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THE BIVARIATE ORTHOGONAL INVERSE EXPANSION AND THE MOMENTS OF ORDER STATISTICS

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1. Introduction

The orthogonal inverse expansion has been introduced in my previous paper [9] to obtain the universal upper bounds and the approximation for the moments of order statistics. For the same purpose two other series have been introduced by David and Johnson [2] and Plackett [6] and the error of the approximation of $E(X_{r/n})$ is evaluated by Saw [8] and Plackett [6].

In this paper we shall derive the universal upper bounds and the approximation for $E(X_{r/n}^i X_{s/n}^j)$ $(i, j=1, 2)$ together with the error of the approximation by means of the bivariate orthogonal inverse expansion.

2. Some preliminaries

First we restate for convenience the following Proposition in [9].

Proposition 1. *Let H be a pre-Hilbert space and $\{\varphi_\nu\}_{\nu=0,1,\cdots}$ be any orthonormal system in H. Put $a_\nu=(f, \varphi_\nu)$ and $b_\nu=(g, \varphi_\nu)$ for any elements f, g in H. Then we have*

$$(2.1)\qquad |(f, g)-\sum_{\nu=0}^{k} a_\nu b_\nu| \leq \{\|f\|^2-\sum_{\nu=0}^{k} a_\nu^2\}^{1/2}\{\|g\|^2-\sum_{\nu=0}^{k} b_\nu^2\}^{1/2},$$

where equality holds if and only if $f, g, \varphi_0, \cdots, \varphi_k$ are linearly dependent.

We also use the following well-known Proposition concerning a bivariate orthonormal system in a rectangular domain. The proof is found, for example, in Courant and Hilbert [1].

Proposition 2. *Let $L^2(0, 1)$ and $L^2(R)$ be the Hilbert spaces of all functions square integrable in the interval $(0,1)$ and the square $R=\{(u, v)| 0\leq u, v\leq 1\}$, respectively. If $\{\varphi_\nu(u)\}_{\nu=0,1,\cdots}$ is a complete orthonormal system in $L^2(0, 1)$, then $\{\varphi_\lambda(u)\varphi_\nu(v)\}_{\lambda,\nu=0,1,\cdots}$ is a complete orthonormal system in $L^2(R)$.*

EXAMPLE 1. Legendre polynomials in (0, 1):

$$\varphi_\nu(u) = \frac{\sqrt{2\nu+1}}{\nu!}\frac{d^\nu}{du^\nu}u^\nu(u-1)^\nu \qquad (\nu = 0, 1, \cdots) \tag{2.2}$$

constitute a complete orthonormal system in $L^2(0, 1)$, so we can get a complete orthonormal system in $L^2(R)$ by Proposition 2.

To get the results corresponding to Example 3 in [9], we decompose $L^2(R)$ into the following four subspaces:

$$\begin{aligned} L^2_{e,e}(R) &= \{f(u,v) \mid f \in L^2(R) \text{ and } f(u,v) = f(1-u,v) = f(u,1-v)\}\,, \\ L^2_{e,0}(R) &= \{f(u,v) \mid f \in L^2(R) \text{ and } f(u,v) = f(1-u,v) = -f(u,1-v)\}\,, \\ L^2_{0,e}(R) &= \{f(u,v) \mid f \in L^2(R) \text{ and } f(u,v) = -f(1-u,v) = f(u,1-v)\}\,, \\ L^2_{0,0}(R) &= \{f(u,v) \mid f \in L^2(R) \text{ and } f(u,v) = -f(1-u,v) = -f(u,1-v)\}\,. \end{aligned} \tag{2.3}$$

Proposition 3. *Let $\{\varphi_\nu(u)\}_{\nu=0,1,\cdots}$ be any complete orthonormal system in $L^2(0, 1)$ satisfying $\varphi_\nu(u)=(-1)^\nu\varphi_\nu(1-u)$. Then*

$$\begin{aligned} &\{\varphi_{2\lambda}(u)\varphi_{2\nu}(v)\}_{\lambda,\nu=0,1,\cdots} \in L^2_{e,e}(R)\,, \quad \{\varphi_{2\lambda}(u)\varphi_{2\nu+1}(v)\}_{\lambda,\nu=0,1,\cdots} \in L^2_{e,0}(R)\,, \\ &\{\varphi_{2\lambda+1}(u)\varphi_{2\nu}(v)\}_{\lambda,\nu=0,1,\cdots} \in L^2_{0,e}(R)\,, \quad \{\varphi_{2\lambda+1}(u)\varphi_{2\nu+1}(v)\}_{\lambda,\nu=0,1,\cdots} \in L^2_{0,0}(R) \end{aligned} \tag{2.4}$$

and each subsystem is complete and orthonormal in the corresponding subspace.

The proof of Proposition 3 is straightforward. The completeness follows from Parseval's equation as in Example 3 of [9].

REMARK 1. Legendre polynomials cited in Example 1 satisfy the assumption of Proposition 3 and will be used in section 4.

Proposition 4. *Suppose that a random variable X has a distribution function $F(x)$ absolutely continuous with respect to the Lebesgue measure and that $\mathrm{E}(X^2)<\infty$. Put $u=F(x)$, and then the inverse function $x(u)=F^{-1}(u)$ is defined almost everywhere u and*

$$x(u)x(v) \in L^2(R)\,. \tag{2.5}$$

If further $F(x)$ is symmetric, then

$$x(u)x(v) \in L^2_{0,0}(R)\,. \tag{2.6}$$

Proof.

$$\begin{aligned} \iint_R [x(u)x(v)]^2 du dv &= \int_0^1 [x(u)]^2 du \int_0^1 [x(v)]^2 dv \\ &= \left(\int_{-\infty}^{\infty} x^2 dF(x)\right)^2 < \infty\,, \end{aligned}$$

which shows (2.5). Symmetricity of $F(x)$ means that $x(u)=-x(1-u)$, which implies (2.6).

