Ma, Z.-M. and Röckner, M.
Osaka J. Math.
37 (2000), 273–314

CONSTRUCTION OF DIFFUSIONS ON CONFIGURATION SPACES

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(Received July 7, 1998)

0. Introduction

Let Γ_X be the configuration space over a finite dimensional Riemannian manifold X, i.e.,

$$\Gamma_X := \{\gamma \subset X \mid \gamma \cap K \text{ is a finite set for every compact } K \subset X\}$$

equipped with the vague topology via the identification $\gamma \equiv \sum_{x\in\gamma} \varepsilon_x$. In [3] a natural "differential geometry" was introduced on Γ_X via a canonical lifting procedure that "lifts" geometric objects on X (as the tangent bundle TX, the Riemannian metric $\langle\ ,\ \rangle_{TX}$, the gradient ∇^X, the divergence div^X, etc.) to Γ_X. In [3] and the subsequent papers [4], [5], [33], [34], [14], this "geometry" on Γ_X has been exploited to analyze Dirichlet forms for various (mixed Poisson or Gibbs) measures μ on Γ_X and associated diffusion processes, to study their ergodic and spectral properties, identify the corresponding intrinsic metric on Γ_X, to prove a Rademacher theorem, to solve stochastic differential equations for infinite interacting particle systems, etc. We refer to [31] for a survey on these results. The present paper supplements the above papers and yields substantial generalizations with completely new applications. Its main results can be summarized as follows:

(1) As announced in all the above references, we provide a complete proof for the existence of diffusions with state space Γ_X and associated with the Dirichlet forms studied there. Such a proof was not given previously in sufficient generality and was so far only carried out in special cases (i.e., if $X = \mathbb{R}^d$ and the underlying measure μ is a Gibbs measure of a certain type) by H. Osada [29] and M. Yoshida [37] (but see also Remark 4.14 below). We thus supplement the more recent papers mentioned above and extend the work initiated in [29] and [37] developing a method which (cf. point (4) below) extends to even infinite dimensional base manifolds X (such as loop spaces).

(2) We generalize the above lifting procedure to arbitrary square field operators $S(\cdot,\cdot)$ acting on a space $\mathcal{D}$ of functions on X (replacing $\langle\nabla^X\cdot,\nabla^X\cdot\rangle_{TX}$ resp. $C_0^\infty(X)$) to obtain square field operators S^Γ acting on the corresponding finitely based functions $\mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$ on Γ_X (cf. Subsections 1.2, 1.3). This yields (pre-)

Dirichlet forms of type

$$\mathcal{E}_{\mu}^{\Gamma}(F,G)=\int_{\Gamma_X} S^{\Gamma}(F,G)\,d\mu\ ;\quad F,G\in\mathcal{F}^{\Gamma}C_b^{\infty}(\mathcal{D}),$$

on $L^2(\Gamma_X;\mu)$ for e.g. probability measures μ on Γ_X. We also replace X by an arbitrary measurable space E and Γ_X by $\overline{\Gamma}_E$, i.e., the space of all *multiple configurations* on E.

(3) We prove closability of the (pre-) Dirichlet forms on Γ_X (or more generally on $\overline{\Gamma}_E$) mentioned under (2) for a large class of measures satisfying a condition on their associated *Campbell measures* (cf. Subsections 2.2, 2.3). We emphasize that in contrast to [3, 4] and [31] we do not need an integration by parts formula.

(4) In case E is a metric space, we prove that the closures of the (pre-) Dirichlet forms under (2) (provided the latter are closable) are *quasi-regular* on $\overline{\Gamma}_E$ in the sense of [25], [8] under very general assumptions (cf. Subsection 4.3), if $\overline{\Gamma}_E$ is given a suitable topology (which is the vague topology in case E is locally compact, cf. Section 3). As a consequence by the main result in [25], [8] (see also [26]) we obtain the corresponding diffusions on $\overline{\Gamma}_E$ (cf. Subsection 4.4).

(5) We show in detail that (2)–(4) apply to the previously studied case where $E = X$ = Riemannian manifold (hence obtain the desired diffusions on Γ_X mentioned under (1)), and also to the case $E = \mathcal{L}(\mathbb{R}^d)$ = free loop space over $\mathbb{R}^d$. $\mathbb{R}^d$ is, however, equipped with a general elliptic metric (with bounded derivatives). We also include a comprehensive exposition of the proof of the closability of the underlying (pre-) Dirichlet form on $\mathcal{L}(\mathbb{R}^d)$, since the corresponding one in the original paper [7] was a little sketchy. Based on this result we then show that the lifted (pre-) Dirichlet form on $\overline{\Gamma}_{\mathcal{L}(\mathbb{R}^d)}$ (cf. (2)) is closable by the method mentioned under (3). As an application of (4) we then obtain corresponding diffusions on $\overline{\Gamma}_{\mathcal{L}(\mathbb{R}^d)}$, i.e., infinite particle systems consisting of loops. We thus complete a programme already announced as forthcoming work in [1, 2].

All the above results depend on (quite weak, checkable) conditions in terms of "downstairs" quantities on E, so are of a "lifted type". Doing this generalized lifting procedure while assuming less structure on the base space E (i.e., assuming merely that E is a measurable or metric space) we loose, of course, the nice geometric structure of $\overline{\Gamma}_E$ which was used essentially in e.g. [3, 4, 5] or [31]. However, as far as the Dirichlet forms and diffusions on $\overline{\Gamma}_E$ are concerned, our framework is sufficient, and particularly designed for applications to cases where the base manifold is infinite dimensional, e.g. E equal to $\mathcal{L}(\mathbb{R}^d)$ or more general *Finsler manifolds* (cf. e.g. [17]).

As far as the measures μ on $\overline{\Gamma}_E$ are concerned, to which the above applies, we cover mixed Poisson measures and Ruelle type measures if $E = X$ is a finite dimensional Riemannian manifold (cf. Subsections 1.4.1, 2.4.1, 4.4.1). If $E = \mathcal{L}(\mathbb{R}^d)$ we only discuss Poisson measures on $\overline{\Gamma}_{\mathcal{L}(\mathbb{R}^d)}$ in detail in this paper (cf. Subsections 1.4.2., 2.4.2, 4.5.2). Corresponding Gibbs states will be treated in a forthcoming paper. One final aim is namely to construct infinite particle systems consisting of loops undergoing very singular interactions.

We should also mention that, since the natural assumption on the underlying measures μ on $\overline{\Gamma}_E$ is that they should only have first moments (cf. Subsection 2.1), in the proof of quasi-regularity (cf. (4) above) we have to perform a careful study of square field operators on L^1 instead of L^2. This is done in Subsection 4.2. One difficulty is that the Banach-Alaoglu resp. Banach-Saks Theorems do not hold on L^1.

The organization of this paper is apparent from the table of contents above. We only would like to add that we have devoted the entire Section 3 to a precise analysis of a suitable topology on $\overline{\Gamma}_E$, in case E is not locally compact. We also make a comparison with another commonly used topology on $\overline{\Gamma}_E$ (cf. the Appendix).

Finally, we would like to mention that the present paper is an extended version of a preprint already finished in Spring 1997 which, in particular, is refered to in the articles [3, 4]. All results have been presented in main talks on several conferences, e.g. in Anogia at the Euro-Conference on "Dirichlet forms and their Applications in Geometry and Stochastics" in June 1997 as well as at the Mathematical Sciences Research Institute in Berkeley in November 1997 within the "MSRI-Year in Stochastic Analysis", and in their final extended form at the Institute Henri Poincaré in Paris in May 1998 at the main conference of the TMR-Project "Stochastic Analysis and its Applications".

1. Lifting of square field operators to configuration spaces

In this section we summarize some very simple considerations concerning the construction of square field operators on configuration spaces, to be used below.

1.1. The framework Let $(E, \mathcal{B})$ be a measurable space such that $\{x\} \in \mathcal{B}$ for all $x \in E$ and $\mathcal{B}$ is countably generated. For a $\mathcal{B}$-measurable function $f : E \to \mathbb{R}$ and a (positive) measure γ on $(E, \mathcal{B})$ we use the notation

$$\langle f, \gamma \rangle := \int f \, d\gamma \tag{1.1}$$

provided the integral makes sense. Below we fix $E_k \in \mathcal{B}$, $k \in \mathbb{N}$, such that

$$E_k \subset E_{k+1} \text{ for all } k \in \mathbb{N} \text{ and } E = \bigcup_{k=1}^{\infty} E_k. \tag{1.2}$$

Let $\mathcal{M}(\{E_k\})$ denote the set of all (positive) measures γ on $(E, \mathcal{B})$ such that $\gamma(E_k)<\infty$ for all $k \in \mathbb{N}$. Let $\overline{\Gamma}_E := \overline{\Gamma}_E(\{E_k\})$ denote the set of all $\gamma \in \mathcal{M}(\{E_k\})$ such that $\gamma(A) \in \mathbb{Z}_+ \cup \{+\infty\}$ for all $A \in \mathcal{B}$. By our assumption on $(E, \mathcal{B})$ we have for all $\gamma \in \overline{\Gamma}_E$ that

$$\gamma = \sum_{x \in \operatorname{supp} \gamma} \gamma(\{x\})\, \varepsilon_x$$

where ε_x denotes Dirac measure in x and $\operatorname{supp} \gamma := \{x \in E \mid \gamma(\{x\}) > 0\}$ is countable. $\overline{\Gamma}_E$ is called *(multiple) configuration space* (over E, $(E_k)_{k \in \mathbb{N}}$).

Let $\mathcal{F}_b(\{E_k\})$ denote the set of all bounded, $\mathcal{B}(E)$-measurable functions $f : E \to \mathbb{R}$ such that there exists $k \in \mathbb{N}$ such that $f = 0$ on E_k^c (where $A^c := E \setminus A$ for $A \subset E$). Let $\mathcal{D} := \mathcal{D}(\{E_k\})$ be a linear subspace of $\mathcal{F}_b(\{E_k\})$ having the following property:

($\mathcal{D}$.1) $\varphi(f_1, \dots, f_N) \in \mathcal{D}$ for all $N \in \mathbb{N}$; $f_1, \dots, f_N \in \mathcal{D}$ and all $\varphi \in C_b^\infty(\mathbb{R}^N)$ with $\varphi(0) = 0$.

Let $S : \mathcal{D} \times \mathcal{D} \to \mathcal{F}_b(\{E_k\})$ be a bilinear map satisfying the following chain rule:

(S.1) For all $N \in \mathbb{N}$; $f_1, \dots, f_N, g_1, \dots, g_N \in \mathcal{D}$, and all $\varphi, \psi \in C_b^\infty(\mathbb{R}^N)$; $x \in E$

$$S(\varphi_0(f_1, \dots, f_N), \psi_0(g_1, \dots, g_N))(x) = \sum_{i,j=1}^{N} \partial_i \varphi(f_1(x), \dots, f_N(x))\, \partial_j \psi(g_1(x), \dots, g_N(x))\, S(f_i, g_j)(x),$$

where $\varphi_0 := \varphi - \varphi(0, \dots, 0)$, $\psi_0 := \psi - \psi(0, \dots, 0)$ and $\partial_i :=$ derivative w.r.t. the i-th coordinate.

As usual we set $S(f) := S(f, f)$, $f \in \mathcal{D}$.

REMARK 1.1. A bilinear form as S above (in particular, if it is symmetric and positive definite) is called a *square field operator* (cf. e.g. [12]).

1.2. Test functions and lifting formula It is now very easy to "lift" S to the following space $\mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$ of functions on the configuration space $\overline{\Gamma}_E$:

(1.3) $$\mathcal{F}^\Gamma C_b^\infty(\mathcal{D}) := \{g(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle) \mid N \in \mathbb{N}; f_1, \dots, f_N \in \mathcal{D}, g \in C_b^\infty(\mathbb{R}^N)\}.$$

We define for $F := g_F(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle)$, $G := g_G(\langle g_1, \cdot\rangle, \dots, \langle g_M, \cdot\rangle) \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$ and $\gamma \in \overline{\Gamma}_E$

(1.4) $$S^\Gamma(F, G)(\gamma) := \sum_{i=1}^{N} \sum_{j=1}^{M} \partial_i g_F(\langle f_1, \gamma\rangle, \dots, \langle f_N, \gamma\rangle)\, \partial_j g_G(\langle g_1, \gamma\rangle, \dots, \langle g_M, \gamma\rangle) \langle S(f_i, g_j), \gamma\rangle$$

and $S^\Gamma(F) := S^\Gamma(F, F)$. However, it is not clear immediately whether S^Γ is well-defined for $F, G \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$, i.e., independent of the representation of $F, G : \overline{\Gamma}_E \to \mathbb{R}$ chosen in (1.4).

1.3. Well-definedness We have the following

Lemma 1.2. *Let* $F \in \mathcal{F}^{\Gamma} C_b^{\infty}(\mathcal{D})$, $\gamma \in \overline{\Gamma}_E$, $x \in E$. *Then*

$$E \ni y \mapsto F(1_{E\setminus\{x\}} \cdot \gamma + \gamma(\{x\})\,\varepsilon_y) - F(1_{E\setminus\{x\}} \cdot \gamma) \in \mathcal{D}.$$

Furthermore, for all $F, G \in \mathcal{F}^{\Gamma} C_b^{\infty}(\mathcal{D})$, $\gamma \in \overline{\Gamma}_E$,

$$\begin{aligned}(1.5)\qquad & S^{\Gamma}(F, G)(\gamma) \\ &= \sum_{x \in \operatorname{supp}\gamma} \gamma(\{x\})^{-1} S\big(F(1_{E\setminus\{x\}} \cdot \gamma + \gamma(\{x\})\,\varepsilon_{\cdot}) - F(1_{E\setminus\{x\}} \cdot \gamma), \\ &\qquad\qquad G(1_{E\setminus\{x\}} \cdot \gamma + \gamma(\{x\})\,\varepsilon_{\cdot}) - G(1_{E\setminus\{x\}} \cdot \gamma)\big)(x).\end{aligned}$$

In particular, S^{Γ} *is well-defined by (1.4) on* $\mathcal{F}^{\Gamma} C_b^{\infty}(\mathcal{D}) \times \mathcal{F}^{\Gamma} C_b^{\infty}(\mathcal{D})$. *(We note here that the sum over* $x \in \operatorname{supp}\gamma$ *has only a finite number of non-zero summands, since* $\gamma \in \overline{\Gamma}_E$; $F, G \in \mathcal{F}^{\Gamma} C_b^{\infty}(\mathcal{D})$*).*

Proof. If $F = g_F(\langle f_1, \cdot\rangle, \ldots, \langle f_N, \cdot\rangle) \in \mathcal{F}^{\Gamma} C_b^{\infty}(\mathcal{D})$, we have

$$\begin{aligned} E \ni y \mapsto\ & F(1_{E\setminus\{x\}}\gamma + \gamma(\{x\})\,\varepsilon_y) - F(1_{E\setminus\{x\}}\gamma) \\ =\ & \varphi_0^{F,x,\gamma}(f_1(y), \ldots, f_N(y)) \in \mathcal{D}, \end{aligned}$$

where for $z_1, \ldots, z_N \in \mathbb{R}$

$$\varphi^{F,x,\gamma}(z_1, \ldots, z_N) := g_F(\langle f_1, 1_{E\setminus\{x\}}\gamma\rangle + \gamma(\{x\})z_1, \ldots, \langle f_N, 1_{E\setminus\{x\}}\gamma\rangle + \gamma(\{x\})z_N).$$

Furthermore, if $G = g_G(\langle g_1, \cdot\rangle, \ldots, \langle g_M, \cdot\rangle) \in \mathcal{F}^{\Gamma} C_b^{\infty}(\mathcal{D})$, then the right hand side of (1.5) is equal to

$$\sum_{x \in \operatorname{supp}\gamma} \gamma(\{x\})^{-1} S\left(\varphi_0^{F,x,\gamma}(f_1, \ldots, f_N), \varphi_0^{G,x,\gamma}(g_1, \ldots, g_M)\right)(x)$$

(with $\varphi^{F,x,\gamma}$, $\varphi^{G,x,\gamma}$ as above) which in turn by (S.1) is the same as

$$\begin{aligned} &\sum_{x \in \operatorname{supp}\gamma} \sum_{i=1}^{N} \sum_{j=1}^{M} \gamma(\{x\})^{-1} \partial_i \varphi^{F,x,\gamma}(f_1(x), \ldots, f_N(x))\ \partial_j \varphi^{G,x,\gamma}(g_1(x), \ldots, g_M(x)) \\ &\qquad\qquad S(f_i, g_j)(x) \\ &= \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{x \in \operatorname{supp}\gamma} \gamma(\{x\})\ \partial_i g_F(\langle f_1, \gamma\rangle, \ldots, \langle f_N, \gamma\rangle)\ \partial_j g_G(\langle g_1, \gamma\rangle, \ldots, \langle g_M, \gamma\rangle) \\ &\qquad\qquad S(f_i, g_j)(x) \\ &= \sum_{i=1}^{N} \sum_{j=1}^{M} \partial_i g_F(\langle f_1, \gamma\rangle, \ldots, \langle f_N, \gamma\rangle)\ \partial_j g_G(\langle g_1, \gamma\rangle, \ldots, \langle g_M, \gamma\rangle)\langle S(f_i, g_j), \gamma\rangle. \qquad \square \end{aligned}$$

REMARK 1.3. We note that by (1.4) (since $1 \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$)

$$S(1, F) = S(F, 1) = 0 \text{ for all } F \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}).$$

It is now easy to see that $(S^\Gamma, \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}))$ inherits a property similar to property (S.1) of (S,$\mathcal{D}$).

Lemma 1.4. *Let $F_1, \dots, F_N, G_1, \dots, G_N \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$, $\varphi, \psi \in C_b^\infty(\mathbb{R}^N)$. Then for all $\gamma \in \overline{\Gamma}_E$*

$$(1.6) \qquad \begin{aligned} &S^\Gamma(\varphi(F_1, \dots, F_N), \psi(G_1, \dots, G_N))(\gamma) \\ &= \sum_{i,j=1}^{N} \partial_i \varphi(F_1(\gamma), \dots, F_N(\gamma)) \; \partial_j \psi(G_1(\gamma), \dots, G_N(\gamma)) \; S^\Gamma(F_i, G_j)(\gamma). \end{aligned}$$

Proof. Immediate by (1.4). □

REMARK 1.5. Let $\sigma \in \mathcal{M}(\{E_k\})$. Suppose there exists $N \in \mathcal{B}$ such that $\sigma(N) = 0$ and that for $f, g \in \mathcal{D}$, $S(f, g)(x)$ is only defined for $x \in N^c$, and only on N^c equal to a function in $\mathcal{F}_b(\{E_k\})$. If also (S.1) only holds for all $x \in N^c$, then nevertheless $S^\Gamma(F, G)(\gamma)$ is well-defined by (1.4) for all $F, G \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$ and $\gamma \in \{\tilde{\gamma} \in \overline{\Gamma}_E \mid \tilde{\gamma}(N) = 0\}$. This follows from the proof of Lemma 1.2. Also the assertion of Lemma 1.4 then still holds for such γ. This is useful to realize, since $\mu(\{\gamma \in \overline{\Gamma}_E \mid \gamma(N) > 0\}) = 0$ for all (positive) measures μ on $(\overline{\Gamma}_E, \sigma(N_A \mid A \in \mathcal{B}))$ where $N_A := \gamma(A)$, $\gamma \in \overline{\Gamma}_E$, such that the measure

$$(1.7) \qquad \mathcal{B} \ni A \mapsto \sigma^\mu(A) := \int_{\overline{\Gamma}_E} \gamma(A) \; \mu(d\gamma)$$

is absolutely continuous w.r.t. σ. Indeed, in this case for all $k \in \mathbb{N}$ and $N_k := N \cap E_k$

$$0 = \int \gamma(N_k) \; \mu(d\gamma) = \sum_{n=0}^{\infty} n \; \mu(\{\gamma \in \overline{\Gamma}_E \mid \gamma(N_k) = n\}).$$

Therefore, $\mu(\{\gamma \in \overline{\Gamma}_E \mid \gamma(N_k) > 0\}) = 0$, hence $\mu(\{\gamma \in \overline{\Gamma}_E \mid \gamma(N) > 0\} = 0$. We shall use this in our applications to the free loop space below.

