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STABLE-LIKE PROCESSES: CONSTRUCTION OF THE TRANSITION DENSITY AND THE BEHAVIOR OF SAMPLE PATHS NEAR $t=0$

AKIRA NEGORO

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Introduction

Let $X=(X_t, \boldsymbol{P}_x; x\in\boldsymbol{R}^d)$ be a d-dimensional pure jump type Markov process associated with the operator $-(-\Delta)^{\alpha(x)/2}(0<\alpha(x)<2)$. Following Bass [1], we call it the stable-like process with exponent $\alpha(x)$. Under a mild regularity condition for $\alpha(x)$, the process is first constructed by Bass [1] and next by Tsuchiya [12]: Bass has done it by showing the uniqueness of solutions to the martingale problem for the operator and Tsuchiya by showing the pathwise uniqueness of solutions to a stochastic differential equation associated with the operator.

In this paper, we will show the existence of a transition density and local Hölder conditions for sample paths of the process X with smooth exponent $\alpha(x)$. For this aim, we want to adapt the theory of pseudo-differential operators to the operator $-(-\Delta)^{-\alpha(x)/2}$, but its symbol $-|\xi|^{\alpha(x)}$ is not smooth. Hence we consider the operator L_Φ which is obtained from $-(-\Delta)^{\alpha(x)/2}$ by cutting off the support of its integral kernel (i.e. Lévy measure) with a positive smooth function Φ (see Section 1 for the precise definition of L_Φ). There exists a pure jump type Markov process X_Φ associated with L_Φ in the same sense as the above. Since L_Φ can be regarded as a pseudo-differential operator of variable order, we introduce a class of such operators and provide the fundamental theorem for algebra and asymptotic expansion formula of their symbols. Next we prove that L_Φ satisfies the (H)-condition (see [7] p.83 for the (H)-condition). These facts allow us to construct a fundamental solution, in the sense of pseudo-differential operators, to the initial-value problem for the equation $\partial_t-L_\Phi=0$. Furthermore, we show that this fundamental solution has a smooth kernel and this gives a transition density of X_Φ. Using a localization argument, we see that X also has a transition density. Finally, using certain estimates for the symbol of the fundamental solution and expanding the method of Khintchine [6] and Blumenthal and Getoor [3], we obtain the local Hölder conditions for sample paths of X; this result is a natural extension of that of

[3] in the case of symmetric stable processes.

Pseudo-differential operators of variable order are treated by Unterberger and Bokobza [14], [15], Unterberger [13], Višik and Eskin [16], [17], Beasuzamy [2] and Leopold [9] [10], etc. They, however, do not treat the initial-value problem for evolution equations with respect to such operators.

Section 1 is devoted to construction of a fundamental solution $E(\cdot)$ to the initial-value probelm for $\partial_t - L_\Phi = 0$ (Theorem 1.3). It implies the existence of a transition density of X_Φ (Theorem 1.6) and also implies the existence of a transition density of X (Theorem 1.7). The (H)-condition follows from Theorem 1.1, which is a key result for the construction of the fundamental solution.

In Section 2, we prove local Hölder conditions for sample paths of X (Theorem 2.1). Lemma 2.1 is an extension of a fundamental result of Khintchine [6]. Lemma 2.2 gives a relation between the symbol of the fundamental solution $E(\cdot)$ and the characteristic function of a random variable used in [3].

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1. Constrcution of the transition density

We begin with introducing some notations. For $n=0, 1, 2, \cdots, \infty$, $\boldsymbol{C}_b^n(\boldsymbol{R}^d)$ is the space of real-valued n times differentiable functions which are defined on $\boldsymbol{R}^d$ and have bounded continuous derivatives up to order n. $\boldsymbol{C}_0^\infty(\boldsymbol{R}^d)$ is the subspace of $\boldsymbol{C}_b^\infty(\boldsymbol{R}^d)$ consisting of those functions with compact support. $\mathcal{S}$ or $\mathcal{S}(\boldsymbol{R}^d)$ denotes the Schwartz class on $\boldsymbol{R}^d$. $\boldsymbol{C}_b^{1,2}([0, \infty)\times\boldsymbol{R}^d)$ denotes the space of real-valued functions on $[0, \infty)\times\boldsymbol{R}^d$ which together with first-derivative in time variable and first two-derivatives in space variables are bounded and continuous. For a bounded function $\alpha(x)$ on $\boldsymbol{R}^d$, set

$$\bar{\alpha} = \sup_{x\in\boldsymbol{R}^d} \alpha(x) \quad \text{and} \quad \underline{\alpha} = \inf_{x\in\boldsymbol{R}^d} \alpha(x) .$$

Let Ω be the space of $\boldsymbol{R}^d$-valued càdlàg functions ω on $[0, \infty)$ and let $X_t\colon \Omega\to\boldsymbol{R}^d$ be the function defined by $X_t(\omega)=\omega(t)$. Let $\mathcal{F}_t$ be the σ-field generated by $\{X_s, s\le t\}$ and $\mathcal{F}=\mathcal{F}_\infty$. Given a positive kernel $\nu(x, dy)$ on $\boldsymbol{R}^d\times(\boldsymbol{R}^d\setminus\{0\})$ satisfying $\int_{\boldsymbol{R}^d\setminus\{0\}}(|y|^2\wedge 1)\,\nu(x, dy)<\infty$, we define the operator L on $\boldsymbol{C}_b^2(\boldsymbol{R}^d)$ by

$$Lf(x) = \int_{\boldsymbol{R}^d\setminus\{0\}} \{f(x+y)-f(x)-\nabla f(x)\cdot y 1_{\{|y|\le 1\}}(y)\}\ \nu(x, dy) ,$$

where $z\cdot y$ is the scalar product in $\boldsymbol{R}^d$, ∇ is the gradient operator and $1_E(\cdot)$ the indicator function of a set E. We say that a probability measure $\boldsymbol{P}$ on $(\Omega, \mathcal{F})$ is a solution to the martingale problem for the operator L starting at x if $\boldsymbol{P}(X_0=x)=1$ and, for every $f\in \boldsymbol{C}_b^{1,2}([0,\infty)\times\boldsymbol{R}^d)$,

$$f(t, X_t)-f(0, X_0)-\int_0^t (\partial_u+L)f(u, X_u)\,du$$

is a $\boldsymbol{P}$-martingale with respect to the filtration $\{\mathcal{F}_t\}$.

In this paper, we will focus our attention on the following type of kernels:

$$\nu(x, dy) = w_{\alpha(x)}|y|^{-(d+\alpha(x))}\,dy\,,$$

where $\alpha(x)$ is of $\boldsymbol{C}_b^\infty(\boldsymbol{R}^d)$ with $0<\underline{\alpha}\leq\alpha(x)\leq\bar{\alpha}<2$, and $w_{\alpha(x)}$ is defined through the Lévy-Khintchine formula

$$|\xi|^{\alpha(x)} = \int_{\boldsymbol{R}^d\setminus\{0\}} \{1-\cos\xi\cdot y\}\, w_{\alpha(x)}\,|y|^{-(d+\alpha(x))}\,dy\,.$$

We note that $w_{\alpha(x)}$ is a positive function of $\boldsymbol{C}_b^\infty(\boldsymbol{R}^d)$. Then the operator L can be regarded as a pseudo-differential operator with symbol $-|\xi|^{\alpha(x)}$; hence, in the following, we will denote the operator L by $-(-\Delta)^{\alpha(x)/2}$. By a result of Bass [1] or Tsuchiya [12], for each starting point, there exists a unique solution to the martingale problem for the opeartor $-(-\Delta)^{\alpha(x)/2}$. Therefore, the family of solutions to the martingale problem defines a Markov process on $\boldsymbol{R}^d$, and it is called the stable-like process with exponent $\alpha(x)$.

The purpose of this section is to show the existence of a transition density of the process. To conclude this, we consider the kernel ν_Φ defined by

$$\nu_\Phi(x, dy) = w_{\alpha(x)}|y|^{-(d+\alpha(x))}\,\Phi(|y|)\,dy\,,$$

where Φ is a function of $\boldsymbol{C}_b^\infty([0,\infty))$ satisfying the conditions:

(1) $0\leq\Phi\leq1$ on $[0,\infty)$,

(2) there exists a real number $r_0>0$ such that $\Phi(t)=1$ for any $t\in[0, r_0]$,

(3) $\Phi(t)=0$ for any $t\in[1,\infty)$.

Let L_Φ denote the operator corresponding to this kernel. Then the uniqueness of solutions to the martingale problem for L_Φ also holds and hence there exists a unique Markov process X_Φ associated with L_Φ in the same sense as the above (cf. [12]). At first, we will construct a transition density of this Markov process and obtain some estimates for the density. Then, using them, we show the existence of a transition density of the original stable-like process.

Now, the operator L_Φ can be regarded as a pseudo-differential operator with symbol p_Φ:

$$p_\Phi(x,\xi) = \int_{\boldsymbol{R}^d\setminus\{0\}} \{\exp(i\xi\cdot y)-1-i\xi\cdot y\}\,\frac{w_{\alpha(x)}\,\Phi(|y|)}{|y|^{d+\alpha(x)}}\,dy\,. \tag{1.1}$$

To adapt the theory of pseudo-differential operators for L_Φ, we start to discuss some properties of the function p_Φ. For a multi-index $n=(n_1, n_2, \cdots, n_d)$, let $\partial_\xi^n=\partial^{n_1}/\partial\xi_1^{n_1}\cdots\partial^{n_d}/\partial\xi_d^{n_d}$ and $D_x^n=(-i)^{|n|}\partial_x^n$, where $|n|=n_1+n_2+\cdots+n_d$.

Theorem 1.1. (1) *p_Φ is of $C_b^\infty(R^d\times R^d)$.*
(2) *For any multi-indices m and n, there exists a constant $C_{m,n}>0$ such that for any $(x, \xi)\in R^d\times R^d$*

$$(1.2)\qquad |\partial_\xi^n D_x^m p_\Phi(x, \xi)|\leq C_{m,n}(|\xi|\vee 1)^{(\alpha(x)-|n|)}(1+\log(|\xi|\vee 1))^{|m|}.$$

(3) *There exist constants $R>0$ and $C_0>0$ such that for any $x\in R^d$ and $|\xi|>R$*

$$(1.3)\qquad |p_\Phi(x, \xi)|\geq C_0|\xi|^{\alpha(x)}.$$

REMARK. If we set $C'_{m,n}=C_{m,n}/C_0$, then

$$(1.4)\qquad \left|\frac{\partial_\xi^n D_x^m p_\Phi(x, \xi)}{p_\Phi(x, \xi)}\right|\leq C'_{m,n}|\xi|^{-|n|}\{1+\log(|\xi|\vee 1)\}^{|m|},$$

for any $x\in R^d$ and $|\xi|>R$. This implies the (H)-condition.

