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INVARIANT SUBRINGS UNDER THE ACTION BY A FINITE GROUP GENERATED BY PSEUDO-REFLECTIONS

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1. Introduction

In this note, let R be a regular local ring with maximal ideal \mathfrak{m} and let k be the residue field of R . Assume that R contains k as a subfield. Let G be a finite subgroup of $\text{Aut}_k R$. Assume that $(|G|, \text{ch } k) = 1$ if k has positive characteristic. Further we assume that G is generated by pseudo-reflections relative to the induced action on the Zariski tangent space $\mathfrak{m}/\mathfrak{m}^2$ of R . (Hence R^G is again a regular local ring and R is a finitely generated R^G -module (c.f. [3]).) For an arbitrary Macaulay local ring B with maximal ideal \mathfrak{n} , we put $r(B) = \dim_{B/\mathfrak{n}} \text{Ext}_B^d(B/\mathfrak{n}, B)$ ($d = \dim B$) and call it the type of B . Recall that B is a Gorenstein local ring if and only if $r(B) = 1$. The aim of this paper is to prove the following

Theorem. *Let \mathfrak{a} be an ideal of R and assume that \mathfrak{a} is stable under the G -action on R . We denote R/\mathfrak{a} by A . Then we have*

- (1) *If A is a Macaulay local ring, then the ring A^G of invariants is again a Macaulay local ring and the inequality $r(A^G) \leq r(A)$ holds.*
- (2) *If A is a complete intersection, then the ring A^G of invariants is again a complete intersection.*

It is known that A^G is a Macaulay local ring if A is a Macaulay local ring (c.f. Proposition 13, [2]).

As a consequence of this theorem we have

Corollary (c.f. Watanabe, [4]). *If A is a Gorenstein local ring, then the ring A^G of invariants is again a Gorenstein local ring.*

2. Proof of the theorem

An R -module M is called an (R, G) -module if the group G acts on the additive group of the module M so that the identity $s(ax) = s(a)s(x)$ holds for every $s \in G$, $a \in R$, and $x \in M$. An R -homomorphism of (R, G) -modules is

called a homomorphism of (R, G) -modules if it is compatible with G -action. For an (R, G) -module M , M^G is an R^G -module and it is contained in the R^G -module M as a direct summand. The projection $\rho_M: M \rightarrow M^G$ is given by $\rho_M(x) = 1/g \cdot \sum_{s \in G} s(x)$ which is called the Reynolds operator for M , where $g = |G|$.

Note that $[]^G$ is an exact functor.

Let N be a finitely generated R -module and let $i \geq 0$ be an integer. We put $\beta_i^R(N) = \dim_k \text{Tor}_i^R(k, N)$ and call it the i -th Betti number of N . Recall that, if the sequence $\cdots \rightarrow F_i \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow N \rightarrow 0$ is a minimal free resolution of N , then the number $\beta_i^R(N)$ is equal to the rank of the R -module F_i .

First we give the following lemma.

Lemma 1. *Let M be an (R, G) -module and assume that M is finitely generated as an R -module. Then there exists an exact sequence*

$$\cdots \rightarrow F_i \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

of (R, G) -modules such that each (R, G) -module F_i is a finitely generated free R -module with $\text{rank}_R F_i = \beta_i^R(M)$.

Proof. We put $r = \beta_0^R(M)$ ($= \dim_k M/mM$). Notice that the sequence $0 \rightarrow mM \rightarrow M \xrightarrow{\varepsilon} M/mM \rightarrow 0$ of (R, G) -modules can be regarded as an exact sequence of G -spaces over the field k . Then, by virtue of Maschke's theorem, we can find an r -dimensional G -subspace V of M so that $\varepsilon(V) = M/mM$. (Recall that $(|G|, \text{ch } k) = 1$ if k has positive characteristic. This is one of our standard assumptions.) Let $\{e_i\}_{1 \leq i \leq r}$ be a k -basis of V and let $[a_{ij}(s)]$ denote the matrix representation of an element s of G relative to this basis. Let F be a finitely generated free R -module of rank r and let $\{X_i\}_{1 \leq i \leq r}$ be an R -free basis of F . We put $s(X_j) = \sum_{i=1}^r a_{ij}(s)X_i$ for every $s \in G$ and for every $1 \leq j \leq r$, and we define a G -action on F by

$$s\left(\sum_{j=1}^r a_j X_j\right) = \sum_{j=1}^r s(a_j) s(X_j)$$

where $s \in G$ and $a_j \in R$. Then the R -module F becomes an (R, G) -module under this action. Moreover, if we define an R -linear map $f: F \rightarrow M$ by $f(X_j) = e_j$ for every $1 \leq j \leq r$, then f is a surjective homomorphism of (R, G) -modules. (Note that $M = \sum_{i=1}^r Re_i$ by Nakayama's lemma.) Inductively we can construct an exact sequence of (R, G) -modules mentioned above.

