $g \in \tilde{\mathcal{C}}$. If $g(t)$ is a deformation of $g$ with sufficiently small domain of $t$, then there exist a 1-parameter family of positive functions $f(t)$ on $M$, a 1-parameter family of diffeomorphisms $\gamma(t)$ of $M$ and a deformation $g(t)$ in $\tilde{C}$ such that $f(0) = f$, $\delta g'(0) = 0$ and $g(t) = f(t)\gamma(t)*g(t)$.

**Proof.** By Theorem 2.5, $g(t)$ is decomposed into $f(t)\bar{g}(t)$, where $\bar{g}(t)$ is a deformation in $\tilde{C}$. Applying Slice theorem [4, Theorem 2.2] to $\bar{g}(t)$, we get $\bar{g}(t) = \gamma(t)*g(t)$, where $g(t)$ is a deformation such that $\delta g'(0) = 0$. Also we easily see that $g(t) \in \tilde{C}$ for each $t$.

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