1.4. Examples Throughout this paper we are mainly interested in the following two case studies to which the above (and all results obtained in the following sections) apply: a) E := a (finite dimensional) Riemannian manifold, b) E := the free loop space. A more comprehensive study of a large class of base spaces E including these two will be done in a forthcoming paper.

1.4.1. Riemannian manifolds Let E be a finite dimensional Riemannian manifold X with tangent bundle TX and inner product $\langle \; , \; \rangle_{TX}$ and let $\mathcal{B}$ be equal to its

Borel σ-field $\mathcal{B}(X)$. Let E_k, $k \in \mathbb{N}$, be open, relatively compact subsets of X such that the closure $\overline{E}_k$ of E_k is contained in E_{k+1} for all $k \in \mathbb{N}$. Let $\mathcal{D} := C_0^\infty(X)$, i.e., the set of all infinitely differentiable real-valued functions on X with compact support. Define

$$S(f,g) := \langle \nabla^X f, \nabla^X g\rangle_{TX} \quad ; \ f, g \in \mathcal{D}.$$

Then $(S^\Gamma, \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}))$ is well-defined on $\overline{\Gamma}_X$ by (1.4) and satisfies the chain rule (1.6).

1.4.2. The free loop space Let us recall the framework of [7]: Let $g := (g_{ij})$ be a uniformly elliptic Riemannian metric with bounded derivatives over $\mathbb{R}^d$ and

$$\Delta_g := (\det g)^{-1/2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left[(\det g)^{1/2}\ g^{ij} \frac{\partial}{\partial x_j}\right]$$

the corresponding Laplacian. Let $p_t(x,y)$, $x, y \in \mathbb{R}^d$, $t \geq 0$, be the associated heat kernel with respect to the Riemannian volume element. Let $W(\mathbb{R}^d)$ denote the set of all continuous paths $\omega : [0,1] \to \mathbb{R}^d$ and let $\mathcal{L}(\mathbb{R}^d) := \{\omega \in W(\mathbb{R}^d) \mid \omega(0) = \omega(1)\}$, i.e., $\mathcal{L}(\mathbb{R}^d)$ is the free loop space over $\mathbb{R}^d$. Let P_1^x be the law of the bridge defined on $\{\omega \in \mathcal{L}(\mathbb{R}^d) \mid \omega(0) = \omega(1) = x\}$ coming from the diffusion on $\mathbb{R}^d$ generated by Δ_g and let

$$\sigma := \int P_1^x p_1(x,x)dx \tag{1.8}$$

be the *Bismut/Høegh-Krohn measure* on $\mathcal{L}(\mathbb{R}^d)$ which is σ-finite, but not finite. We consider $\mathcal{L}(\mathbb{R}^d)$ equipped with the Borel σ-field $\mathcal{B}(\mathcal{L}(\mathbb{R}^d))$ coming from the uniform norm $\|\cdot\|_\infty$ on $\mathcal{L}(\mathbb{R}^d)$ which makes it a Banach space. The tangent space $T_\omega\mathcal{L}(\mathbb{R}^d)$ at a loop $\omega \in \mathcal{L}(\mathbb{R}^d)$ was introduced in [20] as the space of *periodical* vector fields $X_t(\omega) = \tau_t(\omega)\ h(t)$, $t \in [0,1]$, along ω. Here τ denotes the stochastic parallel transport w.r.t. σ associated with the Levi-Civita connection of $(\mathbb{R}^d, g)$ and h belongs to the linear space H_ω consisting of all absolutely continuous maps $h : [0,1] \to T_{\omega(0)}\mathbb{R}^d \equiv \mathbb{R}^d$ such that

$$(h,h)_{\omega(0)} := \int_0^1 g_{\omega(0)}(h'(s), h'(s))\ ds + \int_0^1 |h(s)|^2\ ds < \infty \tag{1.9}$$

and satisfying the holonomy condition $\tau_1(\omega)h(1) = h(0)$ (cf. [20] for details). Note that if we consider $\mathcal{L}(\mathbb{R}^d)$ as continuous maps from S^1 to $\mathbb{R}^d$, this notion is invariant by rotations of S^1 and (1.9) induces an inner product on $T_\omega\mathcal{L}(\mathbb{R}^d)$ which turns it into a Hilbert space. Below we shall also need the Hilbert space $\widetilde{T}_\omega\mathcal{L}(\mathbb{R}^d)$ ($\supset T_\omega\mathcal{L}(\mathbb{R}^d)$) with inner product $(\ ,\ )_H$ which is constructed analogously but without the holonomy condition, i.e., H_ω is replaced by H which denotes the linear space of all absolutely continuous maps $h : [0,1] \to T_{\omega(0)}\mathbb{R}^d \equiv \mathbb{R}^d$ satisfying (1.9). We note that by the uniform ellipticity of g, H is indeed independent of $\omega \in \mathcal{L}(\mathbb{R}^d)$ and the norms

$\|\cdot\|_{\omega(0)} := (\ ,\ )_{\omega(0)}^{1/2}$, $\omega \in \mathcal{L}(\mathbb{R}^d)$, are all equivalent. Let $\mathcal{F}C_0^\infty$ denote the linear span of the set of all functions $u : \mathcal{L}(\mathbb{R}^d) \to \mathbb{R}$ such that there exist $k \in \mathbb{N}$, $f \in C_0^\infty((\mathbb{R}^d)^k)$, $t_1, \dots, t_k \in [0, 1]$ with

$$u(\omega) = f(\omega(t_1), \dots, \omega(t_k)), \quad \omega \in \mathcal{L}(\mathbb{R}^d). \tag{1.10}$$

Note that $\mathcal{F}C_0^\infty$ is dense in $L^2(\sigma) :=(\text{real})L^2(\mathcal{L}(\mathbb{R}^d); \sigma)$. Let $\mathcal{F}C^\infty$, $\mathcal{F}C_b^\infty$ be defined correspondingly with $C^\infty((\mathbb{R}^d)^k)$ resp. $C_b^\infty((\mathbb{R}^d)^k)$ replacing $C_0^\infty((\mathbb{R}^d)^k)$. We define the directional derivative of $u \in \mathcal{F}C^\infty$, u as in (1.10), at $\omega \in \mathcal{L}(\mathbb{R}^d)$ with respect to $X(\omega) \in \widetilde{T}_\omega\mathcal{L}(\mathbb{R}^d)$ by

$$\begin{aligned} \partial_h u(\omega) := \partial_X u(\omega) &:= \sum_{i=1}^k d_i f(\omega(t_1), \dots, \omega(t_k))\, X_{t_i}(\omega) \\ &= \sum_{i=1}^k g_{\omega(t_i)}(\nabla_i f(\omega(t_1), \dots, \omega(t_k)), \tau_{t_i}(\omega)h(t_i)) \end{aligned} \tag{1.11}$$

where $h \in H$ with $X(\omega) = (\tau_t(\omega)h(t))_{t\in[0,1]}$ and ∇_i resp. d_i denotes the gradient (with respect to g) resp. the differential relative to the i-th coordinate of f. Note that if we consider u as a function on $W(\mathbb{R}^d)$ then

$$\partial_X u(\omega) = \frac{d}{ds} u(\omega + sX(\omega))|_{s=0}, \quad \omega \in \mathcal{L}(\mathbb{R}^d). \tag{1.12}$$

Hence $\partial_X u$ is well-defined by (1.11)(i.e., independent of the special representation of u).

Let for $u \in \mathcal{F}C^\infty$ and $\omega \in \mathcal{L}(\mathbb{R}^d)$, $\tilde{D}u(\omega)$ be the unique element in H such that $(\tilde{D}u(\omega), h)_{\omega(0)} = \partial_h u(\omega)$ for all $h \in H$ and let $Du(\omega)$ be its (w.r.t. $(\ ,\ )_{\omega(0)}$) orthogonal projection onto H_ω.

Since H is separable and consists of continuous functions on [0, 1] it follows by the construction of the stochastic parallel transport that there exists $N \in \mathcal{B}(\mathcal{L}(\mathbb{R}^d))$ with $\sigma(N) = 0$ such that both $Du(\omega)$ and $\tilde{D}u(\omega)$ are defined for all $\omega \in N^c$ and all $u \in \mathcal{F}C^\infty$. The measurability of $\omega \mapsto \tilde{D}u(\omega)$ will be discussed in Subsection 2.4.2 below.

For $u \in \mathcal{F}C^\infty$ we have that for all $\omega \in N^c$

$$\|Du(\omega)\|_{\omega(0)} \leq \|\tilde{D}u(\omega)\|_{\omega(0)} \tag{1.13}$$

and if $u(\omega) = f(\omega(s_1), \dots, \omega(s_k))$ then for all $\omega \in N^c$

$$\tilde{D}u(\omega)(s) = \sum_{i=1}^k G(s, s_i)\, \tau_{s_i}(\omega)^{-1} \nabla_i f(\omega(s_1), \dots, \omega(s_k)) \tag{1.14}$$

where G is the Green function of $-(d^2/dt^2) + 1$ with Neumann boundary conditions on [0, 1], i.e.,

$$G(s,u) = \frac{e}{2(e^2-1)}(e^{u+s-1} + e^{1-(u+s)} + e^{|u-s|-1} + e^{1-|u-s|}). \tag{1.15}$$

For $u, v \in \mathcal{F}C^\infty$ define

$$S(u,v)(\omega) := (Du(\omega), Dv(\omega))_{\omega(0)}, \quad \omega \in \mathcal{L}(\mathbb{R}^d). \tag{1.16}$$

For $k \in \mathbb{N}$ let U_k denote the open ball in $\mathbb{R}^d$ of radius k and define open subsets of $\mathcal{L}(\mathbb{R}^d)$ by

$$E_k := \{\omega \in \mathcal{L}(\mathbb{R}^d) \mid \omega(0) \in U_k\}. \tag{1.17}$$

Define $\mathcal{D}$ to be the linear span of the set of all functions $u : \mathcal{L}(\mathbb{R}^d) \to \mathbb{R}$ of the type specified in (1.10), but with $t_1 = 0$. We note that $v_\chi \cdot u \in \mathcal{D}$ for all $u \in \mathcal{F}C_0^\infty$ and $v_\chi(\omega) := \chi(\omega(0))$, $\omega \in \mathcal{L}(\mathbb{R}^d)$, $\chi \in C_0^\infty(\mathbb{R}^d)$. Clearly, $\mathcal{D}$ satifies ($\mathcal{D}$.1) and by (1.13), (1.14) we see that $S(u, v)$ is equal to a function in $\mathcal{F}_b(\{E_k\})$ on N^c. It follows by (1.14) that (S,$\mathcal{D}$) satisfies (S.1) for all $x \in N^c$. (This is even true with $\mathcal{F}C^\infty$ replacing $\mathcal{D}$). Hence Remark 1.5 implies that $S^\Gamma(F, G)(\gamma)$ is well-defined by (1.4) for all $F, G \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$ and all $\gamma \in \overline{\Gamma}_{\mathcal{L}(\mathbb{R}^d)}$ such that $\gamma(N) = 0$.

2. Closability of corresponding (pre-) Dirichlet forms

In this section we consider square field operators S on the base space E and their "lifts" S^Γ to the configuration space $\overline{\Gamma}_E$ as above. But we additionally assume minimum properties of S so that S itself and hence S^Γ lead to symmetric pre-Dirichlet forms $\mathcal{E}^E$ and $\mathcal{E}^\Gamma$ over E, $\overline{\Gamma}_E$ respectively. In particular, we prove that closability of (a "weighted version" of) $\mathcal{E}^E$ implies the closability of $\mathcal{E}^\Gamma$ for a large class of reference measures. Examples and applications are discussed at the end of this section. Let $(E, \mathcal{B})$, $(E_k)_{k\in\mathbb{N}}$, $(S, \mathcal{D})$ (with properties ($\mathcal{D}$.1), (S.1)) and $\overline{\Gamma}_E$, $(S^\Gamma, \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}))$ be as in Section 1.

2.1. Corresponding pre-Dirichlet forms

Below we assume that $(S, \mathcal{D})$ satisfies the following additional conditions:

(S.2) $S(f, f)(x) \geq 0$ for all $f \in \mathcal{D}$, $x \in E$.

(S.3) $S(f, g)(x) = S(g, f)(x)$ for all $f, g \in \mathcal{D}$, $x \in E$.

Lemma 2.1. *$(S^\Gamma, \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}))$ inherits properties* (S.2), (S.3), *i.e.,*

$$S^\Gamma(F, F)(\gamma) \geq 0 \quad \text{for all } F \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}),\ \gamma \in \overline{\Gamma}_E, \tag{2.1}$$

and

$$S^\Gamma(F, G)(\gamma) = S^\Gamma(G, F)(\gamma) \quad \text{for all } F, G \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}),\ \gamma \in \overline{\Gamma}_E. \tag{2.2}$$

Proof. (2.1) follows from (1.5), and (2.2) also by (1.5) or by (1.4). □

Let $\sigma \in \mathcal{M}(\{E_k\})$. Define

$$\mathcal{E}_\sigma^E(f,g) := \int_E S(f,g)(x)\,\sigma(dx); \quad f,g \in \mathcal{D}. \tag{2.3}$$

As before, once we have fixed σ it is enough to assume that (S.1), (S.2), (S.3) hold for all x outside a σ-zero set (cf. Remark 1.5 for details).

Let $\mathcal{B}(\overline{\Gamma}_E) := \sigma(\{N_A \mid A \in \mathcal{B}\})$ (cf. Remark 1.5) and let μ be a probability measure on $(\overline{\Gamma}_E, \mathcal{B}(\overline{\Gamma}_E))$ satisfying the following condition:
(μ.1) $\int_{\overline{\Gamma}_E} \gamma(E_k)\,\mu(d\gamma) < \infty$ for all $k \in \mathbb{N}$.
Then for any $f \in \mathcal{F}_b(\{E_k\})$

$$\int \langle |f|, \gamma\rangle\,\mu(d\gamma) < \infty, \tag{2.4}$$

hence we may define

$$\mathcal{E}_\mu^\Gamma(F,G) := \int_{\overline{\Gamma}_E} S^\Gamma(F,G)(\gamma)\,\mu(d\gamma); \quad F,G \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}). \tag{2.5}$$

Let $\mathcal{D}^\sigma$ (resp. $\mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D})$) denote the σ-classes (resp. the μ-classes) determined by $\mathcal{D}$ (resp. $\mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$). Suppose that the following condition (S.σ) (connecting S and σ) holds:
(S.σ) $S(f,g) = 0$ σ-a.e. for all $f,g \in \mathcal{D}$ such that $f = 0$ σ-a.e.
Then $\mathcal{E}_\sigma^E$ can be defined (representative-wise) on $\mathcal{D}^\sigma \times \mathcal{D}^\sigma$, hence $(\mathcal{E}_\sigma^E, \mathcal{D}^\sigma)$ is a positive definite symmetric bilinear form on $L^2(E;\sigma)$ (:= (real)$L^2(E;\mathcal{B};\sigma)$). Likewise, suppose that the following condition ($S^\Gamma.\mu$) (connecting S^Γ and μ) holds:
($S^\Gamma.\mu$) $S^\Gamma(F,G) = 0$ μ-a.e. for all $F,G \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$ such that $F = 0$ μ-a.e.
Then $(\mathcal{E}_\mu^\Gamma, \mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D}))$ is a positive definite symmetric bilinear form on $L^2(\overline{\Gamma}_E;\mu)$ (:=(real)$L^2(\overline{\Gamma}_E;\mathcal{B}(\overline{\Gamma}_E);\mu)$.

For the notions "closable", "closure" and "Dirichlet form" appearing in the next proposition we refer to [25, Chap. I, Sect. 3 and 4].

Proposition 2.2. (i) *If condition* (S.σ) *holds then,* $(\mathcal{E}_\sigma^E, \mathcal{D}^\sigma)$ *is a pre-Dirichlet form on* $L^2(E;\sigma)$ *(i.e., if* $(\mathcal{E}_\sigma^E, \mathcal{D}^\sigma)$ *is densely defined and closable on* $L^2(E;\sigma)$*, then its closure* $(\mathcal{E}_\sigma^E, D(\mathcal{E}_\sigma^E))$ *is a Dirichlet form).*

(ii) *If condition* ($S^\Gamma.\mu$) *holds, then* $(\mathcal{E}_\mu^\Gamma, \mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D}))$ *is a pre-Dirichlet form on* $L^2(\overline{\Gamma}_E;\mu)$.

Proof. Both assertions (i) and (ii) follow by (S.1) resp. Lemma 1.4 directly from [25, Chap. I, Proposition 4.10] (see also [25, Chap. II, Excerise 2.7]). □

2.2. A class of measures on configuration space We still consider the situation of Subsection 2.1 (i.e., conditions ($\mathcal{D}$.1), (S.1)–(S.3) are still in force and $\sigma \in \mathcal{M}(\{E_k\})$, but assume, in addition, that the probability measure μ on $(\overline{\Gamma}_E, \mathcal{B}(\overline{\Gamma}_E))$ with property (μ.1) satisfies the following condition ($\mu.\sigma$) w.r.t. σ:

($\mu.\sigma$) There exists a $\mathcal{B}(\overline{\Gamma}_E) \otimes \mathcal{B}$-measurable function $\rho: \overline{\Gamma}_E \times E \to \mathbb{R}_+$ such that for all $\mathcal{B}(\overline{\Gamma}_E) \otimes \mathcal{B}$-measurable functions $h: \overline{\Gamma}_E \times E \to \mathbb{R}_+$

$$\int_{\overline{\Gamma}_E} \int_E h(\gamma, x)\ \gamma(dx)\, \mu(d\gamma) = \int_{\overline{\Gamma}_E} \int_E h(\gamma + \varepsilon_x, x)\, \rho(\gamma, x)\, \sigma(dx)\, \mu(d\gamma).$$

REMARK 2.3. (i) The left hand side of the equality in condition ($\mu.\sigma$) is just the integral of h w.r.t. the so-called *Campbell-measure* of μ (see e.g. [28] and the references therein).

(ii) There are many examples for which condition ($\mu.\sigma$) is satisfied. In particular, it holds for Gibbs measures in which we are specially interested. We refer to Subsection 2.4 below for details and references.

