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Let $\mathcal{O}$ be the integral closure of a discrete rank one valuation ring $R$ with maximal ideal $p$ in a finite Galois extension $L$ of the quotient field of $R$. Auslander, Goldman and Rim have proved in [1] and [2] that a crossed product $\Lambda$ over $\mathcal{O}$ with trivial factor sets is a maximal order in $K_n$ if and only if a prime ideal $\mathfrak{p}$ in $\mathcal{O}$ over $p$ is unramified and $\Lambda$ is a hereditary if and only if $\mathfrak{p}$ is tamely ramified. Recently Williamson has generalized those results in [11] to a crossed product $\Lambda$ with any factor sets in $U(\mathcal{O})$, where $U(\mathcal{O})$ means the set of units in $\mathcal{O}$, namely if $\mathfrak{p}$ is tamely ramified, then $\Lambda$ is hereditary and the rank of $\Lambda$ is determined.

In this paper, we shall modify the Williamson’s method by making use of a property of crossed product over a ring.

Let $G$, $S$ and $H$ be the Galois group of $L$, decomposition group of $\mathfrak{p}$ and inertia group of $\mathfrak{p}$, respectively. We denote a crossed product $\Lambda$ with factor sets $\{a_{\sigma,\tau}\}$ in $U(\mathcal{O})$ by $(\alpha_{\sigma,\tau}, G, \mathcal{O})$. Then we shall prove in Theorem 1 that $\Lambda$ is a hereditary order if and only if so is $(\alpha_{\sigma,\tau}, H, \mathcal{O}_H)$ where $\mathfrak{p}_H = \mathfrak{p} \cap \mathcal{O}_H$, and $\mathcal{O}_H$ is the integral closure of $R$ in the inertia field $\mathcal{O}_H$. Using this fact and the structure of hereditary orders [7], [8] we obtain the above results in [1], [2] and [11].

Furthermore, we shall show that $\Lambda$ is hereditary if and only if $\mathfrak{p}$ is tamely ramified under an assumptions that $R/p$ is a perfect field.

Finally, we give a complete description of hereditary orders in a generalized quaternions over rationals in Theorem 3.

1. Reduction theorem

In this paper we always assume that $R$ is a discrete rank one valuation ring with maximal ideal $p$ and $p$ in the characteristic of $R/p$. Let $L$ be a finite Galois extension of the quotient field of $R$ with Galois
group $G$, and $\mathcal{O}$ the integral closure of $R$ in $L$. For a prime ideal $\mathfrak{p}$ in $\mathcal{O}$ over $\mathfrak{p}$ we denote the decomposition group and the inertia group of $\mathfrak{p}$ by $S$ and $H$ and their fields and the integral closure by $L_{\mathcal{O}}$, $L_H$ and $\mathcal{O}_S$, $\mathcal{O}_H$ and so on.

We note that $\mathcal{O}$ is a semi-local Dedekind domain and hence, $\mathcal{O}$ is a principal ideal domain. Let $\{\mathfrak{p}_i\}_{i=1}^I$ be the set of prime ideals in $\mathcal{O}$ and $S_i$ and $H_i$ be decomposition group and inertia group of $\mathfrak{p}_i$. Let $\mathfrak{p}\mathcal{O} = \prod\mathfrak{p}_i^s\mathfrak{p}_i$, where $P = \prod\mathfrak{p}_i$. Since $(\mathcal{O}_i, \mathcal{O}_j) = \mathcal{O}$ for $i \neq j$, $\mathcal{O}/P^n = \mathcal{O}/\mathcal{O}_i^n \bigoplus \cdots \bigoplus \mathcal{O}/\mathcal{O}_i^n$. We note that $(\mathcal{O}/\mathcal{O}_i^n)^\sigma = \mathcal{O}/\mathcal{O}_i^n$ for $\sigma \in G$. Then $\mathcal{O}_H/\mathcal{O}_H^i$ is the separable closure of $R/\mathfrak{p}$ in $\mathcal{O}/\mathcal{O}_i^n$ and $\mathcal{O}_H/\mathcal{O}_H^i$ is a Galois extension of $R/\mathfrak{p}$ with Galois group $S_i/H_i$, (see [10], p. 290).

Let $\Lambda$ be a crossed product over $\mathcal{O}$ with factor sets $\{a_{\sigma, \tau}\}$ in $U(\mathcal{O})$: $\Lambda = (a_{\sigma, \tau}, G, \mathcal{O})$. Since $P^n = P$ for all $\sigma \in G$, $P^n\Lambda = \Lambda P^n$ is a two-sided ideal in $\Lambda$. Let $\Lambda(n) = \Lambda/P^n\Lambda = (a_{\sigma, \tau}, G, \mathcal{O}/\mathcal{O}_i^n) = \mathcal{O}/\mathcal{O}_i^n$ as a module. We put $\Lambda(S_i, n) = (a_{\sigma, \tau}, S_i, \mathcal{O}/\mathcal{O}_i^n)$. Since $\mathfrak{a}_{\sigma, \tau}^{-1}(\mathfrak{u}_{\sigma, \tau}/\mathcal{O}_i^n)\mathfrak{a}_{\sigma, \tau} = \mathfrak{a}_{\sigma, \tau}^{-1}(\mathcal{O}/\mathcal{O}_i^n)$, $\mathfrak{a}_{\sigma}^{-1}\Lambda(S_i, n)\mathfrak{a}_{\tau} = \Lambda(S_i, n)$, where $S_i = \sigma^{-1}S_i\sigma$. Thus we have

$$\Lambda(S_i, n)\mathfrak{a}_{\sigma} = \mathfrak{a}_{\sigma}\Lambda(S_i, n).$$

Let $G = \sigma_i S_i + \sigma_i S_i + \cdots + \sigma_i S_i = S_i \sigma_i + \cdots + S_i \sigma_i$, $\sigma_i S_i = S_i$, since $G$ is a finite group. Then

$$\Lambda(n) = \Lambda(S, n) + \mathfrak{a}_{\sigma_i} \Lambda(S, n) + \cdots + \mathfrak{a}_{\sigma_i} \Lambda(S, n)$$

$$+ \Lambda(S_2, n) + \mathfrak{a}_{\sigma_i} \Lambda(S_2, n) + \cdots + \mathfrak{a}_{\sigma_i} \Lambda(S_2, n)$$

$$+ \cdots$$

$$+ \Lambda(S_g, n) + \mathfrak{a}_{\sigma_i} \Lambda(S_g, n) + \cdots + \mathfrak{a}_{\sigma_i} \Lambda(S_g, n),$$

where $S = S_i$.