3. Universal upper bounds and approximation for $E(X^i_{r/n}X^j_{s/n})$ $(i, j=1, 2)$

The following Theorem is an extension of Theorem 1 in [9] to the bivariate case.

Theorem 1. *Let $X_{i/n}$ be the ith (smallest) order statistic in a random sample of size n with distribution function $F(x)$ absolutely continuous with respect to the Lebesgue measure having mean μ and finite variance σ^2. Let $\{\varphi_\nu(u)\}_{\nu=0,1,\cdots}$ $(\varphi_0(u)=1)$ be any complete orthonormal system in $L^2(0,1)$ and let for any pair r, s $(1\leqq r<s\leqq n)$*

$$a_\lambda = \int_0^1 x(u)\varphi_\lambda(u)du\,, \tag{3.1}$$

$$b_{\lambda,\nu} = \frac{1}{B(r,\,s-r,\,n-s+1)}\iint_{0<u<v<1} u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s}\varphi_\lambda(u)\varphi_\nu(v)dudv, \tag{3.2}$$

then

$$\left|E(X_{r/n}X_{s/n})-\mu E(X_{r/n}+X_{s/n})+\mu^2-\frac{1}{2}\sum_{\lambda,\nu=1}^{k} a_\lambda a_\nu(b_{\lambda,\nu}+b_{\nu,\lambda})\right| \tag{3.3}$$

$$\leqq \{\sigma^4-\sum_{\lambda,\nu=1}^{k} a_\lambda^2 a_\nu^2\}^{1/2}\left\{\frac{B(2r-1,\,2s-2r-1,\,2n-2s+1)}{2[B(r,\,s-r,\,n-s+1)]^2}\right.$$

$$-\frac{B(2r-1,\,2n-2r+1)}{2[B(r,\,n-r+1)]^2}-\frac{B(r+s-1,\,2n-r-s+1)}{B(r,\,n-r+1)B(s,\,n-s+1)}$$

$$\left.-\frac{B(2s-1,\,2n-2s+1)}{2[B(s,\,n-s+1)]^2}+1-\frac{1}{4}\sum_{\lambda,\nu=1}^{k}(b_{\lambda,\nu}+b_{\nu,\lambda})^2\right\}^{1/2},$$

where

$$B(p,\,q,\,r) = \frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(p+q+r)}. \tag{3.4}$$

Proof. It holds that

$$E(X_{r/n}X_{s/n}) = \frac{1}{B(r,\,s-r,\,n-s+1)}\iint_{-\infty<x<y<\infty} xy[F(x)]^{r-1} \tag{3.5}$$

$$\times[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}dF(x)dF(y)$$

$$=\frac{1}{B(r,\,s-r,\,n-s+1)}\iint_{0<u<v<1} x(u)x(v)u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s}dudv$$

$$=\frac{1}{B(r,\,s-r,\,n-s+1)}\iint_{0<v<u<1} x(u)x(v)v^{r-1}(u-v)^{s-r-1}(1-u)^{n-s}dudv$$

$$=\iint_{R} f(u,\,v)g(u,\,v)dudv\,,$$

where

$$f(u, v) = x(u)x(v)\,, \tag{3.6}$$

$$g(u, v) = \begin{cases} \dfrac{1}{2B(r,\ s-r,\ n-s+1)} u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s} & 0<u<v<1, \\ \dfrac{1}{2B(r,\ s-r,\ n-s+1)} v^{r-1}(u-v)^{s-r-1}(1-u)^{n-s} & 0<v<u<1. \end{cases} \tag{3.7}$$

Since

$$\begin{aligned} ||f||^2 &= \iint_R [x(u)x(v)]^2 dudv = (\mu^2+\sigma^2)^2 < \infty\,, \\ ||g||^2 &= \frac{B(2r-1,\ 2s-2r-1,\ 2n-2s+1)}{2[B(r,\ s-r,\ n-s+1)]^2} < \infty\,, \end{aligned} \tag{3.8}$$

we can apply the Proposition 1 to (f, g) in (3.5). Replacing $\{\varphi_\nu\}_{\nu=0,1,\cdots,k}$ by $\{\varphi_0(u)\varphi_\nu(v)\}_{\nu=0,1,\cdots} \cup \{\varphi_\lambda(u)\varphi_0(v)\}_{\lambda=1,2,\cdots} \cup \{\varphi_\lambda(u)\varphi_\nu(v)\}_{\lambda,\nu=1,2,\cdots,k}$ in Proposition 1, we have

$$\begin{aligned} &|\mathrm{E}(X_{r/n}X_{s/n}) - \sum_{\lambda=0}^{\infty} a^*_{\lambda,0} b^*_{\lambda,0} - \sum_{\nu=0}^{\infty} a^*_{0,\nu} b^*_{0,\nu} + a^*_{0,0} b^*_{0,0} - \sum_{\lambda,\nu=1}^{k} a^*_{\lambda,\nu} b^*_{\lambda,\nu}| \\ &\leqq \{||f||^2 - \sum_{\lambda=0}^{\infty} a^{*2}_{\lambda,0} - \sum_{\nu=0}^{\infty} a^{*2}_{0,\nu} + a^{*2}_{0,0} - \sum_{\lambda,\nu=1}^{k} a^{*2}_{\lambda,\nu}\}^{1/2} \{||g||^2 - \sum_{\lambda=0}^{\infty} b^{*2}_{\lambda,0} - \sum_{\nu=0}^{\infty} b^{*2}_{0,\nu} \\ &\quad + b^{*2}_{0,0} - \sum_{\lambda,\nu=1}^{k} b^{*2}_{\lambda,\nu}\}^{1/2}\,, \end{aligned} \tag{3.9}$$

