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STRUCTURE OF A CLASS OF POLYNOMIAL MAPS WITH IN Variant FACTORS

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Let \( \mathbf{R}[x_1, \ldots, x_n] \) be a ring of polynomials of \( n \) \((n \geq 3)\) indeterminates with coefficients in \( \mathbf{R} \). Let \( \phi = (\phi_1, \ldots, \phi_n) \in (\mathbf{R}[x_1, \ldots, x_n])^n \) be a polynomial map from \( \mathbf{R}^n \) to \( \mathbf{R}^n \). Let \( x = (x_1, x_2, \ldots, x_n) \) and define

\[
\lambda_{n,a}(x) := \sum_{i=1}^{n} x_i^2 - a \prod_{i=1}^{n} x_i \in \mathbf{R}[x_1, x_2, \ldots, x_n].
\]

where \( a (\neq 0) \in \mathbf{R} \). We will write \( \lambda \) in stead of \( \lambda_{n,a} \) if no confusion happens. \( \lambda_{n,a} \) is called an invariant factor of \( \phi \) if

\[
\lambda_{n,a} \circ \phi = \lambda_{n,a}.
\]

Now let

\[
G_{n,a} = \{ \phi; \phi \in (\mathbf{R}[x_1, \ldots, x_n])^n, \lambda_{n,a} \circ \phi = \lambda_{n,a} \},
\]

that is, \( G_{n,a} \) is the set of polynomial maps of which invariant factor is \( \lambda_{n,a} \). The main aim of this note is to determine the structure of \( G_{n,a} \).

Let \( \Omega_{n,a} = \{ x \in \mathbf{R}; \lambda_{n,a}(x) = 0 \} \). Then by the equality (1), for any \( n \in \mathbf{N} \),

\[
\phi^n(\Omega_{n,a}) \subset \Omega_{n,a},
\]

that is, \( \Omega_{n,a} \) is an invariant variety of \( \phi^n \), where \( \phi^n \) denotes \( n \)-th iteration of \( \phi \) (see [3]). By using this property, we may investigate the asymptotic dynamical behaviours of iterations of \( \phi \) ([1, 2, 3]). We are led naturally to study the structure of \( G_{n,a} \). In fact, we will prove first that \( G_{n,a} \) is a group, then we will determine the generators of the group.

In the case \( n = 3 \), we have showed the following:

**Theorem 1** ([2]). With the notations above, \( G_{3,1} = (\tau_1, \tau_2, \tau_3, \tau_4) \) is a group generated by \( \tau_1(x, y, z) = (y, x, z), \tau_2(x, y, z) = (z, y, x), \tau_3(x, y, z) = (-x, -y, z), \tau_4(x, y, z) = (x, y, xy - z) \).
The proof of Theorem 1 depends strongly on the reducibility of polynomial $u^2 + v^2 - auv$, but when $n \geq 4$, the corresponding polynomial that we have to treat is irreducible, thus the method for $n = 3$ is failed.

Let $p \in R[x_1, \ldots, x_n]$, $\phi \in (R[x_1, \ldots, x_n])^n$, we denote by $\deg p$ the degree of the polynomial $p$, and define the degree of $\phi$ as $\deg \phi = \sum_{i=1}^n \deg \phi_i$.

Let $S_n$ be the symmetric group on $n$ letters, we have

$S_n := \{(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}); \pi \in S_n \} \simeq S_n$.

So we can denote by $\pi$ the permutation $(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$.

**Lemma 1.** Let $n \geq 3$, $a \neq 0$, $c$ is a constant, then $\lambda_{n,a} + c$ is irreducible.

Proof. If the conclusion of the lemma is not true, then $\lambda_{n,a} + c$ is reducible, i.e. we have the non-trivial factorization of $\lambda_{n,a} + c$

\begin{equation}
\lambda_{n,a} + c = p_1 p_2.
\end{equation}

Thus if we consider $\lambda_{n,a} + c$ as a polynomial of $x_n$ with degree 2, then we have either

