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Osaka University
Throughout this note, $R$ stands for a ring with identity and all modules are unital modules. In this note, for a given module $M$, we say that $M$ has dominant dimension at least $n$, written $\text{dom dim } M \geq n$, if each of the first $n$ terms of the minimal injective resolution of $M$ is flat. Following Morita [5], we call $R$ left (resp. right) QF-3 if $\text{dom dim } _RR \geq 1$ (resp. $\text{dom dim } R_\ell \geq 1$). He showed that if $R$ is left noetherian and left QF-3 then it is also right QF-3. Thus, if $R$ is left and right noetherian, $R$ is left QF-3 if and only if it is right QF-3. Generalizing this, we will prove the following

**Theorem.** Let $R$ be left and right noetherian. For any $n \geq 1$, $\text{dom dim } _RR \geq n$ if and only if $\text{dom dim } R_\ell \geq n$.

In case $R$ is artinian, our dominant dimension coincides with Tachikawa's one [8], and the above theorem has been established (see Tachikawa [9] for details).

In what follows, for a given left or right $R$-module $M$, we denote by $M^*$ the $R$-dual of $M$, by $\epsilon_M: M \to M^{**}$ the usual evaluation map and by $E(M)$ the injective hull of $M$. We denote by $\mathfrak{mod} R$ (resp. $\mathfrak{mod} R^\text{op}$) the category of all finitely generated left (resp. right) $R$-modules, where $R^\text{op}$ stands for the opposite ring of $R$ and right $R$-modules are considered as left $R^\text{op}$-modules.

**1. Preliminaries.** In this section, we recall several known facts which we need in later sections.

**Lemma 1.1.** Let $R$ be right noetherian. For any $N \in \mathfrak{mod} R^\text{op}$ and for any injective left $R$-module $E$, $\text{Hom}_R (\text{Ext}_R^i (N, R), E) \cong \text{Tor}_i^R (N, E)$ for $i \geq 1$.

Proof. See Cartan and Eilenberg [1, Chap. VI, Proposition 5.3].

**Lemma 1.2.** Every finitely presented submodule of a flat module is torsionless.

Proof. See Lazard [4, Théorème 1.2].

**Lemma 1.3.** Let $R$ be right noetherian. Let $E$ be an injective left $R$-module
and suppose that every finitely generated submodule of $E$ is torsionless. Then $E$ is flat.

Proof. See Sato [6, Lemma 1.4]. His argument remains valid in our setting.

**Lemma 1.4.** Let $R$ be left and right noetherian. Suppose that $R$ is left QF-3. An injective left $R$-module $E$ is flat if and only if it is cogenerated by $E_R$.

Proof. Immediate by Lemmas 1.2 and 1.3.

**Lemma 1.5.** Let $R$ be left noetherian. Suppose that $\text{inj} \dim R < \infty$. For a minimal injective resolution $0 \to R \to E_0 \to E_1 \to \cdots$, $E = \bigoplus_{i=0}^\infty E_i$ is an injective cogenerator.

Proof. See Iwanaga [3, Theorem 2]. His argument remains valid in our setting.

2. **Proof of Theorem.** In order to prove the theorem, we need two more lemmas.

**Lemma 2.1.** Let $R$ be left noetherian and $n \geq 1$. For any $M \in \text{mod} \ R$ with $\text{Ext}_R^i(M, R) = 0$ for $1 \leq i \leq n$ and for any $L \in \text{mod} \ R$ with $\text{proj} \dim L = m < n$, $\text{Ext}_R^i(M, L) = 0$ for $1 \leq i \leq n - m$.

Proof. By induction on $m \geq 0$. The case $m = 0$ is clear. Let $m \geq 1$ and let $0 \to K \to P \to L \to 0$ be an exact sequence in $\text{mod} \ R$ with $P$ projective. Since $\text{proj} \dim K = m - 1$, by induction hypothesis $\text{Ext}_R^i(M, K) = 0$ for $1 \leq i \leq n - m + 1$. Applying the functor $\text{Hom}_R(-, -)$ to the above exact sequence, we get $\text{Ext}_R^i(M, L) = \text{Ext}_R^{i+1}(M, K) = 0$ for $1 \leq i \leq n - m$.