Let $p_{i,j}$ be projections of $\Lambda(\mathfrak{a})$ to $\mathfrak{a}_{\sigma_i} \Lambda(S_i, n)$. For a two-sided ideal $\mathfrak{a}$ in $\Lambda(n)$ we have $\mathfrak{a} \supseteq \sum p_{i,j}(\mathfrak{a})$. Since $\mathfrak{a}_{\sigma_i}$ is unit, $p_{i,j}(\mathfrak{a}) = \mathfrak{a}_{\sigma_i} P_{i,j}(\mathfrak{a})$ for all $j$. Let $\mathfrak{a}$ be the unit element in $\Lambda(S, n)$. Then $\Lambda(S_i, n)\mathfrak{a} = 0$ for $i \neq 1$ and $\mathfrak{a}\Lambda(S_i, n) = \mathfrak{a}_{\sigma_i} \Lambda(S_i, n) \Lambda(S, n) = 0$ for $j \neq 1$. Hence, $\mathfrak{a}\Lambda\mathfrak{a} = p_{11}(\mathfrak{a})$. Furthermore, since $S_i = S_i^{\sigma_i}$, $\mathfrak{a}_{\sigma_i}^{-1} p_{i,j}(\mathfrak{a}) \mathfrak{a}_{\sigma_i} = p_{i,j}(\mathfrak{a})^{\sigma_i}$. Therefore,

$$\mathfrak{a} = \sum_{i,j} \mathfrak{a}_{\sigma_i} \mathfrak{a}_{\sigma_i}^{\sigma_i}$$

for a two-sided ideal of $\mathfrak{a}_0$ in $\Lambda(S, n)$. Conversely, the above ideal is a two-sided ideal in $\Lambda(n)$ for a two-sided ideal $\mathfrak{a}_0$ in $\Lambda(S, n)$.

Thus, we have

**Lemma 1.** Let $\Lambda(n)$ and $\Lambda(S, n)$ be as above. Then we have a one-to-one correspondence between two-sided ideals of $\Lambda(n)$ and $\Lambda(S, n)$ as above.

We note that the above correspondence preserves product of ideals.
Next we shall consider $A_s = (a_{x, y}, S, O)$ \((\subset \Lambda = (a_{x, y}, G, S))\), where $S$ is the decomposition group of $\mathfrak{P}$. Since $\mathcal{D}_S$ is contained in the center of $\Lambda$, we may regard $\Lambda_S$ as an order over $\mathcal{O}_S$. Let $\mathfrak{B}_S$ be the prime ideal in $\mathcal{O}_S$ over $\mathfrak{p}$. Since $\mathfrak{B}_S$ is contained in the center of $\Lambda$, we may regard $\Lambda_S$ as an order over $\mathfrak{B}_S$. Let $\mathfrak{B}_S$ be the prime ideal in $\mathfrak{B}_S$ over $\mathfrak{b}$. Then $\mathfrak{B}_S/\mathfrak{B}_S = \mathfrak{O}/\mathfrak{P}$. If we set $\Gamma = (a_{x, y}, S, \mathfrak{B}_S) = (\Lambda_S)_{\mathfrak{B}_S}$, $\Gamma(n) = \Gamma/\mathfrak{P}^n \Gamma' \cong \bar{\Lambda}(S, n)$. In $\Gamma$ we may regard $K = L_S$ and $\mathcal{O} = \mathcal{O}_S$. Let $H$ be the inertia group of a unique prime ideal $\mathfrak{P}$ in $\mathcal{O}$. Then $H$ is a normal subgroup of $S$, (see [10], p. 290) and we have $S = H + \sigma_H H + \cdots + \sigma_H H$. Let $\Gamma_H = (a_{x, y}, H, \mathcal{O})$, then $\mathfrak{P}^n \Gamma H = \Gamma_H \mathfrak{P}^n$. Hence $\Gamma = \Gamma(n) = \Gamma/\mathfrak{P}^n \Gamma \cong \Gamma_H(n) = \Gamma_H$. Furthermore, 

$$\Gamma = \Gamma_H + u_{a_{x, y}} \Gamma_H + \cdots + u_{a_{x, y}} \Gamma_H.$$ 

By a similar argument as above, we have $u_{a_{x, y}} \Gamma_H \alpha = \Gamma_H$. We denote this automorphism by $f_\sigma$. Then the restriction of $f_\sigma$ on $\mathcal{O}/\mathfrak{P}^n$ coincides with $\sigma$. Let $\mathfrak{R}_H$ be the radical of $\Gamma_H$. Then $\mathfrak{R}_H \cong \mathfrak{P}^n \mathfrak{R}_H$. We put $\mathfrak{R} = \mathfrak{R}_H + u_{a_{x, y}} \mathfrak{R}_H + \cdots + u_{a_{x, y}} \mathfrak{R}_H$, then $\mathfrak{R}$ is a two-sided ideal of $\Gamma$ and $\mathfrak{R}^m = \mathfrak{R}_H^m + \cdots + u_{a_{x, y}} \mathfrak{R}_H^m \cong \mathfrak{P}^n \mathfrak{R}$ for some $m$. $\Gamma/\mathfrak{R} = \Gamma_H/\mathfrak{R}_H + u_{a_{x, y}} \Gamma_H/\mathfrak{R}_H + \cdots + u_{a_{x, y}} \Gamma_H/\mathfrak{R}_H$ and $\Gamma_H/\mathfrak{R}_H \cong \mathcal{O}/\mathfrak{P}$. Now we consider a crossed product of $\Gamma_H/\mathfrak{R}_H$ with automorphisms $\{f_\sigma\}$ and factor sets $\{\alpha_\sigma\}$. We define a two-sided $\Gamma_H/\mathfrak{R}_H$-module $\Gamma_H/\mathfrak{R}_H$ as follows: for $x, y \in \Gamma_H/\mathfrak{R}_H$, $x* y = x^\sigma y$ and $y* x = y^\sigma x$, and denote it by $(\sigma, \Gamma_H/\mathfrak{R}_H)$. Since $\Gamma_H/\mathfrak{R}_H$ is semi-simple, $\{\sigma\}$ is the complete set of automorphisms of $\mathcal{O}/\mathfrak{P}$ (see [10], p. 290). Hence $\{f_\sigma\}$ is a complete outer-Galois, namely for any two-sided $\Gamma_H/\mathfrak{R}_H$-module $A \boxplus B$ in $(\sigma, \Gamma_H/\mathfrak{R}_H) A/B$ is not isomorphic to some of those forms in $(1, \Gamma_H/\mathfrak{R}_H)$ if $\sigma = 1$. Therefore, for any two-sided ideal $\mathfrak{A}$ in $\Gamma/\mathfrak{R}$ we have by [3], Theorem 48.2