$p_1 = f_1(x_1, \ldots, x_{n-1})x_n + g_1(x_1, \ldots, x_{n-1}),$

$p_2 = f_2(x_1, \ldots, x_{n-1})x_n + g_2(x_1, \ldots, x_{n-1})$

or

$p_1 = f_1(x_1, \ldots, x_{n-1})x_n^2 + g_1(x_1, \ldots, x_{n-1})x_n + h_2(x_1, \ldots, x_{n-1}),$

$p_2 = f_2(x_1, \ldots, x_{n-1}).$

From the hypothesis $n \geq 3$ and by comparing the degree of two sides of the equality (*), it is easy to see that the factorizations above are impossible for both cases. This proves the lemma. \hfill \Box

Now define $\psi = (x_1, x_2, \ldots, x_{n-1}, a \prod_{i=1}^{n-1} x_i - x_n)$, which will play an important role in the studies of this note. By a direct calculation, we see immediately $\psi \in G_{n,a}$.

**Lemma 2.** Let $\phi \in G_{n,a}$. If $\deg \phi > n$, then there exists $\pi \in S_n$, such that $\deg(\psi \circ \pi \circ \phi) < \deg \phi$.

Proof. Because we can find a permutation $\pi$, such that

$\deg \phi_{\pi(1)} \leq \deg \phi_{\pi(2)} \leq \cdots \leq \deg \phi_{\pi(n)}$. 
we can assume that
\begin{equation}
\deg \phi_n \geq \deg \phi_{n-1} \geq \cdots \geq \deg \phi_1.
\end{equation}

Since $\deg \phi > n$, we have $\deg \phi_n \geq 2$. From the equality (1),
\begin{equation}
\left(\phi_n - a \prod_{i=1}^{n-1} \phi_i\right) \phi_n + \sum_{i=1}^{n-1} \phi_i^2 = \sum_{i=1}^{n} x_i^2 - a \prod_{i=1}^{n} x_i.
\end{equation}

1. If $\deg \phi_n \neq \deg \left(a \prod_{i=1}^{n-1} \phi_i\right)$, then
\[
\deg \left(\phi_n - a \prod_{i=1}^{n-1} \phi_i\right) = \sup \left\{ \deg \phi_n, \ deg \left(a \prod_{i=1}^{n-1} \phi_i\right) \right\}.
\]
Thus by (2) and (3), we have
\[
\deg \lambda_{n,a} = n = \sup \left\{ \deg \phi_n, \ deg \left(a \prod_{i=1}^{n-1} \phi_i\right) \right\} + \deg \phi_n \geq \sum_{i=1}^{n} \deg \phi_i = \deg \phi > n,
\]
This contradiction follows that
\begin{equation}
\deg \phi_n = \deg (\phi_1 \cdots \phi_{n-1}).
\end{equation}

2. If $\deg \phi_n = \deg \phi_{n-1}$, then by (4), we have $\deg \phi_i = 0$, $1 \leq i \leq n - 2$. Thus from the equality (3), there exists constants $c_1$ and $c_2$, such that
\[
\phi_n^2 + \phi_{n-1}^2 - ac_1 \phi_n \phi_{n-1} = \sum_{i=1}^{n} x_i^2 - a \prod_{i=1}^{n} x_i + c_2.
\]
Notice that the left member of the equality above is reducible, but by Lemma 1, the right member of the equality above is irreducible, this contraction yields that
\begin{equation}
\deg \phi_n > \deg \phi_{n-1}.
\end{equation}

3. If $\deg \left(a \prod_{i=1}^{n-1} \phi_i - \phi_n\right) = \deg \phi_n$, then
\begin{equation}
\deg \left(\prod_{i=1}^{n-1} \phi_i\right) \leq \deg \phi_n.
\end{equation}
Using (5), (6) and using the analyses similar to the case 1, we have

\[ n = \deg \lambda = \deg \lambda \circ \phi = 2 \deg \phi_n \geq \sum_{i=1}^{n} \deg \phi_i = \deg \phi > n. \]

This contradiction implies that \( \deg \left( a \prod_{i=1}^{n-1} \phi_i - \phi_n \right) \neq \deg \phi_n \), thus from (4), we have \( \deg \left( a \prod_{i=1}^{n-1} \phi_i - \phi_n \right) < \deg \phi_n \). By the definition of \( \psi \), we obtain finally

\[ \deg(\psi \circ \pi \circ \phi) < \deg \phi. \]

Now, define \( \rho = (-x_1, -x_2, x_3, \ldots, x_n) \).