Proof of Theorem 1.1. In the proof, C denotes different positive constants. Let S^{d-1} be the unit sphere of R^d and s be the uniform measure on S^{d-1}. Since s is invariant under rotation, we have

$$(1.5)\qquad p_\Phi(x, \xi)=\int_0^1\int_{S^{d-1}}(\cos r\theta\cdot\xi-1)\frac{w_{\alpha(x)}\Phi(r)}{r^{1+\alpha(x)}}dr\, s(d\theta);$$

hence

$$(1.6)\qquad \partial_x^m p_\Phi(x, \xi)=\sum_{k=0}^{|m|}a_k(x)\int_0^1\frac{(\log r)^k\Phi(r)}{r^{1+\alpha(x)}}dr\int_{S^{d-1}}(\cos r\theta\cdot\xi-1)\, s(d\theta),$$

where the function $a_k(x)$ is a linear combination of derivatives up to order k of $\alpha(x)$ and $w_{\alpha(x)}$. Then $a_k(x)$ $(k=1, 2, \cdots)$ are of $C_b^\infty(R^d)$. Hence, to obtain the estimate for $\partial_x^m p_\Phi$, it is sufficient to evaluate the following integral:

$$I_k=\int_0^1\frac{(\log r)^k\Phi(r)}{r^{1+\alpha(x)}}dr\int_{S^{d-1}}(\cos r\theta\cdot\xi-1)\, s(d\theta).$$

For $|\xi|\leq 1$, noting $|\cos r\theta\cdot\xi-1|\leq\frac{1}{2}r^2$, we see that

$$|I_k|\leq\frac{1}{2}s(S^{d-1})\int_0^1 r^{1-\alpha(x)}(\log r)^k dr<\infty.$$

When $|\xi|>1$, putting $q=r|\xi|$ and $\tilde{\xi}=\xi/|\xi|$, we can rewrite I_k as follows:

$$\begin{aligned}I_k&=\int_0^{|\xi|}\frac{|\xi|^{\alpha(x)}(\log q-\log|\xi|)^k\Phi(q/|\xi|)}{q^{1+\alpha(x)}}dq\int_{S^{d-1}}(\cos q\theta\cdot\tilde{\xi}-1)\, s(d\theta)\\&=|\xi|^{\alpha(x)}\sum_{j=1}^k\binom{k}{j}(-\log|\xi|)^{k-j}\int_0^{|\xi|}\int_{S^{d-1}}(\log q)^j\Phi(q/|\xi|)\end{aligned}$$

$$\times \frac{1}{q^{1+\alpha(x)}} (\cos q\theta\cdot\tilde{\xi}-1)\, dq\, s(d\theta)\,.$$

Since

$$\begin{aligned}&\left|\int_0^{|\xi|} \frac{\Phi(q/|\xi|)(\log q)^j}{q^{1+\alpha(x)}}(\cos q\theta\cdot\tilde{\xi}-1)\,dq\right| \\ &\quad\le \frac{1}{2}\int_0^1 \frac{\Phi(q/|\xi|)|\log q|^j}{q^{1+\alpha(x)}}\, q^2\,dq + 2\int_1^\infty \frac{\Phi(q/|\xi|)(\log q)^j}{q^{1+\alpha(x)}}\,dq < \infty\,, \\ &\quad |I_k| \le C(|\xi|^{\alpha(x)}\vee 1)(1+\log(|\xi|\vee 1))^k\,.\end{aligned}$$

Hence, we have

$$|\partial_x^m p_\Phi(x,\xi)| \le C(|\xi|^{\alpha(x)}\vee 1)(1+\log(|\xi|\vee 1))^{|m|}\,.$$

From (1.6), it follows that for any $m=(m_1, m_2, \cdots, m_d)$, $n=(n_1, n_2, \cdots, n_d)$ ($|n|\ge 1$) and $(x, \xi)\in \boldsymbol{R}^d\times\boldsymbol{R}^d$

$$\begin{aligned}&\partial_\xi^n\,\partial_x^m\, p_\Phi(x,\xi) \\ &\quad= \sum_{k=1}^{|m|} a_k(x)\int_0^1 \frac{(\log r)^k\,\Phi(r)}{r^{1+\alpha(x)-|n|}}\,dr \int_{S^d} \exp(ir\theta\cdot\xi)\,(i\theta_1)^{n_1}\cdots(i\theta_d)^{n_d}\,s(d\theta)\,.\end{aligned}$$

Therefore, we will estimate the integral:

$$K_{n,k} = \int_0^1 \frac{(\log r)^k\,\Phi(r)}{r^{1+\alpha(x)-|n|}} \int_{S^{d-1}} \exp(ir\theta\cdot\xi)\,(i\theta_1)^{n_1}(i\theta_2)^{n_2}\cdots(i\theta_d)^{n_d}\,s(d\theta)\,.$$

If $|\xi|\le 1$ and $n\ge 2$, then we immediately see that

$$|K_{n,k}| \le \frac{k!\,s(S^{d-1})}{(|n|-\bar{\alpha})^{k+1}} < \infty\,.$$

When $|n|=1$ and $|\xi|\le 1$, noting

$$\int_{S^{d-1}} (i\theta_j)\,s(d\theta) = 0\,,$$

we have

$$\begin{aligned}|K_{n,k}| &\le \left|\int_{S^d} \{\exp(ir\theta\cdot\xi)-1\}\, i\theta_j\, s(d\theta)\right| \int_0^1 \frac{\Phi(r)\,(-\log r)^k}{r^{\bar{\alpha}}}\,dr \\ &\le s(S^{d-1})\int_0^1 r^{1-\bar{\alpha}}(-\log r)^k\,dr < \infty\,.\end{aligned}$$

Next, we consider the case when $|\xi|>1$. We rewrite $K_{n,k}$ in the form:

$$\begin{aligned}K_{n,k} = {} & |\xi|^{\alpha(x)-|n|}\sum_{j=0}^{k}\binom{k}{j}(-\log|\xi|)^{k-j}\int_0^{|\xi|} \frac{(\log q)^j\,\Phi(q/|\xi|)}{q^{1+\alpha(x)-|n|}}\,dq \\ & \times\int_{S^d} \exp(iq\theta\cdot\tilde{\xi})\,(i\theta_1)^{n_1}(i\theta_2)^{n_2}\cdots(i\theta_d)^{n_d}\,s(d\theta).\end{aligned}$$

We will evaluate the integral

$$\tilde{K}_{n,j}=\int_0^{|\xi|}\frac{(\log q)^j\,\Phi(q/|\xi|)}{q^{1+\alpha(x)-|n|}}\,dq \\ \times\int_{S^d}\exp(iq\theta\cdot\xi)\,(i\theta_1)^{n_1}(i\theta_2)^{n_2}\cdots(i\theta_d)^{n_d}\,s(d\theta)\,.$$

We divide $\tilde{K}_{n,j}$ into two parts $\tilde{K}_{n,j}^{(1)}$ and $\tilde{K}_{n,j}^{(2)}$:

$$\tilde{K}_{n,j}^{(1)}=\int_0^{1}\frac{(\log q)^j\,\Phi(q/|\xi|)}{q^{1+\alpha(x)-|n|}}\,dq \\ \times\int_{S^{d-1}}\exp(iq\theta\cdot\xi)\,(i\theta_1)^{n_1}(i\theta_2)^{n_2}\cdots(i\theta_d)^{n_d}\,s(d\theta)$$

and

$$\tilde{K}_{n,j}^{(2)}=\int_1^{|\xi|}\frac{(\log q)^j\,\Phi(q/|\xi|)}{q^{1+\alpha(x)-|n|}}\,dq \\ \times\int_{S^{d-1}}\exp(iq\theta\cdot\xi)\,(i\theta_1)^{n_1}(i\theta_2)^{n_2}\cdots(i\theta_d)^{n_d}\,s(d\theta)\,.$$

Adopting the same method as in estimating of $K_{n,k}$ for $|\xi|\leq 1$, we can show that

$$|\tilde{K}_{n,j}^{(1)}|<\infty \quad \text{if} \quad |n|\geq 1\,.$$

Now, let $\eta=q\xi$. Then

$$\tilde{K}_{n,j}^{(2)}=\int_1^{|\xi|}\frac{(\log q)^j\,\Phi(q/|\xi|)}{q^{1+\alpha(x)-|n|}}\,\{\partial_\eta^n\int_{S^{d-1}}\exp(i\eta\cdot\theta)\,s(d\theta)|_{\eta=q\xi}\}\,dq\,. \tag{1.7}$$

To estimate $\tilde{K}_{n,j}^{(2)}$, we use the following result of Jones ([5] p.9):

$$\int_{S^{d-1}}\exp(i\eta\cdot\theta)\,s(d\theta)=\omega_d\frac{2^\nu\,\Gamma(\nu+1)}{|\eta|^\nu}J_\nu(|\eta|)\,, \tag{1.8}$$

where $\omega_d=2\sqrt{\pi^d}/\Gamma(d/2)$ and J_ν is the Bessel function of index $\nu=(d-2)/2$. Let

$$F_h(\eta)=(\eta/2)^{-(\nu+h)}J_{\nu+h}(|h|)=\sum_{p=0}^{\infty}\frac{(-1)^p}{2^{2p}\,p!\,\Gamma(\nu+p+h+1)}\,|\eta|^{2p}\,.$$

Taking the $|n|$-th derivative of both the sides of (1.8), we have the equation

$$\partial_\eta^n\int_{S^{d-1}}\exp(i\eta\cdot\theta)\,s(d\theta)=\sum_{l}^{[n/2]}C_l\,\eta_1^{n_1-2l_1}\,\eta_2^{n-2l_2}\cdots\eta_d^{n_d-2l_d}\,F_{\nu+|n|-|l|}(\eta),$$

where $i=(l_1, l_2, \cdots, l_d)$, $n=(n_1, n_2, \cdots, n_d)$, $[n/2]=([n_1/2], [n_2/2], \cdots[n_d/2])$ and $[\cdot]$ is Gauss' symbol, C_l is a constant depending on only l; hence

$$\partial_\eta^n\int_{S^{d-1}}\exp(i\eta\cdot\theta)\,s(d\theta) \tag{1.9}$$

$$= \sum_{l}^{[n/2]} C_l \left(\frac{|\eta|}{2}\right)^{-(\nu+k)} J_{\nu+|n|-|l|}(|\eta|)\, \eta_1^{n_1-2l_1} \eta_2^{n_2-2l_2} \cdots \eta_d^{n_d-2l_d}\,.$$

From (1.7) and (1.9), it follows that

$$\tilde{K}_{n,j}^{(2)} = \int_1^{|\xi|} \sum_{l}^{[n/2]} b_l(\xi) \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+1+\nu+2|l|-|n|}} J_{\nu+|n|-|l|}(q)\, dq\,,$$

where $b_l(\xi)$ denotes a polynomial of ξ. Therefore, we have to estimate the integral

$$\int_1^{|\xi|} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+1+2|l|+\nu-|n|}} J_{\nu+|n|-|l|}(q)\, dq\,. \tag{1.10}$$