(3) 
$$\mathfrak{A} = \Sigma u_{a_{\tau}} \mathfrak{A}_0,$$ 

where $\mathfrak{A}_0$ is a two-sided ideal in $\Gamma_H/\mathfrak{R}_H$ and $\mathfrak{A}_0 = \mathfrak{A}_0$ for all $f_\sigma$, and it is a one-to-one correspondence. Hence, $\Gamma/\mathfrak{R}$ is semi-simple, and $\mathfrak{A}$ is the radical of $\Gamma$. From the definition of $f_\sigma$ we have

(4) 
$$(u_\lambda, \lambda)^\sigma = u_{a_{\tau}}^* \bar{\lambda} \bar{a}_{\tau, \sigma} a_{\tau, \sigma}^{-1} v_{\tau, \sigma}$$

for $\sigma \in S$, $\tau \in H$, $\lambda \in \mathcal{O}/\mathfrak{P}$, and $u_{a_{\tau}} \in \Gamma_H/\mathfrak{R}_H$. Furthermore, let $\Gamma_H/\mathfrak{R}_H = \mathfrak{A}_0 \boxplus \cdots \boxplus \mathfrak{A}_k$, where the $\mathfrak{A}_i$'s are simple components of $\Gamma_H/\mathfrak{R}_H$. If we classify those ideals $\mathfrak{A}, \mathfrak{B}$ by a relation

(5) 
$$\mathfrak{A} \sim \mathfrak{B} \text{ if and only if } \mathfrak{A} f_\sigma = \mathfrak{B} \text{ for some } f_\sigma,$$

then the number of maximal two-sided ideals in $\Gamma/\mathfrak{R}$ is equal to this class number.

Thus, we have
Lemma 2. Let $L$ be a Galois extension of the field $K$ with Galois group $G$ such that $S=G$, $\Gamma=(a_{\sigma\tau}, S, \mathcal{O})$, and $\Gamma_H=(a_{\sigma\tau}, H, \mathcal{O})$. If we denote the radicals of $\Gamma$ and $\Gamma_H$ by $\mathcal{R}, \mathcal{R}_H$, then, $\mathcal{R}'=\Sigma\mathcal{R} \mathcal{R}_H \pmod{\mathcal{R}'}$ for some $t<n$, and there exists a one-to-one correspondence between two-sided ideals in $\Gamma/\mathcal{R}$ and $\Gamma_H/\mathcal{R}_H$ which is given by (3) and (4).

Lemma 3. Let $\Omega$ be an order over $R$ in a central simple $K$-algebra $\Sigma$ and $\mathcal{R}$ the radical of $\Omega$. Then $\Omega$ is hereditary if and only if $\mathcal{R}'=\alpha \Omega = \Omega \alpha$ for some $t>0$ and $\alpha \in \Sigma$.

Proof. If $\mathcal{R}'=\alpha \Omega$, then the left (right) order of $\mathcal{R}=\Omega$, and $\mathcal{R}\mathcal{R}'=\alpha^{-1} \Omega$. Hence $\mathcal{R}$ is inversible in $\Omega$, which implies that $\Omega$ is hereditary by [7], Lemma 3.6. The converse is clear by [7], Theorem 6.1.

Theorem 1. Let $R$ be a discrete rank one valuation ring and $K$ its quotient field, and $L$ a Galois extension of $K$ with group $G$. Let $S$ and $H$ be decomposition group and inertia group of a prime ideal $\mathfrak{p}$ in the integral closure $\mathcal{O}$ of $R$ in $L$. Let $\Lambda=(a_{\sigma\tau}, G, \mathcal{O}), \Lambda_S=(a_{\sigma\tau}, S, \mathcal{O}_S)$, and $\Lambda_H=(a_{\sigma\tau}, H, \mathcal{O}_H)$. Then the following statement is equivalent

1) $\Lambda$ is hereditary,
2) $\Lambda_S$ is hereditary,
3) $\Lambda_H$ is hereditary.

In this case the rank of $\Lambda$ is equal to that of $\Lambda_S$ and is equal or less than that of $\Lambda_H$.

Proof. 1)$\rightarrow$2). Let $\mathcal{R}, \mathcal{R}_S$ be the radicals of $\Lambda$ and $\Lambda_S$ and $P$ be the product of the prime ideals as in the beginning. Then $\mathcal{R}'=P \Lambda$. For $n'>t$ we have $\mathcal{R}'_S= \mathcal{R} \mathcal{R}_S \pmod{\mathcal{R}'}$ by Lemma 1 and remark after that. Hence $\mathcal{R}'_S= \mathcal{R} \mathcal{R}_S$ since $\mathcal{R}'_S= \mathcal{R} \mathcal{R}_S$. Therefore, $\Lambda_S$ is hereditary by Lemma 3. The remaining parts are proved similarly by using Lemmas 1, 2, and 3, and a remark before Lemma 2.

If $(|H|, \rho)=1$, then $\Lambda/\mathfrak{p} \Lambda$ is separable by [11], Theorem 1, (see Lemma 4 below) and hence $\Lambda$ is hereditary, where $|H|$ means the order of group $H$. Therefore, we have

Corollary 1. ([11]). If $\mathfrak{p}$ is tamely ramedfied, i.e. $(|H|, \rho)=1$, then $\Lambda=(a_{\sigma\tau}, G, \mathcal{O})$ is hereditary of the same rank as that of $\Lambda_S=(a_{\sigma\tau}, S, \mathcal{O}_S)$ and its rank is equal to the class number of ideals defined by (5).

Corollary 2. ([1, 2]). If $\{a_{\sigma\tau}\} = \{1\}$, then $\Lambda$ is hereditary if and only if a prime ideal $\mathfrak{p}$ in $\mathcal{O}$ over $\mathfrak{p}$ is tamely ramedified. In this case the rank of $\Lambda$ is equal to the ramification index of $\mathfrak{p}$.