Lemma 2. *Let M be an (R, G) -module and assume that M is finitely generated as an R -module. Then the inequality $\beta_i^R(M) \geq \beta_i^{R^G}(M^G)$ holds for every integer $i \geq 0$.*

Proof. Let $\cdots \rightarrow F_i \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ be an exact sequence of (R, G) -modules obtained for M by Lemma 1. Since R is a finitely generated free R^G -module and since F_i^G is a direct summand of F_i as an R^G -module, we see that F_i^G is a finitely generated free R^G -module for every integer $i \geq 0$. Therefore, as the sequence $\cdots \rightarrow F_i^G \rightarrow \cdots \rightarrow F_1^G \rightarrow F_0^G \rightarrow M^G \rightarrow 0$ of R^G -modules is exact, to prove this lemma we have only to show that $\text{rank}_R F \geq \text{rank}_{R^G} F^G$ for every (R, G) -module F which is finitely generated and free as an R -module. We put $r = \text{rank}_{R^G} F^G$ and let $\{x_i\}_{1 \leq i \leq r}$ be an R^G -free basis of F^G . We denote by K the quotient field of R and consider $K \otimes_R F$ as a (K, G) -module naturally (*i.e.*, We define $s(c \otimes x) = s(c) \otimes s(x)$ for $s \in G$, $c \in K$, and $x \in F$).

Now assume that $r > \text{rank}_R F$. Then $\{1 \otimes x_i\}_{1 \leq i \leq r}$ is not linearly independent over K and so we may express, without loss of generality, $1 \otimes x_1 = \sum_{i=2}^r c_i \otimes x_i$ for some $c_i \in K$. Let s be an element of G . Then, as $s(x_i) = x_i$ for every $1 \leq i \leq r$, we have $1 \otimes x_1 = \sum_{i=2}^r s(c_i) \otimes x_i$. Thus we see that $1 \otimes x_1 = \sum_{i=2}^r (1/g \cdot \sum_{s \in G} s(c_i)) \otimes x_i$ where $g = |G|$. Now we put $b_i = 1/g \cdot \sum_{s \in G} s(c_i)$. Then, since $b_i \in K^G$ and since K^G coincides with the quotient field of R^G , we can find a non-zero element a of R^G so that $ab_i \in R^G$ for every $2 \leq i \leq r$. Therefore there is an identity $ax_1 = \sum_{i=2}^r a_i x_i$ in F^G where $a_i = ab_i$. But this is impossible, since $\{x_i\}_{1 \leq i \leq r}$ is linearly independent over R^G . Thus we conclude that $r \leq \text{rank}_R F$.

Proof of the theorem.

First consider (1) and suppose that A is a Macaulay local ring. It is known that A^G is a Macaulay local ring (c.f. Proposition 13, [2]). We put $s = \dim R - \dim A^G$. (Note that $s = \dim R^G - \dim A^G$, since $\dim R^G = \dim R$ and $\dim A^G = \dim A$.) Then we have, by Lemma 3.5 of [1], that $\beta_s^R(A) = r(A)$. Similarly we have $\beta_s^{R^G}(A^G) = r(A^G)$, since R^G is a regular local ring by the standard assumption. Thus we conclude that $r(A^G) \leq r(A)$ by Lemma 2.

Now consider (2) and suppose that A is a complete intersection. Then $\beta_1^R(A) = s$ and hence $\beta_s^{R^G}(A^G) \leq s$ by Lemma 2. On the other hand, since R^G is a regular local ring and since $s = \dim R^G - \dim A^G$, we know that $\beta_1^{R^G}(A^G) \geq s$. Therefore $\beta_1^{R^G}(A^G) = s$ and this implies that A^G is again a complete intersection.

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