(iii) Condition ($\mu.\sigma$) implies that the *mean* or *intensity measure* σ^μ of μ is absolutely continuous w.r.t. σ, more precisely

$$\sigma^\mu(A) := \int_{\overline{\Gamma}_E} \gamma(A)\ \mu(d\gamma) = \int_E 1_A(x) \int_{\overline{\Gamma}_E} \rho(\gamma, x)\ \mu(d\gamma)\ \sigma(dx).$$

The following is a generalization of [14, Theorem 5.1] (see also [31, Theorem 6.9]). The proof, however, after our preparations is more or less the same. We include it for the reader's convenience.

Theorem 2.4. *Let* $(S, \mathcal{D})$, $(S^\Gamma, \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}))$, $\sigma \in \mathcal{M}(\{E_k\})$, *and* μ *be as above (so that conditions* ($\mathcal{D}$.1), (S.1)–(S.3), (μ.1), ($\mu.\sigma$) *are satisfied). Then the following assertions hold.*

(i) *Let* $F, G \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$. *Then*

$$\begin{aligned}(2.6)\qquad &\int_{\overline{\Gamma}_E} S^\Gamma(F, G)(\gamma)\, \mu(d\gamma) \\ &= \int_{\overline{\Gamma}_E} \int_E S(F(\gamma + \varepsilon_\cdot) - F(\gamma), G(\gamma + \varepsilon_\cdot) - G(\gamma))\ \rho(\gamma, x)\, \sigma(dx)\, \mu(d\gamma).\end{aligned}$$

(ii) *Suppose that for* μ*-a.e.* $\gamma \in \overline{\Gamma}_E$ *condition* ($S.\sigma_\gamma$) *(i.e.,* ($S.\sigma$) *with* σ *replaced by* $\sigma_\gamma := \rho(\gamma, \cdot) \cdot \sigma$) *holds. Then condition* ($S^\Gamma.\mu$) *holds as well and hence* $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^{\infty,\mu}(\mathcal{D}))$ *is well-defined and a pre-Dirichlet form on* $L^2(\overline{\Gamma}_E; \mu)$.

Proof. (i): By (1.4) and condition $(\mu.\sigma)$, if $F = g_F(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle)$, $G = g_G(\langle g_1, \cdot\rangle, \dots, \langle g_M, \cdot\rangle)$, we have

$$
\begin{aligned}
(2.7)\qquad & \int_{\overline{\Gamma}_E} S^\Gamma(F, G)(\gamma)\ \mu(d\gamma) \\
&= \int_{\overline{\Gamma}_E} \int_E \sum_{i=1}^N \sum_{j=1}^M \partial_i g_F(\langle f_1, \gamma\rangle, \dots, \langle f_N, \gamma\rangle)\ \partial_j g_G(\langle g_1, \gamma\rangle, \dots, \langle g_M, \gamma\rangle) \\
&\qquad S(f_i, g_j)(x)\ \gamma(dx)\ \mu(d\gamma) \\
&= \int_{\overline{\Gamma}_E} \int_E \sum_{i=1}^N \sum_{j=1}^M \partial_i g_F(\langle f_1, \gamma\rangle + f_1(x), \dots, \langle f_N, \gamma\rangle + f_N(x)) \\
&\qquad \partial_j g_G(\langle g_1, \gamma\rangle + g_1(x), \dots, \langle g_M, \gamma\rangle + g_M(x)) \\
&\qquad S(f_i, g_j)(x)\ \rho(\gamma, x)\ \sigma(dx)\ \mu(d\gamma).
\end{aligned}
$$

By (S.1) the latter is clearly equal to the right hand side of (2.6). (We note that because of $(\mathcal{D}.1)$, $F(\gamma + \varepsilon_\cdot) - F(\gamma)$ and $G(\gamma + \varepsilon_\cdot) - G(\gamma)$ are both in $\mathcal{D}$).

(ii): If $F \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$ such that $F = 0$ μ-a.e., then by condition $(\mu.\sigma)$ for all $k \in \mathbb{N}$

$$
\begin{aligned}
(2.8)\qquad 0 &= \int_{\overline{\Gamma}_E} \gamma(E_k)\ |F(\gamma)|\ \mu(d\gamma) \\
&= \int_{\overline{\Gamma}_E} \int_E 1_{E_k}(x)\ |F(\gamma + \varepsilon_x)|\ \sigma_\gamma(dx)\ \mu(d\gamma).
\end{aligned}
$$

Since $k \in \mathbb{N}$ was arbitrary, it follows that for μ-a.e. $\gamma \in \overline{\Gamma}_E$

$$F(\gamma + \varepsilon_x) = 0 \quad \text{for } \sigma_\gamma\text{-a.e. } x \in E,$$

hence for μ-a.e. $\gamma \in \overline{\Gamma}_E$

$$F(\gamma + \varepsilon_x) - F(\gamma) = 0 \quad \text{for } \sigma_\gamma\text{-a.e. } x \in E.$$

Since by assumption $(S.\sigma_\gamma)$ holds for μ-a.e. $\gamma \in \overline{\Gamma}_E$, this implies that the right hand side of (2.6) applied to $F := G$ is zero, hence by (2.6)

$$S^\Gamma(F, F) = 0 \quad \mu\text{-a.e.},$$

and the assertion follows by the Cauchy-Schwarz inequality and Proposition 2.2 (ii). □

2.3. Lifting of closability In this subsection we additionally assume that for μ-a.e. $\gamma \in \overline{\Gamma}_E$ condition $(S.\sigma_\gamma)$ and, furthermore, that the following condition (C) holds:

(C) There exists a σ^{μ}-integrable function $w : E \to (0,1]$ such that $(\mathcal{E}^{E}_{w\cdot\sigma_\gamma}; \mathcal{D}^{\sigma_\gamma})$ defined by

$$\mathcal{E}^{E}_{w\cdot\sigma_\gamma}(f,g) := \int_E S(f,g)\; w\; d\sigma_\gamma;\;\; f,g \in \mathcal{D}^{\sigma_\gamma},$$

is closable on $L^2(E; w\,\sigma_\gamma)$ for μ-a.e. $\gamma \in \overline{\Gamma}_E$.

REMARK 2.5. By conditions (μ.1), ($\mu.\sigma$) it immediately follows that there exists $N \in \mathcal{B}(\overline{\Gamma}_E)$ such that $\mu(N) = 0$ and $\sigma_\gamma \in \mathcal{M}(\{E_k\})$ for all $\gamma \in N^c$. Therefore, in (C) $\mathcal{E}^{E}_{w\cdot\sigma_\gamma}(f,f) < \infty$ for all $f \in \mathcal{D}^{\sigma_\gamma}$, $\gamma \in N^c$.

We are now prepared to prove the main result of this subsection, whose proof is an adaption of [14, Theorem 6.3] resp. [31, Theorem 6.13] to our more general situation here.

Theorem 2.6. *Let* $(S, \mathcal{D})$, $(S^\Gamma, \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}))$, $\sigma \in \mathcal{M}(\{E_k\})$, *and* μ *as above (so that conditions* ($\mathcal{D}$.1), (S.1)–(S.3), (μ.1), ($\mu.\sigma$), (S.σ_γ) *for* μ*-a.e.* $\gamma \in \overline{\Gamma}_E$*, and condition* (C) *are satisfied). Then* $(\mathcal{E}^\Gamma_\mu, \mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D}))$ *is closable on* $L^2(\overline{\Gamma}_E;\mu)$*. In particular, if* $\mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D})$ *is dense in* $L^2(\overline{\Gamma}_E;\mu)$*, then its closure* $(\mathcal{E}^\Gamma_\mu, D(\mathcal{E}^\Gamma_\mu))$ *is a Dirichlet form on* $L^2(\overline{\Gamma}_E;\mu)$.

Proof. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D})$ such that $F_n \to 0$ in $L^2(\overline{\Gamma}_E;\mu)$ as $n \to \infty$ and

$$\mathcal{E}^\Gamma_\mu(F_n - F_m, F_n - F_m) \underset{n,m\to\infty}{\longrightarrow} 0. \tag{2.9}$$

We have to show that

$$\mathcal{E}^\Gamma_\mu(F_{n_k}, F_{n_k}) \underset{k\to\infty}{\longrightarrow} 0 \tag{2.10}$$

for some subsequence $(n_k)_{k\in\mathbb{N}}$. Let $(n_k)_{k\in\mathbb{N}}$ be a subsequence such that

$$\left(\int_{\overline{\Gamma}_E} F_{n_k}^2\, d\mu\right)^{1/2} + \mathcal{E}^\Gamma_\mu(F_{n_{k+1}} - F_{n_k}, F_{n_{k+1}} - F_{n_k})^{1/2} < \frac{1}{2^k} \quad \text{for all } k \in \mathbb{N}. \tag{2.11}$$

Then

$$\begin{aligned}\infty &> \sum_{k=1}^{\infty} \mathcal{E}^\Gamma_\mu(F_{n_{k+1}} - F_{n_k}, F_{n_{k+1}} - F_{n_k})^{1/2} \\ &\geq \int_{\overline{\Gamma}_E} \sum_{k=1}^{\infty} \left(\int_E S((F_{n_{k+1}} - F_{n_k})(\gamma + \varepsilon_\cdot) - (F_{n_{k+1}} - F_{n_k})(\gamma))(x)\; \sigma_\gamma(dx)\right)^{1/2} \mu(d\gamma)\end{aligned}$$

where we used (2.6). Consequently,

$$\sum_{k=1}^{\infty} \mathcal{E}^E_{\sigma_\gamma}(u^{(\gamma)}_{n_{k+1}} - u^{(\gamma)}_{n_k}, u^{(\gamma)}_{n_{k+1}} - u^{(\gamma)}_{n_k})^{1/2} < \infty \quad \text{for } \mu\text{-a.e. } \gamma \in \overline{\Gamma}_E, \tag{2.12}$$

where for $k \in \mathbb{N}$, $\gamma \in \overline{\Gamma}_E$

$$u^{(\gamma)}_{n_k}(x) := F_{n_k}(\gamma + \varepsilon_x) - F_{n_k}(\gamma), \quad x \in E.$$

(Recall that $u_{n_k} \in \mathcal{D}$ by ($\mathcal{D}$.1)). (2.12) and the fact that w from condition (C) is bounded, imply that for μ-a.e. $\gamma \in \overline{\Gamma}_E$

$$\mathcal{E}^E_{w\sigma_\gamma}(u^{(\gamma)}_{n_k} - u^{(\gamma)}_{n_l}, u^{(\gamma)}_{n_k} - u^{(\gamma)}_{n_l}) \underset{k,l\to\infty}{\longrightarrow} 0. \tag{2.13}$$

CLAIM 1. For μ-a.e. $\gamma \in \overline{\Gamma}_E$

$$\int_E (u^{(\gamma)}_{n_k})^2 \, w \, d\sigma_\gamma \underset{k\to\infty}{\longrightarrow} 0.$$

To prove Claim 1 we first note that for μ-a.e. $\gamma \in \overline{\Gamma}_E$

$$\int w \, d\sigma_\gamma < \infty$$

as follows immediately from condition ($\mu.\sigma$) (taking $h(\gamma, x) := w(x)$ for $x \in E$, $\gamma \in \overline{\Gamma}_E$), since $w \in L^1(E; \sigma^\mu)$. Therefore, for μ-a.e. $\gamma \in \overline{\Gamma}_E$

$$\int_E F^2_{n_k}(\gamma) w \, d\sigma_\gamma = F^2_{n_k}(\gamma) \int w \, d\sigma_\gamma \underset{k\to\infty}{\longrightarrow} 0. \tag{2.14}$$

Furthermore, by ($\mu.\sigma$)

$$\begin{aligned}
&\int_{\overline{\Gamma}_E}\int_E F^2_{n_k}(\gamma + \varepsilon_x) \, w(x) \, \sigma_\gamma(dx) \, (1 + \langle w, \gamma\rangle)^{-1} \mu(d\gamma) \\
&= \int_{\overline{\Gamma}_E} F^2_{n_k}(\gamma) \int_E \frac{w(x)}{1 + \langle w, \gamma\rangle - w(x)} \, \gamma(dx) \, \mu(d\gamma) \\
&\le \int_{\overline{\Gamma}_E} F^2_{n_k}(\gamma) \, \mu(d\gamma) < \frac{1}{2^k} \, ,
\end{aligned}$$

since the integral w.r.t. γ is dominated by 1 for all $\gamma \in \overline{\Gamma}_E$ and because of (2.11). Hence

$$\infty > \sum_{k=1}^{\infty} \left(\int_{\overline{\Gamma}_E}\int_E F^2_{n_k}(\gamma + \varepsilon_x) \, w(x) \, \sigma_\gamma(dx) \, (1 + \langle w, \gamma\rangle)^{-1} \mu(d\gamma) \right)^{1/2}$$

$$\geq \int_{\overline{\Gamma}_E} \sum_{k=1}^{\infty} \left(\int_E F_{n_k}^2(\gamma + \varepsilon_x) \; w(x) \; \sigma_\gamma(dx) \right)^{1/2} (1 + \langle w, \gamma \rangle)^{-1} \; \mu(d\gamma).$$

But $\langle w, \gamma \rangle < \infty$ for μ-a.e. $\gamma \in \overline{\Gamma}_E$, since $w \in L^1(E; \sigma^\mu)$. Hence we conclude that for μ-a.e. $\gamma \in \overline{\Gamma}_E$

$$\int F_{n_k}^2(\gamma + \varepsilon_x) \; w(x) \; \sigma_\gamma(dx) \underset{k \to \infty}{\longrightarrow} 0. \tag{2.15}$$

Now Claim 1 follows by (2.14) and (2.15).

CLAIM 2. For μ-a.e. $\gamma \in \overline{\Gamma}_E$

$$S(u_{n_k}^{(\gamma)}) \underset{k \to \infty}{\longrightarrow} 0 \quad \sigma_\gamma\text{-a.e.}$$

To prove Claim 2 we note that by Claim 1 and (2.13) assumption (C) implies that for μ-a.e. $\gamma \in \overline{\Gamma}_E$

$$\mathcal{E}_{w\sigma_\gamma}^E(u_{n_k}^{(\gamma)}, u_{n_k}^{(\gamma)}) \underset{k \to \infty}{\longrightarrow} 0.$$

Hence Claim 2 follows from (2.12) since $w > 0$.

From Claim 2 we now easily deduce (2.10) by (2.6) and Fatou's Lemma as follows:

$$\begin{aligned} \mathcal{E}_\mu^\Gamma(F_{n_k}, F_{n_k}) &\leq \int_{\overline{\Gamma}_E} \liminf_{l \to \infty} \int_E S(u_{n_k}^{(\gamma)} - u_{n_l}^{(\gamma)})(x) \; \sigma_\gamma(dx) \; \mu(d\gamma) \\ &\leq \liminf_{l \to \infty} \; \mathcal{E}_\mu^\Gamma(F_{n_k} - F_{n_l}, F_{n_k} - F_{n_l}), \end{aligned}$$

which by (2.9) can be made arbitrarily small for k large enough. □

REMARK 2.7. (i) All above results are obviously valid in the more general situation described in Remark 1.5.

(ii) If $(\mathcal{E}_\mu^\Gamma, \mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D}))$ is closable on $L^2(\overline{\Gamma}_E; \mu)$, then S^Γ extends to all of $D(\mathcal{E}_\mu^\Gamma) \times \mathcal{D}(\mathcal{E}_\mu^\Gamma)$. We shall use this fact below without further notice.

2.4. Examples

2.4.0. Poisson measures Let $\sigma \in \mathcal{M}(\{E_k\})$ and $\mu := \pi_\sigma$ be the Poisson measure on $(\overline{\Gamma}_E, \mathcal{B}(\overline{\Gamma}_E))$ with intensity σ, i.e., π_σ is the unique probability measure on $(\overline{\Gamma}_E, \mathcal{B}(\overline{\Gamma}_E))$ such that

$$\int e^{\langle f, \gamma \rangle} \; \pi_\sigma(d\gamma) = e^{\int (e^f - 1) d\sigma} \quad \text{for all } f \in \mathcal{F}_b(\{E_k\}). \tag{2.16}$$

Then it is well-known that $\mu := \pi_\sigma$ satisfies conditions (μ.1) (since $\sigma^\mu = \sigma$) and ($\mu.\sigma$) with $\rho(\gamma, x) = 1$ for all $x \in E$, $\gamma \in \overline{\Gamma}_E$. Condition ($\mu.\sigma$) is just the so-called Mecke identity (cf. [27, Satz 3.1]) in this case.

2.4.1. Riemannian manifolds Consider the situation described in Subsection 1.4.1 with $(E_k)_{k\in\mathbb{N}}$ and (S,$\mathcal{D}$) as given there, and let σ be any positive Radon measure on $(X, \mathcal{B}(X))$. Clearly, conditions (S.2) and (S.3) hold in this case.

a) $\mu := \pi_\sigma$

Then Theorem 2.4 (i) applies and, clearly, if every $x \in \operatorname{supp}\sigma$ (:= the largest closed subset A of X such that $\sigma(X \setminus A) = 0$) is an accumulation point of $\operatorname{supp}\sigma \setminus \{x\}$ (which is e.g. the case if σ is absolutely continuous w.r.t. m), then condition (S.σ) holds. Hence Theorem 2.4 (ii) applies.

In order to satisfy condition (C), e.g., the following condition will do:

($\sigma.m$) There exists $\rho := d\sigma/(dm)$, where m is the volume element on X, such that either

(i) $\sqrt{\rho} \in H^{1,2}_{\mathrm{loc}}(X;m)$ (i.e., $\sqrt{\rho}$ is locally in the Sobolev space of order 1 in $L^2(X;m)$).

(ii) For m-a.e. $x \in \{\rho > 0\}$ there exists $\varepsilon > 0$ such that

$$\int_{\{y \mid |y-x|<\varepsilon\}} \frac{1}{\rho(y)}\, m(dy) < \infty$$

(cf. [25, Chap. II, Subsections 1a) and 2a)] whose results immediately generalize to manifolds). Condition ($\sigma.m$) implies that (C) holds for any continuous strictly positive $w : X \to (0, 1]$, which is σ-integrable, and hence Theorem 2.6 applies. Since, clearly, $\mathcal{D}$ is dense in $L^2(X;\sigma)$ and $\mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$ is dense in $L^2(\overline{\Gamma}_X;\pi_\sigma)$, the closures $(\mathcal{E}_\sigma^X, D(\mathcal{E}_\sigma^X))$ and $(\mathcal{E}_{\pi_\sigma}^\Gamma, D(\mathcal{E}_{\pi_\sigma}^\Gamma))$ are Dirichlet forms on $L^2(X;\sigma)$, $L^2(\overline{\Gamma}_X;\pi_\sigma)$ respectively.

b) $\mu :=$ Gibbs measure

This case has been discussed in all detail in [14, Sect. 6], and, in particular, in [31, Sect. 6] to which we refer for definitions and exact conditions (which are generally very weak) implying that Theorems 2.4 and 2.6 apply in these cases. Here again $w : X \to (0, 1]$ can be taken to be any continuous strictly positive function, which is σ^μ-integrable. In particular, we generalize the closability results in [29] and [37] (cf. [31, Remark 7.5]).