Proof. $\{a_{\sigma\tau}\} = \{1\}$, then $\Sigma=(a_{\sigma\tau}, G, L)=K_n$. We assume that $\Lambda$ is
hereditary, then \( \Lambda_H \) is also hereditary by Theorem 1. \( \Lambda_H L = (L_H)_h \), where \( h = |H| \), \((\mathcal{O}_H)_h\) is a maximal order in \( \Lambda_H L \). Furthermore, the composition length of left ideals of \((\mathcal{O}_H)_h\) modulo the radical \((\mathcal{P}_H)_h\) is equal to \( h \), which is invariant for hereditary orders in \( \Lambda_H L \) by [10], Corollary to Lemma 2.5. On the other hand \([\Lambda_H/\mathcal{P}_H]: \mathcal{O}/\mathcal{P}] = h\). Hence, \( \mathcal{P}_H \Lambda_H \) is the radical and \( \Lambda_H/\mathcal{P}_H \Lambda_H \) is semi-simple which is a group ring of \( H \) over \( \mathcal{O}/\mathcal{P} \). Therefore, \(|H|, \rho = 1\). In this case \( \mathfrak{M} = (\sum \mathfrak{M}_h) \cdot \mathcal{O}/\mathcal{P} \) is a two-sided ideal in \( \Lambda_H/\mathcal{P}_H \Lambda_H \) which is invariant under automorphisms \( f_\sigma \) of (4). \( \mathfrak{M} \) is a minimal two-sided ideal in \( \Lambda_H/\mathcal{P}_H \Lambda_H \) which is invariant under \( f_\sigma \). Hence, \( \Lambda_S/\mathfrak{M} \approx \sum \mathfrak{M}_h \mathfrak{A} \) for some maximal ideal \( \mathfrak{M} \) in \( \Lambda_S \). Furthermore, since \( \Lambda_S \) is principal \( 2^3 \), \( \Lambda_S/\mathfrak{M} \approx \Lambda_S/\mathfrak{M}' \) for any maximal ideal \( \mathfrak{M}' \) in \( \Lambda_S \) by [10], Theorem 4.1. Therefore, there exists \( h \) two-sided ideals in \( \Lambda_H/\mathcal{P}_H \Lambda_H \) which is invariant under \( f_\sigma \), since \([\mathfrak{M}: \mathcal{O}/\mathcal{P}] = 1\).

By the same argument as in the proof of Theorem 1 we have

**Proposition 1.** We assume that \( R/\mathfrak{p} \) is a perfect field, and we use the same notations as in Theorem 1. Let \( V \) be the second ramification group \( V \) and \( \Lambda_V = \langle v, \sigma \rangle \). Then \( \Lambda \) is hereditary if and only if so is \( \Lambda_V \).

**Proof.** By virtue of Theorem 1 we may assume \( G = H \). Let \( G = V + \sigma V + \cdots + \sigma^p V \). Then \( \Lambda = \Lambda_V + u_\sigma \Lambda_V + \cdots + u_p \Lambda_V \). Since \( V \) is a normal subgroup of \( G \) by [10], p. 295, an inner-automorphism by \( u_\sigma \) in \( \Lambda \) reduces an automorphism \( f_\sigma \) in \( \Lambda_V \). Let \( \mathfrak{M} \) be the radical of \( \Lambda \) and \( \mathfrak{M} = \mathfrak{M}_V + u_\sigma \mathfrak{M} + \cdots + u_p \mathfrak{M} \). We shall show that \( \mathfrak{M} \) is the radical of \( \Lambda \). By assumption that \( R/\mathfrak{p} \) is perfect, \( \Lambda/\mathfrak{M} \) is separable. Therefore, there exist \( x_i \), \( y_i \) in \( \Lambda \) such that \( \sum x_i y_i = 1 \) and \( \sum x_i \otimes y_i = \sum x_i \otimes (y_i \lambda) \), where \( \lambda \rightarrow (y) \) gives an anti-isomorphism of \( \Lambda \) to \( \Lambda^* \). Furthermore, we note that \(|G/V| = t\) is a relative prime to \( p \) by [10], p. 296. Let \( \theta = 1/t(\sum a^{-1} x_i \otimes \lambda) = 1/t(\sum a^{-1} x_i \otimes (y_i \lambda) \otimes \mu) \). Then \( 1/t(\sum a^{-1} x_i \otimes \lambda) \) and \( \sum a^{-1} x_i \otimes (y_i \lambda) \otimes \mu = 1 \). We show that \( (\gamma \otimes 1^*) (1 \otimes \eta^*) \) is zero for any \( \eta \). Let \( \gamma \) be in \( \Lambda \). \((\gamma \otimes 1^*) \theta = 1/t(\sum a^{-1} x_i \otimes (u_i \gamma) \otimes (y_i \lambda) ) \). Since \( \sum x_i \otimes (y_i \lambda) \), we obtain \( \sum x_i \otimes (y_i \lambda) \otimes \mu = \sum x_i \otimes (y_i \lambda) \). Therefore, \((\gamma \otimes 1^*) (1 \otimes \eta^*) \theta = 0 \).}

2) See the definition in [10].
However, we obtain $\alpha_{\sigma, \tau}^{-1} - \alpha_{\tau, \sigma}^{-1} = \alpha_{\sigma, \tau}^{-1} - \alpha_{\tau, \sigma}^{-1}$ by the relation of $\alpha_{\sigma, \tau}$. Hence, $((\alpha_{\sigma} \otimes 1)^* - (1 \otimes \alpha_{\sigma}^*)) \theta = 0$. Therefore, $(\alpha_{\sigma} \gamma \otimes 1^*) - (1 \otimes (\alpha_{\sigma} \gamma)^*) \theta = (\alpha_{\sigma} \otimes 1^*) (\gamma \otimes 1 - 1 \otimes \gamma^*) \theta + (1 \otimes \gamma^*) (\alpha_{\sigma} \otimes 1 - 1 \otimes \alpha_{\sigma}^*) \theta = 0$. Thus we have proved that $\mathcal{R}$ is the radical of $\Lambda$. We can prove the proposition similarly to Theorem 1 by Lemma 3.

2. Tamely ramification

In this section we always assume that $R/p$ is a perfect field.

**Theorem 2.** Let $L$ be a Galois extension of $K$ with Galois group $G$, and $\Lambda = (a_{\sigma, \tau}, G, \mathcal{O})$ a crossed product with a factor set $\{a_{\sigma, \tau}\}$ in $U(\mathcal{O})$. We assume $R/p$ is a perfect field. Then $\Lambda$ is hereditary if and only if every prime ideal $\mathfrak{P}$ in $\mathcal{O}$ over $p$ is tamely ramified, where $U(\mathcal{O})$ is the set of unit elements in $\mathcal{O}$.