where $a^*_{\lambda,\nu}$ and $b^*_{\lambda,\nu}$ are the Fourier coefficients of f and g in R, that is,

$$\begin{aligned} a^*_{\lambda,\nu} &= \iint_R x(u)x(v)\varphi_\lambda(u)\varphi_\nu(v)dudu = a_\lambda a_\nu\,, \\ b^*_{\lambda,\nu} &= \iint_R g(u, v)\varphi_\lambda(u)\varphi_\nu(v)dudv \\ &= \frac{1}{2B(r,\ s-r,\ n-s+1)} \iint_{0<u<v<1} u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s} \\ &\quad \times [\varphi_\lambda(u)\varphi_\nu(v) + \varphi_\lambda(v)\varphi_\nu(u)]dudv \\ &= \frac{b_{\lambda,\nu} + b_{\nu,\lambda}}{2}\,, \end{aligned} \tag{3.10}$$

because of (3.1) and (3.2). This implies

$$\begin{aligned} a^*_{0,0} &= a_0^2 = \mu^2\,, \qquad & a^*_{\lambda,\nu} &= a^*_{\nu,\lambda}\,, \\ b^*_{0,0} &= 1\,, & b^*_{\lambda,\nu} &= b^*_{\nu,\lambda}\,, \end{aligned} \tag{3.11}$$

and hence

$$\sum_{\nu=0}^{\infty} a_{0,\nu}^{*2} = \sum_{\lambda=0}^{\infty} a_{\lambda,0}^{*2} = \mu^2 \sum_{\nu=0}^{\infty} a_\nu^2 = \mu^2(\mu^2+\sigma^2) . \tag{3.12}$$

We calculate each term in (3.9). If we put $\lambda=0$ in (3.2) and transform the variable u to t by $u=vt$, then

$$\begin{aligned} b_{0,\nu} &= \frac{1}{B(r,\, s-r,\, n-s+1)} \int_0^1 v^{s-1}(1-v)^{n-s}\varphi_\nu(v)dv \int_0^1 t^{r-1}(1-t)^{s-r-1}dt \\ &= \frac{1}{B(s,\, n-s+1)} \int_0^1 v^{s-1}(1-v)^{n-s}\varphi_\nu(v)dv , \end{aligned} \tag{3.13}$$

and similarly

$$b_{\lambda,0} = \frac{1}{B(r,\, n-r+1)} \int_0^1 u^{r-1}(1-u)^{n-r}\varphi_\lambda(u)du . \tag{3.14}$$

We have already met with (3.13) in Theorem 1 and 2 in [9] in dealing with $\mathrm{E}(X_{s/n})$. From (3.10), (3.13), (3.14) and the completeness of $\{\varphi_\nu(u)\}_{\nu=0,1,\cdots}$ in $L^2(0, 1)$, it follows that

$$\begin{aligned} \sum_{\nu=0}^{\infty} a_{0,\nu}^{*} b_{0,\nu}^{*} &= \frac{\mu}{2} \sum_{\nu=0}^{\infty} a_\nu (b_{0,\nu}+b_{\nu,0}) \\ &= \frac{\mu}{2} \mathrm{E}(X_{r/n}+X_{s/n}) , \end{aligned} \tag{3.15}$$

$$\begin{cases} \displaystyle\sum_{\lambda=0}^{\infty} b_{\lambda,0}^2 = \frac{B(2r-1,\, 2n-2r+1)}{[B(r,\, n-r+1)]^2} , \\ \displaystyle\sum_{\nu=0}^{\infty} b_{0,\nu}^2 = \frac{B(2s-1,\, 2n-2s+1)}{[B(s,\, n-s+1)]^2} , \\ \displaystyle\sum_{\nu=0}^{\infty} b_{0,\nu} b_{\nu,0} = \frac{B(r+s-1,\, 2n-r-s+1)}{B(r,\, n-r+1)B(s,\, n-s+1)} , \end{cases} \tag{3.16}$$

whence we can calculate

$$\sum_{\nu=0}^{\infty} b_{0,\nu}^{*2} = \sum_{\lambda=0}^{\infty} b_{\lambda,0}^{*2} = \frac{1}{4} \left(\sum_{\lambda=0}^{\infty} b_{\lambda,0}^2 + 2 \sum_{\nu=0}^{\infty} b_{0,\nu} b_{\nu,0} + \sum_{\nu=0}^{\infty} b_{0,\nu}^2\right) . \tag{3.17}$$

Substituting (3.11), (3.12) and (3.15)~(3.17) into (3.9), we can get Theorem 1.

Corollary 1. *For any distribution absolutely continuous with respect to the Lebesgue measure with mean zero and variance one and for any* r, s $(1 \leqq r < s \leqq n)$,

$$|\mathrm{E}(X_{r/n}X_{s/n})| \leqq \Big\{\frac{B(2r-1,\, 2s-2r-1,\, 2n-2s+1)}{2[B(r,\, s-r,\, n-s+1)]^2} \tag{3.18}$$

$$-\frac{B(2r-1,\,2n-2r+1)}{2[B(r,\,n-r+1)]^2}-\frac{B(r+s-1,\,2n-r-s+1)}{B(r,\,n-r+1)B(s,\,n-s+1)}$$
$$-\frac{B(2s-1,\,2n-2s+1)}{2[B(s,\,n-s+1)]^2}+1\Big\}^{1/2} \qquad (=a,\ say),$$

and equality holds if and only if

$$(3.19)\qquad x(u)x(v)=\pm\frac{1}{a}\Big\{1+g(u,\,v)-\frac{u^{r-1}(1-u)^{n-r}+v^{r-1}(1-v)^{n-r}}{2B(r,\,n-r+1)}$$
$$-\frac{u^{s-1}(1-u)^{n-s}+v^{s-1}(1-v)^{n-s}}{2B(s,\,n-s+1)}\Big\},$$

where $g(u, v)$ *is defined by* (3.7).

