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Author(s)	Hoshino, Mitsuo
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ON DOMINANT DIMENSION OF NOETHERIAN RINGS

Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

MITSUO HOSHINO

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Throughout this note, R stands for a ring with identity and all modules are unital modules. In this note, for a given module M , we say that M has *dominant dimension* at least n , written $\text{dom dim } M \geq n$, if each of the first n terms of the minimal injective resolution of M is flat. Following Morita [5], we call R left (resp. right) QF -3 if $\text{dom dim } {}_R R \geq 1$ (resp. $\text{dom dim } R_R \geq 1$). He showed that if R is left noetherian and left QF -3 then it is also right QF -3. Thus, if R is left and right noetherian, R is left QF -3 if and only if it is right QF -3. Generalizing this, we will prove the following

Theorem. *Let R be left and right noetherian. For any $n \geq 1$, $\text{dom dim } {}_R R \geq n$ if and only if $\text{dom dim } R_R \geq n$.*

In case R is artinian, our dominant dimension coincides with Tachikawa's one [8], and the above theorem has been established (see Tachikawa [9] for details).

In what follows, for a given left or right R -module M , we denote by M^* the R -dual of M , by $\varepsilon_M: M \rightarrow M^{**}$ the usual evaluation map and by $E(M)$ the injective hull of M . We denote by $\text{mod } R$ (resp. $\text{mod } R^{op}$) the category of all finitely generated left (resp. right) R -modules, where R^{op} stands for the opposite ring of R and right R -modules are considered as left R^{op} -modules.

1. Preliminaries. In this section, we recall several known facts which we need in later sections.

Lemma 1.1. *Let R be right noetherian. For any $N \in \text{mod } R^{op}$ and for any injective left R -module E , $\text{Hom}_R(\text{Ext}_R^i(N, R), E) \simeq \text{Tor}_i^R(N, E)$ for $i \geq 1$.*

Proof. See Cartan and Eilenberg [1, Chap. VI, Proposition 5.3].

Lemma 1.2. *Every finitely presented submodule of a flat module is torsionless.*

Proof. See Lazard [4, Théorème 1.2].

Lemma 1.3. *Let R be right noetherian. Let E be an injective left R -module*

and suppose that every finitely generated submodule of E is torsionless. Then E is flat.

Proof. See Sato [6, Lemma 1.4]. His argument remains valid in our setting.

Lemma 1.4. *Let R be left and right noetherian. Suppose that R is left QF-3. An injective left R -module E is flat if and only if it is cogenerated by $E_{(R)}$.*

Proof. Immediate by Lemmas 1.2 and 1.3.

Lemma 1.5. *Let R be left noetherian. Suppose that $\text{inj dim } R_R = n < \infty$. For a minimal injective resolution $0 \rightarrow R \rightarrow E_0 \rightarrow E_1 \rightarrow \dots$, $E = \bigoplus_{i=0}^n E_i$ is an injective cogenerator.*

Proof. See Iwanaga [3, Theorem 2]. His argument remains valid in our setting.

2. Proof of Theorem. In order to prove the theorem, we need two more lemmas.

Lemma 2.1. *Let R be left noetherian and $n \geq 1$. For any $M \in \text{mod } R$ with $\text{Ext}_R^i(M, R) = 0$ for $1 \leq i \leq n$ and for any $L \in \text{mod } R$ with $\text{proj dim } L = m < n$, $\text{Ext}_R^i(M, L) = 0$ for $1 \leq i \leq n - m$.*

Proof. By induction on $m \geq 0$. The case $m = 0$ is clear. Let $m \geq 1$ and let $0 \rightarrow K \rightarrow P \rightarrow L \rightarrow 0$ be an exact sequence in $\text{mod } R$ with P projective. Since $\text{proj dim } K = m - 1$, by induction hypothesis $\text{Ext}_R^i(M, K) = 0$ for $1 \leq i \leq n - m + 1$. Applying the functor $\text{Hom}_R(M, -)$ to the above exact sequence, we get $\text{Ext}_R^i(M, L) \simeq \text{Ext}_R^{i+1}(M, K) = 0$ for $1 \leq i \leq n - m$.