2.4.2. The free loop space Consider the situation described in Subsection 1.4.2 with $(E_k)_{k\in\mathbb{N}}$ and (S,$\mathcal{D}$) as given there. Clearly, conditions (S.2), (S.3) hold in this case. Then by the definition (see (1.8)) of σ we have that $\sigma \in \mathcal{M}(\{E_k\})$. In order to define $\mathcal{E}_\sigma^E$ (with $E := \mathcal{L}(\mathbb{R}^d)$) by (2.3) we first have to show that $\omega \mapsto S(u, v)(\omega)$ is $\mathcal{B}(\mathcal{L}(\mathbb{R}^d))$-measurable for all $u, v \in \mathcal{D}$. We shall, in fact, prove more, namely that we can construct the direct integral of H_ω, $\omega \in \mathcal{L}(\mathbb{R}^d)$, i.e.,

$$\mathcal{H} := \int_{\mathcal{L}(\mathbb{R}^d)}^{\oplus} H_\omega\ \sigma(d\omega),$$

in the sense of [13, Chap. II, Sect. 1] and that $Du \in \mathcal{H}$ for all $u \in \mathcal{D}$. To this end we first note that for all $\omega \in N^c$ (cf. Subsection 1.4.1 for the definition of N) every $h \in H$ has a unique decomposition

$$h(t) = \hat{h}(\omega)(t) + a(\omega) \cdot t, \quad t \in [0, 1], \tag{2.17}$$

with $a(\omega) := h(1) - \tau_1(\omega)^{-1}h(0)$ and $\hat{h}(\omega)(t) := h(t) - a(\omega) \cdot t$. Clearly, $\hat{h}(\omega) \in H_\omega$. Furthermore, obviously (since $\tau_1(\omega)$ is a bijection)

$$H_\omega \cap \{[0, 1] \ni t \mapsto a \cdot t \mid a \in \mathbb{R}^d\} = \{0\},$$

and both these subspaces of H are closed w.r.t. $\|\cdot\|_{\omega(0)}$. Hence the map $P_\omega : H \to H_\omega$ defined by

$$P_\omega h := \hat{h}(\omega), \quad h \in H,$$

with $\hat{h}(\omega)$ as in (2.17) is continuous. Defining $P_\omega : H \to H_\omega$ by $P_\omega :\equiv 0$ for $\omega \in N$, we obtain that for all $h_1, h_2 \in H$

$$\mathcal{L}(\mathbb{R}^d) \ni \omega \mapsto (P_\omega h_1, P_\omega h_2)_{\omega(0)}$$

is $\mathcal{B}(\mathcal{L}(\mathbb{R}^d))$-measurable.

Now let $\{\tilde{h}_n \in H \mid n \in \mathbb{N}\}$ be a dense set in H (w.r.t. $\|\cdot\|_{\omega(0)}$; recall that $\|\cdot\|_{\omega(0)}$, $\omega \in \mathcal{L}(\mathbb{R}^d)$, are all equivalent). Then $\{P_\omega \tilde{h}_n \mid n \in \mathbb{N}\}$ is a dense subset of H_ω (w.r.t. $\|\cdot\|_{\omega(0)}$). Setting $h_n(\omega) := P_\omega \tilde{h}_n$ for $\omega \in N^c$ and $h_n(\omega) := 0$ for all $\omega \in N$, $n \in \mathbb{N}$, we obtain that for all $n, m \in \mathbb{N}$

$$\mathcal{L}(\mathbb{R}^d) \ni \omega \mapsto (h_n(\omega), h_m(\omega))_{\omega(0)}$$

is $\mathcal{B}(\mathcal{L}(\mathbb{R}^d))$-measurable. We fix $(h_n(\omega))_{n\in\mathbb{N}}$, $\omega \in \mathcal{L}(\mathbb{R}^d)$, for the rest of this section. Now we define a *measurable field of vectors* $\mathfrak{M}$ in the sense of [13, Chap. II] as follows: let $\mathfrak{M}$ be the set of all maps ("sections") $V : \mathcal{L}(\mathbb{R}^d) \to \bigcup_{\omega\in\mathcal{L}(\mathbb{R}^d)} H_\omega$, such that $V(\omega) \in H_\omega$ (where $H_\omega := \{0\}$ for $\omega \in N$) and for all $n \in \mathbb{N}$

$$\mathcal{L}(\mathbb{R}^d) \ni \omega \mapsto (V(\omega), h_n(\omega))_{\omega(0)}$$

is $\mathcal{B}(\mathcal{L}(\mathbb{R}^d))$-measurable. Let

$$\mathcal{H} := \int_{\mathcal{L}(\mathbb{R}^d)}^{\oplus} H_\omega \, \sigma(d\omega) \tag{2.18}$$

be the corresponding *direct integral* of $(H_\omega, (\ ,\ )_{\omega(0)})_{\omega\in\mathcal{L}(\mathbb{R}^d)}$ in the sense of [13, Chap. II, Sect. 1], i.e., all $V \in \mathfrak{M}$ such that

$$\|V\|_{\mathcal{H}}^2 := \int_{\mathcal{L}(\mathbb{R}^d)} \|V(\omega)\|_{\omega(0)}^2 \, \sigma(d\omega) < \infty. \tag{2.19}$$

Note that by an obvious variant of the classical Riesz-Fischer theorem $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is complete. Let $\Pi_\omega : H \to H_\omega$ denote the (w.r.t. $(\ ,\ )_{\omega(0)}$) orthogonal projection. Then, clearly,

$$\mathcal{L}(\mathbb{R}^d) \ni \omega \mapsto (\Pi_\omega h, h_n(\omega))_{\omega(0)} = (h, h_n(\omega))_{\omega(0)}$$

is $\mathcal{B}(\mathcal{L}(\mathbb{R}^d))$-measurable for all $h \in H$, $n \in \mathbb{N}$. That is, $\omega \mapsto \Pi_\omega h$ belongs to $\mathfrak{M}$. Hence for $u \in \mathcal{D}$ it easily follows from (1.14) that $\omega \mapsto Du(\omega) = \Pi_\omega(\tilde{D}u(\omega))$ belongs to $\mathfrak{M}$. Since $\sigma \in \mathcal{M}(\{E_k\})$, (1.14) implies that

$$\int_{\mathcal{L}(\mathbb{R}^d)} (\tilde{D}u(\omega), \tilde{D}u(\omega))^2_{\omega(0)}\, \sigma(d\omega) < \infty.$$

Since for all $\omega \in N^c$

$$(Du(\omega), Du(\omega))_{\omega(0)} \leq (\tilde{D}u(\omega), \tilde{D}u(\omega))_{\omega(0)},$$

we hence conclude that also $\omega \mapsto Du(\omega)$ is in $\mathcal{H}$ for all $u \in \mathcal{D}$. So, we may define (as in (2.3))

$$\mathcal{E}^E_\sigma(u, v) := \int_{\mathcal{L}(\mathbb{R}^d)} S(u, v)\, d\sigma; \quad u, v \in \mathcal{D}.$$

Next, let us show that (S.σ) holds. We recall the following result from [23, 24] (see also [19] for another proof) which we need below in an essential way.

Proposition 2.8. *For all $n \in \mathbb{N}$ there exists $\beta_n \in \bigcap_{p\geq 1} L^p(\mathcal{L}(\mathbb{R}^d); \sigma)$ such that for all $u, v \in \mathcal{D}$*

$$\text{(2.20)} \quad \begin{aligned} &\int \partial_{h_n(\omega)} u(\omega)\ v(\omega)\ \sigma(d\omega) \\ &= -\int u(\omega)\, \partial_{h_n(\omega)} v(\omega)\ \sigma(d\omega) - \int u(\omega) v(\omega) \beta_n(\omega)\ \sigma(d\omega) \end{aligned}$$

where $\partial_{h_n(\omega)}$ is defined by (1.11) *with h replaced by $h_n(\omega)$.*

As an immediate consequence we obtain:

Lemma 2.9. *Condition (S.σ) holds. In particular, S respects σ-classes of $\mathcal{D}$ and $(\mathcal{E}^E_\sigma, \mathcal{D}^\sigma)$ is a well-defined positive definite symmetric bilinear form on $L^2(\mathcal{L}(\mathbb{R}^d);\sigma)$.*

Proof. Let $u_1, u_2 \in \mathcal{D}$ such that $u_0 := u_1 - u_2 = 0$ σ-a.e. Let $n \in \mathbb{N}$. Then by Proposition 2.8 for all $n \in \mathbb{N}$, $v \in \mathcal{D}$,

$$\int \partial_{h_n(\omega)} u_0(\omega)\ v(\omega)\ \sigma(d\omega) = 0\ .$$

Since $\mathcal{D}$ is dense in $L^2(\mathcal{L}(\mathbb{R}^d);\sigma)$, it follows that $\partial_{h_n(\omega)}u_0(\omega) = 0$ for σ-a.e. $\omega \in \mathcal{L}(\mathbb{R}^d)$. Hence for all $n \in \mathbb{N}$ and σ-a.e. $\omega \in \mathcal{L}(\mathbb{R}^d)$

$$\begin{aligned}(2.21)\qquad & (Du_0(\omega), h_n(\omega))_{\omega(0)} \\ &= (\tilde{D}u_0(\omega), h_n(\omega)) \\ &= \partial_{h_n(\omega)}u_0(\omega) = 0.\end{aligned}$$

Therefore, $Du_0 = 0$ as an element in $\mathcal{H}$. Thus, $S(u_0, v) = 0$ σ-a.e. for all $v \in \mathcal{D}$. □

Next we show that $(\mathcal{E}_\sigma^E, \mathcal{D}^\sigma)$ is closable on $L^2(\mathcal{L}(\mathbb{R}^d);\sigma)$. In fact, this was claimed already in [7], but the proof was a little sketchy. So, we give the details here.

Proposition 2.10. (i) *$(\mathcal{E}_\sigma^E, \mathcal{D}^\sigma)$ is closable on $L^2(\mathcal{L}(\mathbb{R}^d);\sigma)$.*
(ii) *If $(\mathcal{E}_\sigma^E, D(\mathcal{E}_\sigma^E))$ denotes the closure of $(\mathcal{E}_\sigma^E, \mathcal{D}^\sigma)$ on $L^2(\mathcal{L}(\mathbb{R}^d);\sigma)$ and if $w \in D(\mathcal{E}_\sigma^E)$, w bounded, $w > 0$ σ-a.e., then also $(\mathcal{E}_{w\cdot\sigma}^E, \mathcal{D}^\sigma)$ is closable on $L^2(\mathcal{L}(\mathbb{R}^d); w\cdot\sigma)$.*

Proof. (i): Let $u_n \in \mathcal{D}$, $n \in \mathbb{N}$, such that $u_n \to 0$ in $L^2(\mathcal{L}(\mathbb{R}^d);\sigma)$ as $n \to \infty$ and $\mathcal{E}_\sigma^E(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$. Since $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ (cf. (2.18), (2.19)) is complete, there exists $V \in \mathcal{H}$ such that

$$\lim_{n\to\infty} \|Du_n - V\|_{\mathcal{H}} = 0.$$

We have to show that

$$(2.22)\qquad V = 0 \qquad \text{(as an element of } \mathcal{H}).$$

But by (2.21) and Proposition 2.8 for all $k \in \mathbb{N}$, $u \in \mathcal{D}$

$$\begin{aligned}&\int \Big(u(\omega)h_k(\omega), V(\omega)\Big)_{\omega(0)} \sigma(d\omega) \\ &= \lim_{n\to\infty} \int u(\omega)\Big(h_k(\omega), Du_n(\omega)\Big)_{\omega(0)} \sigma(d\omega) \\ &= \lim_{n\to\infty} \int u(\omega)\partial_{h_k(\omega)}u_n(\omega)\ \sigma(d\omega) \\ &= -\lim_{n\to\infty} \int \Big(\partial_{h_k(\omega)}u(\omega) + u(\omega)\beta_k(\omega)\Big)\ u_n(\omega)\ \sigma(d\omega) \\ &= 0.\end{aligned}$$

But, clearly $\omega \mapsto u(\omega)\ h_k(\omega)$, $k \in \mathbb{N}$, $u \in \mathcal{D}$, form a total set in $\mathcal{H}$. So, (2.22) follows.

(ii): By assertion (i) and by approximation (2.20) extends to all $u \in D(\mathcal{E}_\sigma^E)$. Replacing u in (2.20) by $u \cdot w$ ($\in \mathcal{D}(\mathcal{E}_\sigma^E)$) and using the product rule, we obtain that (2.20) holds for the measure $w \cdot \sigma$ instead of σ with β_n replaced by $\beta_n + \partial_{h_n(\cdot)}w/w$.

Now the assertion follows by exactly the same arguments as in the proof of assertion (i). □

Let now $\mu := \pi_\sigma$ be the Poisson measure on $\overline{\Gamma}_E$ with intensity measure σ. By Subsection 2.4.0 we know that $(\mu.1)$ and $(\mu.\sigma)$ are satisfied with $\sigma_\gamma = \sigma$ for all $\gamma \in \overline{\Gamma}_E$. So, if $(S^\Gamma, \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}))$ is defined by (1.4) (i.e., as in Subsection 1.4.2), then Theorem 2.4 applies. Furthermore, by [9, Lemma 5.2.1] there exists a strictly positive, bounded continuous function $w \in D(\mathcal{E}_\sigma^E)$. Hence, by Proposition 2.10 also (C) holds. Hence also Theorem 2.6 applies to $(\mathcal{E}_{\pi_\sigma}^\Gamma, \mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D}))$ given by (2.5).

REMARK 2.11. (i) So far, we have only considered the case μ = Poisson measure π_σ. Similarly, we can consider mixed Poisson measures here (and likewise in Subsection 2.4.1), i.e.,

$$\mu = \int_{\mathbb{R}_+} \pi_{z\sigma}\ \lambda(dz),$$

where λ is a probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that

$$\int_{\mathbb{R}_+} z\,\lambda(dz) < \infty.$$

(ii) We can also consider Gibbs measures on $\overline{\Gamma}_{\mathcal{L}(\mathbb{R}^d)}$. Candidates for such have been constructed in [21]. But the claim made in the latter reference that they are actually Gibbs measures is yet unproved, since the proof for this given there contains a substantial gap. More details on this will be discussed in [6].

3. Topology for configuration spaces

Our next aim is to find conditions (e.g. on $(\mathcal{E}_\sigma^E, D(\mathcal{E}_\sigma^E))$) that imply the quasi-regularity of $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ on $\overline{\Gamma}_E$. For this we need a separable topology on $\overline{\Gamma}_E$ which can be metrized by a complete metric. The construction thereof is the purpose of this section.

3.1. A metric on $\mathcal{M}(E)$ In this section we assume that (E, ρ) is a separable metric space. Let $\mathcal{B}(E)$ denote the corresponding Borel σ-field and $\mathcal{B}_b(E)$ the set of all bounded $\mathcal{B}(E)$-measurable real functions on E. Let $\mathcal{M}(E)$ denote the set of all finite positive measures on $\mathcal{B}(E)$, and $\mathcal{M}_1(E)$ the subset of $\mathcal{M}(E)$ consisting of all probability measures. Recall that the Prohorov metric p on $\mathcal{M}_1(E)$ for $\gamma_1, \gamma_2 \in \mathcal{M}_1(E)$ is defined by

$$p(\gamma_1, \gamma_2) := \inf\{\varepsilon > 0 \mid \gamma_1(A) \le \gamma_2(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathcal{B}(E)\} \tag{3.1}$$

where $A^\varepsilon := \{x \in E \mid \rho(x, A) < \varepsilon\}$. It is known that p induces the topology of weak convergence which is separable since (E, ρ) is separable. p is equivalent to the following metric β

(3.2) $$\beta(\gamma_1, \gamma_2) := \sup\left\{\left|\int f\,d\gamma_1 - \int f\,d\gamma_2\right| \;\middle|\; f \in \mathcal{B}_b(E) \text{ such that } \|f\|_{BL} \le 1\right\}$$

where $\|f\|_{BL} := \sup_{x\in E} |f(x)| + \sup_{x\neq y} |f(x) - f(y)|\, \rho(x, y)^{-1}$. See e.g. [15, Sect. 11.3] for details.

Below for any $\mathcal{B}(E)$-measurable function f on E and any Borel measure γ on E we write $\bar{f}(\gamma) := \int f\,d\gamma$ provided the integral makes sense. For $A \in \mathcal{B}(E)$ and $\varepsilon > 0$, we set

(3.3) $$g_{A,\varepsilon}(x) := \frac{1}{1+\varepsilon}(\varepsilon - \rho(x, A) \wedge \varepsilon)\ ,\ \ \forall x \in E.$$

We now define for $\gamma_1, \gamma_2 \in \mathcal{M}(E)$

(3.4) $$\bar{\rho}_0(\gamma_1, \gamma_2) := \sup\{|\bar{g}_{A,\varepsilon}(\gamma_1) - \bar{g}_{A,\varepsilon}(\gamma_2)| \mid A \in \mathcal{B}(E),\ \ \varepsilon > 0\}.$$

Note that in (3.4) it is enough to take the supremum over all closed subsets A of E since $g_{A,\varepsilon} = g_{\bar{A},\varepsilon}$ for all $A \in \mathcal{B}(E)$ (here $\bar{A}$ stands for the closure of A).

Lemma 3.1. *If $\gamma_1, \gamma_2 \in \mathcal{M}_1(E)$, then*

$$p(\gamma_1, \gamma_2) \le 2\,\bar{\rho}_0(\gamma_1, \gamma_2)^{1/2} \quad \textit{and} \quad \bar{\rho}_0(\gamma_1, \gamma_2) \le \beta(\gamma_1, \gamma_2) \le 2\ p(\gamma_1, \gamma_2).$$

Proof. See the proof of Theorem 11.3.3. and Corollary 11.6.5 in [15]. □

Theorem 3.2. *$\bar{\rho}_0$ is a metric on $\mathcal{M}(E)$ and the $\bar{\rho}_0$-topology coincides with the (separable) topology of weak convergence on $\mathcal{M}(E)$. Moreover, $\bar{\rho}_0$ is complete on $\mathcal{M}(E)$ if ρ is complete on E.*

Proof. Obviously $\bar{\rho}_0$ is symmetric and satisfies the triangle inequality. Moreover, $\bar{\rho}_0$ has the following properties:

(3.5) $\bar{\rho}_0(\gamma_1 + \gamma_3, \gamma_2 + \gamma_3) = \bar{\rho}_0(\gamma_1, \gamma_2)$, for all $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{M}(E)$;

(3.6) $\bar{\rho}_0(c\,\gamma_1, c\,\gamma_2) = c\,\bar{\rho}_0(\gamma_1, \gamma_2)$, for all $\gamma_1, \gamma_2 \in \mathcal{M}(E)$, $c \in \mathbb{R}^+$;

(3.7) $\bar{\rho}_0(\gamma_1, \gamma_2) \ge |\gamma_1(E) - \gamma_2(E)|$, for all $\gamma_1, \gamma_2 \in \mathcal{M}(E)$;

(3.8) $\bar{\rho}_0(c_1\gamma, c_2\gamma) = |c_1 - c_2|\gamma(E)$, for all $c_1, c_2 \in \mathbb{R}^+$, $\gamma \in \mathcal{M}(E)$.