Proof. We get (3.18) from Theorem 1 by excluding the terms corresponding to $\{\varphi_\lambda(u)\varphi_\nu(v)\}_{\lambda,\nu=1,2,\cdots,k}$ in (3.9). Equality holds from Proposition 1 if and only if

$$(3.20)\qquad x(u)x(v)=\alpha+\beta g(u,\,v)+\sum_{\nu=1}^{\infty}\gamma_\nu\varphi_\nu(u)+\sum_{\nu=1}^{\infty}\delta_\nu\varphi_\nu(v),$$

for some constants α, β, γ_ν and $\delta_\nu(\nu=1, 2, \cdots)$. Integrating (3.20) by u and v, we get

$$(3.21)\qquad \begin{aligned}&\alpha+\beta\int_0^1 g(u,\,v)du+\sum_{\nu=1}^{\infty}\delta_\nu\varphi_\nu(v)=0,\\ &\alpha+\beta\int_0^1 g(u,\,v)dv+\sum_{\nu=1}^{\infty}\gamma_\nu\varphi_\nu(u)=0,\\ &\alpha+\beta=0.\end{aligned}$$

From (3.7) we have

$$(3.22)\qquad \int_0^1 g(u,\,v)du=\frac{1}{2B(r,\,s-r,\,n-s+1)}\Big\{\int_0^v u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s}du$$
$$+\int_v^1 v^{r-1}(u-v)^{s-r-1}(1-u)^{n-s}du\Big\}$$
$$=\frac{v^{r-1}(1-v)^{n-r}}{2B(r,\,n-r+1)}+\frac{v^{s-1}(1-v)^{n-s}}{2B(s,\,n-s+1)},$$

$$(3.23)\qquad \int_0^1 g(u,\,v)dv=\frac{u^{r-1}(1-u)^{n-r}}{2B(r,\,n-r+1)}+\frac{u^{s-1}(1-u)^{n-s}}{2B(s,\,n-s+1)}.$$

From (3.21)~(3.23) and $\|x(u)x(v)\|^2=1$, we can determine α, β, $\sum_{\nu=1}^{\infty}\gamma_\nu\varphi_\nu(u)$ and $\sum_{\nu=1}^{\infty}\delta_\nu\varphi_\nu(v)$. Substituting these relations into (3.20), we have (3.19).

In particular, when $r=n-1$ and $s=n$ in Corollary 1, we get

$$(3.24)\qquad |\mathrm{E}(X_{n-1/n}X_{n/n})|\leqq\frac{n-2}{2}\sqrt{\frac{n-1}{2n-1}}.$$

This exhibits a universal upper bound for $E(X_{n-1/n}X_{n/n})$ when the distribution is required to have mean zero and variance one. By the way, under the same assumption as above, the upper bound due to Gumbel [3] and Hartley and David [4] is

$$|E(X_{n/n})| \leqq \sqrt{\frac{n-1}{2n-1}}\,. \tag{3.25}$$

Some numerical values for $n=5$ in Corollary 1 are shown in the second column of Table 1. The third column is calculated from Corollary 2 (i.e. universal upper bounds for symmetric population). The last three columns give the values of $E(X_{r/5}X_{s/5})$ from the uniform distribution in the interval $(-\sqrt{3}, \sqrt{3})$, exponential distribution with the density $e^{-(x+1)}$ $(x \geqq -1)$, and the standard normal distribution, the values being calculated from Sarhan and Greenberg [7].

Table 1. Some special values of $E(X_{r/5}X_{s/5})$ and the upper bounds.

(r, s)	upper bound		true distribution		
	any distribution	symmetric distribution	uniform	exponential	normal
(4, 5)	1	0.8696	0.8571	0.8272	0.8000
(3, 5)	0.6667	0.2795	0.1429	−0.0645	0.1482
(2, 5)	0.8729	0.6512	−0.5714	−0.6006	−0.4699
(1, 5)	1.3244	1.3025	−1.2857	−0.9867	−1.2783
(3, 4)	0.6667	0.4812	0.2857	−0.4003	0.2084
(2, 4)	0.4543	0.3660	−0.1429	−0.0533	−0.0951

Now we shall consider, as in Moriguti [5], the case when the distribution is known to be symmetric. This additional information is expected to reduce the upper bound as is the case with $E(X_{r/n})$ in [9]. For this purpose we shall define

$$I(p_1, p_2, p_3, q_1, q_2) = \iint\limits_{\substack{0<u<v<1\\ u+v<1}} u^{p_1-1}v^{p_2-1}(1-v)^{p_3-1}(v-u)^{q_1-1}(1-u-v)^{q_2-1}dudv\,, \tag{3.26}$$

where p_1, p_2, p_3, q_1, q_2 are positive integers.