**Lemma 2.2.** Let $R$ be left and right noetherian. Suppose that $R$ is left QF-3. For any $n \geq 2$, $\text{dom} \ dim R^n$ if and only if for an $M \in \text{mod} \ R$, $M^* = 0$ implies $\text{Ext}_R^i(M, R) = 0$ for $1 \leq i \leq n - 1$.

Proof. Let $0 \to R \to E_0 \to E_1 \to \cdots$ be a minimal injective resolution. For any $i \geq 1$ we have an exact sequence of functors

$\text{Hom}_R(-, E_{i-1}) \to \text{Hom}_R(-, \text{Im} f_i) \to \text{Ext}_R^i(-, R) \to 0$.

"Only if" part. For a given $M \in \text{mod} \ R$ with $M^* = 0$, by Lemma 1.2 $\text{Hom}_R(M, E_i) = 0$ for $1 \leq i \leq n - 1$. Thus $\text{Hom}_R(M, \text{Im} f_i) = 0$, and by the above exact sequence $\text{Ext}_R^i(M, R) = 0$ for $1 \leq i \leq n - 1$.

"If" part. By induction on $i \geq 0$, we show that $E_i$ is flat for $0 \leq i \leq n - 1$. By assumption, $E_0$ is flat. Let $1 \leq i \leq n - 1$ and suppose that $E_{i-1}$ is flat. For a given $M \in \text{mod} \ R$ with $M^* = 0$, we claim $\text{Hom}_R(M, \text{Im} f_i) = 0$. We have
Ext_k^i(M, R) = 0. Also, by Lemma 1.2 \( \hom_k(M, E_{i-1}) = 0 \). Thus by the above exact sequence \( \hom_k(M, \im f_i) = 0 \). Hence \( \im f_i \) is cogenerated by \( E_k(R) \), and by Lemma 1.4 \( E_i \) is flat.

We are now in a position to prove the theorem. It suffices to prove the "only if" part.

"Only if" part of Theorem. The case \( n=1 \) is due to Morita [5, Theorem 1]. Let \( n \geq 2 \). Note that \( R \) is left and right \( QF-3 \). Replacing \( R \) with \( R^* \) in Lemma 2.2, it suffices to show that for any \( N \in \text{mod } R^* \) with \( N^* = 0 \) we have \( \ext_k^i(N, R) = 0 \) for \( 1 \leq i \leq n-1 \). For a given \( N \in \text{mod } R^* \) with \( N^* = 0 \), we claim first that \( \ext_k^i(N, R)^* = 0 \) for \( i \geq 1 \). For any \( i \geq 1 \), by Lemma 1.1 \( \hom_k(\ext_k^i(N, R), E_k(R)) = \tor_k(N, E_k(R)) = 0 \), thus \( \ext_k^i(N, R)^* = 0 \). Hence by Lemma 2.2 \( \ext_k^i(\ext_k^i(N, R), R) = 0 \) for \( i \geq 1 \) and \( 1 \leq j \leq n-1 \). Now, by induction on \( i \geq 1 \), we show that \( \ext_k^i(N, R) = 0 \) for \( 1 \leq i \leq n-1 \). Let \( \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow N \rightarrow 0 \) be an exact sequence in \( \text{mod } R^* \) with the \( P_i \) projective and put \( N_i = \im f_i \). Since \( N^* = 0 \), we have an exact sequence

\[
0 \rightarrow P_0^* \xrightarrow{\beta_i} N_i^* \xrightarrow{\alpha_i} \ext_k^i(N, R) \rightarrow 0.
\]