Using the asymptotic expansion formula for Bessel functions (cf. [4] p.230), we obtain

$$\begin{aligned}
&\int_1^{|\xi|} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+1+\nu+2|l|-|n|}} J_{|n|+\nu-|l|}(q)\, dq \\
&= \frac{(2/\pi)^{1/2}}{\Gamma(\nu+|n|-|l|+1/2)} \sum_{k=0}^{N-1} \binom{\nu+|n|-|l|+1/2}{k} \frac{\Gamma(\nu+|n|-|l|+k+1/2)}{2^k} \\
&\times \int_1^{|\xi|} \frac{(-1)^{k/2}(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+3/2+k+\nu+2|l|-|n|}} \left\{ \begin{matrix} \cos\{q-(\nu+|n|-|l|)\,\pi/2-\pi/4\} \\ \sin\{q-(\nu+|n|-|l|)\,\pi/2-\pi/4\} \end{matrix} \right\} dq \\
&+ \int_1^{|\xi|} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+3/2+p+\nu+2|l|-|n|}} O(q^{-p-1/2})\, dq\,.
\end{aligned}$$

If N is a sufficiently large integer,

$$\int_1^{\infty} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+3/2+N+\nu+2|l|-|n|}} O(q^{-p-1/2})\, dq < \infty\,.$$

Thus, it is sufficient to prove the boundedness of the integrals:

$$\int_1^{|\xi|} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+s}} \left\{ \begin{matrix} \cos(q+c\pi) \\ \sin(q+c\pi) \end{matrix} \right\} dq \quad (j=0,1,\cdots,k)\,. \tag{1.11}$$

Repeating the integration by parts and using the property $\Phi^{(l)}(1)=0$ $(l=0,1,2,\cdot)$, we see that the integrals of the type (1.11) are represented by a linear combination of the following formula:

$$\begin{aligned}
&\pm(\alpha(x)+s)\cdots(\alpha(x)+s+u-1)\, \frac{1}{|\xi|^v} \int_1^{|\xi|} \frac{\Phi^{(v)}(q/|\xi|)(\log q)^j}{q^{\alpha(x)+s+u}} \left\{ \begin{matrix} \cos(q+c\pi) \\ \sin(q+c\pi) \end{matrix} \right\} dq \\
&+c\cos(q+c\pi)\ (\text{or } c\sin(q+c\pi)) \quad (j, u, v = 0, 1, 2, \cdots)\,.
\end{aligned}$$

Therefore, it is enough to show the boundedness of the integral with the form:

$$\int_1^{|\xi|} \frac{\Phi^{(v)}(q/|\xi|)(\log q)^j}{q^{\underline{\alpha}+v+s+u}}\, dq\,;$$

it is easily verified by the use of the integration by parts. Consequently, we prove the assertions (1) and (2). Next, we show the assertion (3). From (1.8), we see that

$$\begin{aligned}|p_\Phi(x,\xi)| &= |\xi|^{\alpha(x)} w_{\alpha(x)} \int_0^{|\xi|} \frac{\Phi(q/|\xi|)}{q^{1+\alpha(x)}}\, dq \int_{S^{d-1}} \{1-\exp(iq\theta\cdot\hat{\xi})\}\, s(d\theta) \\ &= |\xi|^{\alpha(x)} w_{\alpha(x)}\, \omega_d \int_0^{|\xi|} \frac{\Phi(q/|\xi|)}{q^{1+\alpha(x)}} \{1-\Gamma(\nu+1)\sum_{p=0}^{\infty} \frac{(-1)^p}{2^{2p}\, p!\,\Gamma(\nu+p+1)} q^{2p}\}\, dq \\ &= |\xi|^{\alpha(x)} w_{\alpha(x)}\, \omega_d\, \Gamma(\nu+1) \int_0^{|\xi|} \frac{\Phi(q/|\xi|)}{q^{1+\alpha(x)}} \Big\{\frac{q^2}{2^2\,\Gamma(\nu+2)} \\ &\quad - \sum_{p=2}^{\infty} \frac{(-1)^p q^{2p}}{2^{2p}\, p!\,\Gamma(\nu+p+1)}\Big\}\, dq\,.\end{aligned}$$

The convergence radius of the power series $\sum_{p=2}^{\infty}(-1)^p q^{2p}/2^{2p} p!\,\Gamma(\nu+p+1)$ is infinite and it is equal to zero at $q=0$. Hence, there is a sufficiently small number $q_0>0$ such that, for any $q\in[0, q_0]$,

$$\frac{q^2}{2^2\,\Gamma(\nu+2)} - \sum_{p=2}^{\infty} \frac{(-1)^{p-1} q^{2p}}{2^{2p}\, p!\,\Gamma(\nu+p+1)} > \frac{q^2}{2^3\,\Gamma(\nu+2)}\,.$$

Therefore,

$$|p_\Phi(x,\xi)| \geq |\xi|^{\alpha(x)} w_{\alpha(x)} \frac{\omega_d\,\Gamma(\nu+1)}{2^3\,\Gamma(\nu+2)} \int_0^{q_0} q^{1-\alpha(x)}\, dq \quad \text{for any } \xi \text{ with } |\xi|>R=\frac{q_0}{r_0}\,;$$

hence the assertion (3) is verified. Consequently Theorem 1.1 is proved.

Since L_Φ can be regarded as a pseudo-differential operator of variable order, extending the theory for pseudo-differential operator of constant order, we prepare a general theory for such operators of variable order in the following. In what follows, for simplicity, we let

$$p_{(m)}^{(n)}(x,\xi) = \partial_\xi^n D_x^m\, p(x,\xi)$$

and, in particular,

$$p^{(l)}(x,\xi) = p_{(0)}^{(l)}(x,\xi) \quad \text{and} \quad p_{(l)}(x,\xi) = p_{(l)}^{(0)}(x,\xi)\,.$$

DEFINITION 1.1. *Let ζ be a bounded function on $\boldsymbol{R}^d$.*

(1) *We say that a function $p(x,\xi)$ of $C^\infty(\boldsymbol{R}^d\times\boldsymbol{R}^d)$ is a symbol of the class $S_{\rho,\delta}^{\zeta}(0\leq\delta\leq\rho\leq1,\ \delta<1)$, if for any multi-indices m and n, there exists a constant $C_{m,n}$ such that*

$$|p_{(m)}^{(n)}(x,\xi)| \leq C_{m,n} \langle\xi\rangle^{\zeta(x)+\delta|m|-\rho|n|} \tag{1.12}$$

for any $(x, \xi) \in \boldsymbol{R}^d \times \boldsymbol{R}^d$, *where* $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. *We set*

$$S^{-\infty} = \bigcap_{-\infty<\theta<\infty} S^{\theta}_{\rho,\delta} \quad \text{and} \quad S^{\infty}_{\rho,\delta} = \bigcup_{-\infty<\theta<\infty} S^{\theta}_{\rho,\delta} . \tag{1.13}$$

(2) *We say that a linear operator* $P: \mathcal{S}(\boldsymbol{R}^d)\to\mathcal{S}(\boldsymbol{R}^d)$ *is a pseudo-differential operator with symbol* $p(x, \xi)$ *of class* $S_{\rho,\delta}$, *if Pu can be represented by*

$$Pu(x) = \int \exp(ix\cdot\xi)\, p(x, \xi)\, \hat{u}(\xi)\, \check{d}\xi \quad \text{for} \quad u\in\mathcal{S}(\boldsymbol{R}^d)\,, \tag{1.14}$$

where $\check{d}\xi=(1/2\pi)^d\, d\xi$, *and* $\hat{u}$ *is the Fourier transform of* u. *In this case, we write* $P=p(x, D_x)\in S^{\zeta}_{\rho,\delta}$, *and we also denote the symbol* $p(x, \xi)$ *of* P *by* $\sigma(P)(x, \xi)$. *Moreover the semi-norms* $|p|^{\zeta}_{k}$ $(k=1, 2, \cdots)$ *are defined by*

$$|p|^{\zeta}_{k} = \max_{|m+n|\le k}\ \sup_{(x,\xi)\in \boldsymbol{R}\times\boldsymbol{R}^d} \{|p^{(n)}_{(m)}(x, \xi)|\langle\xi\rangle^{-(\zeta(x)+\delta|m|-\rho|n|)}\} .$$

DEFINITION 1.2. (1) *We say that a function* $a(\eta, y)$ *of* $C^\infty(\boldsymbol{R}^d\times\boldsymbol{R}^d)$ *belongs to the class* $\mathcal{A}^{\theta}_{\delta,\kappa}$ $(-\infty<\theta<\infty,\ 0\le\delta<1,\ 0\le\kappa)$, *if for any multi-indices m and n, there exists a canstant* $C_{m,n}$ *such that*

$$|\partial^m_\eta\, \partial^n_y\, a(\eta, y)| \le C_{m,n}\langle\eta\rangle^{\theta+\delta|n|}\langle y\rangle^{\kappa} .$$

We set

$$\mathcal{A} = \bigcup_{0\le\delta<1}\ \bigcup_{-\infty<\theta<\infty}\ \bigcup_{\kappa\ge0} \mathcal{A}^{\theta}_{\delta,\kappa} .$$

(3) *For an element* $a(\eta, y)$ *of* $\mathcal{A}$, *we define the oscillatory integral* $Os[e^{-iy\cdot\eta}\, a]$ *by*

$$\begin{aligned} Os[e^{iy\cdot\eta}\, a] &= Os\text{-}\iint \exp(-i\eta\cdot y)\, a(\eta, y)\, \check{d}\eta\, dy \\ &= \lim_{\varepsilon\to0} \iint \exp(-i\eta\cdot y)\, \chi(\varepsilon\eta, \varepsilon y)\, a(\eta, y)\, \check{d}\eta\, dy\,, \end{aligned}$$

where $\chi\in\mathcal{S}(\boldsymbol{R}^d\times\boldsymbol{R}^d)$ *and* $\chi(0, 0)=1$.

Theorem 1.2. *Assume that* $0\le\delta<\rho\le1$.
(1) *Let* $\zeta_j(j=1, 2)$ *be a bounded function on* $\boldsymbol{R}^d$ *and* $P_j=p_j(x, D_x)\in S^{\zeta_j}_{\rho,\delta}(j=1, 2)$. *Then* $P=P_1\cdot P_2$ *belongs to* $S^{\zeta_1+\zeta_2}_{\rho,\delta}$ *with symbol* $p(x, \xi)$:

$$p(x, \xi) = Os\text{-}\iint \exp(-i\eta\cdot y)\, p_1(x, \xi+\eta)\, p_2(x+y, \xi)\, \check{d}\eta\, dy \tag{1.15}$$

and it has the asymptotic expansion formula:

$$p(x, \xi) - \sum_{|l|<N} \frac{1}{l!}\, p_1^{(l)}(x, \xi)\, p_{2(l)}(x, \xi) \in S^{\zeta_1+\zeta_2-N(\rho-\delta)}_{\rho,\delta} \tag{1.16}$$

for any integer $N\ge1$.
(2) *Let* $P=p(x, D_x)\in S^{\zeta}_{\rho,\delta}$. *We define* P^* *by*

$$(Pu, v) = (u, P^*v) \quad \text{for} \quad u, v\in\mathcal{S}(\boldsymbol{R}^d) .$$

Then $P^*(x, D_x)$ *is a pseudo-differential operator of the class* $S^{\zeta}_{\rho,\delta}$ *and its symbol* $p^*(x, \xi)$ *is given by*

$$p^*(x, \xi) = Os-\iint \exp(-i\eta\cdot y)\overline{p(x+y, \xi+\eta)}\, \check{d}\eta\, dy\,,$$

and it has the asymptotic expansion formula:

$$(1.17)\qquad p^*(x, \xi) - \sum_{|l|<N} \frac{(-1)^{|l|}}{l!} \overline{p^{(l)}_{(l)}(x, \xi)} \in S^{\zeta-N(\rho-\delta)}_{\rho,\delta}$$

for any integer $N \geq 1$.