Lemma 1. *The following relations for $I(p_1, p_2, p_3, q_1, q_2)$ hold:*

$$I(p_1, p_2, p_3, q_1, q_2) = \sum_{i=0}^{p_2-1}\sum_{j=0}^{p_3-1}\binom{p_2-1}{i}\binom{p_3-1}{j} \times B(q_1+i, q_2+j, p_1+p_2+p_3-i-j-2)2^{-(p_1+p_2+p_3-i-j-2)}\,, \tag{3.27}$$

$$(3.28)\qquad I(p_1, p_2, p_3, q_1, q_2) = I(p_1, p_3, p_2, q_2, q_1).$$

Proof. Binomial expansion of $v^{p_2-1}(1-v)^{p_3-1}=\{(v-u)+u\}^{p_2-1}\{(1-u-v)+u\}^{p_3-1}$ yields

$$I(p_1, p_2, p_3, q_1, q_2) = \sum_{i=0}^{p_2-1}\sum_{j=0}^{p_3-1}\binom{p_2-1}{i}\binom{p_3-1}{j} \times \iint_{\substack{0<u<v<1\\ u+v<1}} u^{p_1+p_2+p_3-i-j-3}(v-u)^{q_1+i-1}(1-u-v)^{q_2+j-1}du dv.$$

After transforming the variable v to t by $v=u+(1-2u)t$, we can see that the right side of the equation is equal to

$$\sum_{i=0}^{p_2-1}\sum_{j=0}^{p_3-1}\binom{p_2-1}{i}\binom{p_3-1}{j}\int_0^{\frac{1}{2}} u^{p_1+p_2+p_3-i-j-3}(1-2u)^{q_1+q_2+i+j-1}du \times\int_0^1 t^{q_1+i-1}(1-t)^{q_2+j-1}dt,$$

which proves (3.27). (3.28) is obvious from (3.27).

Theorem 2. *Let $X_{i/n}$ be the ith (smallest) order statistic in a random sample of size n with symmetric distribution $F(x)$ absolutely continuous with respect to the Lebesgue measure with finite variance σ^2. Let $\{\varphi_\nu(u)\}_{\nu=0,1,\cdots}(\varphi_0(u)=1)$ be any complete orthonormal system in $L^2(0, 1)$ satisfying $\varphi_\nu(u)=(-1)^\nu\varphi_\nu(1-u)$ for $\nu=1, 2, \cdots$. Putting a_λ and $b_{\lambda,\nu}$ as in* (3.1) *and* (3.2), *we have for any r, s $(1\leqq r<s\leqq n)$*

$$(3.29)\qquad |\mathrm{E}(x_{r/n}X_{s/n})-\frac{1}{2}\sum_{\lambda,\nu=0}^{k} a_{2\lambda+1}a_{2\nu+1}(b_{2\lambda+1,2\nu+1}+b_{2\nu+1,2\lambda+1})|$$
$$\leqq \{\sigma^4-\sum_{\lambda,\nu=0}^{k} a_{2\lambda+1}^2 a_{2\nu+1}^2\}^{1/2}\{A_1-B_1-\frac{1}{4}\sum_{\lambda,\nu=0}^{k}(b_{2\lambda+1,2\nu+1}+b_{2\nu+1,2\lambda+1})^2\}^{1/2},$$

where

$$(3.30)\qquad A_1=\frac{B(2r-1, 2s-2r-1, 2n-2s+1)+B(n+r-s, n+r-s, 2s-2r-1)}{8[B(r, s-r, n-s+1)]^2},$$
$$B_1=\frac{I(2r-1, n-s+1, n-s+1, s-r, s-r)+2I(n+r-s, r, n-s+1, s-r, s-r)+I(2n-2s+1, r, r, s-r, s-r)}{8[B(r, s-r, n-s+1)]^2}$$

and $I(p_1, p_2, p_3, q_1, q_2)$ is defined by (3.27).

Proof. Since (3.5)～(3.8) hold also in this case for $\mu=0$, we substitute $\{\varphi_{2\lambda}(u)\varphi_\nu(v),\ \varphi_{2\lambda+1}(u)\varphi_{2\nu}(v)\}_{\lambda,\nu=0,1,\cdots}\cup\{\varphi_{2\lambda+1}(u)\varphi_{2\nu+1}(v)\}_{\lambda,\nu=0,1,\cdots,k}$ for $\{\varphi_\nu\}_{\nu=0,1,\cdots,k}$ in Proposition 1 to get

$$(3.31)\qquad |E(X_{r/n}X_{s/n})-\sum_{\lambda,\nu=0}^{\infty}(a_{2\lambda,\nu}^{*}b_{2\lambda,\nu}^{*}+a_{2\lambda+1,2\nu}^{*}b_{2\lambda+1,2\nu}^{*})-\sum_{\lambda,\nu=0}^{k}a_{2\lambda+1,2\nu+1}^{*}b_{2\lambda+1,2\nu+1}^{*}|$$
$$\leqq\{\|f\|^2-\sum_{\lambda,\nu=0}^{\infty}(a_{2\lambda,\nu}^{*2}+a_{2\lambda+1,2\nu}^{*2})-\sum_{\lambda,\nu=0}^{k}a_{2\lambda+1,2\nu+1}^{*2}\}^{1/2}$$
$$\times\{\|g\|^2-\sum_{\lambda,\nu=0}^{\infty}(b_{2\lambda,\nu}^{*2}+b_{2\lambda+1,2\nu}^{*2})-\sum_{\lambda,\nu=0}^{k}b_{2\lambda+1,2\nu+1}^{*2}\}^{1/2},$$

where $a_{\lambda,\nu}^{*}$ and $b_{\lambda,\nu}^{*}$ are defined by (3.10). Since $x(u)x(v)\in L_{0,0}^{2}(R)$ by Proposition 4, from Proposition 3 and (3.10) we have

$$(3.32)\qquad a_{2\lambda,\nu}^{*}=a_{2\lambda+1,2\nu}^{*}=0$$

and by Proposition 2 and (3.7), (3.10)

$$(3.33)\qquad \|g\|^2-\sum_{\lambda,\nu=0}^{\infty}(b_{2\lambda,\nu}^{*2}+b_{2\lambda+1,2\nu}^{*2})=\sum_{\lambda,\nu=0}^{\infty}b_{2\lambda+1,2\nu+1}^{*2}.$$