Lemma 2.2. *Let R be left and right noetherian. Suppose that R is left QF-3. For any $n \geq 2$, $\text{dom dim } R_R \geq n$ if and only if for an $M \in \text{mod } R$, $M^* = 0$ implies $\text{Ext}_R^i(M, R) = 0$ for $1 \leq i \leq n - 1$.*

Proof. Let $0 \rightarrow R \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \rightarrow \dots$ be a minimal injective resolution. For any $i \geq 1$ we have an exact sequence of functors

$$\text{Hom}_R(-, E_{i-1}) \rightarrow \text{Hom}_R(-, \text{Im } f_i) \rightarrow \text{Ext}_R^i(-, R) \rightarrow 0.$$

“Only if” part. For a given $M \in \text{mod } R$ with $M^* = 0$, by Lemma 1.2 $\text{Hom}_R(M, E_i) = 0$ for $1 \leq i \leq n - 1$. Thus $\text{Hom}_R(M, \text{Im } f_i) = 0$, and by the above exact sequence $\text{Ext}_R^i(M, R) = 0$ for $1 \leq i \leq n - 1$.

“If” part. By induction on $i \geq 0$, we show that E_i is flat for $0 \leq i \leq n - 1$. By assumption, E_0 is flat. Let $1 \leq i \leq n - 1$ and suppose that E_{i-1} is flat. For a given $M \in \text{mod } R$ with $M^* = 0$, we claim $\text{Hom}_R(M, \text{Im } f_i) = 0$. We have

$\text{Ext}_R^i(M, R) = 0$. Also, by Lemma 1.2 $\text{Hom}_R(M, E_{i-1}) = 0$. Thus by the above exact sequence $\text{Hom}_R(M, \text{Im } f_i) = 0$. Hence $\text{Im } f_i$ is cogenerated by $E({}_R R)$, and by Lemma 1.4 E_i is flat.

We are now in a position to prove the theorem. It suffices to prove the “only if” part.

“Only if” part of Theorem. The case $n = 1$ is due to Morita [5, Theorem 1]. Let $n \geq 2$. Note that R is left and right QF-3. Replacing R with R^{op} in Lemma 2.2, it suffices to show that for any $N \in \text{mod } R^{op}$ with $N^* = 0$ we have $\text{Ext}_R^i(N, R) = 0$ for $1 \leq i \leq n - 1$. For a given $N \in \text{mod } R^{op}$ with $N^* = 0$, we claim first that $\text{Ext}_R^i(N, R)^* = 0$ for $i \geq 1$. For any $i \geq 1$, by Lemma 1.1 $\text{Hom}_R(\text{Ext}_R^i(N, R), E({}_R R)) = \text{Tor}_i^R(N, E({}_R R)) = 0$, thus $\text{Ext}_R^i(N, R)^* = 0$. Hence by Lemma 2.2 $\text{Ext}_R^j(\text{Ext}_R^i(N, R), R) = 0$ for $i \geq 1$ and $1 \leq j \leq n - 1$. Now, by induction on $i \geq 1$, we show that $\text{Ext}_R^i(N, R) = 0$ for $1 \leq i \leq n - 1$. Let $\dots \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} N \rightarrow 0$ be an exact sequence in $\text{mod } R^{op}$ with the P_i projective and put $N_i = \text{Im } f_i$. Since $N^* = 0$, we have an exact sequence

$$0 \rightarrow P_0^* \xrightarrow{\beta_1} N_1^* \xrightarrow{\alpha_1} \text{Ext}_R^1(N, R) \rightarrow 0.$$