Proof. If $\mathfrak{P}$ is tamely ramified, then $\Lambda$ is hereditary by Corollary 1. We assume that $\Lambda$ is hereditary. Then by virtue of Proposition 1 we may assume that $G$ is equal to the second ramification group $V$. Since the elements of $G$ operate trivially on $\mathcal{O}/\mathfrak{P}$, $\Lambda = \Lambda/\mathfrak{P}\Lambda = \bar{\mathcal{O}} + a_{\sigma} \bar{\mathcal{O}} + \cdots + a_{\bar{\sigma}} \bar{\mathcal{O}}$ is a generalized group ring. Furthermore, from a relation on a factor set we have $a_{\sigma} \tau = A_{\sigma} = A_{\sigma}^* A_{\sigma}$, where $A' = \Pi_{\beta} a_{\beta, \sigma}$. Since $R/p = \mathcal{O}/\mathfrak{P}$ is perfect and $G$ is a $p$-group by [10], p. 296, we have $\bar{a}_{\sigma, \tau} = A_{\sigma} A_{\tau}/A_{\tau}$, $A_{\sigma} \in \bar{\mathcal{O}}$. Therefore, $\Lambda$ is a group ring of $G$ over $\bar{\mathcal{O}}$. As well known (see [5], p. 435), the radical $\mathcal{R}$ of $\Lambda$ is equal to $\Sigma(1 - \bar{a}) \bar{\mathcal{O}}$ and $\Lambda/\mathcal{R} = \bar{\mathcal{O}}$. Hence $\Lambda$ is a unique maximal order by [2], Theorem 3.11. Let $\sigma$ be an element in $G$. $(u_{\sigma})^t = u_{\sigma} C_{\sigma^t}; C_{\sigma^t} \in U(\mathcal{O})$. Hence, if we replace a basis $\{u_{\sigma}\}$ by $\{u_{\sigma}^t\}; u_{\sigma}^t = (u_{\sigma})^t$, and $u_{\sigma}^t = u_{\sigma}$ if $\tau \notin (\sigma)$, we may assume $a_{\sigma^t, \sigma} = 1$ if $i + j \leq |\sigma| = n$ and $a_{\sigma^t, \sigma} = a$ if $i + j = n$, where $a$ is a unit element in $\mathcal{O}$. It is clear that $a$ is an element of the $(\sigma)$-fixed subfield $L_{\sigma^t}$ of $L$. Since $\mathcal{R} = \Sigma(1 - \bar{u}_{\sigma}) \bar{\mathcal{O}}, (1 - u_{\sigma}) \in \mathcal{R}$. $(1 - u_{\sigma})(1 + u_{\sigma} + u_{\sigma^2} + \cdots + u_{\sigma^{n-1}}) = 1 - a \in \mathcal{R}$. Hence $1 - a \in \mathcal{R} \wedge \mathcal{O}_{(\sigma)} = \mathcal{O}_{(\sigma)}$. Furthermore, every one-sided ideal in $\Lambda$ is a two-sided ideal and a power of $\mathcal{R}$ by [2], Theorem 3.11. Since $(1 - u_{\sigma}) \Lambda \subseteq \mathcal{O} \Lambda, (1 - u_{\sigma}) \Lambda \subseteq \mathcal{O} \Lambda$. Put $\mathfrak{B} = (\pi)$. Then $\pi = (1 - u_{\sigma}) \Sigma u_{\sigma} x_{\sigma} = \Sigma a_{\sigma, \tau} x_{\sigma} = x_{\sigma} = x_{\sigma} = x_{\sigma^2} = \cdots = x_{\sigma^{n-1}}$. Therefore, $x_{\sigma}(1 - a) = \pi$. However, $(1 - a) \equiv 0 \pmod{\mathfrak{B}_{(\sigma)}}$. Therefore, $\mathfrak{B}$ is unramified over $\mathfrak{B}_{(\sigma)}$ which implies $|\sigma| = 1$. Hence $\mathcal{V} = (1)$, which has proved the theorem.

**Corollary 3.** Let $\Lambda = (a_{\sigma, \tau}, G, \mathcal{O})$. Then $\Lambda$ is hereditary if and only if $\Lambda/\mathcal{P}\Lambda$ is semisimple, where $P = \Pi \mathfrak{P}_{(\sigma)}$.

Proof. It is clear from Theorems 1 and 2 and the proof of Proposition 1.
**Proposition 2.** Let $\Lambda = (a_{\sigma, \tau}, G, \mathcal{O})$ and $t$ the ramification index of a maximal order $\Omega$ in $\Lambda K : (N(\Omega)'/p\Omega)$. We assume that $R/p$ is perfect. If $\Lambda$ is a hereditary order of rank $r$, then the ramification index of $\mathcal{O}$ is equal to $rt$, where $N(\Omega)$ means the radical of $\Omega$.

Proof. If $\Lambda$ is hereditary, then $N(\Lambda)=PA$ by Corollary 3. Hence, $N(\Lambda)^e=p\Lambda$. Therefore, $e=rt$ by [7], Theorem 6.1.

**Corollary 4.** Let $\Lambda = (a_{\sigma, \tau}, G, \mathcal{O})$ be a hereditary order. Then $\Lambda \approx \Gamma = (b_{\sigma, \tau}, G, \mathcal{O})$ if and only if $\Lambda K \approx \Gamma K$.

Proof. Since $\Lambda$ is hereditary, $\mathcal{O}$ is tamely ramified. If $\Lambda K \approx \Gamma K$, then $\Lambda \approx \Gamma$ by Proposition 2 and [8], Corollary 4.3.

**Corollary 5.** Let $\Lambda = (a_{\sigma, \tau}, G, \mathcal{O})$ and $e$ the ramification index of $\mathcal{O}$ over $\mathfrak{p}$. Then $\Lambda$ is a hereditary order of rank $e$ if and only if $(e, p)=1$ and a maximal order in $\Lambda K$ is unramified.

**Corollary 6.** We assume $\Lambda = (a_{\sigma, \tau}, H, \mathcal{O})$ is hereditary and a maximal order in $\Lambda K$ is unramified. Then $\Lambda$ is a minimal hereditary order$^3$.

Proof. Let $\Omega$ be a maximal order in $\Lambda K$. Put $\Omega / N(\Omega) = \Delta_m$ and $[\Delta : R/\mathfrak{p}]=s$, where $\Delta$ is a division ring. Since $N(\Omega)/N(\Omega) = \Omega / N(\Omega)$, we obtain $m^s = [\Omega / p\Omega : R/\mathfrak{p}]=|\Lambda / p\Lambda : R/\mathfrak{p}| = |H|^s$. The ranker of $\Lambda \leq m$ by [8], Corollary to Lemma 2.5. Hence $r = |H| = m\sqrt{s} > r\sqrt{s}$ by Proposition 2. Therefore, $s=1$ and $m=|H|=r$. Hence, $\Lambda$ is minimal by [8], Corollary to Lemma 2.5.

**Remark 1.** If $R$ is complete and $R/\mathfrak{p}$ is finite, then we obtain, as well known (cf. [6]), that the ramification index of a maximal order in $\Sigma = (a_{\sigma, \tau}, G, L)$ is equal to the index of $\Sigma$.