Hence the essential part of this proof consists in calculating $\sum_{\lambda,\nu=0}^{\infty}b_{2\lambda+1,2\nu+1}^{*2}$. From (3.10) we have

$$b_{2\lambda+1,2\nu+1}^{*}$$
$$=\frac{1}{2B(r,s-r,n-s+1)}\Big\{\iint_{0<u<v<1}u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s}\varphi_{2\lambda+1}(u)\varphi_{2\nu+1}(v)dudv$$
$$+\iint_{0<u<v<1}u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s}\varphi_{2\lambda+1}(v)\varphi_{2\nu+1}(u)dudv\Big\}.$$

Transforming the variable (u, v) to $(1-v, 1-u)$ in the second integral, we can rewrite the right-hand side as

$$\frac{1}{2B(r,s-r,n-s+1)}\iint_{0<u<v<1}\{(u^{r-1}(1-v)^{n-s}+u^{n-s}(1-v)^{r-1}\}$$
$$\times(v-u)^{s-r-1}\varphi_{2\lambda+1}(u)\varphi_{2\nu+1}(v)dudv,$$

which becomes after transforming (u, v) to $(1-u, 1-v)$

$$\frac{1}{2B(r,s-r,n-s+1)}\iint_{0<v<u<1}\{(1-u)^{r-1}v^{n-s}+(1-u)^{n-s}v^{r-1}\}$$
$$\times(u-v)^{s-r-1}\varphi_{2\lambda+1}(u)\varphi_{2\nu+1}(v)dudv.$$

So if we put

$$(3.34)\quad \xi(u,v)=\begin{cases}\dfrac{1}{4B(r,s-r,n-s+1)}\{u^{r-1}(1-v)^{n-s}+u^{n-s}(1-v)^{r-1}\}(v-u)^{s-r-1} \\ \qquad\qquad 0<u<v<1, \\ \dfrac{1}{4B(r,s-r,n-s+1)}\{(1-u)^{r-1}v^{n-s}+(1-u)^{n-s}v^{r-1}\}(u-v)^{s-r-1} \\ \qquad\qquad 0<v<u<1,\end{cases}$$

$$(3.35)\qquad \eta(u, v)=\frac{1}{2}\{\xi(u, v)-\xi(u, 1-v)\},$$

then we can see $\xi(u, v)=\xi(v, u)=\xi(1-u, 1-v)$ and $\eta(u, v)\in L^2_{0,0}(R)$. Since

$$(3.36)\qquad b^*_{2\lambda+1,2\nu+1}=\iint_R \eta(u, v)\varphi_{2\lambda+1}(u)\varphi_{2\nu+1}(v)dudv,$$

from Proposition 3 we have

$$(3.37)\qquad \sum_{\lambda,\nu=0}^{\infty} b^{*2}_{2\lambda+1,2\nu+1}=\int_0^1\int_0^1[\eta(u, v)]^2dudv$$
$$=\frac{1}{2}\int_0^1\int_0^1[\xi(u, v)]^2dudv-\frac{1}{2}\int_0^1\int_0^1\xi(u, v)\xi(u,1-v)dudv.$$

After some calculation the last two integrals are expressed as follows:

$$(3.38)\qquad \begin{aligned}&\int_0^1\int_0^1[\xi(u, v)]^2dudv=2A_1,\\ &\int_0^1\int_0^1\xi(u, v)\xi(u, 1-v)dudv=2B_1.\end{aligned}$$

Substituting (3.32), (3.33), (3.37) and (3.38) into (3.31), we have (3.29).

Similarly we can get

$$(3.39)\qquad \begin{aligned}&\sum_{\lambda,\nu=0}^{\infty} b^{*2}_{2\lambda,2\nu}=\frac{1}{2}\int_0^1\int_0^1[\xi(u, v)]^2dudv+\frac{1}{2}\int_0^1\int_0^1\xi(u, v)\xi(u, 1-v)dudv,\\ &\sum_{\lambda,\nu=0}^{\infty} b^{*2}_{2\lambda,2\nu+1}=\sum_{\lambda,\nu=0}^{\infty} b^{*2}_{2\lambda+1,2\nu}\\ &=\frac{B(2r-1, 2s-2r-1, 2n-2s+1)-B(n+r-s, n+r-s, 2s-2r-1)}{8[B(r, s-r, n-s+1)]^2},\end{aligned}$$

which is available to check (3.33) and will be used in the proof of Theorem 3.

Corollary 2. *For any symmetric distribution absolutely continuous with respect to the Lebesgue measure with mean zero and variance one,*

$$(3.40)\qquad \begin{aligned}|\mathrm{E}(X_{r/n}X_{s/n})|\leqq &\frac{1}{2\sqrt{2}B(r, s-r, n-s+1)}\\ &\times\{B(2r-1, 2s-2r-1, 2n-2s+1)\\ &+B(n+r-s, n+r-s, 2s-2r-1)\\ &-I(2r-1, n-s+1, n-s+1, s-r, s-r)\\ &-2I(n+r-s, r, n-s+1, s-r, s-r)\\ &-I(2n-2s+1, r, r, s-r, s-r)\}^{1/2}\quad (=b,\ say),\end{aligned}$$

where $I(p_1, p_2, p_3, q_1, q_2)$ *is defined by* (3.27), *and equality holds if and only if*

$$x(u)x(v) = \pm\frac{1}{b}\eta(u, v), \tag{3.41}$$

where $\eta(u, v)$ *is defined by* (3.35).

Proof. We get (3.40) from Theorem 2 by excluding the term corresponding to $\{\varphi_{2\lambda+1}(u)\varphi_{2\nu+1}(v)\}_{\lambda,\nu=0,1,\cdots,k}$ in (3.31). An analogous argument as in Corollary 1 leads us to (3.41).