Since $\text{Ext}_R^1(\text{Ext}_R^1(N, R), R) = 0$, α_1 splits. On the other hand, since $\text{Ext}_R^1(N, R)^* = 0$, $\text{Hom}_R(\text{Ext}_R^1(N, R), N_1^*) = 0$. Thus $\text{Ext}_R^1(N, R) = 0$. Next, let $1 < i \leq n - 1$ and suppose that $\text{Ext}_R^j(N, R) = 0$ for $1 \leq j \leq i - 1$. We have an exact swquence

$$0 \rightarrow P_0^* \rightarrow \dots \rightarrow P_{i-1}^* \xrightarrow{\beta_i} N_i^* \xrightarrow{\alpha_i} \text{Ext}_R^i(N, R) \rightarrow 0.$$

Since $\text{Ext}_R^j(\text{Ext}_R^i(N, R), R) = 0$ for $1 \leq j \leq n - 1$, and since $\text{proj dim Im } \beta_i \leq i - 1 < n - 1$, by Lemma 2.1 $\text{Ext}_R^1(\text{Ext}_R^i(N, R), \text{Im } \beta_i) = 0$. Thus α_i splits. On the other hand, $\text{Ext}_R^i(N, R)^* = 0$ implies $\text{Hom}_R(\text{Ext}_R^i(N, R), N_i^*) = 0$. Hence $\text{Ext}_R^i(N, R) = 0$.

3. Left exactness of the double dual. In this section, we establish the relation between the dominant dimension of a left and right noetherian ring R and the behavior of the functor $(\)^{**}: \text{mod } R \rightarrow \text{mod } R$. Compare our results with Colby and Fuller [2, Theorems 1 and 2].

Proposition 3.1. *Let R be left and right noetherian. Then R is left QF-3 if and only if the functor $(\)^{**}: \text{mod } R \rightarrow \text{mod } R$ preserves monomorphisms.*

This is an immediate consequence of Morita [5, Theorem 1] and the following lemmas.

Lemma 3.2. *Let R be left noetherian and right QF-3. For any monomorphism $\alpha: M \rightarrow L$ with $M, L \in \text{mod } R$, α^{**} is monic.*

Proof. For a given exact sequence $0 \rightarrow M \xrightarrow{\alpha} L \rightarrow K \rightarrow 0$ in $\text{mod } R$, we claim $(\text{Cok } \alpha^*)^* = 0$. By Lemma 1.1 $\text{Hom}_R(\text{Ext}_R^1(K, R), E(R_R)) \simeq \text{Tor}_1^f(E(R_R), K) = 0$. Since $\text{Cok } \alpha^*$ is imbedded into $\text{Ext}_R^1(K, R)$, we get $\text{Hom}_R(\text{Cok } \alpha^*, E(R_R)) = 0$. Thus $(\text{Cok } \alpha^*)^* = 0$, and α^{**} is monic.

Lemma 3.3. *Let R be right noetherian. Suppose that for any monomorphism $\alpha: M \rightarrow L$ with $M, L \in \text{mod } R$ α^{**} is monic. Then R is left QF-3.*

Proof. For a given $M \in \text{mod } R$ with $M \subset E({}_R R)$, we claim that M is torsionless. Replacing M with $M + R$ if necessary, we may assume $R \subset M$. Denote by ι the inclusion $R \hookrightarrow M$. Since ι^{**} is monic, so is $\iota^{**} \circ \varepsilon_R = \varepsilon_M \circ \iota$. Thus $R \cap \text{Ker } \varepsilon_M = 0$, which implies $\text{Ker } \varepsilon_M = 0$. Hence by Lemma 1.3 $E({}_R R)$ is flat.

Now we can prove the following

Proposition 3.4. *Let R be left and right noetherian. Then $\text{dom dim } {}_R R \geq 2$ if and only if the functor $(\)^{**}: \text{mod } R \rightarrow \text{mod } R$ is left exact.*

Proof. ‘‘Only if’’ part. For a given exact sequence $0 \rightarrow M \xrightarrow{\alpha} L \xrightarrow{\beta} K \rightarrow 0$ in $\text{mod } R$, we claim $(\text{Cok } \alpha^*)^* = 0 = \text{Ext}_R^1(\text{Cok } \alpha^*, R)$. Note that $\text{dom dim } R_R \geq 2$. By Lemma 3.2, α^{**} is monic. Thus $(\text{Cok } \alpha^*)^* = 0$, and by Lemma 2.2 $\text{Ext}_R^1(\text{Cok } \alpha^*, R) = 0$. Hence the following sequence is exact:

$$0 \rightarrow M^{**} \xrightarrow{\alpha^{**}} L^{**} \xrightarrow{\beta^{**}} K^{**}.$$