Indeed, (3.5) and (3.6) can be verified straightforward. (3.7) is verified by noticing that $g_{E,\varepsilon} = \varepsilon/(1+\varepsilon)$ for all $\varepsilon > 0$, which also implies (3.8). Suppose now $\bar{\rho}_0(\gamma_1, \gamma_2) = 0$. By (3.7) we have $\gamma_1(E) = \gamma_2(E)$. Without loss of generality we may assume $\gamma_1(E) = c > 0$. Let $\gamma_1' = c^{-1}\gamma_1$, and $\gamma_2' = c^{-1}\gamma_2$. Applying (3.6) and Lemma 3.1 we see that $\gamma_1' = \gamma_2'$, consequently $\gamma_1 = \gamma_2$, which shows that $\bar{\rho}_0$ is a metric on $\mathcal{M}(E)$. In the remainder of the proof we fix a probability measure $\mu \in \mathcal{M}_1(E)$. For any element $\gamma \in \mathcal{M}(E)$ we set $\gamma^\# := (\gamma + \mu)/(\gamma(E) + 1)$. Then applying (3.5)–(3.8) we see that the following two statements are equivalent:

(a) $\bar{\rho}_0(\gamma_n, \gamma) \longrightarrow 0$ when $n \to \infty$,

(b) $\bar{\rho}_0(\gamma_n^\#, \gamma^\#) \longrightarrow 0$ and $\gamma_n(E) \longrightarrow \gamma(E)$ when $n \to \infty$.

But by Lemma 3.1 the above statement (b) is equivalent to

(c) $\gamma_n^\#$ converges weakly to $\gamma^\#$ and $\gamma_n(E) \longrightarrow \gamma(E)$.

Consequently, (a) is equivalent to γ_n converges weakly to γ. In other words, the $\bar{\rho}_0$-topology coincides with the topology of weak convergence on $\mathcal{M}(E)$.

Suppose now (E, ρ) is complete. Then by Lemma 3.1 $\mathcal{M}_1(E)$ is a $\bar{\rho}_0$-closed subset of $\mathcal{M}(E)$. Let γ_n, $n \in \mathbb{N}$, be a $\bar{\rho}_0$-Cauchy sequence in $\mathcal{M}(E)$. Then $\gamma_n^\#$, $n \in \mathbb{N}$, is a $\bar{\rho}_0$-Cauchy sequence in $\mathcal{M}_1(E)$ and $\gamma_n(E)$, $n \in \mathbb{N}$, is a Cauchy sequence of real numbers. Let $\gamma^\#$ be the limit of $\gamma_n^\#$ and c be the limit of $\gamma_n(E)$, $n \in \mathbb{N}$, respectively. Then $\gamma_n \underset{n\to\infty}{\longrightarrow} c\, \gamma^\#$ weakly. Thus $\bar{\rho}_0$ is complete. □

Corollary 3.3. *There exists a countable family of pairs (A_j, ε_j), $j \in \mathbb{N}$, with A_j being closed subsets of E and ε_j strictly positive numbers such that for all $\gamma, \gamma' \in \mathcal{M}(E)$*

$$\bar{\rho}_0(\gamma, \gamma') = \sup_j |\bar{g}_{A_j,\varepsilon_j}(\gamma) - \bar{g}_{A_j,\varepsilon_j}(\gamma')|. \tag{3.9}$$

Proof. Let $\{\gamma_l \mid l \in \mathbb{N}\}$ be a countable dense subset of $\mathcal{M}(E)$. For each pair (γ_l, γ_m) we can find a sequence of pairs $(A_{l,m,n}, \varepsilon_{l,m,n})_{n\in\mathbb{N}}$ with $A_{l,m,n}$ being closed subsets of E and $\varepsilon_{l,m,n}$ being strictly positive numbers such that

$$\bar{\rho}_0(\gamma_l, \gamma_m) = \sup_n |\bar{g}_{A_{l,m,n},\varepsilon_{l,m,n}}(\gamma_l) - \bar{g}_{A_{l,m,n},\varepsilon_{l,m,n}}(\gamma_m)|.$$

Rearrange $\{(A_{l,m,n}, \varepsilon_{l,m,n}) \mid l, m, n \in \mathbb{N}\}$ by $\{(A_j, \varepsilon_j) \mid j \in \mathbb{N}\}$. Then (A_j, ε_j), $j \in \mathbb{N}$, is as desired. □

3.2. $\{E_k\}$-vague topology From now on we shall deal with locally finite Borel measures on a separable metric space E. Recall that a positive Borel measure ν on E is said to be *locally finite* if for each $x \in E$ there exists an open neighbourhood U of x such that $\nu(U) < \infty$. If E is complete, every locally finite Borel measure is a Radon measure on E (cf. [35]).

Let us call $(E_k)_{k\in\mathbb{N}}$ an *exhausting sequence* if $(E_k)_{k\in\mathbb{N}}$ is an increasing sequence of open sets such that $\bigcup_{k\in\mathbb{N}} E_k = E$. $(E_k)_{k\in\mathbb{N}}$ will be called a *well-exhausting sequence* if, in addition to the above property, there exists a sequence $(\delta_k)_{k\in\mathbb{N}}$ of strictly positive numbers such that

$$E_k^{\delta_k} \subset E_{k+1}, \quad \forall k \in \mathbb{N}. \tag{3.10}$$

Let $(E_k)_{k\in\mathbb{N}}$ be an exhausting sequence. We shall write $\gamma \in \mathcal{M}(\{E_k\})$ if γ is a positive Borel measure and $\gamma(E_k) < \infty$ for all $k \in \mathbb{N}$. Note that γ is locally finite if and only if $\gamma \in \mathcal{M}(\{E_k\})$ for some exhausting sequence $(E_k)_{k\in\mathbb{N}}$. If, in addition, E is locally compact, then we can find a well-exhausting sequence $(E_k)_{k\in\mathbb{N}}$ such that any locally finite measure γ is in $\mathcal{M}(\{E_k\})$. In the general case we have the following result:

Lemma 3.4. *For any exhausting sequence $(A_k)_{k\in\mathbb{N}}$, there exists a well-exhausting sequence $(E_k)_{k\in\mathbb{N}}$ such that*

$$\mathcal{M}(\{A_k\}) \subset \mathcal{M}(\{E_k\}).$$

Proof. Let $E_k := \{x \in E \mid \rho(x, A_k^c) > 1/2^k\}$, and let $\delta_k := 2^{-(k+1)}$. Then $E_k \subset A_k$ and (3.10) is fulfilled for each k. Hence one can check that $(E_k)_{k\in\mathbb{N}}$ is as desired. □

From now on we fix a well-exhausting sequence $(E_k)_{k\in\mathbb{N}}$. For example, $E_k := \{x \in E \mid \rho(x, x_0) < k\}$ for some fixed point $x_0 \in E$. We shall write $f \in C_0(\{E_k\})$ if $f \in C_b(E)$ and $\operatorname{supp} f \subset E_k$ for some $k \in \mathbb{N}$. Note that if $f \in C_0(\{E_k\})$, then $\bar{f}(\gamma)$ is well-defined for all $\gamma \in \mathcal{M}(\{E_k\})$.

DEFINITION 3.5. Let $\gamma, \gamma_n \in \mathcal{M}(\{E_k\})$, $n \in \mathbb{N}$. We say that $(\gamma_n)_{n\in\mathbb{N}}$ *converges $\{E_k\}$-vaguely* to γ, if

$$|\bar{f}(\gamma_n) - \bar{f}(\gamma)| \longrightarrow 0\ , \quad \forall f \in C_0(\{E_k\}). \tag{3.11}$$

Note that if E is a locally compact space and E_k is relatively compact for all k, then $\mathcal{M}(\{E_k\})$ is exactly the family of all Radon measures on E and $\{E_k\}$-vague convergence coincides with the usual vague convergence for Radon measures. In this case it is well-known that there is a complete metric on $\mathcal{M}(\{E_k\})$ which induces the vague topology (which is separable). However, we have to modify this metric to cover non-locally compact spaces E, as we shall do below.

By our assumption $(E_k)_{k\in\mathbb{N}}$ is a well-exhausting sequence, hence there exists a sequence of positive numbers $(\delta_k)_{k\in\mathbb{N}}$ satisfying (3.10). We set $\phi_k := ((1+\delta_k)/\delta_k)g_{E_k,\delta_k}$ (cf. (3.3)), fix a function $\zeta \in C_b^\infty(\mathbb{R})$ such that $0 \le \zeta \le 1$ on $[0,\infty)$, $\zeta(t) = t$ on $[-1/2, 1/2]$, $\zeta' > 0$, $\zeta'' \le 0$, and define for $\gamma_1, \gamma_2 \in \mathcal{M}(\{E_k\})$

$$\bar{\rho}(\gamma_1, \gamma_2) := \sup_{k\in\mathbb{N}} c_k\, \zeta(\bar{\rho}_0(\phi_k \cdot \gamma_1, \phi_k \cdot \gamma_2)), \tag{3.12}$$

where $c_k \in \mathbb{R}_+$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} c_k = 0$.

Theorem 3.6. *$\bar{\rho}$ is a metric on $\mathcal{M}(\{E_k\})$, the corresponding induced topology is separable and a sequence converges w.r.t. this topology if and only if it converges*

$\{E_k\}$-vaguely. Moreover, $\bar{\rho}$ is a complete metric on $\mathcal{M}(\{E_k\})$ if ρ is complete on E.

Proof. Suppose that $\bar{\rho}(\gamma_1, \gamma_2) = 0$ for some $\gamma_1, \gamma_2 \in \mathcal{M}(\{E_k\})$. Then $\phi_k \cdot \gamma_1 = \phi_k \cdot \gamma_2$ for all $k \in \mathbb{N}$. Letting $k \to \infty$ we get $\gamma_1 = \gamma_2$, which implies that $\bar{\rho}$ is a metric on $\mathcal{M}(\{E_k\})$. Let $\gamma, \gamma_n \in \mathcal{M}(\{E_k\})$, $n \in \mathbb{N}$. One can easily check that the following statements (a) – (d) are equivalent:

(a) $\bar{\rho}(\gamma_n, \gamma) \underset{n\to\infty}{\longrightarrow} 0$.

(b) $\bar{\rho}_0(\phi_j \cdot \gamma_n, \phi_j \cdot \gamma) \underset{n\to\infty}{\longrightarrow} 0$, $\forall j \in \mathbb{N}$.

(c) $\bar{f}(\phi_j \cdot \gamma_n) \underset{n\to\infty}{\longrightarrow} \bar{f}(\phi_j \cdot \gamma)$, $\forall f \in C_b(E)$, $j \in \mathbb{N}$.

(d) $\bar{f}(\gamma_n) \underset{n\to\infty}{\longrightarrow} \bar{f}(\gamma)$, $\forall f \in C_0(\{E_k\})$.

Indeed, (a) $\Leftrightarrow$ (b) follows directly from (3.12), (b) $\Leftrightarrow$ (c) follows from Theorem 3.2. (c) $\Rightarrow$ (d) follows from the fact that $f\phi_j = f$ if $\operatorname{supp} f \subset E_j$. (d) $\Rightarrow$ (c) follows from the fact that $f\phi_j \in C_0(\{E_k\})$ for all $f \in C_b(E)$ and all $j \in \mathbb{N}$. Clearly, the equivalence of (a) and (d) means that the topology induced by $\bar{\rho}$ is compatible with $\{E_k\}$-vague convergence for sequences. Also, since $\phi_j\phi_{j+1} = \phi_j$, the equivalence (a) $\Leftrightarrow$ (b) implies that $\bar{\rho}$ is separable since $\bar{\rho}_0$ is a separable metric on $\mathcal{M}(E)$. It remains to check the last assertion. To this end we now assume that ρ is a complete metric on E, then $\bar{\rho}_0$ is complete on $\mathcal{M}(E)$. Let γ_n, $n \in \mathbb{N}$, be a $\bar{\rho}$-Cauchy sequence in $\mathcal{M}(\{E_k\})$. Then for each $j \in \mathbb{N}$, $(\phi_j \cdot \gamma_n)_{n\in\mathbb{N}}$ is a $\bar{\rho}_0$-Cauchy sequence in $\mathcal{M}(E)$. Hence there exists $\gamma^{(j)} \in \mathcal{M}(E)$ such that $\bar{\rho}_0(\phi_j \cdot \gamma_n, \gamma^{(j)}) \underset{n\to\infty}{\longrightarrow} 0$. Note that $\phi_{j+k}\phi_j = \phi_j$ for all $j, k \in \mathbb{N}$. Hence by the uniqueness of the $\bar{\rho}_0$-limit one can easily check that

$$\phi_j \cdot \gamma^{(j+k)} = \gamma^{(j)}, \quad \forall j, k \in \mathbb{N}. \tag{3.13}$$

In particular, (3.13) implies $\gamma^{(j)} \leq \gamma^{(j+1)}$ for all $j \in \mathbb{N}$. Hence one may define a positive Borel measure γ by

$$\gamma := \gamma^{(1)} + \sum_{j=1}^{\infty}(\gamma^{(j+1)} - \gamma^{(j)}).$$

Now (3.13) implies $\phi_j \cdot \gamma = \gamma^{(j)}$. Consequently, $\gamma \in \mathcal{M}(\{E_k\})$ and $\bar{\rho}_0(\phi_j \cdot \gamma_n, \phi_j \cdot \gamma) = \bar{\rho}_0(\phi_j \cdot \gamma_n, \gamma^{(j)}) \underset{n\to\infty}{\longrightarrow} 0$ for all $j \in \mathbb{N}$. Hence $\bar{\rho}(\gamma_n, \gamma) \underset{n\to\infty}{\longrightarrow} 0$ and $\bar{\rho}$ is complete on $\mathcal{M}(\{E_k\})$. □

DEFINITION 3.7. The topology induced by (any) $\bar{\rho}$ as in (3.12) on $\mathcal{M}(\{E_k\})$ is called the $\{E_k\}$-vague topology.

3.3. The induced metric on $\overline{\Gamma}_E$ Let E and $\mathcal{M}(\{E_k\})$ be as in the previous section. Let Γ_E be the subset of $\overline{\Gamma}_E$ consisting of all γ such that $\gamma(\{x\}) \leq 1$ for all $x \in E$.

For a subset A of E we denote by ∂A the boundary of A. For $\gamma \in \mathcal{M}(\{E_k\})$, we set

$$O_\gamma := \{A \in E \mid A \text{ open}, \ A \subset E_k \text{ for some } k \text{ and } \gamma(\partial A) = 0\}.$$

Lemma 3.8. *For any $\gamma \in \mathcal{M}(\{E_k\})$, O_γ forms a base for the topology of E.*

Proof. For $x \in E$ and $\alpha > 0$ we write

$$B_\alpha(x) := \{y \in E \mid \rho(x, y) < \alpha\}.$$

For any $x \in E$ and $\delta > 0$, we can always find $k \in \mathbb{N}$ and $0 < \alpha \leq \delta$ such that $B_\alpha(x) \subset E_k$ and $\gamma(\partial B_\alpha(x)) = 0$. Clearly, the totality of such $B_\alpha(x)$ forms a topological base of E. □

Proposition 3.9. *$\overline{\Gamma}_E$ is an $\{E_k\}$-vaguely closed subset of $\mathcal{M}(\{E_k\})$.*

Proof. Let $\bar{\rho}(\gamma_n, \gamma) \to 0$ with $\gamma_n \in \overline{\Gamma}_E$, for all $n \in \mathbb{N}$. Then $\gamma_n(A) \to \gamma(A)$ for all $A \in O_\gamma$. Hence $\gamma(A) \in \overline{\mathbb{N}}$ for $A \in O_\gamma$. Thus by monotone convergence $\gamma(A) \in \overline{\mathbb{N}}$ for all open sets A. Applying the monotone class argument we see that $\gamma(A) \in \overline{\mathbb{N}}$ for all Borel subsets A of E_k, $k \in \mathbb{N}$. Hence $\gamma \in \overline{\Gamma}_E$. □

From now on we give $\overline{\Gamma}_E$ the trace topology induced by $\bar{\rho}$.

Proposition 3.10. *Γ_E is a G_δ set in $\overline{\Gamma}_E$. More precisely, let*

$$U_k := \{\gamma \in \overline{\Gamma}_E \mid \gamma(\{x\}) \leq 1 \textit{ for all } x \in \overline{E}_k\} \ (\overline{E}_k := \textit{closure of } E_k).$$

Then $\Gamma_E = \bigcap_{k \in \mathbb{N}} U_k$ and each U_k is open in $\overline{\Gamma}_E$.

Proof. Clearly, $\Gamma_E = \bigcap_{k \in \mathbb{N}} U_k$. Let $\gamma \in U_k$ We set

$$\beta_1 := \inf\{\rho(x, y) \mid x, y \in \overline{E}_k, x \neq y, \ \gamma(\{x\}) = \gamma(\{y\}) = 1\},$$
$$\beta_2 := \inf\{\rho(x, \overline{E}_k) \mid x \in E_k^{\delta_k} \setminus \overline{E}_k, \ \gamma(\{x\}) \geq 1\}$$

where δ_k is specified by (3.10). Let $\alpha := (\beta_1 \wedge \beta_2 \wedge \delta_k)/5$. Note that $\alpha > 0$ since $\gamma(E_k^{\delta_k}) < \infty$. For each $x \in \overline{E}_k$ we define a function f_x by

$$f_x(y) := \frac{1+\alpha}{\alpha} \, g_{B_\alpha(x),\alpha}(y), \tag{3.14}$$

where $g_{B_\alpha(x),\alpha}$ is specified by (3.3). Then $f_x(x) = 1$ and $\int_E f_x(y)\gamma(dy) \leq 1$. Let $\gamma' \in \overline{\Gamma}_E$ with $\bar{\rho}(\gamma, \gamma') < 2^{-k-2}\alpha/(1+\alpha)$. Then we have for $x \in E$

$$\gamma'(\{x\}) \leq \int_E f_x(y)\gamma'(dy) \leq \frac{1+\alpha}{\alpha} 2^{k+1} \rho_0(\gamma, \gamma') + \int_E f_x(y)\gamma(dy) \leq \frac{3}{2}.$$

Therefore, $\gamma'(\{x\}) \leq 1$, i.e., $\gamma' \in U_k$. Hence U_k is open in $\overline{\Gamma}_E$. □

Proposition 3.11. *Assume that every $x \in E$ is an accumulation point of $E \setminus \{x\}$, then Γ_E is $\bar{\rho}$-dense in $\overline{\Gamma}_E$. (Hence our notation is justified.)*

Proof. Let $\gamma \in \overline{\Gamma}_E$ and $\alpha > 0$. Take $k \in \mathbb{N}$ large enough so that $2^{-(k-1)} < \alpha$. There are at most finitely many distinct points, say $\{x_i \mid 1 \leq i \leq n\}$, such that $x_i \in E_{k+1}$ and $\gamma(\{x_i\}) > 0$. Suppose that $\gamma(\{x_i\}) = n_i$. Since E is connected we can find n_i different and distinct points $\{x_{ij} \mid 1 \leq j \leq n_i\}$ in E_{k+1} such that $\rho(x_i, x_{ij}) < \alpha/(2N)$, where $N := \gamma(E_{k+1})$. Furthermore, we may assume that x_{ij}, $1 \leq i \leq n$, $1 \leq j \leq n_i$, are all different. We now set $\gamma' := \sum_{i,j} \varepsilon_{x_{ij}}$, $\varepsilon_{x_{ij}}$ being the Dirac measure at the point x_{ij}. Then $\gamma' \in \Gamma_E$. Due to the fact that $|g_{A,\varepsilon}(x) - g_{A,\varepsilon}(y)| \leq \rho(x, y)$ for all $x, y \in E$ and all $A \in \mathcal{B}(E)$, $\varepsilon > 0$, one can check that

$$\bar{\rho}_0(\phi_j \cdot \gamma, \phi_j \cdot \gamma') < \alpha, \quad \text{for all } 1 \leq j \leq k,$$

hence $\bar{\rho}(\gamma, \gamma') < \alpha$. Since $\gamma \in \overline{\Gamma}_E$ and $\alpha > 0$ were arbitrary, Γ_E is $\bar{\rho}$-dense in $\overline{\Gamma}_E$. □

4. Quasi-regularity and diffusions

In this section we shall specify conditions that ensure the quasi-regularity of the Dirichlet form $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ on $L^2(\overline{\Gamma}_E; \mu)$ from Theorem 2.6. Let (E, ρ) be a complete separable metric space. We adopt the notation from Sections 1 and 2. We fix a well-exhausting sequence $(E_k)_{k \in \mathbb{N}}$ with $(\delta_k)_{k \in \mathbb{N}}$ as in (3.10). Let $\overline{\Gamma}_E$ be equipped with the $\{E_k\}$-vague topology. Let $(S, \mathcal{D})$ be as in Subsection 1.1 satisfying conditions $(\mathcal{D}.1)$, (S.1) and also (S.2), (S.3) from Subsection 2.1. We assume, in addition, that $\mathcal{D}$ consists of continuous functions. Let μ be a probability measure on $\mathcal{B}(\overline{\Gamma}_E)$ satisfying $(\mu.1)$ and let $\sigma^\mu \in \mathcal{M}(\{E_k\})$ be defined as

$$\sigma^\mu(A) := \int_{\overline{\Gamma}_E} \gamma(A)\, \mu(d\gamma), \quad A \in \mathcal{B}(E). \tag{4.1}$$

Let $(S^\Gamma, \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}))$ be as specified in (1.4) and assume that condition $(S^\Gamma.\mu)$ from Subsection 2.1 holds and that $(\mathcal{E}_\mu^\Gamma, \mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D}))$ is closable on $L^2(\overline{\Gamma}_E; \mu)$ (see Theorem 2.6 for sufficient conditions). Let $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ denote its closure. As before, once we have fixed the measure σ^μ it is enough to assume that (S.1), (S.2), (S.3) hold outside a set of σ^μ-measure zero (cf. Remark 1.5 for details).