It is to be noted that the upper bound for $E(X_{r/n}X_{s/n})$ is identical to the one for $E(X_{n-s+1/n}X_{n-r+1/n})$ both in Corollary 1 and 2. The simplest form in Corollary 2 appears when $r=1$ and $s=n$, that is,

$$|E(X_{1/n}X_{n/n})| \leqq \frac{n}{2}\left\{\frac{n-1}{2(2n-3)} - \frac{1}{\binom{2n-2}{n-1}}\right\}^{1/2}. \tag{3.42}$$

This is the universal upper bound for $E(X_{1/n}X_{n/n})$ when the distribution is required to be symmetric with mean zero and variance one. The corresponding upper bound in Corollary 1 is

$$|E(X_{1/n}X_{n/n})| \leqq n\left\{\frac{n-1}{4(2n-3)} - \frac{(n-1)^2}{n^2(2n-1)} - \frac{1}{(2n-1)\binom{2n-2}{n-1}}\right\}^{1/2}. \tag{3.43}$$

This is the universal upper bound when the distribution is required only to have mean zero and variance one.

The third column in Table 1 is calculated by (3.40), which, in comparison with the second, shows the effect of the restriction to symmetric populations.

A quite analogous argument as in the proof of Theorem 1 and 2 leads us to the following Theorem, the proof of which may be found in [10].

Theorem 3. *Suppose* $E(X^4)<\infty$, *then under the same assumption and notation as in Theorem 2 we have*

$$\begin{aligned}(3.44)\quad & |E(X^2_{r/n}X^2_{s/n}) - \sigma^2 E(X^2_{r/n}+X^2_{s/n}) + \sigma^4 - \frac{1}{2}\sum_{\lambda,\nu=1}^{\infty} a'_{2\lambda}a'_{2\nu}(b_{2\lambda,2\nu}+b_{2\nu,2\lambda})| \\ & \leqq \{(E(X^4)-\sigma^2)^2 - \sum_{\lambda,\nu=1}^{k} a'^2_{2\lambda}a'^2_{2\nu}\}^{1/2}\{A_2+B_2-C_2+1 \\ & -\frac{1}{4}\sum_{\lambda,\nu=1}^{k}(b_{2\lambda,2\nu}+b_{2\nu,2\lambda})^2\}^{1/2},\end{aligned}$$

where

$$(3.45)\qquad a'_\lambda = \int_0^1 [x(u)]^2 \varphi_\lambda(u)du\,,$$

$$A_2 = A_1\,,$$
$$B_2 = B_1\,,$$

$$(3.46)\qquad C_2 = \frac{B(2r-1,\,2n-2r+1)+B(n,\,n)}{4[B(r,\,n-r+1)]^2} + \frac{B(2s-1,\,2n-2s+1)+B(n,\,n)}{4[B(s,\,n-s+1)]^2} + \frac{B(r+s-1,\,2n-r-s+1)+B(n+r-s,\,n-r+s)}{2B(r,\,n-r+1)B(s,\,n-s+1)}\,.$$

Theorem 4. *Under the same assumption and notation as in Theorem* 3, *the following two inequalities hold for any* r, s $(1 \leqq r < s \leqq n)$.

$$(3.47)\qquad |\mathrm{E}(X^2_{r/n}X_{s/n}) - \sigma^2\mathrm{E}(X_{s/n}) - \sum_{\lambda,\nu=0}^{k} a'_{2\lambda+2}a_{2\nu+1}b_{2\lambda+2,2\nu+1}|$$
$$\leqq \{\sigma^2\mathrm{E}(X^4) - \sigma^6 - \sum_{\lambda,\nu=0}^{k} a'^2_{2\lambda+2}a^2_{2\nu+1}\}^{1/2}\{A_3 - B_3 - \sum_{\lambda,\nu=0}^{k} b^2_{2\lambda+2,2\nu+1}\}^{1/2}\,,$$

$$(3.48)\qquad |\mathrm{E}(X_{r/n}X^2_{s/n}) - \sigma^2\mathrm{E}(X_{r/n}) - \sum_{\lambda,\nu=0}^{k} a_{2\lambda+1}a'_{2\nu+2}b_{2\lambda+1,2\nu+2}|$$
$$\leqq \{\sigma^2\mathrm{E}(X^4) - \sigma^2 - \sum_{\lambda,\nu=0}^{k} a^2_{2\lambda+1}a'^2_{2\nu+2}\}^{1/2}\{A_4 - B_4 - \sum_{\lambda,\nu=0}^{k} b^2_{2\lambda+1,2\nu+2}\}^{1/2}\,,$$

where

(3.49)
$$\begin{cases} A_3 = \dfrac{B(2r-1,\,2s-2r-1,\,2n-2s+1)}{4[B(r,\,s-r,\,n-s+1)]^2} - \dfrac{B(2s-1,\,2n-2s+1)-B(n,\,n)}{2[B(s,\,n-s+1)]^2}\,, \\ B_3 = \dfrac{I(2r-1,\,n-s+1,\,n-s+1,\,s-r,\,s-r)-I(2n-2s+1,\,r,\,r,\,s-r,\,s-r)}{4[B(r,\,s-r,\,n-s+1)]^2}\,, \\ A_4 = \dfrac{B(2r-1,\,2s-2r-1,\,2n-2s+1)}{4[B(r,\,s-r,\,n-s+1)]^2} - \dfrac{B(2r-1,\,2n-2r+1)-B(n,\,n)}{2[B(r,\,n-r+1)]^2}\,, \\ B_4 = -B_3\,. \end{cases}$$