Proof. By Theorem 3.1 in Chap. 2 of [7], we obtain that

$$(1.18)\qquad p(x, \xi) - \sum_{|l|<N} \frac{1}{l!} p_1^{(l)}(x, \xi)\, p_{2(l)}(x, \xi) \in S^{\zeta_1+\zeta_2-N(\rho-\delta)}_{\rho,\delta}\,.$$

Moreover, noting that, when $|l|=0$, $p_1(x, \xi)\, p_2(x, \xi)$ is the symbol with variable order $\zeta_1(x)+\zeta_2(x)$ and, when $|l|\geqq 1$, the order of $p_1^{(l)}(x, \xi)\, p_{2(l)}(x, \xi)$ is $\zeta_1(x)+\zeta_2(x)-|l|(\rho-\delta)$, we have

$$p \in S^{\zeta_1+\zeta_2}_{\rho,\delta}\,.$$

Therefore the assertion (1) holds. In the same way as the above, we can verify the assertion (2).

Definition 1.3. *We say that a sequence* $\{p_k\}_{k\geq 1}$ *of* $S^{\zeta}_{\rho,\delta}$ *converges weakly to* $p \in S^{\zeta}_{\rho,\delta}$ *as* $k\to\infty$ *if, for each* $h\geq 1$, *there is a constant* M_h *such that* $|p|^{\zeta}_h < M_h$, *and, for any multi-indices m and n, we have*

$$(1.19)\qquad p^{(n)}_{k(m)} \to p^{(n)}_{(m)} \quad as\ k\to\infty \quad on\ \boldsymbol{R}^d\times\boldsymbol{R}^d\,.$$

Definition 1.4. *Let* $\boldsymbol{I}$ *be an interval of* $\boldsymbol{R}^1$ *and* $\boldsymbol{V}$ *be a Fréchet space. For a mapping* ϕ: $\boldsymbol{I}\to\phi(t)\in\boldsymbol{V}$, *we write* $\phi\in\mathscr{B}^{|m|}(\boldsymbol{I}, \boldsymbol{V})$ *if* ϕ *is* $|m|$*-times continuously differentiable in* $\boldsymbol{I}$ *in the topology of* $\boldsymbol{V}$ *and each derivative* $D^l_t\phi$ *is bounded* $(|l|\leq|m|)$.

From Theorem 1.1, we see that L_Φ is a pseudo-differential operator of the class $S^{\alpha}_{1,\delta}$, where δ is any positive number less than 1. Now we will construct a fundamental solution in the sense of pseudo-differential operators to the initial-value peoblem for the evolution equation with respect to L_Φ:

$$(1.20)\qquad \begin{aligned} &\{\partial_t - L_\Phi\}\, u = f \quad \text{in} \quad (0, T)\,, \\ &\lim_{t\to 0} u(t) = \phi \quad \text{in} \quad L_2(\boldsymbol{R}^d)\,. \end{aligned}$$

By virtue of Theorems 1.1 and 1.2, we can adapt the argument used in the proof of Theorem 2.1 in Section 2 of Chap. 8 in [8] to the proof of the next theorem.

Theerem 1.3. *There exists a fundamental solution $E(\cdot)$ to the initial-value problem for the evolution equation* (1.20) *such that it satisfies the following conditions: for each $T>0$,*

(1)

$$E(t)=e(t,x,D_x)\in\mathcal{B}^0((0,T];S^0_{1,\delta})\cap\mathcal{B}^1((0,T];S^\alpha_{1,\delta}) \tag{1.21}$$

and, for any $t_0\in(0,T)$,

$$E(t)\in\mathcal{B}^1([t_0,T];S^{-\infty})\equiv\bigcap_{-\infty<\kappa<\infty}\mathcal{B}^1([t_0,T];S^\kappa_{1,\delta}); \tag{1.22}$$

(2) *for any $t\in(0,T)$,*

$$(\partial_t-L_\Phi)E(t)=0; \tag{1.23}$$

(3)

$$e(t,x,\xi)\to 1 \text{ in } S^0_{1,\delta} \text{ weakly as } t\to 0; \tag{1.24}$$

(4)

$$r_0(t,x,\xi)\equiv e(t,x,\xi)-\exp(tp_\Phi(x,\xi))\to 0 \text{ in } S^{-(1-\delta)}_{1,\delta} \text{ weakly as } t\to 0 \tag{1.25}$$

and

$$r_0(t,x,\xi)/t\in\mathcal{B}^0((0,T];S^{\alpha-(1-\delta)}_{1,\delta}). \tag{1.26}$$

Proof. Let $e_0(t,x,\xi)=\exp(tp_\Phi(x,\xi))$. Then this function satisfies the equation:

$$\{\partial_t-p_\Phi(x,\xi)\}\,e_0(t,x,\xi)=0 \qquad e_0(0,x,\xi)=1. \tag{1.27}$$

Furthermore, for any multi-indices m and n,

$$\partial_\xi^n D_x^m e_0(t,x,\xi)=\sum_{k=1}^{|m+n|}t^k((p_\Phi)_k)^{(n)}_{(m)}(x,\xi)\,e_0(t,x,\xi), \tag{1.28}$$

where

$$((p_\Phi)_k)^{(n)}_{(m)}=\sum C^{n^1,n^2,\dots,n^k}_{m^1,m^2,\dots,m^k}\,p_{\Phi}{}^{(n^1)}_{(m^1)}(x,\xi)\,p_{\Phi}{}^{(n^2)}_{(m^2)}(x,\xi)\cdots p_{\Phi}{}^{(n^k)}_{(m^k)}(x,\xi)$$

and the summation is taken over multi-indices m^j and n^j ($j=1,2,\dots,k$) such that $\sum_{j=1}^k m^j=m$, $\sum_{j=1}^k n^j=n$ and $C^{n^1,n^2,\dots,n^k}_{m^1,m^1,\dots,m^k}$ denotes a constant depending only on m^j and n^j ($j=1,2,\dots,k$). From (1.3), there exists a constant $C_1>0$ such that for any $(x,\xi)\in\boldsymbol{R}^d\times\boldsymbol{R}^d$

$$|p_\Phi(x,\xi)|>C_0\langle\xi\rangle^{\alpha(x)}-C_1$$

Therefore, putting $C=\exp(-TC_1)$, we have, for any $(t, x, \xi)\in(0, T]\times\boldsymbol{R}^d\times\boldsymbol{R}^d$,

$$e_0(t, x, \xi)\leq C\exp(-tC_0\langle\xi\rangle^{\alpha(x)}) . \tag{1.29}$$

Since $(t\langle\xi\rangle^{\alpha(x)})^k\exp(-tC_0\langle\xi\rangle^{\alpha(x)})$ is bounded in (t, x, ξ) of $(0, T]\times\boldsymbol{R}^d\times\boldsymbol{R}^d$, there exists a constant $C'_{m,n}$ such that

$$|\partial_\xi^n D_x^m e_0(t, x, \xi)|\leq C'_{m,n}\langle\xi\rangle^{-|n|+\delta|m|} \tag{1.30}$$

for any $(t, x, \xi)\in(0, T]\times\boldsymbol{R}^d\times\boldsymbol{R}^d$. Hence

$$\begin{aligned} &|\partial_\xi^n D_x^m \partial_t e_0(t, x, \xi)| \\ &\leq \sum_{k=0}^{|m+n|} C_{0,m,n,k}\, t^k\langle\xi\rangle^{(k+1)\alpha(x)-|n|+\delta|m|}\exp(-tC_0\langle\xi\rangle^{\alpha(x)}) \end{aligned} \tag{1.31}$$

for any $(t, x, \xi)\in(0, T]\times\boldsymbol{R}^d\times\boldsymbol{R}^d$, where $C_{0,m,n,k}$ is a constant depending only on m, n, and k. These estimates (1.30) and (1.31) yield that

$$e_0\in\mathscr{B}^0((0, T]; \boldsymbol{S}^0_{1,\delta})\cap\mathscr{B}^1((0, T]; \boldsymbol{S}^\alpha_{1,\delta}) ,$$

and it is clear that $e_0\to 0$ weakly as $t\to 0$.
We can define $\{e_j(t)\}_{j=1}^\infty$ and $\{q_j(t)\}_{j=1}^\infty$ $(0\leq t\leq T)$ inductively by

$$q_j(t)=\sum_{k=0}^{j-1}\sum_{|x|+k=j}\frac{1}{n!}p_\Phi^{(n)}(x, \xi)\, e_{k(n)}(t, x, \xi) \quad (j\geq 1) \tag{1.32}$$

and

$$\begin{aligned} \{\partial_t-p_\Phi(x, \xi)\}\, e_j(t, x, \xi) &= q_j(t, x, \xi) \\ e_j(0, x, \xi) &= 0 \quad (j\geq 1) . \end{aligned} \tag{1.33}$$

Then the solution $e_j(t, x, \xi)$ of (1.33) has the form:

$$e_j(t, x, \xi)=e_0(t, x, \xi)\int_0^t\frac{q_j(s, x, \xi)}{e_0(s, x, \xi)}ds . \tag{1.34}$$