We shall see that the following condition implies that $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ is quasi-regular:

(Q) There exist $\chi_j \in \mathcal{D}$, $\chi_j \geq 0$, $j \in \mathbb{N}$, and $f_{ln} : E \to \mathbb{R}$, $l, n \in \mathbb{N}$, continuous such that:

  (i) $\sup_{l \in \mathbb{N}} f_{ln} = \rho(\cdot, y_n)$ for all $n \in \mathbb{N}$ and some dense subset $\{y_n \mid n \in \mathbb{N}\}$ of E.

(ii) There exists $C \in (0,\infty)$ such that for all $j, l, n \in \mathbb{N}$ and all $\varphi \in C_b^\infty(\mathbb{R})$

$$\chi_j \cdot (\varphi \circ f_{ln}) \in \mathcal{D} \text{ and } S(\chi_j \cdot (\varphi \circ f_{ln})) \leq C \sup(\|\varphi'\|_\infty, \|\varphi\|_\infty)^2 (\chi_j + S(\chi_j)^{1/2})^2.$$

(iii) For all $k \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $\chi_j = 1$ on E_k.

We shall see below that (Q) can even be relaxed (cf. condition ($\overline{\text{Q}}$) in Subsection 4.3) and that (Q) (resp. ($\overline{\text{Q}}$)) can be easily checked in applications (cf. Subsection 4.5).

We note that since by condition ($\mathcal{D}$.1) the set $\mathcal{D}$ forms an algebra, condition (Q) (resp. ($\overline{\text{Q}}$)) and a monotone class argument immediately implies that $\mathcal{D}$ and $\mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$ is dense in $L^p(E;\sigma^\mu)$, $L^p(\overline{\Gamma}_E;\mu)$ respectively for all $p \in [1,\infty)$.

4.1. Definition and a general criterion for quasi-regularity Under condition (Q) (resp. ($\overline{\text{Q}}$) in Subsection 4.3) the definition of quasi-regularity given in [25, Chap. IV, Def. 3.1] obviously simplifies as follows: $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ on $L^2(\overline{\Gamma}_E;\mu)$ is quasi-regular if and only if the following condition holds:

(Q.1) There exists an $\mathcal{E}_\mu^\Gamma$-nest $(K_n)_{n\in\mathbb{N}}$ consisting of compact sets in $\overline{\Gamma}_E$.

We recall that a sequence $(A_n)_{n\in\mathbb{N}}$ of closed subsets of $\overline{\Gamma}_E$ is called an $\mathcal{E}_\mu^\Gamma$-nest, if

$$\{F \in D(\mathcal{E}_\mu^\Gamma) \mid F = 0 \text{ on } \overline{\Gamma}_E \setminus A_n \text{ for some } n \in \mathbb{N}\}$$

is dense in $D(\mathcal{E}_\mu^\Gamma)$ w.r.t. the norm

$$\|\cdot\|_{\mathcal{E}_{\mu,1}^\Gamma} := \mathcal{E}_{\mu,1}^\Gamma(\cdot,\cdot)^{1/2} := \left(\mathcal{E}_\mu^\Gamma(\cdot,\cdot) + (\cdot,\cdot)_{L^2(\overline{\Gamma}_E;\mu)}\right)^{1/2}.$$

The following proposition provides an easy-to-check condition for the quasi-regularity of $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$, which is that one needs a suitable "weakly differentiable" metric on $\overline{\Gamma}_E$:

Proposition 4.1. *Suppose there exists a bounded complete metric $\bar{\rho}$ on $\overline{\Gamma}_E$ generating the (separable) $\{E_k\}$-vague topology such that for all $\gamma \in \overline{\Gamma}_E$, $\bar{\rho}(\cdot,\gamma) \in D(\mathcal{E}_\mu^\Gamma)$ and $S^\Gamma(\bar{\rho}(\cdot,\gamma)) \leq \eta$ μ-a.e. for some $\eta \in L^1(\overline{\Gamma}_E;\mu)$ (independent of γ). Then* (Q.1) *holds.*

Proof. The assertion follows directly from the proof of [32, Theorem 3.4]. For the reader's convenience we repeat the main argument here: Let $\gamma_n \in \overline{\Gamma}_E$, $n \in \mathbb{N}$, such that $\{\gamma_n \mid n \in \mathbb{N}\}$ is dense in $\overline{\Gamma}_E$ w.r.t. the $\{E_k\}$-vague topology. For each $n \in \mathbb{N}$ define

$$F_n := \inf_{1\leq i\leq n} \bar{\rho}(\cdot,\gamma_i).$$

By [32, Lemma 3.2] we know that

$$S^\Gamma(F_n) \leq \sup_{1\leq i\leq n} S^\Gamma(\bar{\rho}(\cdot,\gamma_i)) \leq \eta \text{ for all } n \in \mathbb{N}.$$

Hence by the Banach-Saks Theorem (see e.g. [25, Appendix A, Theorem 2.2])

$$G_N := \frac{1}{N}\sum_{k=1}^{N} F_{n_k} \underset{N\to\infty}{\longrightarrow} F \in D(\mathcal{E}_\mu^\Gamma) \text{ w.r.t. } \|\cdot\|_{\mathcal{E}^\Gamma_{\mu,1}}$$

for some subsequence $(n_k)_{k\in\mathbb{N}}$. Hence by [25, Chap. III, Proposition 3.5] there exists an $\mathcal{E}_\mu^\Gamma$-nest $(A_n)_{n\in\mathbb{N}}$ such that

$$G_N \underset{N\to\infty}{\longrightarrow} F \text{ uniformly on } A_n \text{ for all } n \in \mathbb{N}.$$

Since $F_k(\gamma) \downarrow 0$ as $k \to \infty$ for all $\gamma \in \overline{\Gamma}_E$, it follows that

$$F_k \underset{k\to\infty}{\longrightarrow} 0 \text{ uniformly on } A_n \text{ for all } n \in \mathbb{N}.$$

This implies that each A_n is uniformly $\bar{\rho}$-bounded. Since A_n is closed, $\bar{\rho}$ is complete, and $\bar{\rho}$ generates the topology of $\overline{\Gamma}_E$, it follows that each A_n is compact. □

In Subsection 4.3 below we shall prove that (Q) (resp. ($\overline{\text{Q}}$)) implies that the metric $\bar{\rho}$ on Γ_E defined in (3.12) for properly chosen c_k, $k \in \mathbb{N}$, can be taken as the metric $\bar{\rho}$ in Proposition 4.1

First we need some preparations.

4.2. A larger class of cylinder functions In this subsection we do *not* use condition (Q) (resp. ($\overline{\text{Q}}$)). We consider the following norm $|\cdot|_\Gamma$ on $\mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D})$:

$$|F|_\Gamma := \left(\int S^\Gamma(F)\, d\mu\right)^{1/2} + \int |F|\, d\mu, \quad F \in \mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D}).$$

Let $\mathcal{V}$ be the completion of $(\mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D}), |\cdot|_\Gamma)$. The inclusion map $i : (\mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D}), |\cdot|_\Gamma) \subset (L^1(\overline{\Gamma}_E;\mu), \|\cdot\|_{L^1(\overline{\Gamma}_E;\mu)})$ extends uniquely to a continuous linear map $\bar{i}$ from $\mathcal{V}$ to $L^1(\overline{\Gamma}_E;\mu)$.

Lemma 4.2. *$\bar{i} : \mathcal{V} \longrightarrow L^1(\overline{\Gamma}_E;\mu)$ is one-to-one, i.e., $\mathcal{V} \hookrightarrow L^1(\overline{\Gamma}_E;\mu)$. Furthermore, S^Γ extends uniquely to a bilinear continuous map from $(\mathcal{V}, |\cdot|_\Gamma) \times (\mathcal{V}, |\cdot|_\Gamma)$ to $L^1(\overline{\Gamma}_E;\mu)$.*

Proof. Let $F_n \in \mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D})$, $n \in \mathbb{N}$, be a $|\cdot|_\Gamma$-Cauchy-sequence such that $F_n \longrightarrow 0$ in $L^1(\overline{\Gamma}_E;\mu)$ as $n \to \infty$. By the Cauchy-Schwarz inequality

$$S^\Gamma(F_n - F_m) \geq (S^\Gamma(F_n)^{1/2} - S^\Gamma(F_m)^{1/2})^2$$

for all $n, m \in \mathbb{N}$, hence $(S(F_n))_{n\in\mathbb{N}}$ is a Cauchy-sequence in $L^1(\overline{\Gamma}_E;\mu)$, hence converges to some Y in $L^1(\overline{\Gamma}_E;\mu)$. We have to show that $Y = 0$. So, let $\varphi \in C_b^\infty(\mathbb{R})$ such that $\varphi(0) = 0$ and $\varphi'(0) = 1$. Then $\varphi(F_n) \underset{n\to\infty}{\longrightarrow} 0$ in $L^2(\overline{\Gamma}_E; \mu)$ and by Lemma 1.4

$$\begin{aligned} & S^\Gamma(\varphi(F_n) - \varphi(F_m)) \\ &= \varphi'(F_n)^2 \, S^\Gamma(F_n) + \varphi'(F_m)^2 \, S^\Gamma(F_m) \\ &\quad + \varphi'(F_n)\varphi'(F_m) \, S(F_n - F_m) \\ &\quad - \varphi'(F_n)\varphi'(F_m) \, (S(F_n) + S(F_m)) \\ &\underset{n,m\to\infty}{\longrightarrow} 0 \quad \text{in } L^1(\overline{\Gamma}_E; \mu). \end{aligned}$$

Hence (because $(\mathcal{E}_\mu^\Gamma, \mathcal{F}^\Gamma C_b^{\infty,\mu}(\mathcal{D}))$ is closable on $L^2(\overline{\Gamma}_E; \mu)$) we have that in $L^1(\overline{\Gamma}_E; \mu)$

$$\begin{aligned} 0 &= \lim_{n\to\infty} S^\Gamma(\varphi(F_n)) = \lim_{n\to\infty} \varphi'(F_n)^2 \, S^\Gamma(F_n) \\ &= \varphi'(0)^2 \, Y = Y \end{aligned}$$

where we used Lemma 1.4 again. □

Lemma 4.3. *Let $f \in \mathcal{D}$. Then $\langle f, \cdot\rangle \in \mathcal{V}$ and $S^\Gamma(\langle f, \cdot\rangle) = \langle S(f), \cdot\rangle$. In particular, condition $(S.\sigma^\mu)$ holds.*

Proof. Let $n \in \mathbb{N}$ and $\varphi_n \in C_b^\infty(\mathbb{R})$ such that $0 \le \varphi_n' \le 1$, $\varphi_n(t) = t$ for $t \in [-n, n]$ and $\varphi_n(t) = (n+1)\operatorname{sign} t$ on $\mathbb{R} \setminus [-n-2, n+2]$. Then $\varphi_n(\langle f, \cdot\rangle) \in \mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$, $\varphi_n(\langle f, \cdot\rangle) \to \langle f, \cdot\rangle$ in $L^1(\overline{\Gamma}_E; \mu)$ as $n \to \infty$ and by definition for all $n \in \mathbb{N}$

$$S^\Gamma(\varphi_n(\langle f, \cdot\rangle)) = \varphi_n'(\langle f, \cdot\rangle)^2 \langle S(f), \cdot\rangle,$$

which converges to $\langle S(f), \cdot\rangle$ in $L^1(\overline{\Gamma}_E; \mu)$. Hence the assertion follows from Lemma 4.2 resp. assumption $(S^\Gamma.\mu)$. □

Now we can consider the following norm on $\mathcal{D}^{\sigma^\mu}$:

$$|f|_E := \left(\int S(f)\, d\sigma^\mu\right)^{1/2} + \int |f| \, d\sigma^\mu, \quad f \in \mathcal{D}^{\sigma^\mu}.$$

Note that by Lemma 4.3

$$|f|_E = |\langle f, \cdot\rangle|_\Gamma \quad \text{for all } f \in \mathcal{D}^{\sigma^\mu}. \tag{4.2}$$

Let $\overline{\mathcal{D}}$ denote the completion of $\mathcal{D}^{\sigma^\mu}$ w.r.t. $|\cdot|_E$. Again the inclusion map $i:(\mathcal{D}^{\sigma^\mu}, |\cdot|_E) \subset (L^1(E;\sigma^\mu), \|\cdot\|_{L^1(E;\sigma^\mu)})$ extends uniquely to a continuous linear map $\bar{i}$ from $\overline{\mathcal{D}}$ to $L^1(E;\sigma^\mu)$.

Lemma 4.4. *$\bar{i} : \overline{\mathcal{D}} \to L^1(E;\sigma^\mu)$ is one-to-one, i.e., $\overline{\mathcal{D}} \subset L^1(E;\sigma^\mu)$. Furthermore, S extends uniquely to a bilinear continuous map from $(\overline{\mathcal{D}}, |\cdot|_E) \times (\overline{\mathcal{D}}, |\cdot|_E)$ to*

$L^1(E;\sigma^\mu)$ *satisfying* (S.1)–(S.3) *with* $\mathcal{D}$ *replaced by* $\overline{\mathcal{D}}$.

Proof. Let $f_n \in \mathcal{D}^{\sigma^\mu}$, $n \in \mathbb{N}$, be a $|\cdot|_E$-Cauchy sequence such that $f_n \to 0$ in $L^1(E;\sigma^\mu)$ as $n \to \infty$. By (4.2) it follows that $(\langle f_n, \cdot\rangle)_{n\in\mathbb{N}}$ is a $|\cdot|_\Gamma$-Cauchy sequence in $\mathcal{V}$ such that $\langle f_n, \cdot\rangle \to 0$ in $L^1(\overline{\Gamma}_E;\mu)$ as $n \to \infty$. Hence Lemma 4.2 and (4.2) imply that $|f_n|_E \to 0$ as $n \to \infty$. The remaining parts of the assertion then easily follow from (4.2). □

REMARK 4.5. Of course, the above arguments "to close square field operators" are entirely standard. We repeated them here because usually the underlying space is L^2, whereas we had to work on L^1.

Finally, we can now prove the result we are aiming for, i.e., that an enlarged space of cylinder functions is contained in $D(\mathcal{E}_\mu^\Gamma)$.

Define

$$\begin{aligned}(4.3)\qquad \mathcal{F}^\Gamma C_b^\infty(\overline{\mathcal{D}}) := \{g(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle) \mid\ & N \in \mathbb{N},\\ & f_1, \dots, f_N \ \ \sigma^\mu\text{-versions of elements in } \overline{\mathcal{D}},\\ & g \in C_b^\infty(\mathbb{R}^N)\}.\end{aligned}$$

Proposition 4.6. $\mathcal{F}^\Gamma C_b^{\infty,\mu}(\overline{\mathcal{D}}) \subset D(\mathcal{E}_\mu^\Gamma)$ *and* (1.4) *holds for* S^Γ *with* $\mathcal{D}$ *replaced by* $\overline{\mathcal{D}}$. *Furthermore,* $\langle f, \cdot\rangle \in \mathcal{V}$ *and* $S^\Gamma(\langle f, \cdot\rangle) = \langle S(f), \cdot\rangle$ *for all* $f \in \overline{\mathcal{D}}$.

Proof. Let $F = g_F(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle) \in \mathcal{F}^\Gamma C_b^\infty(\overline{\mathcal{D}})$. Then for $1 \le j \le N$ there are $f_j^{(n)} \in \mathcal{D}$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} |f_j - f_j^{(n)}|_E = 0,$$

hence by (4.2)

$$\lim_{n\to\infty} |\langle f_j, \cdot\rangle - \langle f_j^{(n)}, \cdot\rangle|_\Gamma = 0.$$

Consequently,

$$F^{(n)} := g_F(\langle f_1^{(n)}, \cdot\rangle, \dots, \langle f_N^{(n)}, \cdot\rangle) \underset{n\to\infty}{\longrightarrow} F \text{ in } L^2(\overline{\Gamma}_E;\mu)$$

and for all $n, m \in \mathbb{N}$

$$S^\Gamma(F^{(n)} - F^{(m)}) =$$

$$\sum_{i,j=1}^{N} \Big[\partial_i g_F(\langle f_1^{(n)}, \cdot\rangle, \dots, \langle f_N^{(n)}, \cdot\rangle)\, \partial_j g_F(\langle f_1^{(n)}, \cdot\rangle, \dots, \langle f_N^{(n)}, \cdot\rangle)\, \langle S(f_i^{(n)}, f_j^{(n)}), \cdot\rangle$$

$$+\partial_i g_F(\langle f_1^{(m)}, \cdot\rangle, \dots, \langle f_N^{(m)}, \cdot\rangle)\, \partial_j g_F(\langle f_1^{(m)}, \cdot\rangle, \dots, \langle f_N^{(m)}, \cdot\rangle)\, \langle S(f_i^{(m)}, f_j^{(m)}), \cdot\rangle$$

$$-2\, \partial_i g_F(\langle f_1^{(n)}, \cdot\rangle, \dots, \langle f_N^{(n)}, \cdot\rangle)\, \partial_j g_F(\langle f_1^{(m)}, \cdot\rangle, \dots, \langle f_N^{(m)}, \cdot\rangle)\, \langle S(f_i^{(n)}, f_j^{(m)}), \cdot\rangle \Big]$$

$$\longrightarrow 0 \text{ in } L^1(\Gamma_E; \mu) \text{ as } n, m \to \infty,$$

since $S(f_i^{(n)}, f_j^{(m)}) \longrightarrow S(f_i, f_j)$ for all $1 \le i, j \le N$ in $L^1(E; \sigma^\mu)$ as $n, m \to \infty$ by Lemma 4.4. The second part of the assertion is now obvious, while the last part follows as in the proof of Lemma 4.3. □

The following result, which will be used below, is well-known for square field operators on L^2 (cf. e.g. [32, Lemma 3.2]). We need it, however, on L^1. Since the proof is slightly different, we include it here.