Finally we shall generalize Corollary 2.

The following lemma is well known. However we shall give a proof for a completeness, (cf. [11], Theorem 1).

**Lemma 4.** Let $K$ be a commutative ring and $G$ a finite group which operates on $K$ trivially. \{a_{\sigma, \tau}\} is a factor set in the unit elements of $K$. Then a generalized group ring $(a_{\sigma, \tau}, G, K)$ is separable over $K$ if and only if $Kn=K$, where $n=|G|$.

Proof. Let $\psi$ be a $K$-homomorphism of $\Lambda$ to $\Lambda \otimes \Lambda^* = \Lambda^*$:

$$\psi(u_\sigma) = \sum u_\sigma \otimes u_\sigma^* k(\sigma, \tau, \rho), \quad k(\sigma, \tau, \rho) \in K.$$  

Then $\psi$ is left $\Lambda^*$-homomorphic if and only if

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$^3$ See the definition in [8], § 2.
\[ a_\eta, k(\sigma, \tau, \rho) = a_\eta, k(\eta \sigma, \eta \tau, \rho) \]
\[ a_\rho, k(\sigma, \tau, \rho) = a_\eta, k(\sigma \eta, \tau, \rho \eta) \]
for any \( \eta \in G \).

From (6) we have \( k(1, \tau, \rho) = \varphi^{-1}(\rho, \rho \tau, \tau) \). If \( \Lambda \) is separable over \( K \),
then there exists a \( \Lambda^e \)-homomorphism \( \psi \) of \( \Lambda \) to \( \Lambda^e \) such that \( \varphi \psi = I \),
where \( \varphi : \Lambda^e \to \Lambda ; \varphi(x \otimes y^*) = xy \). Hence
\[ 1 = \psi(1) = \sum u_{\alpha} \alpha a_{\alpha} k(1, \tau, \rho) = u_{\alpha} \sum a_{\alpha} \alpha a_{\alpha} k(1, 1, 1) \].
If we replace \( \rho, \sigma \) and \( \tau \) by \( \eta, \eta \) and \( \eta \) in the relation of factor sets,
then we have \( a_{\eta, \eta^{-1}} = a_{\eta^{-1}, \eta} \), where we assume \( a_{\eta, 1} = a_{1, \eta} = 1 \). Hence \( 1 = \eta \). The converse is given by [11], Theorem 1. (cf. the proof of Proposition 1).

**Proposition 3.** We assume that \( \Lambda = (\sigma, \tau, G, \mathcal{O}) \) is an order in a matric \( K \)-algebra over \( K \) and \( R/p \) is not necessarily perfect. Then \( \Lambda \) is hereditary if and only if \( \mathfrak{B} \) is tamely ramified. In this case the rank of \( \Lambda \) is equal to the ramification index of \( \mathfrak{B} \).

**Proof.** We assume that \( \Lambda \) is hereditary. Since \( \{a_{\sigma, \tau}\} \) is similar to the unit factor set in \( L, \Lambda_H = (\sigma, \tau, H, \mathcal{O}) \) is in \( (K)_{H} \). We know similarly to the proof of Corollary 2 that \( N(\Lambda_H) = p\Lambda_H \). Hence, \( \Lambda_H = \Lambda_H / \mathfrak{p} = \mathfrak{O} + \mathfrak{a}_\rho \mathfrak{O} + \cdots + \mathfrak{a}_\rho \mathfrak{O} \) is semi-simple. However, since \( \Omega / N(\Omega) = (R/p)_{H} \) for a maximal order \( \Omega \) in \( (K)_{H} \), \( \Lambda = \Sigma(R/p)_{m_i} \) by [7], Theorem 4.6. Hence, \( \Lambda \) is separable. Therefore, \((|H|, p) = 1 \) by Lemma 4.

**3. Hereditary orders in a generalized quaternions**

Finally, we shall determine all the hereditary orders in a generalized quaternions. Let \( Z \) be the ring of integers and \( K \) the field of rationals. Let \( d \) be an integer which is not divided by any quadrate and \( L = K(\sqrt{d}) \). Then the Galois group \( G = \{1, g\} \) and \( (\sqrt{d})^* = -\sqrt{d} \). For any integer \( a \) we have \( \Sigma = (a, G, L) = K + Kg + K\sqrt{d} + Kg\sqrt{d} \) with relations \( g^2 = a, (\sqrt{d})^2 = d, \) and \( g\sqrt{d} = -\sqrt{d} g \). We have determined all hereditary orders in [9], Theorem 1.2 in the case \( a = -1 \).

We use the same argument here as that in [9], § 1.

First we shall determine the types of maximal orders over \( Z_p \).

**Proposition 4.** Let \( R \) be the ring of \( p \)-adic integers, \( L = K(\sqrt{d}) \) and \( \Lambda = (a, G, \mathcal{O}) \). We denote the radical of \( \Lambda \) by \( \mathfrak{p} \) and \( \Lambda / \mathfrak{p} \) by \( \bar{\Lambda} \). Then

1) If \( p = 2, d \equiv 1 \) \( (\mod 4) \), then \( \Lambda \) is a maximal order such that \( \bar{\Lambda} = (R/2)_2 \).
2) If \( p = 2, d \equiv 2, 3 \) \( (\mod 4) \), then \( \Lambda \) is not hereditary.
3) If \( p = 2, d \equiv 0 \) \( (\mod p) \), then \( \Lambda \) is a maximal order such that \( \bar{\Lambda} = (R/p)_2 \).
4) If \( p \not= 2, d \equiv 0 \) \( (\mod p) \).
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a) \((a/p)^{p}=1\), then \(\Lambda\) is a hereditary order of rank two.
b) \((a/p)=-1\), then \(\Lambda\) is a unique maximal order.

Proof. We shall consider the following three cases.

1) \(H=1\). Then i) \(p=\mathfrak{p}, \mathfrak{P}_{2}\) and \(S=H\), ii) \(p=\mathfrak{P}_{2}\) and \(S=G\). Since \(\mathfrak{P}\) is unramified, \(\Lambda\) is maximal order by Theorem 1. In the case i) \(\mathfrak{O}/\mathfrak{p}\mathfrak{O}=\mathfrak{O}/\mathfrak{P}_{1}+\mathfrak{O}/\mathfrak{P}_{2}\), and \(\Lambda\) is a maximal order such that \(\Lambda/p\Lambda=(R/p)_{2}\). The case ii) \(\Lambda/p\Lambda=\mathfrak{O}/\mathfrak{P}_{1}+g\mathfrak{O}/\mathfrak{P}_{2}\). Since \(G=S\), \(\Lambda/p\Lambda\) is not commutative and hence, \(\Lambda\) is not a unique maximal.