‘‘If’’ part. By Lemma 3.3, $E({}_R R)$ is flat. For a given $M \in \text{mod } R$ with $M \subset E({}_R R)/R$, we claim that M is torsionless. There is some $L \in \text{mod } R$ such that $L \subset E({}_R R)$ and $M = L/R$. By Lemma 1.2, L is torsionless. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \rightarrow & R & \longrightarrow & L & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow \wr & & \downarrow \varepsilon_L & & \downarrow \varepsilon_M \\ 0 & \rightarrow & R^{**} & \rightarrow & L^{**} & \rightarrow & M^{**}. \end{array}$$

Since ε_L is monic, so is ε_M . Thus by Lemma 1.4 $E(E({}_R R)/R)$ is flat.

4. Remarks. In this final section, we make some remarks on noetherian rings of finite self-injective dimension.

The following proposition is essentially due to Iwanaga [3].

Proposition 4.1. *Let R be left noetherian. Suppose that $\text{inj dim } {}_R R < \infty$ and that the last non-zero term of the minimal injective resolution of ${}_R R$ is flat. Then R is quasi-Frobenius.*

Proof. Suppose to the contrary that ${}_R R$ is not injective. Put $n = \text{inj dim } {}_R R$

and let $0 \rightarrow {}_R R \rightarrow E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_n \rightarrow 0$ be a minimal injective resolution. There is a torsion theory $(\mathcal{I}, \mathcal{F})$ in $\text{mod } R$ such that \mathcal{F} consists of the modules $M \in \text{mod } R$ with $\text{Ext}_R^n(M, R) = 0$. Note that \mathcal{I} contains a simple module L . Since E_n is flat, and since $\text{Hom}_R(L, E_n) \simeq \text{Ext}_R^n(L, R) \neq 0$, by Lemma 1.2 L is torsionless, which implies $L \in \mathcal{F}$, a contradiction.

Proposition 4.2. *Let R be left noetherian. Suppose that $\text{inj dim } R_R < \text{dom dim } {}_R R$. Then $E({}_R R)$ is an injective cogenerator.*

Proof. Let $0 \rightarrow {}_R R \rightarrow E_0 \rightarrow E_1 \rightarrow \dots$ be a minimal injective resolution and put $E = \bigoplus_{i=0}^n E_i$, where $n = \text{inj dim } R_R$. By Lemma 1.5 E is an injective cogenerator. Thus, since E is flat, by Lemma 1.2 every $M \in \text{mod } R$ is torsionless, namely $E({}_R R)$ is an injective cogenerator.

The next proposition generalizes Sumioka [7, Theorem 5].

Proposition 4.3. *Let R be left and right noetherian and $n \geq 1$. Suppose that $\text{inj dim } {}_R R \leq n \leq \text{dom dim } {}_R R$. For a minimal injective resolution $0 \rightarrow {}_R R \rightarrow E_0 \rightarrow E_1 \rightarrow \dots$, $E = \bigoplus_{i=0}^n E_i$ is an injective cogenerator if and only if $\text{inj dim } R_R \leq n$.*

Proof. “Only if” part. Since E_i is flat for $0 \leq i \leq n-1$, and since $E_i = 0$ for $i > n$, E_n and thus E have weak dimension at most n . Thus by Lemma 1.1 $\text{Hom}_R(\text{Ext}_R^{n+1}(N, R), E) \simeq \text{Tor}_{n+1}^R(N, E) = 0$ for all $N \in \text{mod } R^{op}$. Hence, since E is an injective cogenerator, $\text{inj dim } R_R \leq n$.

“If” part. By Lemma 1.5.

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Institute of Mathematics
University of Tsukuba
Ibaraki, 305 Japan