Lemma 4.7. *Let $f, g \in \overline{\mathcal{D}}$. Then:*

(i) $|f| \in \overline{\mathcal{D}}$ *and*

$$|S(|f|, g)| \le |S(f, g)| \quad \sigma^\mu\text{-a.e.}$$

(ii) $f \vee g := \sup(f, g) \in \overline{\mathcal{D}}$, $f \wedge g := \inf(f, g) \in \overline{\mathcal{D}}$ *and* $S(f \vee g) \vee S(f \wedge g) \le S(f) \vee S(g)$.

(iii) $(f - a)^+ \wedge c \in \overline{\mathcal{D}}$ *for all* $c \in [0, \infty]$, $a \in \mathbb{R}$, *and*

$$S((f - a)^+ \wedge c) \le S(f),$$

where $f^+ := f \vee 0$.

(iv) *Let* $\chi \in \overline{\mathcal{D}}$ *and let* $u_n : E \to \mathbb{R}$ *be* $\mathcal{B}$*-measurable such that* $\chi u_n \in \overline{\mathcal{D}} \cap L^2(E; \sigma^\mu)$, $n \in \mathbb{N}$, $u_n \to u$ σ^μ*-a.e. as* $n \to \infty$, *and there exists* $\kappa \in (0, \infty)$ *such that*

$$|\chi u_n|_{E,2} := \left(\int (S(\chi u_n) + (\chi u_n)^2)\, d\sigma^\mu \right)^{1/2} \le \kappa.$$

Then $\chi u \in \overline{\mathcal{D}}$ *and*

$$\frac{1}{N} \sum_{k=1}^{N} \chi u_{n_k} \longrightarrow \chi u \text{ as } N \to \infty \text{ w.r.t. } |\cdot|_{E,2}$$

for some subsequence $(n_k)_{k \in \mathbb{N}}$. *In particular,*

$$S(\chi u) \le \limsup_{n \to \infty} S(\chi u_n).$$

(v) *Let $\chi \in \overline{\mathcal{D}} \cap L^2(E;\sigma^\mu)$ and let $u : E \to \mathbb{R}$ be $\mathcal{B}(E)$-measurable such that for all $\varphi \in C_b^\infty(\mathbb{R})$, $\chi \cdot (\varphi \circ u) \in \overline{\mathcal{D}}$ and for some $C \in (0,\infty)$ (independent of φ)*

$$S(\chi \cdot (\varphi \circ u)) \le C \sup(\|\varphi'\|_\infty, \|\varphi\|_\infty)^2 (\chi + S(\chi)^{1/2})^2.$$

Then $\chi \cdot ((u-a)^+ \wedge c) \in \overline{\mathcal{D}}$ for all $c \in [0,\infty)$, $a \in \mathbb{R}$, and

$$S(\chi((u-a)^+ \wedge c)) \le (c \vee 1)^2 C(\chi + S(\chi)^{1/2})^2.$$

Proof. (i): For $n \in \mathbb{N}$ let $\varphi_n \in C^\infty(\mathbb{R})$ such that $\varphi_n(t) \to |t|$ as $n \to \infty$ for all $t \in \mathbb{R}$, $|\varphi_n'| \le 1$, $\varphi_n(0) = 0$, $\varphi_n'(t) \to \operatorname{sign} t$ as $n \to \infty$ for all $t \in \mathbb{R}$. (E.g. for $n \in \mathbb{N}$ take $\delta_n \in C_0^\infty(\mathbb{R})$ such that $\delta_n \ge 0$, $\int \delta_n \, dt = 1$, $\delta_n(t) = \delta_n(-t)$ for all $t \in \mathbb{R}$, $\operatorname{supp} \delta_n \subset]-1/n, 1/n[$. Then define

$$\varphi_n(t) := \int |t-s| \; \delta_n(s) \; ds - \int |s| \; \delta_n(s) \; ds, \quad t \in \mathbb{R}.)$$

Then $\varphi_n(f) \in \overline{\mathcal{D}}$ by Lemma 4.4 and $\varphi_n(f) \to |f|$ as $n \to \infty$ in $L^1(E;\sigma^\mu)$ (because $\varphi_n(f) \le |f|$ for all $n \in \mathbb{N}$). Furthermore, for all $n, m \in \mathbb{N}$ by (S.1)

$$S(\varphi_n(f) - \varphi_m(f)) = (\varphi_n'(f) - \varphi_m'(f))^2 \; S(f).$$

Consequently, $(\varphi_n(f))_{n\in\mathbb{N}}$ is a $|\cdot|_E$-Cauchy sequence, so Lemma 4.4 implies that $|f| \in \overline{\mathcal{D}}$. Furthermore, by (S.1) we have in $L^1(E;\sigma^\mu)$

$$\begin{aligned}|S(|f|, g)| &= \lim_{n\to\infty} |S(\varphi_n(f), g)| \\ &= \lim_{n\to\infty} |\varphi_n'(t) \; S(f,g)| \le |S(f,g)|.\end{aligned}$$

(ii): Using that $f \vee g = (1/2)(f + g + |f-g|)$ and $f \wedge g = -\sup(-f,-g)$, the proof of (ii) can be deduced from (i) by exactly the same arguments as used in the proof of Lemma 3.2 in [32].

(iii): For $n \in \mathbb{N}$ we choose $\tilde\varphi_n \in C_b^\infty(\mathbb{R})$ such that $\tilde\varphi_n(t) = t$ for $t \in [0,c)$, $\tilde\varphi_n(t) = -1/n$ for $t \in (-\infty, -2/n]$, $\tilde\varphi_n(t) = c + 1/n$ for $t \in (c + 2/n, \infty)$, and $0 \le \tilde\varphi_n' \le 1$. Let $\varphi_n(t) := \tilde\varphi_n(t-a) - \tilde\varphi_n(-a)$, $t \in \mathbb{R}$. Then $\varphi_n(f) \to (f-a)^+ \wedge c$ in $L^1(E;\sigma^\mu)$ as $n \to \infty$, and hence exactly the same arguments as in the proof of assertion (i) imply (iii).

(iv): Let $\mathcal{D}_2 := \{\chi u \in \overline{\mathcal{D}} \cap L^2(E;\sigma^\mu) \mid u : E \to \mathbb{R},\ \mathcal{B}(E)\text{-measurable}\}$ and let $\overline{\mathcal{D}}_2$ be the completion of $\mathcal{D}_2$ w.r.t.

$$|\chi u|_{E,2} := \left(\int (S(\chi u) + (\chi u)^2) \, d\sigma^\mu\right)^{1/2}.$$

Then the embedding $i : (\mathcal{D}_2, |\cdot|_{E,2}) \subset (L^2(E;\sigma^\mu), \|\cdot\|_{L^2(E;\sigma^\mu)})$ uniquely extends to a continuous linear map $\bar i : \overline{\mathcal{D}}_2 \to L^2(E;\sigma^\mu)$. $\bar i$ is one-to-one, i.e., $\overline{\mathcal{D}}_2 \subset L^2(E;\sigma^\mu)$.

(Indeed, if $\chi u_n \in \mathcal{D}_2$, $n \in \mathbb{N}$, such that $\chi u_n \to 0$ as $n \to \infty$ in $L^2(E;\sigma^\mu)$ and $S(\chi(u_n - u_m)) \to 0$ as $n, m \to \infty$ in $L^1(E;\sigma^\mu)$, then $\chi u_n \to 0$ as $n \to \infty$ in $L^1(E;\sigma^\mu)$, since $\chi \in \mathcal{F}_b(\{E_k\})$. Hence by Lemma 4.4, $S(\chi u_n) \to 0$ as $n \to \infty$ in $L^1(E;\sigma^\mu)$.) Therefore, assertion (iv) follows immediately from the Banach-Saks Theorem applied to the Hilbert space $(\overline{\mathcal{D}}, |\cdot|_{E,2})$.

(v): For $n \in \mathbb{N}$ we choose $\tilde{\varphi}_n$ as in the proof of (iii). Then $\chi\tilde{\varphi}_n(u) \to \chi \cdot ((u - a)^+ \wedge c)$ in $L^2(E;\sigma^\mu)$ as $n \to \infty$ and by assumption for all $n \in \mathbb{N}$

$$S(\chi\tilde{\varphi}_n(u)) \leq (c \vee 1)^2 C \; (\chi + S(\chi)^{1/2})^2.$$

Hence the assertions follows by (iv). □

4.3. A weakly differentiable metric on $\overline{\Gamma}_E$ Below we shall use a weaker version of condition (Q), namely with $\mathcal{D}$ in (Q)(ii) replaced by the bigger set $\overline{\mathcal{D}}$ defined in the previous subsection. We shall refer to this version as condition $(\overline{\text{Q}})$. We recall that ϕ_k, $k \in \mathbb{N}$, in (3.12) was defined as

$$\phi_k := \frac{1 + \delta_k}{\delta_k} g_{E_k, \delta_k} \tag{4.4}$$

where g_{E_k,δ_k} is as defined in (3.3) and E_k, δ_k is as in the introductory part of this section. In this subsection we shall prove the following result.

Proposition 4.8. *Assume condition $(\overline{\text{Q}})$ holds. Set*

$$c_k := \left(1 + \int \tilde{\chi}_{j_k}^2 \, d\sigma^\mu\right)^{-1/2} 2^{-k/2}, \quad k \in \mathbb{N},$$

where $\tilde{\chi}_{j_k}$ is as in Lemma 4.10 *below, and as in* (3.12) *define*

$$\bar{\rho}(\gamma_1, \gamma_2) := \sup_{k \in \mathbb{N}} c_k \; \zeta(\bar{\rho}_0(\phi_k \cdot \gamma_1, \phi_k \cdot \gamma_2)). \tag{4.5}$$

Then $\bar{\rho}$ satisfies the conditions of Proposition 4.1 *with*

$$\eta := \sup_{k \in \mathbb{N}} \left(2^{-k} \left(1 + \int \tilde{\chi}_{j_k}^2 \, d\sigma^\mu\right)^{-1} \langle \tilde{\chi}_{j_k}^2, \cdot \rangle \right).$$

As a consequence of Propositions 4.1 (see also the discussion preceding it) and 4.8 we have:

Corollary 4.9. *If condition $(\overline{\text{Q}})$ holds, then $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ is quasi-regular.*

For the proof of Proposition 4.8 we need several lemmas. We assume $(\overline{\text{Q}})$ to hold further on.

Lemma 4.10. *Let $\varepsilon > 0$, $k \in \mathbb{N}$, and $A \subset E$, A closed. Let $j_k \in \mathbb{N}$ such that $\chi_{j_k} = 1$ on E_{k+1} (where χ_j , $j \in \mathbb{N}$, is as in assumption $(\overline{\mathrm{Q}})$). Then $\phi_k\, g_{A,\varepsilon} \in \overline{\mathcal{D}}$ and*

$$S(\phi_k\, g_{A,\varepsilon}) \leq \tilde{\chi}_{j_k}^2$$

where $\tilde{\chi}_{j_k} := 2\delta_k^{-1}\chi_{j_k}\left[S(\chi_{j_k})^{1/2} + C(\chi_{j_k} + S(\chi_{j_k})^{1/2})\right]^2$ with C as in assumption $(\overline{\mathrm{Q}})$.

Proof.

CLAIM. Let $j \in \mathbb{N}$. Then $\chi_j\, g_{A,\varepsilon} \in \overline{\mathcal{D}}$ and

$$S(\chi_j g_{A,\varepsilon}) \leq \left(\frac{\varepsilon \vee 1}{1+\varepsilon}\right)^2 \left[S(\chi_j)^{1/2} + C(\chi_j + S(\chi_j)^{1/2})\right]^2 \quad \sigma^{\mu}\text{-a.e.}$$

To prove the Claim we note that by Lemma 4.7 (ii), (v) and condition $(\overline{\mathrm{Q}})$ (see also Lemma 4.4) for all $\varepsilon \in (0, \infty)$, $n, N \in \mathbb{N}$

$$g_{Nn} := \chi_j(\varepsilon - \sup_{l \leq N} f_{ln}^+ \wedge \varepsilon) \in \overline{\mathcal{D}}$$

and

$$S(g_{Nn}) \leq (\varepsilon \vee 1)^2[S(\chi_j)^{1/2} + C(\chi_j + S(\chi_j)^{1/2})]^2 \quad \sigma^{\mu}\text{-a.e.}$$

Let $n \in \mathbb{N}$. Since by condition $(\overline{\mathrm{Q}})$, $\lim_{N\to\infty} g_{Nn}(x) = \chi_j(x)(\varepsilon - \rho(x, y_n) \wedge \varepsilon)$ for all $x \in E$ and in $L^2(E;\sigma^{\mu})$, we conclude by Lemma 4.7 (iv) that $\chi_j(\varepsilon - \rho(\cdot, y_n) \wedge \varepsilon) \in \overline{\mathcal{D}}$ and that

$$S(\chi_j(\varepsilon - \rho(\cdot, y_n) \wedge \varepsilon)) \leq (\varepsilon \vee 1)^2 \left[S(\chi_j)^{1/2} + C(\chi_j + S(\chi_j)^{1/2})\right]^2 \; \sigma^{\mu}\text{-a.e.}$$

By exactly the same arguments we then obtain that the same holds for $\chi_j(\varepsilon - \rho(\cdot, y) \wedge \varepsilon)$ for all $y \in E$. Now pick a dense subset $\{x_n \mid n \in \mathbb{N}\}$ of A and consider for $N \in \mathbb{N}$

$$\begin{aligned} u_N &:= \sup_{n \leq N} \chi_j(\varepsilon - \rho(\cdot, x_n) \wedge \varepsilon) \\ &= \chi_j(\varepsilon - \inf_{n \leq N} \rho(\cdot, x_n) \wedge \varepsilon). \end{aligned}$$

Then by Lemma 4.7 (ii) and the above, $u_N \in \overline{\mathcal{D}}$ and

$$S(u_N) \leq (\varepsilon \vee 1)^2 \left[S(\chi_j)^{1/2} + C(\chi_j + S(\chi_j)^{1/2})\right]^2.$$

Since $u_N(x) \to \chi_j(x)(\varepsilon - \rho(x, A) \wedge \varepsilon)$ as $N \to \infty$ for all $x \in E$ and in $L^2(E;\sigma^{\mu})$, the Claim follows by the definition of $g_{A,\varepsilon}$ (cf. (3.3)).

Since $\chi_{j_k} = 1$ on E_{k+1}, we have that

$$\phi_k g_{A,\varepsilon} = \frac{1+\delta_k}{\delta_k}\left(\chi_{j_k} g_{E_k,\delta_k}\right)\cdot\left(\chi_{j_k} g_{A,\varepsilon}\right).$$

Hence by the Claim and (S.1) (cf. Lemma 4.4)

$$S(\phi_k g_{A,\varepsilon}) \le 4\left(\frac{\delta_k \vee 1}{\delta_k}\right)^2\left(\frac{\varepsilon\vee 1}{1+\varepsilon}\right)^2 \chi_{j_k}^2\ \left[S(\chi_{j_k})^{1/2} + C(\chi_{j_k} + S(\chi_{j_k})^{1/2})\right]^2, \tag{4.6}$$

and the assertion follows. □

Lemma 4.11. *Let $k \in \mathbb{N}$, $\gamma_0 \in \overline{\Gamma}_E$, and set*

$$F_k(\gamma) := \zeta(\bar\rho_0(\phi_k\cdot\gamma, \phi_k\cdot\gamma_0)), \quad \gamma\in\overline{\Gamma}_E$$

(with $\bar\rho_0$ as in (3.9)*). Then $F_k \in D(\mathcal{E}_\mu^\Gamma)$ and*

$$S^\Gamma(F_k) \le \langle\tilde\chi_{j_k}^2, \cdot\rangle \ \mu\text{-a.e.}$$

Proof. Let $A \subset E$, A closed. By Lemma 4.10 and Proposition 4.6 $|\langle\phi_k\, g_{A,\varepsilon}, \cdot\rangle - \langle\phi_k\, g_{A,\varepsilon}, \gamma_0\rangle| \in D(\mathcal{E}_\mu^\Gamma)$ and

$$S^\Gamma(|\langle\phi_k g_{A,\varepsilon}, \cdot\rangle - \langle\phi_k g_{A,\varepsilon}, \gamma_0\rangle|) \le \langle\tilde\chi_{j_k}^2, \cdot\rangle \quad \mu\text{-a.e.}$$

(since $1 \in D(\mathcal{E}_\mu^\Gamma)$ and $S^\Gamma(1) = 0$ by Remark 1.3). Hence by Proposition 4.6 and [32, Lemma 3.2] for all $n\in\mathbb{N}$, $\zeta(\sup_{j\le n}|\langle\phi_k g_{A_j,\varepsilon_j}, \cdot\rangle - \langle\phi_k g_{A_j,\varepsilon_j}, \gamma_0\rangle|) \in D(\mathcal{E}_\mu^\Gamma)$ and

$$S^\Gamma(\zeta(\sup_{j\le n}|\langle\phi_k g_{A_j,\varepsilon_j}, \cdot\rangle - \langle\phi_k g_{A_j,\varepsilon_j}, \gamma_0\rangle|)) \le \langle\tilde\chi_{j_k}^2, \cdot\rangle \quad \mu\text{-a.e.} \tag{4.7}$$

where (A_j, ε_j), $j\in\mathbb{N}$, are as specified in Corollary 3.3. Since $\zeta(\sup_{j\le n}|\langle\phi_k g_{A_j,\varepsilon_j}, \gamma\rangle - \langle\phi_k g_{A_j,\varepsilon_j}, \gamma_0\rangle|) \to F_k(\gamma)$ as $n\to\infty$ for all $\gamma\in\overline{\Gamma}_E$, and in $L^2(\overline{\Gamma}_E;\mu)$, (4.7) and the Banach-Saks Theorem implies the assertion. □

Proof of Proposition 4.8. Let $\gamma_0 \in \overline{\Gamma}_E$. Then by Lemma 4.11 for all $k\in\mathbb{N}$

$$\begin{aligned} S^\Gamma(c_k F_k) &\le 2^{-k}(1+\int \tilde\chi_{j_k}^2\, d\sigma^\mu)^{-1}\langle\tilde\chi_{j_k}^2, \cdot\rangle \\ &\le \eta \quad \mu\text{-a.e.}, \end{aligned}$$

thus by [32, Lemma 3.2] for all $n\in\mathbb{N}$

$$S^\Gamma(\sup_{k\le n} c_k F_k) \le \eta \quad \mu\text{-a.e.}$$

But $\sup_{k\le n} c_k F_k \to \bar{\rho}(\cdot, \gamma_0)$ as $n \to \infty$ μ-a.e. and in $L^2(\overline{\Gamma}_E;\mu)$. Hence the assertion follows by the Banach-Saks Theorem, since

$$\int \eta \, d\mu \le \sum_{k=1}^{\infty} 2^{-k} \left(1 + \int \tilde{\chi}_{j_k}^2 \, d\sigma^{\mu}\right)^{-1} \int \tilde{\chi}_{j_k}^2 \, d\sigma^{\mu} < \infty. \qquad \square$$

4.4. Corresponding diffusions The following is now easy to prove:

Proposition 4.12. *Assume that condition* $(\overline{\mathrm{Q}})$ *holds. Then* $(\mathcal{E}_{\mu}^{\Gamma}, D(\mathcal{E}_{\mu}^{\Gamma}))$ *has the local property (i.e.,* $\mathcal{E}_{\mu}^{\Gamma}(F, G) = 0$ *provided* $F, G \in D(\mathcal{E}_{\mu}^{\Gamma})$ *with* $\operatorname{supp}(|F|\mu) \cap \operatorname{supp}(|G|\mu) = \emptyset$).