We will show the following estimate:

$$|e_{j(m)}^{(n)}(t, x, \xi)|\leq\begin{cases} C_{j,m,n}\langle\xi\rangle^{-j(1-\delta)-|n|+\delta|m|} \\ C'_{j,m,n}\, t\langle\xi\rangle^{\alpha(x)-j(1-\delta)-|n|+\delta|m|}\end{cases} \tag{1.35}$$

for any $(t, x, \xi)\in(0, T]\times\boldsymbol{R}^d\times\boldsymbol{R}^d (j\geq 1)$, where $C_{j,m,n}$ and $C'_{j,m,n}$ are constants depending only on j, m and n. In fact, assume that the inequality

$$\begin{aligned} &\left|\left(\frac{q_j(t, x, \xi)}{e_0(t, x, \xi)}\right)_{(m)}^{(n)}\right| \\ &\leq\tilde{C}_{j,m,n}\langle\xi\rangle^{\alpha(x)}\sum_{k=1}^{2j-1}(t\langle\xi\rangle^{\alpha(x)})^k\langle\xi\rangle^{-j(1-\delta)-|n|+\delta|m|} \\ &((t, x, \xi)\in(0, T]\times\boldsymbol{R}^d\times\boldsymbol{R}^d) \end{aligned} \tag{1.36}$$

holds for $j \leq j_0 - 1$. Then, combining (1.34) with (1.36), we have

$$(1.37)\quad \left|\left(\frac{e_{j_0-1}(t, x, \xi)}{e_0(t, x, \xi)}\right)^{(n)}_{(m)}\right| \leq C_{j_0-1,m,n} \sum_{k=2}^{2(j_0-1)} (t\langle\xi\rangle^{\alpha(x)})^k \langle\xi\rangle^{-(j_0-1)(1-\delta)-|n|+\delta|m|}$$

for any $(t, x, \xi) \in (0, T] \times \boldsymbol{R}^d \times \boldsymbol{R}^d$. Note that

$$(1.38)\quad \begin{aligned} &\left|\left(\frac{q_{j_0}(s, x, \xi)}{e_0(s, x, \xi)}\right)^{(n)}_{(m)}\right| \\ &\leq \sum_{|l|=1} \left|\left(\frac{p_\Phi^{(l)}(x, \xi)\, e_{j_0-1(l)}(t, x, \xi)}{e_0(t, x, \xi)}\right)^{(n)}_{(m)}\right| + \tilde{C}_{j_0,m,n} \sum_{|l|=1} \left|\left(\frac{(q_{j_0-1}(t, x, \xi))^{(l)}_{(l)}}{e_0(t, x, \xi)}\right)^{(n)}_{(m)}\right| \\ &\leq \sum_{|l|=1} \left|\left(p_\Phi^{(l)}(x, \xi) \left(\frac{e_{j_0-1}(t, x, \xi)}{e_0(t, x, \xi)}\right)_{(l)}\right)^{(n)}_{(m)}\right| \\ &+ \sum_{|l|=1} \left|\left(tp_\Phi^{(l)}(x, \xi)\, p_{\Phi(l)}(x, \xi) \frac{e_{j_0-1}(t, x, \xi)}{e_0(t, x, \xi)}\right)^{(n)}_{(m)}\right| \\ &+ \tilde{C}_{j_0,m,n} \sum_{|l|=1} \left|\left(tp(x, \xi)_{\Phi(l)} \frac{q_{j_0-1}(t, x, \xi)}{e_0(t, x, \xi)}\right)^{(n+1)}_{(m)}\right| \\ &+ \tilde{C}_{j_0,m,n} \left|\left(\frac{q_{j_0-1}(t, x, \xi)}{e_0(t, x, \xi)}\right)^{(n+1)}_{(m+1)}\right| \end{aligned}$$

for any $(t, x, \xi) \in (0, T] \times \boldsymbol{R}^d \times \boldsymbol{R}^d$. Then, from (1.34), we see that the inequality (1.36) holds for $j = j_0$. Thus, by induction, it holds for any $j \geq 0$. Hence, from (1.29), (1.34) and (1.38) for $j = j_0$, we see that the first inequality of (1.35) holds when $j = j_0$. Moreover, writing $(t\langle\xi\rangle^{\alpha(x)})^k = (t\langle\xi\rangle^{\alpha(x)})(t\langle\xi\rangle^{\alpha(x)})^{k-1}$ and using a similar argument to the above, we obtain the second inequality of (1.35). This means that

$$(1.39)\quad e_j(t, x, \xi) \in \mathscr{B}^0([0, T]; S_{1,\delta}^{-j(1-\delta)}) \cap \mathscr{B}^1([0, T]; S_{1,\delta}^{\alpha-j(1-\delta)}).$$

Next, put $E_j(t) = e_j(t, x, D_x)$ $(j \geq 0)$. Then, by Theorem 1.2, we can write

$$(1.40)\quad \begin{aligned} &\sigma(L_\Phi E_j(t))(x, \xi) \\ &= p_\Phi(x, \xi)\, e_j(t, x, \xi) + \sum_{0<|l|<N-j} \frac{1}{l!} p_\Phi^{(l)}(x, \xi)\, e_{j(l)}(t, x, \xi) \\ &+ r_{N,j}(t, x, \xi) \quad (j = 0, 1, 2, \cdots N-1). \end{aligned}$$

From Theorem 1.1 and 1.2, the first inequality of (1.35) and (1.40), we find that

$$(1.41)\quad r_{N,j}(t) \in \mathscr{B}^0((0, T]; S_{1,\delta}^{\alpha-N(1-\delta)}) \quad j = 1, 2, \cdots.$$

Similarly, replacing the first inequality of (1.35) by the second one of (1.35), we have

$$r_{N,j}(t)/t \in \mathscr{B}^0((0, T]; S^{2\alpha-N(1-\delta)}_{1,\delta}) \quad j = 1, 2, \cdots. \tag{1.42}$$

From the above discussion, we have a sequence $\{e_j\}_{j=0}^{\infty}$ of symbols satisfying $e_j \in S^{j(1-\delta)}_{1,\delta}$. Therefore, we can construct an operator

$$\tilde{E}(t) = \tilde{e}(t, x, D_x) \in S^0_{1,\delta} \tag{1.43}$$

with an analogous argument used in Theorem A.1 of [8] (p.238–239). Indeed, let ψ be a function of $C_0^\infty((0, \infty))$ with

$$0 \leq \psi(t) \leq 1, \quad \psi(t) = 0 \ (0 < t \leq 1) \quad \text{and} \quad \psi(t) = 1 \quad (t \geq 2).$$

Putting $\psi_j(\xi) = \psi(\varepsilon_j |\xi|)$ $(j = 1, 2, \cdots)$ for any sequence $\{\varepsilon_j\}_{j\geq 1}$ of positive numbers, we have the estimate

$$|\partial_\xi^n D_x^m(e_j(t, x, \xi)\, \psi_j(\xi))| \leq \begin{cases} C_{j,m,n} \langle\xi\rangle^{-j(1-\delta)+\delta|m|-|n|} \\ C_{j,m,n}\, \varepsilon_j \langle\xi\rangle^{-j(1-\delta)+\delta|m|-|n|+1} \end{cases}$$

for any $(t, x, \xi) \in (0, T] \times R^d \times R^d$ and any multi-indices m and n. Now, we inductively choose the sequence $\{\varepsilon_j\}_{j\geq 1}$ satisfying

$$0 < \varepsilon_j \leq 2^{-j} (\max_{|m+n|\leq j} (C_{j,m,n}))^{-1}$$

and

$$1 > \varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \cdots \to 0,$$

and define the symbol $\tilde{e}$ by

$$\tilde{e}(t, x, \xi) = e_0(t, x, \xi) + \sum_{j=1}^{\infty} e_j(t, x, \xi)\, \psi_j(\xi)$$

for any $(t, x, \xi) \in (0, T] \times R^d \times R^d$. Then the symbol $\tilde{e}$ satisfies the following properties:

(i)

$$\tilde{e}(t, x, \xi) - \sum_{j=0}^{N-1} e_j(t, x, \xi) \in \mathscr{B}^0((0, T]; S^{-N(1-\delta)}_{1,\delta}) \cap \mathscr{B}^1((0, T]; S^{\alpha-N(1-\delta)}_{1,\delta}), \tag{1.44}$$

(ii)

$$\tilde{e}(t) \to 1 \text{ and } \tilde{e}(t) - \sum_{j=0}^{N-1} e_j(t) \to 0 \text{ weakly in } S^0_{1,\delta} \tag{1.45}$$

as $t \to 0$ for any $N \geq 1$ (see [8] in detail). Let $R(t) = (\partial_t - L_\Phi)\tilde{E}(t)$. For any positive integer N, we rewrite $R(t)$ in the form

$$R(t) = (\partial_t - L_\Phi)(\sum_{j=0}^{N-1} E_j(t)) + (\partial_t - L_\Phi)(\tilde{E}(t) - \sum_{j=0}^{N-1} E_j(t)). \tag{1.46}$$

Then from Theorem 1.2 and (1.44), we see that, for any positive integer N,

$$(\partial_t - L_\Phi)\,(\tilde{E}(t) - \sum_{j=0}^{N-1} E_j(t)) \in \mathscr{B}^0((0, T]; S_{1,\delta}^{\alpha-N(1-\delta)})\,. \tag{1.47}$$

Moreover, it follows from (1.32), (1.33) and (1.40) that

$$\begin{aligned} \sigma((\partial_t - L_\Phi)\,(\sum_{j=0}^{N-1} E_j(t)))\,(x, \xi) &= \sum_{j=0}^{N-1} (\partial_t - p_\Phi(x, \xi))\, e_j(t, x, \xi) \\ &\quad - \sum_{j=1}^{N-1} \sum_{|l|+k=j, k<j} \frac{1}{l!} p_\Phi^{(l)}(x, \xi)\, e_{k(l)}(t, x, \xi) - \sum_{i=0}^{N-1} r_{N,j}(t, x, \xi) \\ &= -\sum_{j=0}^{N-1} r_{N,j}(t, x, \xi) \end{aligned} \tag{1.48}$$

for any positive integer N and $(t, x, \xi) \in (0, T] \times \boldsymbol{R}^d \times \boldsymbol{R}^d$. Therefore, (1.41) and (1.42) yield that

$$\begin{aligned} (\partial_t - L_\Phi)\,(\sum_{j=0}^{N-1} E_j(t)) &\in \mathscr{B}^0((0, T]; S_{1,\delta}^{\alpha-N(1-\delta)}) \\ &\quad \cap \mathscr{B}^1((0, T]; S_{1,\delta}^{2\alpha-N(1-\delta)})\,. \end{aligned} \tag{1.49}$$

Hence, it follows from (1.47) and (1.49) that

$$R(t) \in \mathscr{B}^0((0, T]; S^{-\infty})\,. \tag{1.50}$$

Now, let $\{W_\nu(t)\}_{\nu \geq 1}$ be a sequence of operators defined by

$$W_1(t) = -R(t)$$

and

$$W_\nu(t) = \int_0^t W_1(t-s)\, W_{\nu-1}(s)\, ds\,.$$

Then, using the same method as in the proof of Theorem 2.1 in Chap. 8 of [8], we see that

$$\sigma(W(t))\,(x, \xi) = \sum_{\nu=1}^{\infty} \sigma(W_\nu(t))\,(x, \xi)$$

converges in the topology of $\mathscr{B}^0((0, T]; S^{-\infty})$. If we set

$$E(t) = \tilde{E}(t) + \int_0^t \tilde{E}(t-s)\, W(s)\, ds\,, \tag{1.51}$$

then we have

$$(\partial_t - L_\Phi)\, E(t) = R(t) + W(t) + \int_0^t R(t-s)\, W(s)\, ds = 0$$

for any $t \in (0, T]$. We get (1.21) from (1.44) and (1.50). The relations (1.24) and (1.25) follow from (1.45) and (1.51). Moreover, with the same argument as in Theorem 2.1 in Chap. 8 of [8], we see that, for any positive number $t_0 \in (0, T]$,

$$e_j(t) \in \mathscr{B}^1([t_0, T]; S^{-\infty}) \quad j = 1, 2, \cdots .$$

The proof of Theorem 1.3 is complete.