Since \( \ext_k^i(\ext_k^i(N, R), R) = 0 \), \( \alpha_i \) splits. On the other hand, since \( \ext_k^i(N, R)^* = 0 \), \( \hom_k(\ext_k^i(N, R), N^*_i) = 0 \). Thus \( \ext_k^i(N, R) = 0 \). Next, let \( 1 < i \leq n-1 \) and suppose that \( \ext_k^i(N, R) = 0 \) for \( 1 \leq j \leq i-1 \). We have an exact sequence

\[
0 \rightarrow P_0^* \rightarrow \cdots \rightarrow P_{i-1}^* \xrightarrow{\beta_i} N_i^* \xrightarrow{\alpha_i} \ext_k^i(N, R) \rightarrow 0.
\]

Since \( \ext_k^i(\ext_k^i(N, R), R) = 0 \) for \( 1 \leq j \leq n-1 \), and since \( \proj \dim \im \beta_i \leq i-1 < n-1 \), by Lemma 2.1 \( \ext_k^i(\ext_k^i(N, R), \im \beta_i) = 0 \). Thus \( \alpha_i \) splits. On the other hand, \( \ext_k^i(N, R)^* = 0 \) implies \( \hom_k(\ext_k^i(N, R), N^*_i) = 0 \). Hence \( \ext_k^i(N, R) = 0 \).

3. Left exactness of the double dual. In this section, we establish the relation between the dominant dimension of a left and right noetherian ring \( R \) and the behavior of the functor \( (\_)^{**} : \text{mod } R \rightarrow \text{mod } R \). Compare our results with Colby and Fuller [2, Theorems 1 and 2].

**Proposition 3.1.** Let \( R \) be left and right noetherian. Then \( R \) is left \( QF-3 \) if and only if the functor \( (\_)^{**} : \text{mod } R \rightarrow \text{mod } R \) preserves monomorphisms.

This is an immediate consequence of Morita [5, Theorem 1] and the following lemmas.

**Lemma 3.2.** Let \( R \) be left noetherian and right \( QF-3 \). For any monomorphism \( \alpha : M \rightarrow L \) with \( M, L \in \text{mod } R \), \( \alpha^{**} \) is monic.
Proof. For a given exact sequence $0 \to \alpha_* M \to L \to K \to 0$ in mod $R$, we claim $(\text{Cok } \alpha^*)^* = 0$. By Lemma 1.1, $\text{Hom}_R(\text{Ext}_k(K, R), E(R_{\pi})) = \text{Tor}^k(E(R_{\pi}), K) = 0$. Since \text{Cok } \alpha^* is imbedded into $\text{Ext}_k(K, R)$, we get $\text{Hom}_R(\text{Cok } \alpha^*, E(R_{\pi})) = 0$. Thus $(\text{Cok } \alpha^*)^* = 0$, and $\alpha^{**}$ is monic.

**Lemma 3.3.** Let $R$ be right noetherian. Suppose that for any monomorphism $\alpha: M \to L$ with $M, L \in \text{mod } R$, $\alpha^{**}$ is monic. Then $R$ is left QF-3.

Proof. For a given $M \in \text{mod } R$ with $M \subset E(R_R)$, we claim that $M$ is torsionless. Replacing $M$ with $M + R$ if necessary, we may assume $R \subset M$. Denote by $\iota$ the inclusion $R \hookrightarrow M$. Since $\iota^{**}$ is monic, so is $\iota^{**} \circ \varepsilon_M = \varepsilon_{M/R}$. Thus $R \cap \ker \varepsilon_M = 0$, which implies $\ker \varepsilon_M = 0$. Hence by Lemma 1.3, $E(R_R)$ is flat.

Now we can prove the following

**Proposition 3.4.** Let $R$ be left and right noetherian. Then $\text{dom dim } R_R \geq 2$ if and only if the functor $(\ )^{**}: \text{mod } R \to \text{mod } R$ is left exact.