Proof. By the Cauchy-Schwarz inequality it suffices to prove the following:

CLAIM. Let $F \in D(\mathcal{E}_{\mu}^{\Gamma})$. Then

$$S^{\Gamma}(F) = 0 \ \mu\text{-a.e. on } \overline{\Gamma}_E \setminus \operatorname{supp}(|F| \cdot \mu).$$

To prove the claim we take $G \in D(\mathcal{E}_{\mu}^{\Gamma})$ such that $0 \le G \le 1_{\overline{\Gamma}_E \setminus \operatorname{supp}(|F|\cdot\mu)}$ and $G > 0$ μ-a.e. on $\overline{\Gamma}_E \setminus \operatorname{supp}(|F| \cdot \mu)$. Such a function G exists by [25, Chap. V, Proposition 1.7] since $(\mathcal{E}_{\mu}^{\Gamma}, D(\mathcal{E}_{\mu}^{\Gamma}))$ is quasi-regular. (In our case here we can actually prove by similar arguments as in the previous subsection that $\bar{\rho}(\cdot, A) \in D(\mathcal{E}_{\mu}^{\Gamma})$ for all $A \subset \overline{\Gamma}_E$, so we can take $G := \bar{\rho}(\cdot, \operatorname{supp}(|F| \cdot \mu)) \wedge 1$.) Then $GF = 0$, hence

$$0 = S^{\Gamma}(GF, F) = G S^{\Gamma}(F) + F \ S^{\Gamma}(G, F),$$

therefore, $S^{\Gamma}(F) = 0$ on $\{G > 0\} = \overline{\Gamma}_E \setminus \operatorname{supp}(|F| \cdot \mu)$. $\square$

As a consequence of Proposition 4.12, Corollary 4.9, and [25, Chap.IV, Theorem 3.5, and Chap.V, Theorem 1.11] we now obtain the main result of this section.

Theorem 4.13. *Assume that condition* $(\overline{\mathrm{Q}})$ *holds. Then there exists a conservative (strong Markov) diffusion process*

$$\mathbf{M} = (\Omega, \mathbf{F}, (\mathbf{F}_t)_{t\ge0}, (\Theta_t)_{t\ge0}, (\mathbf{X}_t)_{t\ge0}, (\mathbf{P}_{\gamma})_{\gamma\in\overline{\Gamma}_E})$$

on $\overline{\Gamma}_E$ *(cf. [16]) which is properly associated with* $(\mathcal{E}_{\mu}^{\Gamma}, D(\mathcal{E}_{\mu}^{\Gamma}))$, *i.e., for all ($\mu$-versions of)* $F \in L^2(\overline{\Gamma}_E;\mu)$ *and all* $t > 0$ *the function*

$$\gamma \mapsto p_t F(\gamma) := \int_{\Omega} F(\mathbf{X}_t) \, d\mathbf{P}_{\gamma}, \quad \gamma \in \overline{\Gamma}_E, \tag{4.8}$$

is an $\mathcal{E}^{\Gamma}_{\mu}$*-quasi-continuous version of* $\exp(tH^{\Gamma}_{\mu})F$*, where* H^{Γ}_{μ} *is the generator of* $(\mathcal{E}^{\Gamma}_{\mu}, D(\mathcal{E}^{\Gamma}_{\mu}))$ *(cf.* [25, Chap. I, Sect. 2]). **M** *is up to* μ*-equivalence unique (cf.* [25, Chap. IV, Sect. 6]). *In particular,* **M** *is* μ*-symmetric (i.e.,* $\int G\ p_t F d\mu = \int F\ p_t G\ d\mu$ *for all* $F, G : \overline{\Gamma}_E \to \mathbb{R}_+, \mathcal{B}(\overline{\Gamma}_E)$*-measurable) and has* μ *as an invariant measure.*

4.5. Examples

4.5.1. Riemannian manifolds Consider the situation described in Subsections 1.4.1 resp. 2.4.1. We have already seen there that all assumptions except for $(\overline{\mathrm{Q}})$ made in this section are fulfilled, if $\mu = \pi_\sigma$, resp. if μ is a Gibbs measure of Ruelle type. To show that Corollary 4.9, Proposition 4.12, and Theorem 4.13 apply we have to verify condition $(\overline{\mathrm{Q}})$ in this case.

Let us assume that our manifold $E := X$ is complete, and let ρ be the distance function on X coming from the Riemannian metric. Then the existence of χ_j, $j \in \mathbb{N}$, as in (Q) are guaranteed by the well-known Gaffney Lemma (cf. [11]). The functions $f_k \in \mathcal{D}$, $k \in \mathbb{N}$, in (Q) can be taken to be in $C^1_b(X)$. Their existence is well-known for large classes of (infinite dimensional) Finsler manifolds (cf. [17, Sect. 2] and references therein). Clearly, then $\chi_j f_k \in \mathcal{D}$ for all $j, k \in \mathbb{N}$, and (Q) (ii) follows from the chain rule. Hence even condition (Q) holds in this case.

REMARK 4.14. The above result generalizes corresponding results in [29], [37]. But even in the special cases considered there our result is not covered by theirs, since our Dirichlet forms a priori have a *smaller* domain. So, our quasi-regularity implies theirs, but not vice versa. Whether the two domains actually coincide, is presently an open problem.

4.5.2. The free loop space Consider the situation described in Subsections 1.4.2 resp. 2.4.2. Also here we already know that all our assumptions except for $(\overline{\mathrm{Q}})$ are fulfilled e.g. if μ is the Poisson measure on $\overline{\Gamma}_{\mathcal{L}(\mathbb{R}^d)}$ (or if μ is one of the more general measures described in Remark 2.11). That $(\overline{\mathrm{Q}})$ is valid can also be derived from Proposition 2.8 in [17]. But its proof is based on a number of references containing substantial work on Finsler manifolds and, particularly, general classes of mapping spaces as examples of such (cf. [10], [30], and [22]). So, we prefer to give a short direct and complete proof here.

Let $k \in \mathbb{N}$ and let E_k (and U_k) be as in (1.17) and $\delta_k := 1/2$. Let $\tilde{\chi}_k \in C^\infty_0(U_{k+1})$ such that $\tilde{\chi}_k \geq 0$ and $\tilde{\chi}_k = 1$ on U_k. For $j \in \mathbb{N}$ define $\chi_j : \mathcal{L}(\mathbb{R}^d) \to \mathbb{R}_+$ by

$$\chi_j(\omega) := \tilde{\chi}_j(\omega(0)), \quad \omega \in \mathcal{L}(\mathbb{R}^d).$$

Clearly, $\chi_j \in \mathcal{D}$ and $(\overline{\mathrm{Q}})$ (iii) holds. Furthermore, let $\{\omega_n \mid n \in \mathbb{N}\}$ be a dense subset of $\mathcal{L}(\mathbb{R}^d)$ w.r.t. the uniform norm $\|\cdot\|_\infty$, and define for $k, n \in \mathbb{N}$, $i \in \{1, \ldots, d\}$

$$f_{i,k,n}(\omega) := \varphi(x^i(\omega(s_k) - \omega_n(s_k))), \quad \omega \in \mathcal{L}(\mathbb{R}^d),$$

where $\varphi \in C_b^\infty(\mathbb{R})$ is a fixed odd function such that $|\varphi| \leq 2$, $0 \leq \varphi' \leq 1$, $\varphi'' \leq 0$ on $[0, \infty)$, and $\varphi(x) = x$ for $x \in [-1, 1]$. Furthermore, $\{s_k \mid k \in \mathbb{N}\}$ is a dense subset of $[0, 1]$ and $x^i : \mathbb{R}^d \to \mathbb{R}$, $1 \leq i \leq d$, are the standard coordinate functions (cf. [7], [32, Subsection 4b)]). By [9, Lemma 5.2.2] it follows that $(\overline{\mathrm{Q}})$ (ii) holds for $f_{l,n}$, $l, n \in \mathbb{N}$, if $\{f_{i,k,n} \mid i, k, n \in \mathbb{N}\}$ is renumbered appropriately. But obviously for all $n \in \mathbb{N}$, $\omega \in \mathcal{L}(\mathbb{R}^d)$

$$\sup_{\substack{1 \leq i \leq d \\ k \in \mathbb{N}}} f_{i,k,n}(\omega) = \|\omega - \omega_n\|_\infty \wedge 1.$$

So, $(\overline{\mathrm{Q}})$ (i) holds for $\rho(\omega, \omega') := \|\omega - \omega'\|_\infty \wedge 1$, $\omega, \omega' \in \mathcal{L}(\mathbb{R}^d)$. Taking this uniformly equivalent metric, we see that $(\overline{\mathrm{Q}})$ holds and thus Corollary 4.9, Proposition 4.12, and Theorem 4.13 apply.

Appendix

In this appendix we want to compare the $\{E_k\}$-vague topology on Γ_E introduced in Subsection 3.3 and the usual topology on configuration spaces described e.g. in [36]. To this end let us set

$$\Gamma_{E_k} := \{\gamma \in \Gamma_E \mid \gamma(E \setminus E_k) = 0\}, \quad k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, there is a natural projection map $p_k : \Gamma_E \to \Gamma_{E_k}$ defined by $p_k\gamma := I_{E_k} \cdot \gamma$. The family of p_k determines an injection map $I : \Gamma_E \to \prod_{k \in \mathbb{N}} \Gamma_{E_k}$ by setting $I(\gamma) := (p_k\gamma)_{k \in \mathbb{N}}$. We give each Γ_{E_k} the topology induced by $\bar{\rho}$ and give $\prod_{k \in \mathbb{N}} \Gamma_{E_k}$ the product topology. Then I induces a topology (which below will be called the I^*-topology) on Γ_E. For $k \in \mathbb{N}$ and $n \in \mathbb{N}$, we set

$$\Gamma_{E_k}^{(n)} := \{\gamma \in \Gamma_{E_k} \mid \gamma(E_k) = n\}.$$

Lemma A.1. *$\Gamma_{E_k}^{(n)}$ is both $\bar{\rho}$-open and $\bar{\rho}$-closed in Γ_{E_k}.*

Proof. Let $\gamma_j \in \Gamma_{E_k}^{(n)}$, $j \in \mathbb{N}$, and $\bar{\rho}(\gamma_j, \gamma) \xrightarrow[j \to \infty]{} 0$ for some $\gamma \in \Gamma_{E_k}$. Then

$$\gamma(E_k) = \int_E \phi_{k+1}\, d\gamma = \lim_{j \to \infty} \int_E \phi_{k+1}\, d\gamma_j = n,$$

where ϕ_{k+1} is specified in (3.12). Hence $\Gamma_{E_k}^{(n)}$ is closed. Let $\gamma \in \Gamma_{E_k}^{(n)}$ and $\gamma' \in \Gamma_{E_k}$ be such that $\bar{\rho}(\gamma, \gamma') < 2^{-k-2}$. Then by (3.12) and (3.7) we have

$$|\gamma(E_k) - \gamma'(E_k)| \leq \bar{\rho}_0(\phi_{k+1} \cdot \gamma, \phi_{k+1} \cdot \gamma') \leq 2^{k+1}\, \bar{\rho}(\gamma, \gamma') < \frac{1}{2}.$$

Hence $n - 1/2 \leq \gamma'(E_k) \leq n + 1/2$. But $\gamma'(E_k)$ must be an integer. Therefore $\gamma'(E_k) = n$

and $\gamma' \in \Gamma_{E_k}^{(n)}$. This means that $\{\gamma' \in \Gamma_{E_k} \mid \bar{\rho}(\gamma, \gamma') < 2^{-k-2}\} \subset \Gamma_{E_k}^{(n)}$. Since $\gamma \in \Gamma_{E_k}^{(n)}$ was arbitrary, $\Gamma_{E_k}^{(n)}$ is $\bar{\rho}$-open in Γ_{E_k}. □

For $\gamma \in \Gamma_{E_k}^{(n)}$ let $\{\chi_i(\gamma) \mid 1 \leq i \leq n\}$ be the n distinct points in E_k such that $\gamma(\{\chi_i(\gamma)\}) = 1$ for all $1 \leq i \leq n$. In [36] a metric on $\Gamma_{E_k}^{(n)}$ has been defined by

$$d_k^{(n)}(\gamma, \gamma') := \inf_{\sigma \in \Lambda_n} \sum_{i=1}^{n} \rho(\chi_i(\gamma), \chi_{\sigma(i)}(\gamma')) \quad \forall \gamma, \gamma' \in \Gamma_{E_k}^{(n)},$$

where Λ_n is the permutation group on $\{1, 2, \ldots, n\}$.

Lemma A.2. *$\bar{\rho}$ and $d_k^{(n)}$ generate the same topology on $\Gamma_{E_k}^{(n)}$.*

Proof. Since $\bar{\rho}$ restricted to $\Gamma_{E_k}^{(n)}$ is equivalent to $\bar{\rho}_0$. We need only to compare $\bar{\rho}_0$ and $d_k^{(n)}$. Note that for $g_{A,\varepsilon}$ as in (3.4), we have $|g_{A,\varepsilon}(x) - g_{A,\varepsilon}(y)| \leq \rho(x, y)$ for all $x, y \in E$. Hence for $\gamma, \gamma' \in \Gamma_{E_k}^{(n)}$

$$|\bar{g}_{A,\varepsilon}(\gamma) - \bar{g}_{A,\varepsilon}(\gamma')| = \left| \sum_{i=1}^{n} g_{A,\varepsilon}(\chi_i(\gamma)) - \sum_{i=1}^{n} g_{A,\varepsilon}(\chi_i(\gamma')) \right| \leq d_k^{(n)}(\gamma, \gamma').$$

Consequently,

$$\bar{\rho}_0(\gamma, \gamma') \leq d_k^{(n)}(\gamma, \gamma'), \quad \forall \gamma, \gamma' \in \Gamma_{E_k}^{(n)}. \tag{A.1}$$

On the other hand, for fixed $\gamma \in \Gamma_{E_k}^{(n)}$ we set

$$\beta := \frac{1}{5} \inf\{\rho(\chi_i(\gamma), \chi_j(\gamma)) \mid i \neq j\}.$$

For $\alpha < \beta$ let $f_x(\omega)$ be defined as in (3.14). Let $\gamma' \in \Gamma_{E_k}^{(n)}$, $\bar{\rho}_0(\gamma, \gamma') < \alpha/\{2(1 + \alpha)\}$. Then we have for each i

$$\begin{aligned} &\left| 1 - \sum_{j=1}^{n} f_{\chi_i(\gamma)}(\chi_j(\gamma')) \right| \\ &= \left| \int f_{\chi_i(\gamma)}\, d\gamma - \int f_{\chi_i(\gamma)}\, d\gamma' \right| \\ &\leq \frac{1 + \alpha}{\alpha} \bar{\rho}_0(\gamma, \gamma') \leq \frac{1}{2}. \end{aligned} \tag{A.2}$$

Therefore, there must be some j satisfying

$$\rho(\chi_i(\gamma), \chi_j(\gamma')) < 2\alpha.$$

Otherwise, we would have $\sum_{i=1}^n f_{\chi_i(\gamma)}(\chi_j(\gamma')) = 0$, contradicting (A.2).

Since (A.2) holds for each $1 \leq i \leq n$, hence $d_k^{(n)}(\gamma,\gamma') \leq 2n\alpha < 3n\alpha$, which means that

$$\left\{\gamma' \in \Gamma_{E_k}^{(n)} \mid \bar{\rho}_0(\gamma,\gamma') < \frac{\alpha}{2(1+\alpha)}\right\} \subset \{\gamma' \in \Gamma_{E_k}^{(n)} \mid d_k^{(n)}(\gamma,\gamma') < 3n\alpha\}.$$

Since $\alpha < \beta$ was arbitrary, the $\bar{\rho}_0$-topology is stronger than the $d_k^{(n)}$-topology. This together with (A.1) implies that $\bar{\rho}_0$ and $d_k^{(n)}$ generate the same topology on $\Gamma_{E_k}^{(n)}$. □

The above Lemmas A.1 and A.2 imply that the I^*-topology on Γ_E coincides with the topology described in [36]. We now give some remarks on the comparison between the I^*-topology and $\bar{\rho}$-topology on Γ_E.

Remark A.3. (i) It is easy to see that the $\bar{\rho}$-topology is weaker (i.e., contains less opens sets) than the I^*-topology. But they do not coincide. For example, let $x \in \partial E_k$ for some k and take $x_j \in E_k$, $j \in \mathbb{N}$, such that $\rho(x_j, x) \underset{j\to\infty}{\longrightarrow} 0$. Then $(\varepsilon_{x_j})_{j\in\mathbb{N}}$ converges to ε_x weakly and $\bar{\rho}(\varepsilon_{x_j}, \varepsilon_x) \underset{j\to\infty}{\longrightarrow} 0$. But $(\varepsilon_{x_j})_{j\in\mathbb{N}}$ does not converge to ε_x in the I^*-topology. Because we have $p_k\varepsilon_{x_j} = \varepsilon_{x_j} \in \Gamma_{E_k}^{(1)}$ and $p_k\varepsilon_x = 0 \in \Gamma_{E_k}^{(0)}$, hence $(p_k\varepsilon_{x_j})_{j\in\mathbb{N}}$ and $p_k\varepsilon v_x$ are in two disconnected components of Γ_{E_k} (cf. Lemma A.1).

(ii) On the other hand the Portemanteau Theorem implies that the I^*-topology and $\bar{\rho}$-topology coincide on $\Gamma_E^0 := \{\gamma \in \Gamma_E \mid \gamma(\partial E_k) = 0 \text{ for all } k \in \mathbb{N}\}$. Note that in the classical case $\Gamma_E \setminus \Gamma_E^0$ is negligible w.r.t. Poisson measure.

(iii) Since both $\Gamma(E)$ equipped with the $\bar{\rho}$-topology and $\Gamma(E)$ equipped with the I^*-topology are (topological) Lusin spaces and since the latter topology is finer than the first, it follows that the corresponding Borel σ-fields coincide (cf. [35]).

ACKNOWLEDGEMENT. We would like to thank Sergio Albeverio and Yuri Kondratiev for fruitful discussions about this paper. Financial support by the German Science Foundation (DFG) through the Sonderforschungsbereich 343, by the Ministry of Science and Education of Nordrhein–Westfalen and the EU–TMR Project, No. ERB–FMRX–CT96–0075 is gratefully acknowledged.

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