Let $H_s(-\infty < s < \infty)$ be the Sobolev space with the norm $\|\cdot\|_s$ (see [7] p.116 for the definition). Then, using the L_2-boundedness theorem (cf. [7], Chap. 2, Theorem 4.1), we have

Theorem 1.4. *Let ζ be a bounded function on R^d and $P = p(x, D_x) \in S^{\zeta}_{\rho,\delta}(\delta < \rho)$. Then, for any $s \in R$, P defines a continuous mapping $P: H_{s+\bar{\zeta}} \to H_s$ and there exist an integer k and a constant C such that*

$$\text{(1.52)} \qquad \|Pu\|_s \leq C\,|p|^{\zeta}_k\,\|u\|_{s+\bar{\zeta}} \quad \textit{for} \quad u \in H_{s+\bar{\zeta}} .$$

It is well-known that if κ and s are real numbers and $p_j \to p$ in $S^{\kappa}_{\rho,\delta}$ weakly as $j \to \infty$, then

$$\text{(1.53)} \qquad p_j(X, D_x)\,u \to p(X, D_x)\,u \text{ in } H_s \text{ as } j \to \infty \quad \text{for} \quad u \in H_{s+\kappa}$$

(cf. [7] p.157). Immediately, from Theorem 1.3, Theorem 1.4, and (1.53), we get the following theorem.

Theorem 1.5. *Let $E(\cdot)$ be the same one as in Theorem* 1.3 *and let s be any real number. Then, for $\phi \in H_s$, $u(\cdot) = E(\cdot)\phi$ belongs to $\mathscr{B}^0([0, T]; H_s) \cap \mathscr{B}^1((0, T]; H_{s-\bar{\alpha}})$ for each $T > 0$ and is a solution to the initial-value problem for the evolution equation* (1.20).

Now, we state the main theorems in this paper.

Theorem 1.6. *Let $e(t, x, \xi)$ be the symbol of the fundamental solution $E(t)$ given by Theorem* 1.3. *Then, the function defied by*

$$\text{(1.54)} \qquad K(t, x, y) = \int \exp(i(x-y)\cdot\xi)\, e(t, x, \xi)\, d\!\!{}^{-}\xi$$

($t \in (0, \infty)$, $x, y \in R^d$) is a transition density of the Markov process X_Φ.

Proof. Let $\phi \in C_0^\infty(R^d)$ and $u(t, x) = E(t)\,\phi(x)$. Then $u(t)$, $\partial_t u(t)$ and $L_\Phi \phi$ belong to $\mathcal{S}$. From Theorem 1.3, Theorem 1.5 and (1.53), we see that, for any $s \in R$,

$$\lim_{t \to 0} u(t) = \phi \quad \text{in} \quad H_s ,$$

$$\partial_t u(t)|_{t=0} = \lim_{t \to 0} \partial_t u(t) = \lim_{t \to 0} L_\Phi u(t) = L_\Phi \phi \quad \text{in} \quad H_{s-\bar{\alpha}} .$$

Noting that for any multi-index m and any real number $s>|m|+d/2$

$$|\partial_x^m u(t,x)-\partial_x^m \phi(x)| \le \left|\int \langle\xi\rangle^{-2(s-|m|)}\, \check{d}\xi\right|^{1/2} \|u(t)-\phi\|_s ,$$

we have $\partial_x^m u(t)\to\partial_x^m\phi$ uniformly on $\boldsymbol{R}^d$ as $t\to 0$. Similarly, we have $\partial_t u(t)\to L_\Phi\phi$ uniformly on $\boldsymbol{R}^d$ as $t\to 0$. These facts imply that $u\in \boldsymbol{C}_b^{1,2}([0,T]\times\boldsymbol{R}^d)$. Put $f(s,x)=u(t-s,x)$ $(0\le s\le t)$. Then, $f\in\boldsymbol{C}_b^{1,2}([0,t]\times\boldsymbol{R}^d)$ and f satisfies

$$\begin{cases}\partial_s f(s,x) = -L_\Phi f(s,x) & (0\le s<t)\\ f(t,x)=\phi(x)\,. \end{cases} \tag{1.55}$$

Let $\boldsymbol{P}_x$ be a solution to the martingale problem for L_Φ starting at x. Then

$$f(t,X_t)-f(0,x) = \int_0^t \{\partial_s f(s,X_s) + L_\Phi f(s,X_s)\}\, ds + \text{a } \boldsymbol{P}_x\text{-martingale.} \tag{1.56}$$

Using (1.55) and (1.56), we have

$$u(t,x)=\boldsymbol{E}_x[\phi(X_t)]\,. \tag{1.57}$$

On the other hand, from Theorems 1.3 and 3.3 in Chap. 2 of [7], it follows that

$$u(t,x)=\int_{\boldsymbol{R}^d} K(t,x,y)\,\phi(y)\,dy \quad \text{for} \quad t>0 \text{ and } x\in\boldsymbol{R}^d\,. \tag{1.58}$$

Since (1.57) and (1.58) hold for any $\phi\in\boldsymbol{C}_0^\infty(\boldsymbol{R}^d)$, we see that the function $K(t,x,y)$ $(t>0,\ x,y\in\boldsymbol{R}^d)$ is a transition density of the Markov process X_Φ.

Theorem 1.7. *Let $\{\boldsymbol{P}(t,x,\Gamma);\ t\ge 0,\ x\in\boldsymbol{R}^d,\ \Gamma\in\mathcal{B}(\boldsymbol{R}^d)\}$ be the transition function of the stable-like process with exponent $\alpha(x)$. Then, for each $(t,x)\in(0,\infty)\times\boldsymbol{R}^d$, $\boldsymbol{P}(t,x,dy)$ has a density with respect to Legesgue measure.*

Proof. We first show that the short time behavior of the process X coincides with that of the process X_Φ. Using polar decomposition, we rewrite ν and ν_Φ in the following forms:

$$\nu(x;dy) = 1_{(0,r_0]}(r)\,\frac{w_{\alpha(x)}}{r^{1+\alpha(x)}}\,dr\, s(d\theta) + 1_{(r_0,\infty)}(r)\,\frac{w_{\alpha(x)}}{r^{1+\alpha(x)}}\,dr\, s(d\theta)$$

and

$$\nu_\Phi(x,dy) = 1_{(0,r_0]}(r)\,\frac{w_{\alpha(x)}}{r^{1+\alpha(x)}}\,dr\, s(d\theta) + 1_{(r_0,\infty)}(r)\,\frac{w_{\alpha(x)}\,\Phi(r)}{r^{1+\alpha(x)}}\,dr\, s(d\theta)\,,$$

where r_0 is the same constant as in the definition of the cut-off function Φ. We set

$$G_1(x;\lambda)=\int_\lambda^{r_0}\frac{w_{\alpha(x)}}{r^{1+\alpha(x)}}\,dr\quad(\lambda>0)\,,$$

$$G_2(x;\lambda)=\int_\lambda^\infty g(x)^{-1}\frac{w_{\alpha(x)}}{r^{1+\alpha(x)}}\,dr\quad(\lambda>r_0)$$

and

$$G_{\Phi,2}(x;\lambda)=\int_\lambda^\infty g_\Phi(x)^{-1}\frac{w_{\alpha(x)}\,\Phi(r)}{r^{1+\alpha(x)}}\,dr\quad(\lambda>r_0)\,,$$

where $g(x)=\int_{r_0}^\infty w_{\alpha(x)}/r^{1+\alpha(x)}\,dr$ and $g_\Phi(x)=\int_{r_0}^\infty w_{\alpha(x)}\,\Phi(r)/r^{1+\alpha(x)}\,dr$. In the following, $\hat{G}(x,\cdot)$ denotes the right continuous inverse function of $G(x,\cdot)$, that is,

$$\hat{G}(x,l)=\inf\{\lambda>0\colon G(x,\lambda)\le l\}\,.$$

Let

$$\boldsymbol{U}_1=(0,\infty)\times\boldsymbol{S}^{d-1},\quad \boldsymbol{U}_2=(-1,0)\times\boldsymbol{S}^{d-1}\quad\text{and}\quad \boldsymbol{U}=\boldsymbol{U}_1\cup\boldsymbol{U}_2\,.$$

We denote a generic element of $\boldsymbol{U}$ as $u=(l,\theta)$. Now, let $\{p(t)\}$ be a stationary Poisson point process defined on a probability space $(\Omega,\mathcal{F},\boldsymbol{P})$ with values in $\boldsymbol{U}$ and the characteristic measure $n(du)=dl\,s(d\theta)$. $N_p(ds\times du)$ denotes the counting measure defined by $\{p(t)\}$ and $\tilde{N}_p(ds\times du)=N_p(ds\times du)-dsn(du)$. If we set $a(x,u)=a(x,l)=\hat{G}_1(x,l)$, $b(x,u)=b(x,l)=g(x)\,\hat{G}_2(x,l+1)$ and $b_\Phi(x,u)=b_\Phi(x,l)=g_\Phi(x)\,\hat{G}_{\Phi,2}(x,l+1)$, then the processes X and X_Φ starting at x are respectively realized as solutions of the stochastic differential equations with jumps:

$$X(t)=x+\int_0^t\int_{U_1}a(X(s_-),u)\,\tilde{N}_p(ds\times du)$$
$$+\int_0^t\int_{U_2}b(X(s_-),u)\,N_p(ds\times du)\,,$$
$$X_\Phi(t)=x+\int_0^t\int_{U_1}a(X_\Phi(s_-),u)\,\tilde{N}_p(ds\times du)$$
$$+\int_0^t\int_{U_2}b_\Phi(X_\Phi(s_-),u)\,N_p(ds\times du)\,.$$

Since the coefficient $a(x,u)$ satisfies the Lipschitz condition with respect to the measure $n(du)$ (see [12]), they have unique solutions in the pathwise sense. For specifying the starting point u of the processes, we denote them by $X(t,x)$ and $X_\Phi(t,x)$, respectively. Let $\sigma=\inf\{t>0\colon N_p((0,t]\times U_2)=1\}$. Then for $t<\sigma$

$$X(t)=x+\int_0^t\int_{U_1}a(X_\Phi(s_-),u)\,\tilde{N}_p(ds\times du)$$

and

$$X_\Phi(t)=x+\int_0^t\int_{U_1}a(X_\Phi(s^-),u)\,\tilde{N}_p(ds\times du)\,,$$

because, for $A_1\subset\boldsymbol{U}_1$ and $A_2\subset\boldsymbol{U}_2$, the Poisson processes $N_p((0,t]\times A_1)$ and

$N_p((0, t]\times A_2)$ almost surely do not jump simultaneously. Therefore

$$P(1_{\{t<\sigma\}}X(t, x) = 1_{\{t<\sigma\}}X_\Phi(t, x),\, t\geq 0) = 1\,.$$

We next show the absolute continuity of the transition probability of X. Let $\sigma_0=0$ and

$$\sigma_n = \inf\,\{t>\sigma_{n-1};\, N_p(\{t\}\times U_2) = 1\} \quad (n = 1, 2, \cdots)\,.$$

Then $\sigma_1=\sigma$ and $P(\sigma_n=t)=0$ for each $t>0$. Therefore, for each $t>0$, $x\in \boldsymbol{R}^d$ and Borel set Γ of $\boldsymbol{R}^d$,

$$\begin{aligned}
P(t, x, \Gamma) &= P(X(t, x)\in\Gamma)\\
&= \sum_{n=0}^{\infty} P(X(t, x)\in\Gamma;\, \sigma_n\leq t<\sigma_{n+1})\\
&= \sum_{n=0}^{\infty} P(X(t, x)\in\Gamma;\, \sigma_n< t<\sigma_{n+1})\\
&= \sum_{n=0}^{\infty} E[1_{\{\sigma_n<t\}}\, P(X(t-s, y)\in\Gamma;\, t-s<\sigma)\,|_{s=\sigma_n,\, y=X(\sigma_n, x)}]\\
&= \sum_{n=0}^{\infty} E[1_{\{\sigma_n<t\}}\, P(X_\Phi(t-s, y)\in\Gamma;\, t-s<\sigma)\,|_{s=\sigma_n,\, y=X(\sigma_n, x)}]\,.
\end{aligned}$$

Hence, if the Lebesgue measure of Γ is equal to zero,

$$P(t, x, \Gamma) = 0$$

for any $t>0$ and $x\in\boldsymbol{R}^d$; consequently we have the conclusion.