Proof. "Only if" part. For a given exact sequence $0 \to \alpha_* M \to L \to K \to 0$ in mod $R$, we claim $(\text{Cok } \alpha^*)^* = 0 = \text{Ext}_k(\text{Cok } \alpha^*, R)$. Note that $\text{dom dim } R_R \geq 2$. By Lemma 3.2, $\alpha^{**}$ is monic. Thus $(\text{Cok } \alpha^*)^{**} = 0$, and by Lemma 2.2, $\text{Ext}_k(\text{Cok } \alpha^*, R) = 0$. Hence the following sequence is exact:

$$0 \to M^{**} \xrightarrow{\alpha^{**}} L^{**} \xrightarrow{\beta^{**}} K^{**}.$$ 

"If" part. By Lemma 3.3, $E(R_R)$ is flat. For a given $M \in \text{mod } R$ with $M \subset E(R_R)/R$, we claim that $M$ is torsionless. There is some $L \in \text{mod } R$ such that $L \subset E(R_R)$ and $M = L/R$. By Lemma 1.2, $L$ is torsionless. We have the following commutative diagram with exact rows:

$$
\begin{array}{cccc}
0 & \to & R & \to & L & \to & M & \to & 0 \\
& & \downarrow \iota & & \downarrow \varepsilon_L & & \downarrow \varepsilon_M & \\
0 & \to & R^{**} & \to & L^{**} & \to & M^{**} & &
\end{array}
$$

Since $\varepsilon_L$ is monic, so is $\varepsilon_M$. Thus by Lemma 1.4, $E(E(R_R)/R)$ is flat.

4. Remarks. In this final section, we make some remarks on noetherian rings of finite self-injective dimension.

The following proposition is essentially due to Iwanaga [3].

**Proposition 4.1.** Let $R$ be left noetherian. Suppose that $\text{inj dim } R_R < \infty$ and that the last non-zero term of the minimal injective resolution of $R_R$ is flat. Then $R$ is quasi-Frobenius.

Proof. Suppose to the contrary that $R_R$ is not injective. Put $n = \text{inj dim } R_R$
and let $0 \to {}_R R \to E_0 \to E_1 \to \cdots \to E_\infty \to 0$ be a minimal injective resolution. There is a torsion theory $(\mathcal{T}, \mathcal{D})$ in $\text{mod } R$ such that $\mathcal{T}$ consists of the modules $M \in \text{mod } R$ with $\text{Ext}_R^1(M, R) = 0$. Note that $\mathcal{T}$ contains a simple module $L$. Since $E_n$ is flat, and since $\text{Hom}_R(L, E_n) = \text{Ext}_R^1(L, R) \neq 0$, by Lemma 1.2 $L$ is torsionless, which implies $L \in \mathcal{T}$, a contradiction.

**Proposition 4.2.** Let $R$ be left noetherian. Suppose that $\text{inj dim } {}_R R < \text{dom dim } {}_R R$. Then $E( {}_R R)$ is an injective cogenerator.

Proof. Let $0 \to {}_R R \to E_0 \to E_1 \to \cdots$ be a minimal injective resolution and put $E = \bigoplus_{i=0}^n E_i$, where $n = \text{inj dim } {}_R R$. By Lemma 1.5 $E$ is an injective cogenerator. Thus, since $E$ is flat, by Lemma 1.2 every $M \in \text{mod } R$ is torsionless, namely $E( {}_R R)$ is an injective cogenerator.

The next proposition generalizes Sumioka [7, Theorem 5].

**Proposition 4.3.** Let $R$ be left and right noetherian and $n \geq 1$. Suppose that $\text{inj dim } {}_R R \leq n \leq \text{dom dim } {}_R R$. For a minimal injective resolution $0 \to {}_R R \to E_0 \to E_1 \to \cdots$, $E = \bigoplus_{i=0}^n E_i$ is an injective cogenerator if and only if $\text{inj dim } {}_R R \leq n$.

Proof. “Only if” part. Since $E_i$ is flat for $0 \leq i \leq n - 1$, and since $E_i = 0$ for $i > n$, $E_n$ and thus $E$ have weak dimension at most $n$. Thus by Lemma 1.1 $\text{Hom}_R(\text{Ext}_R^{n+1}(N, R), E) = \text{Tor}_R^{n+1}(N, E) = 0$ for all $N \in \text{mod } R^{op}$. Hence, since $E$ is an injective cogenerator, $\text{inj dim } {}_R R \leq n$.

“If” part. By Lemma 1.5.

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**References**


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