Proof. We shall sketch the proof of (3.47), leaving the details to [10]. From (3.5) we have

$$\mathrm{E}(X^2_{r/n}X_{s/n}) = \frac{1}{B(r,\,s-r,\,n-s+1)} \iint_{0<u<v<1} [x(u)]^2 x(v) u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s}dudv\,,$$

which is written by transforming the variable (u, v) to $(1-u, 1-v)$ as

$$\frac{-1}{B(r,\,s-r,\,n-s+1)} \iint_{0<v<u<1} [x(u)]^2 x(v)(1-u)^{r-1}(u-v)^{s-r-1}v^{n-s}dudv\,.$$

Putting

$$f_1(u, v) = [x(u)]^2 x(v),$$

(3.50)
$$g_1(u, v) = \begin{cases} \dfrac{1}{2B(r, s-r, n-s+1)} u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s} & 0<u<v<1, \\ \dfrac{-1}{2B(r, s-r, n-s+1)} (1-u)^{r-1}(u-v)^{s-r-1}v^{n-s} & 0<v<u<1, \end{cases}$$

we have from Proposition 1

(3.51)
$$\Big| \mathrm{E}(X^2_{r/n}X_{s/n}) - \sum_{\nu=0}^{\infty} c_{0,2\nu+1}d_{0,2\nu+1} - \sum_{\lambda,\nu=0}^{k} c_{2\lambda+2,2\nu+1}d_{2\lambda+2,2\nu+1} \Big|$$
$$\leqq \{\|f_1\|^2 - \sum_{\nu=0}^{\infty} c^2_{0,2\nu+1} - \sum_{\lambda,\nu=0}^{k} c^2_{2\lambda+2,2\nu+1}\}^{1/2}$$
$$\times \{\sum_{\lambda,\nu=0}^{\infty} d^2_{2\lambda,2\nu+1} - \sum_{\nu=0}^{\infty} d^2_{0,2\nu+1} - \sum_{\lambda,\nu=0}^{k} d^2_{2\lambda+2,2\nu+1}\}^{1/2},$$

where

(3.52)
$$c_{\lambda,\nu} = \int_0^1\int_0^1 [x(u)]^2 x(v)\varphi_\lambda(u)\varphi_\nu(v)dudv = a'_\lambda a_\nu,$$
$$d_{2\lambda,2\nu+1} = \int_0^1\int_0^1 g_1(u, v)\varphi_{2\lambda}(u)\varphi_{2\nu+1}(v)dudv = \frac{1}{2}\int_0^1\int_0^1 \{g_1(u, v) - g_1(u, 1-v)\}\varphi_{2\lambda}(u)\varphi_{2\nu+1}(v)dudv.$$

Since $g_1(u, v) - g_1(u, 1-v) \in L^2_{e,0}(R)$, we have

(3.53)
$$\sum_{\lambda,\nu=0}^{\infty} d^2_{2\lambda,2\nu+1} = \frac{1}{4}\|g_1(u, v) - g_1(u, 1-v)\|^2,$$

which after some calculation turns out to be equal to

(3.54)
$$\frac{B(2r-1, 2n-2s+1, 2s-2r-1) - I(2r-1, n-s+1, n-s+1, s-r, s-r) + I(2n-2s+1, r, r, s-r, s-r)}{4[B(r, s-r, n-s+1)]^2}.$$

We can also see

(3.55)
$$\sum_{\nu=0}^{\infty} d^2_{0,2\nu+1} = \frac{B(2s-1, 2n-2s+1) - B(n, n)}{2[B(s, n-s+1)]^2}.$$

Substituting (3.52), (3.54) and (3.55) into (3.51), we have (3.47).

4. The values of $\mathrm{E}(X_{r/n}X_{s/n})$ in normal sample

As an application of Theorem 2, we shall calculate $\mathrm{E}(X_{r/n}X_{s/n})$ for the standard normal population. Adopting Legendre polynomials as $\varphi_\nu(u)$ in Theorem 2 and putting

$$\varphi_\lambda(u) = \sum_i \alpha_{\lambda,i} u^i , \tag{4.1}$$

we have

$$\varphi_\lambda(u)\varphi_\nu(v) = \sum_{i,j} (-1)^j \alpha_{\lambda,i}\alpha_{\nu,j} u^i (1-v)^j , \tag{4.2}$$

$$\begin{aligned} a_{\lambda,\nu} &= \sum_{i,j} \frac{\alpha_{\lambda,i}\alpha_{\nu,j}}{(i+1)(j+1)} \mathrm{E}(X_{i+1/i+1})\ \mathrm{E}(X_{j+1/j+1}) , \\ b_{\lambda,\nu} &= \sum_{i,j} (-1)^j \alpha_{\lambda,i}\alpha_{\nu,j} \frac{\Gamma(r+i)\Gamma(n-s+1+j)\Gamma(n+1)}{\Gamma(r)\Gamma(n-s+1)\Gamma(n+i+j+1)} . \end{aligned} \tag{4.3}$$

Some numerical values of $\alpha_{\lambda,i}$ and $\mathrm{E}(X_{i+1/i+1})$ are shown in [9]. From these relations we get Table 2.

Table 2. Values of $\mathrm{E}(X_{r/5}X_{s/5})$ for standard normal population.

(r, s)	first approximation $a_1^2 b_{11}$	second approximation $a_1^2 b_{11} + \frac{1}{2} a_1 a_3 (b_{13} + b_{31})$	exact value
(4, 5)	0.818 ± 0.043	0.7990 ± 0.0193	0.8000
(3, 5)	0.136 ± 0.071	0.1494 ± 0.0364	0.1482
(2, 5)	-0.546 ± 0.093	-0.4676 ± 0.0075	−0.4699
(1, 5)	-1.228 ± 0.062	-1.2798 ± 0.0050	−1.2783
(3, 4)	0.273 ± 0.115	0.2013 ± 0.0411	0.2084
(2, 4)	-0.136 ± 0.100	-0.0844 ± 0.0415	−0.0951

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