2. The Behavior of Sample Paths near $t=0$

In this section, we investigate the behavior of sample paths of the stable-like process $X=(X(t), P_x)$ with exponent $\alpha(x)$. At first, we state the main result in this section.

Theorem 2.1. *Let x be an arbitrarily fixed point.*
(1) *If $\alpha(x)<\beta$, then*

$$P_x(\lim_{t\to 0} |X(t)-x|/t^{1/\beta} = 0) = 1\,. \tag{2.1}$$

(2) *If $\alpha(x)>\beta>0$, then*

$$P_x(\limsup_{t\to 0} |X(t)-x|/t^{1/\beta} = \infty) = 1\,. \tag{2.2}$$

We provide two lemmas for the proof of this theorem. The first lemma is a modification of Khintchine's result [6]. It is obtained only for processes with stationary independent increments. However a stable-like process is not such a process in general. Accordingly we modify Khintchine's result in the following form, where, for simplicity, we restrict the consideration to conservative processes.

Lemma 2.1. *Let $Y=(Y(t), \boldsymbol{P}_x)$ be a standard process on $\boldsymbol{R}^d$ and let h be a non-decreasing positive funciton on $(0, \lambda)$ with $\lim_{t\downarrow 0} h(t)=0$, where λ is a positive number. $U_r(x)$ is the open ball with center x and radius r. $\boldsymbol{P}_c^{U_r(x)}(\cdot)$ $(c>0)$ is the function defined on $(0, \lambda)$ by*

$$\boldsymbol{P}_c^{U_r(x)}(t) = \sup_{y\in U_r(x)} \boldsymbol{P}_y(|Y(t)-y|>ch(t)) . \tag{2.3}$$

Let x_0 be a point of $\boldsymbol{R}^d$. If there exist positive numbers c_0 and r such that

$$\int_0^\lambda \boldsymbol{P}_c^{U_r(x_0)}(t)/t\, dt<\infty \tag{2.4}$$

for any $c\in(0, c_0)$, then

$$\boldsymbol{P}_{x_0}(\lim_{t\to 0} |Y(t)-x_0|/h(t) = 0) = 1 . \tag{2.5}$$

Proof. Let U_j be the open ball with center x_0 and radius $jr/3$ $(j=1, 2, 3)$. It is clear that, for any positive number a and $t_1\in[0, t]$,

$$\begin{aligned}&\boldsymbol{P}_x(|Y(t)-x|>a)\\ &\quad\le \boldsymbol{P}_x(|Y(t_1)-x|>a/2)+\boldsymbol{P}_x(|Y(t)-Y(t_1)|>a/2,\ |Y(t_1)-x|\le a/2) .\end{aligned}$$

By the Markov property of Y, we get

$$\begin{aligned}\sup_{x\in U_j} &\boldsymbol{P}_x(|Y(t)-x|>a)\\ &\le \sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t_1)-x|>a/2)\\ &\quad+\sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t-t_1)-x|>a/2)\end{aligned} \tag{2.6}$$

for any $a\in(0, r/3)$, $t_1\in[0, t]$ and $j=1, 2$. In particular,

$$\sup_{x\in U_j} \boldsymbol{P}_x(|Y(t)-x|>a)\le 2 \sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t/2)-x|>a/2) \tag{2.7}$$

for any $a\in(0, r/3)$ and $j=1, 2$. In the same way as the above, we have, for any $a>0$ and $t_1, t_2, t_3\in[0, t]$ $(t_1<t_2<t_3)$,

$$\begin{aligned}\boldsymbol{P}_x(|Y(t)-x|>a)&\le \boldsymbol{P}_x(|Y(t_1)-x|>a/4)\\ &\quad+\boldsymbol{P}_x(|Y(t_2)-Y(t_1)|>a/4,\ |Y(t_1)-x|\le a/4)\\ &\quad+\boldsymbol{P}_x(|Y(t_3)-Y(t_2)|>a/4,\ |Y(t_2)-x|\le a/2)\\ &\quad+\boldsymbol{P}_x(|Y(t)-Y(t_3)|>a/4,\ |Y(t_3)-x|\le 3a/4) .\end{aligned}$$

Furthermore, using the Markov property again, we obtain

$$\sup_{x\in U_j} \boldsymbol{P}_x(|Y(t)-x|>a) \tag{2.8}$$

$$\leq \sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t_1)-x|>a/4)$$
$$+\sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t_2-t_1)-x|>a/4)$$
$$+\sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t_3-t_2)-x|>a/4)$$
$$+\sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t-t_3)-x|>a/4)$$

for any $a\in(0, r/3)$, $t_1, t_2, t_3\in[0, t]$ $(t_1<t_2<t_3)$ and $j=1, 2$, and particularly

(2.9) $$\sup_{x\in U_j} \boldsymbol{P}_x(|Y(t)-x|>a)\leq 4 \sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t/4)-x|>a/4)$$

for any $a\in(0, r/3)$, and $j=1, 2$. Next, we will show that, for any positive number c less than c_0,

(2.10) $$\sup_{x\in U_1} \boldsymbol{P}_x(|Y(t)-x|>ch(t/4))\to 0 \quad \text{as } t\to 0\,.$$

In fact, let $ch(t)/4<r/3$ and $t\in(0, \lambda)$. Then, it follows from (2.6) that

(2.11) $$\boldsymbol{P}^{U_2}_{c/4}(t)=\sup_{x\in U_2} \boldsymbol{P}_x(|Y(t)-x|>\frac{c}{4}h(t))$$
$$\leq \sup_{x\in U_3} \boldsymbol{P}_x(|Y(t_1)-x|>\frac{c}{8}h(t))$$
$$+\sup_{x\in U_2} \boldsymbol{P}_x(|Y(t-t_1)-x|>\frac{c}{8}h(t))$$
$$\leq \sup_{x\in U_3} \boldsymbol{P}_x(|Y(t_1)-x|>\frac{c}{8}h(t_1))$$
$$+\sup_{x\in U_3} \boldsymbol{P}_x(|Y(t-t_1)-x|>\frac{c}{8}h(t-t_1))$$

for any $t_1\in[0, t]$. Hence, if $t\in(0, \lambda)$ and $ch(t)/4<r/3$,

(2.12) $$\boldsymbol{P}^{c/4}_{U_2}(t)\leq \boldsymbol{P}^{U_3}_{c/8}(t_1)+\boldsymbol{P}^{U_3}_{c/8}(t-t_1) \quad \forall t_1\in[0, t]\,.$$

Moreover, if $t\in(0, \lambda)$ and $ch(t)/4<r/3$, then

(2.13) $$\begin{aligned}&\boldsymbol{P}^{U_2}_{c/4}(t)\\ &=\frac{1}{\log 2}\int_{t/2}^{t} \boldsymbol{P}^{U_2}_{c/4}(t)\frac{ds}{s}\leq\frac{1}{\log 2}\int_{t/2}^{t} \{\boldsymbol{P}^{U_3}_{c/8}(s)+\boldsymbol{P}^{U_3}_{c/8}(t-s)\}\frac{ds}{s}\\ &\leq\frac{1}{\log 2}\int_{t/2}^{t} \boldsymbol{P}^{U_3}_{c/8}(s)\frac{ds}{s}+\frac{1}{\log 2}\int_{t/2}^{t} \boldsymbol{P}^{U_3}_{c/8}(t-s)\frac{ds}{t-s}\\ &\leq\frac{1}{\log 2}\int_{0}^{t} \boldsymbol{P}^{U_3}_{c/8}(s)\frac{ds}{s}\,.\end{aligned}$$

Thus

$$\sup_{x\in U_1} \boldsymbol{P}_x(|Y(t)-x|>ch(t/4))\leq\frac{4}{\log 2}\int_0^t \boldsymbol{P}^{U_3}_{c/8}(s)\frac{ds}{s}$$

for $t\in(0,\lambda)$ with $ch(t)/4<r/3$. Under the condition (2.4), this means (2.10). Let c and t be positive numbers satisfying $ch(t/4)<r/6$ and $t\in(0,\lambda)$, and let $\sigma_{c,t}$ be the hitting time defined by

$$\sigma_{c,t}=\inf\{s>0:\ |Y(s)-x_0|>ch(t/4)\}\ .$$

Then, the strong Markov property of Y yields that

$$\begin{aligned}(2.14)\quad \boldsymbol{P}_{x_0}(|Y(t)-x_0|>\frac{c}{2}h(t/4)) &\geq \boldsymbol{P}_{x_0}(\sigma_{c,t}\leq t,\ |Y(t)-Y(\sigma_{c,t})|\leq\frac{c}{3}h(t/4)) \\ &= \int_{\{\sigma_{c,t}\leq t\}}\boldsymbol{P}_y(|Y(t-s)-y|\leq\frac{c}{3}h(t/4))|_{s=\sigma_{c,t},y=Y(\sigma_{c,t})}\,d\boldsymbol{P}_{x_0} \\ &\geq \int_{\{\sigma_{c,t}\leq t,Y(\sigma_{c,t})\in U_1\}}\boldsymbol{P}_y(|Y(t-s)-y|\leq\frac{c}{3}h(t/4))|_{s=\sigma_{c,t},y=Y(\sigma_{c,t})}\,d\boldsymbol{P}_{x_0}\ .\end{aligned}$$

On the other hand, by virtue of (2.10), we can find a sufficiently small $t>0$ satisfying

$$(2.15)\qquad \inf_{x\in U_1}\boldsymbol{P}_x(|Y(t)-x|\leq\frac{c}{3}h(t))>\frac{1}{2}\ .$$