2) \(G=S=H\), \(p=2\) and \(a\equiv 1 \text{ (mod 2)}\). In this case 2 is ramified and hence, \(\Lambda\) is not hereditary by Theorem 3.

3) \(G=S=H\), and \(p=2\). Then \(p=\mathfrak{P}_{2}\) and \(\Lambda/\mathfrak{P}_{1}\Lambda=R/p+(R/p)\mathfrak{g}\). Since \(\mathfrak{P}\) is tamey ramified, \(\mathfrak{P}\Lambda=\mathfrak{R}\) by the remark before Corollary 1, and \(\Lambda\) is hereditary. Let \(\mathfrak{A}\) be a two-sided ideal in \(\Lambda\). If \(\mathfrak{A}\) is proper, then \(\mathfrak{A}=(1+g\mathfrak{g})R/p\) and \(d\mathfrak{g}=1\) for some \(y\in\mathfrak{S}=R/p\), and conversely. Therefore, if \((a/p)=1\) then \(\Lambda\) is a hereditary order of rank 2 and if \((a/p)=-1\), then \(\Lambda\) is a unique maximal order. The proposition is trivial from the well known facts of quadratic field.

If we set \(g=i\) and \(\sqrt{-d}=j\), then \(\Sigma=(a, G, L)\) is a generalized quaternions over the field \(K\) of rationals. For any element \(x=x_{1}+x_{2}i+x_{3}j+x_{4}ij\) we define

\[N(x) = x_{1}^{2} - ax_{2}^{2} - dx_{3}^{2} + adx_{4}^{2}.\]

Let \(\Omega\) be a maximal order over \(R\) with basis \(u_{1}, u_{2}, u_{3}\) and \(u_{4}\). We call an element \(x=x_{1}u_{1}\) in \(\Omega\) normalized if \((x_{1}, \ldots, x_{4})=1\).

We note that if \(\Sigma\) contains at least two maximal orders, then \(\hat{\Sigma}\) is a matrix ring over \(\hat{K}\) where \(\hat{}\) means the completion with respect to \(p\), (cf. [9], Lemma 1.4).

In order to use the same argument as in the proof of [9], Theorem 1.2 we need

**Lemma 6.** 1) If either \(p=2\), \(d\equiv 3 \text{ (mod 4)}\) and \(a\equiv 1 \text{ (mod 4)}\) or \(p=2\), \(d\equiv 2 \text{ (mod 4)}\), and \(a\equiv 1 \text{ (mod 8)}\), then there exists a maximal order \(\Omega\) such that \(\Omega=(R/\mathfrak{p})_{2}\). 2) If \(p=2\), \(d\equiv 2 \text{ (mod 4)}\), \(a\equiv 1 \text{ (mod 4)}\) and \(a\equiv 1 \text{ (mod 8)}\), then there exists a unique maximal order. 3) If \(p=2\), \(d\equiv 0 \text{ (mod p)}\) and \((a/p)=1\), then there exists a maximal order \(\Omega\) such that \(\Omega=(R/p)_{2}\), where \(\Omega\) means the factor ring of \(\Omega\) modulo its radical.

Proof. Let \(\Omega=\mathfrak{O}+(1/2)(1+g)\mathfrak{O}=R+R_{2}+R_{1}/2(1+i)+R(1/2)(j+ij)\), where \(i=g\) and \(j=\sqrt{-d}\). We denote \((1/2)(1+i)\) and \((1/2)(j+ij)\) by \(h\) and \(l\). Then we obtain by the direct computations that

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4) Legendre's symbol,
\[ jh = i - l, \quad hj = l, \quadjl = d(1 - h), \quad lj = dh, \quad hl = l + jr, \quad lh = -rj, \quad h^2 = h + r \quad \text{and} \quad I^2 = dr, \]

where \( a = 1 + 4r, \quad r \in R. \)

1) \( d = 3 \pmod{4}. \) Let \( N(\Omega) \) be the radical of \( \Omega \) and \( x = x_1 + x_2 j + x_r h + x_i f \in N(\Omega) / 2\Omega. \) Then \( x_j + j x = x_d + x_z j. \) If \( x_j \equiv 0 \pmod{2}, \) then we may assume \( 1 + j \in N(\Omega). \) Then \( 0 \equiv (1 + j)l + l(1 + j) \equiv d \pmod{2}, \) which is a contradiction. Hence, we know \( N(\Omega) = 2\Omega \) by the similar argument for \( x_1, x_2. \) Since \( \Omega / N(\Omega) \) is not commutative by (7), \( \Omega / N(\Omega) = (R / 2)^2 \) and \( \Omega \) is a maximal order (cf. [9], Lemma 1.3).

2) \( d = 2 \pmod{4}. \) From (7) we obtain \( N(\Omega) = \Lambda j. \) If \( r \equiv 0 \pmod{2}, \) then \( \Omega / N(\Omega) = (R / 2) h + (R / 2)(1 + h). \) Hence \( \Omega \) is a hereditary order of rank two. Let \( \Omega_0 = R + Rj + Rh + R(1/2). \) It is clear that \( \Omega_0 \supseteq \Lambda \) and \( \Omega_0 \) is a ring. Hence \( \Omega_0 \) is a maximal order by [7], Theorems 1.7 and 3.3. If \( r \equiv 0 \pmod{2}, \) then \( \Omega / N(\Omega) \) is a field and hence \( \Omega \) is a unique maximal order.

3) In this case \( \Lambda \) is hereditary. Let \( \Omega = R + Ri + Rj + R(1/p)(j + yij), \) where \( ay^2 = 1 + px, \quad x \in R. \) It is clear that \( \Omega \supseteq \Lambda. \) We shall show that \( \Omega \) is a ring. \((1/p)(j + yij)^p = (d/p)x \in \Omega, \) and \((1/p)(j + yij)^i = -(x/y)j - (1/yp)(j + yij) \in \Omega, \) and \((1/p)(j + yij)^j = (d/p)(1 + ky) \in \Omega. \) Therefore, \( \Omega \) is a maximal order as above.