Therefore, from (2.14) and (2.15), it follows that for sufficiently small $t>0$

$$(2.16)\qquad \boldsymbol{P}_{x_0}(\sigma_{c,t}\leq t,\ Y(\sigma_{c,t})\in U_1)\leq 2\boldsymbol{P}_{x_0}(|Y(t)-x_0|>\frac{c}{2}h(t/4))\ .$$

Set $\tau=\inf\{s>0:\ |Y(s)-Y(s_-)|>r/6\}$. Then

$$\begin{aligned}(2.17)\qquad &\boldsymbol{P}_{x_0}(\sigma_{c,t}\leq t<\tau) \\ &\leq \boldsymbol{P}_{x_0}(\sigma_{c,t}\leq t<\tau,\ Y(\sigma_{c,t})\in U_1) \\ &\leq \boldsymbol{P}_{x_0}(\sigma_{c,t}\leq t,\ Y(\sigma_{c,t})\in U_1)\ .\end{aligned}$$

It follows from (2.16) and (2.17) that if $ch(t/4)<r/6$ and t is sufficiently small, then

$$(2.18)\qquad \boldsymbol{P}_{x_0}(\sigma_{c,t}\leq t<\tau)\leq 2\boldsymbol{P}_{x_0}(|Y(t)-x_0|>\frac{c}{2}h(t/4))\ .$$

Now, put

$$w_m=\boldsymbol{P}_{x_0}(\sup_{2^{-(m+1)}\leq t\leq 2^{-m}}|Y(t)-x_0|/h(t)>\varepsilon,\ 2^{-m+1}<\tau)\ ,$$

where ε is any small positive number. It follows from the increasing property of h that

$$(2.19)\qquad w_m\leq\boldsymbol{P}_{x_0}(\sup_{2^{-(m+1)}\leq t\leq 2^{-m}}|Y(t)-x_0|>\varepsilon h(2^{-(m+1)}),\ 2^{-m+1}<\tau)\ .$$

Let m be a sufficiently large integer and choose θ_m as any number greater than 2^{-m}. The relationship (2.19) implies that

$$w_m \leq \boldsymbol{P}_{x_0}\big(\sup_{0\leq t\leq \theta_m} |Y(t)-x_0|>\varepsilon h(2^{-(m+1)}), 2^{-m+1}<\tau\big).$$

If $\theta_m \in (2^{-m}, 2^{-m+1})$, then

$$\text{(2.20)} \qquad \begin{aligned} w_m &\leq \boldsymbol{P}_{x_0}\big(\sup_{0\leq t\leq \theta_m} |Y(t)-x_0|>\varepsilon h(\theta_m/4), 2^{-m+1}<\tau\big) \\ &\leq \boldsymbol{P}_{x_0}(\sigma_{\varepsilon,\theta_m}\leq \theta_m<\tau). \end{aligned}$$

Therefore, from (2.9), (2.16), (2.17), (2.18) and (2.20), we have

$$w_m \leq 8\boldsymbol{P}^{U_2}_{\varepsilon/8}(\theta_m/4)$$

for any $\theta_m \in (2^{-m}, 2^{-m+1})$. Let $\theta_m = 2^{-z}$ and integrate both the sides of the last inequality with respect to z from $m-1$ to m. Then, for sufficiently large integer m, we have

$$w_m \leq 8\int_{m-1}^{m} \boldsymbol{P}^{U_2}_{\varepsilon/8}(2^{-z}/4)\, dz = \frac{8}{\log 2}\int_{2^{-(m+2)}}^{2^{-(m+1)}} \boldsymbol{P}^{U_2}_{\varepsilon/8}(u)\frac{dz}{u}.$$

Under the condition (2.4), this relationship implies that the series $\sum w_m$ converges. By virtue of the Borel-Cantelli lemma, this means that

$$\text{(2.21)} \qquad \boldsymbol{P}_{x_0}\big(\limsup_{m\to\infty} \{\sup_{2^{-(m+1)}\leq t\leq 2^{-m}} |Y(t)-x_0|/h(t)>\varepsilon, 2^{-m+1}<\tau\}\big) = 0.$$

Accordingly, for convenience sake, set

$$F_m = \{\sup_{2^{-(m+1)}\leq t\leq 2^{-m}} |Y(t)-x_0|/h(t)>\varepsilon\}, \quad \text{and} \quad G_m = \{\tau>2^{-m+1}\}.$$

Then, noting that

$$\boldsymbol{P}_{x_0}(\liminf_{m\to\infty} (F_m\cap G_m)^c) = \boldsymbol{P}_{x_0}(\cup_{N=0}^{\infty}\{(\cap_{m>N}(F_m^c\cap G_m))\cup(\cap_{m>N} G_m^c)\})$$

and $\boldsymbol{P}_{x_0}(\liminf_{m\to\infty} G_m^c)=0$, from (2.21), we obtain

$$\boldsymbol{P}_{x_0}(\liminf_{m\to\infty} F_m^c)\geq \boldsymbol{P}_{x_0}(\liminf_{m\to\infty} F_m^c\cap G_m) = 1;$$

hence (2.5) holds. The proof is complete.

Lemma 2.2. *Let γ be a positive number. The characteristic function $\phi_t^\gamma(x,\cdot)$ of the random variable $t^{-1/\gamma}(X_\Phi(t)-x)$ admits the representation*

$$\text{(2.22)} \qquad \phi_t^\gamma(x,\eta) = e(t, x, t^{-1/\gamma}\eta)$$

for any $(t,x,\eta)\in(0,\infty)\times\boldsymbol{R}^d\times\boldsymbol{R}^d$, where $e(t,x,\xi)$ is the symbol of $E(t)$.

Proof. From Theorem 1.6, we get

$$\begin{aligned} &\phi_t^\gamma(x,\eta) \\ &= \int_{\boldsymbol{R}^d} \exp(i\eta\cdot t^{-1/\gamma}(y-x))\, K(t,x,y)\, dy \end{aligned}$$

$$= Os-\iint \exp(-iz\cdot\mu)\, e(t, x, \mu+t^{-1/\gamma}\eta)\, dz d\mu$$

for any $(t, x, \eta)\in(0, \infty)\times \boldsymbol{R}^d\times \boldsymbol{R}^d$. Using the fact that $Os\,[\exp(-iy\cdot\mu)\, a(y)]= a(0)$ for any $a\in\mathcal{A}$, we obtain (2.22).

Proof of Theorem 2.1. As is shown in the proof of Theorem 1.7, the short time behavior of sample paths of the stable-like process X coincides with that of the process X_Φ. Hence we prove the theorem replacing X by X_Φ. At first, we will show (2.1). Choose real numbers ν, κ satisfying $\alpha(x)<\nu<\kappa<\beta$. Let T be a positive number and let g_κ be the continuous density of d-dimensional symmetric stable distribution of index κ, $(0<\kappa\leq 2)$, that is,

$$\exp(-|\xi|^\kappa)=\int_{\boldsymbol{R}^d}\exp(iy\cdot\xi)\, g_\kappa(y)\, dy \quad \text{for} \quad \xi\in\boldsymbol{R}^d\,. \tag{2.23}$$

Set

$$A(t, x)=\int_{\boldsymbol{R}^d}\exp(-|y-x|^\kappa)\, K(t, x, y)\, dy \tag{2.24}$$

for any $(t, x)\in(0, \infty)\times\boldsymbol{R}^d$. From the definition of $K(t, x, y)$, (2.23) and (2.24), we have

$$A(t, x)=\int_{\boldsymbol{R}^d} e(t, x, \xi)\, g_\kappa(\xi)\, d\xi \quad \text{for} \quad \forall (t, x)\in(0, \infty)\times\boldsymbol{R}^d\,.$$

From (4) in the Theorem 1.3, we see that for any $(t, x, \xi)\in(0, T]\times\boldsymbol{R}^d\times\boldsymbol{R}^d$.

$$|1-\exp(tp_\Phi(x, \xi))|/t\leq |p_\Phi(x, \xi)|\leq C\langle\xi\rangle^{\alpha(x)} \tag{2.25}$$

and

$$|r_0(t, x, \xi)|/t\leq C\langle\xi\rangle^{\alpha(x)}\,. \tag{2.26}$$

Put $\mathcal{D}_\nu=\{z\colon \alpha(z)<\nu\}$. Then, from (2.23), (2.25) and (2.26), we obtain

$$\begin{aligned}&\frac{1}{t}\,|1-A(t, z)|\\ &\leq C\int_{\boldsymbol{R}^d}\langle\xi\rangle^{\alpha(z)} g_\kappa(\xi)\, d\xi\leq C\int_{\boldsymbol{R}^d}\langle\xi\rangle^{\nu} g_\kappa(\xi)\, d\xi\equiv\Lambda_{\kappa,\nu}<\infty\end{aligned}$$

for any $(t, z)\in(0, T]\times\mathcal{D}_\nu$. Using the same argument as in [3], we have, for sufficiently small δ,

$$\boldsymbol{P}_z(|X_\Phi(t)-z|^\kappa>\delta)\leq\frac{2\Lambda_{\kappa,\nu}\, t}{\delta} \tag{2.27}$$

for any $(t, z)\in[0, T]\times\mathcal{D}_\nu$. Let

$$\boldsymbol{P}_c^{\mathcal{D}_\nu}(t)=\sup_{z\in\mathcal{D}_\nu}\boldsymbol{P}_z(|X_\Phi(t)-z|>ct^{1/\beta})\,. \tag{2.28}$$

Then, by (2.27), the relation (2.28) implies that for sufficiently small $t>0$

$$P_c^{\mathcal{D}_v}(t) \leq 2\Lambda_{\kappa,v} c^{-\kappa} t^{1-\kappa/\beta} .$$

By Lemma 2.1, this means that

$$P_x(\lim_{t\to 0} |X_\Phi(t)-x|/t^{1/\beta} = 0) = 1 \quad \text{if} \quad \alpha(x)<\beta .$$

Therefore, the assertion (2.1) holds. Next, we establish the relation (2.2). Choose γ satisfying $\beta<\gamma<\alpha(x)$. Let $\{\xi_n\}_{n\geq 0}$ be a sequence of points in $\boldsymbol{R}^d$ with $|\xi_n|\to\infty$ as $n\to\infty$. Put $t_n=|\xi_n|^{-\gamma}$, and $\tilde{\xi}_n=\xi_n/|\xi_n|$ $(n=1, 2, \cdots)$. Noting that $|\xi_n|^{-\gamma}|p_\Phi(x, \xi_n)|\to\infty$ as $n\to\infty$, from (4) in Theorem 1.3 and Lemma 2.2, we see that

$$\lim_{n\to\infty} \phi^\gamma_{t_n}(x, \tilde{\xi}_n) = 0 . \tag{2.29}$$

Using the same argument as in [3], we also see that (2.29) implies

$$P_x(\limsup_{t\to\infty} |X_\Phi(t, x)-x|/t^{1/\beta} = \infty) = 1 \quad \text{if} \quad \beta<\alpha(x) .$$

Hence, the assertion (2.2) holds.

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Department of Mathematics
Faculty of Liberal Arts
Shizuoka University
Ohya, Shizuoka 422, Japan