Next, we consider a case of \( a \equiv 1 \pmod{4} \) and \( p = 2. \)

**Lemma 7.** We consider the following conditions

i) \( a = 3 \pmod{8}, \quad d = 2 \pmod{4}, \) but \( d = 2 \pmod{8}. \)

ii) \( a = 3 \pmod{8}, \) and \( d = 2 \pmod{8}. \)

iii) \( a = 7 \pmod{8}, \) and \( d = 2 \pmod{4} \) but \( d = 2 \pmod{8}. \)

iv) \( a = 7 \pmod{8}, \) and \( d = 2 \pmod{8}. \)

v) \( a = 1 \pmod{4}, \) and \( d = 3 \pmod{4}. \)

If one of i) and iv) is satisfied, then there is a maximal order \( \Omega \) such that \( \Omega / N(\Omega) = (R / 2)^2. \) If one of ii), iii) and v) is satisfied, then there exists a unique maximal order.

**Proof.** We shall show this lemma by a direct computation. Thus, we give here only a sketch of the proof.

Put \( i = g, \quad j = \sqrt{d} \) and \( H = 1/2(1 + i + j), \quad L = 1/2(i + i + ij). \) Let \( \Lambda = R + Ri + RH + RL. \) If we set \( a = 1 + 2r, \) \( d = 2 + 4k \) where \( r \equiv 1 \pmod{4}, \) \( k \equiv 0 \pmod{2}, \) we have

\[ i^2 = 1 + 2r, \quad H^2 = k + (1 + r)/2 + H, \quad L^2 = -(1/2)(1 + r) - (1 + 2r)k + L, \]

\[ iH = L + r, \quad Hi = 1 + r + i - L, \quad iL = -ri + (1 + 2r)H, \quad Li = 1 + 2r \]

\[ +(1 + r)i = -(1 + 2r)H, \quad LH = r + ((1 + r)/2 + k)i - rH + L, \] (8) and
\[ HL = -(k + (1 + r)/2)i + (1 + r)H. \]

In cases i) and iv) we can show directly that \( N(\bar{\Lambda}) = \bar{\Lambda}(\bar{i} + \bar{1}) \) and \( \bar{\Lambda}/\bar{\Lambda}(1 + i) \approx (R/2)\bar{H} \oplus (R/2)(1 + \bar{H}) \), \( \bar{H}(1 + \bar{H}) = 0 \), where \( \bar{\Lambda} = \Lambda/2\Lambda \). Since \((1-i)(1+i) = 1 - a = -2r, r \equiv 0 \pmod{2}, \) \( \Lambda(1 + i) \equiv 2\Lambda \). Hence \( N(\Lambda) = \Lambda(1 + i) \), which implies that \( \Lambda \) is a hereditary order of rank two. Therefore, there exists a maximal order as in the lemma.

In cases ii) and iii) we obtain similarly that \( \Lambda/\Lambda(1 + i) \approx (R/2)\bar{H} + (R/2)(1 + \bar{H}) \) and \( \bar{H}^2 = 1 + \bar{H}, \) \( (1 + \bar{H})^2 = \bar{H}, \) \( \bar{H}(1 + \bar{H}) = 1 \). Hence, \( \Lambda \) is a unique maximal order.

In case v) we put \( i = 1/2(1 + i + j + ij) \) and \( \Lambda = R + Ri + Rj + Rt \). Then by the same argument in [9], Lemma 1.3 we can show that \( N(\Lambda) = \Lambda(1 + i) \) and \( \Lambda/\Lambda(1 + i) \) is a field. Hence, \( \Lambda \) is a unique maximal order.

From Proposition 4, Lemmas 6 and 7 and the proof of [9], Theorem 1.2 we have

**Theorem 4.** Let \( R \) be a ring of \( \wp \)-adic integers, \( K \) the field of rationals and \( L = K(\sqrt{d}) \). For a unit element \( a \) in \( R \), \( \Sigma = (a, G, L) \) is a generalized quaternions and \( \Lambda = (a, G, \Sigma) \). Then every hereditary order over \( R \) in \( \Sigma \) is isomorphic to one of the following:

1. \( \Lambda \) (unique maximal) if \( \wp = 2, d \equiv 0 \pmod{\wp}, \) \( a/\wp = -1 \).
2. \( \Omega_1 = R + R\sqrt{d} + R(1/2)(1 + g) + (1/2)(\sqrt{d} + g\sqrt{d}) \) (unique maximal) if \( \wp = 2, d \equiv 2 \pmod{4}, a \equiv 1 \pmod{4} \) and \( a \equiv 1 \pmod{8} \).
3. \( \Lambda \) (maximal), \( \Lambda \cap \alpha^{-1}\Lambda\alpha \) if either a) \( \wp = 2, d \equiv 1 \pmod{4} \) or b) \( \wp = 2, d \equiv 0 \pmod{\wp} \).
4. \( \Omega \) (maximal), \( \Gamma_1 = R + Rg + RH + RL \) if one of i) and iv) in Lemma 8 is valid.
5. \( \Gamma_1 \) (unique maximal) if one of ii), iii) and iv) in Lemma 8 is valid.
6. \( \Omega_2 = R + Rg + R\sqrt{d} + Rt \) (unique maximal) if \( \wp = 2, d \equiv 3 \pmod{4}, \) and \( a \equiv 1 \pmod{4} \).
7. \( \Omega_3 = R + R\sqrt{d} + R(1/2)(1 + g) + R(1/4)(\sqrt{d} + g\sqrt{d}) \) (maximal), \( \Gamma_3 = R + R\sqrt{d} + R(1/2)(1 + g) + R(1/4)(\sqrt{d} + g\sqrt{d}) \) if \( \wp = 2, d \equiv 0 \pmod{4}, \) and \( a \equiv 1 \pmod{8} \).
8. \( \Omega_4 \) (maximal), \( \Omega_4 \cap \alpha^{-1}\Omega\alpha \) if either a) \( \wp = 2, d \equiv 3 \pmod{4}, a \equiv 1 \pmod{4} \) or b) \( \wp = 2, d \equiv 2 \pmod{4} \) and \( a \equiv 1 \pmod{8} \).
9. \( \Omega_5 = R + Rg + R\sqrt{d} + R(1/\wp)(\sqrt{d} + yg\sqrt{d}) \) (maximal), \( \Lambda \) if \( \wp = 2, d \equiv 0 \pmod{\wp} \) and \( a/\wp = 1 \).
Where $\mathcal{O}$ means the integral closure of $R$ in $L$ and $\alpha$ is a normalized element with respect to the basis of a maximal order and $N(\alpha)=pq$, $(p,q) = 1$ and $a\gamma^2 \equiv 1 \pmod{\wp}$, $H=(1/2)(1+g\sqrt{d})$, $L=(1/2)(1+\sqrt{d}+g\sqrt{d})$, $t=\frac{1}{2}(1+g+\sqrt{d}+g\sqrt{d})$, and $\wp=(p)$.

Remark 2. A maximal order $\Omega$ in 4) is any ring which contains properly $\Lambda$.

References