**Lemma 3.** Let \( \mathcal{L}_n = \{ \phi \in G_n; \deg \phi_i = 1, 1 \leq i \leq n \} \), then \( \mathcal{L}_n \) is a group generated by \( S_n \) and \( \rho \).

**Proof.** Since \( \deg \phi_i = 1 \), we can write \( \phi_i := \phi_i(x_1, \ldots, x_n) = h_i(x_1, \ldots, x_n) + c_i \), where \( h_i \) are homogeneous linear polynomials of \( x_1, \ldots, x_n \), and \( c_i \in \mathbb{R} \) are constants. By the equality (1),

\[ \sum_{i=1}^{n} (h_i + c_i)^2 - a \prod_{i=1}^{n} (h_i + c_i) = \sum_{i=1}^{n} x_i^2 - a \prod_{i=1}^{n} x_i. \]

By comparing the coefficients of the terms of degree \( n \) of the two sides of (7), we have \( h_i = d_i x_{\pi(i)} \), where \( d_i \in \mathbb{R}, \pi \in S_n \). By comparing the coefficients of the terms of degree \( n - 1 \), we have \( c_i = 0 \). By comparing the coefficients of the square terms, we have \( d_i^2 = 1, 1 \leq i \leq n \). Finally notice that \( |\{i; d_i = -1, 1 \leq i \leq n\}| \in 2\mathbb{N} \) and notice the role of the action of \( \rho \), we obtain this lemma. \( \square \)

**Lemma 4.** Let \( \phi \in G_{n,a} \). Then for any \( i, 1 \leq i \leq n \), we have \( \deg \phi_i \geq 1 \). Moreover, there exists \( \varphi \in \langle \psi, S_n \rangle \), such that

\[ \deg(\varphi \circ \phi)_1 = \cdots = \deg(\varphi \circ \phi)_n = 1. \]

**Proof.** We prove the lemma by induction. By Theorem 4 of §3 of [1], the lemma holds for \( n = 3 \). Now suppose that the conclusions of the lemma are true for the positive integers less than \( n (n \geq 4) \).

If \( \deg \phi > n \), by using Lemma 2 repeatedly, we can decrease the degree of \( \phi \) by using \( \psi \) and a suitable \( \pi \in S_n \), and the degree of each component of \( \phi \) does not increase. Thus we can assume that \( \deg \phi \leq n \).

If the conclusion of the lemma is not true, then there exists some \( \phi_i \), being \( \phi_n \) without losing generality, such that \( \phi_n \equiv c \), where \( c \) is a constant. If \( c = 0 \), by the
equality (1),

\[ \phi_1^2 + \cdots + \phi_{n-1}^2 = x_1^2 + \cdots + x_n^2 - ax_1x_2 \cdots x_n. \]

Notice that the left member of the equality (8) is always non-negative. But for \( n \geq 3 \),
we can choose \( x_1, \ldots, x_n \), such that the right member of the equality (8) is strictly
negative. thus \( c \neq 0 \).

Now let \( \phi = (\phi_1, \ldots, \phi_{n-1}, c), c \neq 0 \). Define

\[
\begin{align*}
\phi_i^{(j)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) &= \\
\phi_i(x_1, \ldots, x_{j-1}, c, x_{j+1}, \ldots, x_n) &\in \mathbb{R}[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n], \\
\phi^{(j)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) &= \\
(\phi_1^{(j)}, \ldots, \phi_{n-1}^{(j)}) &\in (\mathbb{R}[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n])^{n-1}.
\end{align*}
\]

where \( 1 \leq j \leq n \).

From (1) and \( \phi_n = c \), we check directly for any \( j, 1 \leq j \leq n \)

\[ \lambda_{n-1, ca} \circ \phi^{(j)} = \lambda_{n-1, ca}. \]

Since \( \deg \phi \leq n \), we have \( \deg \phi^{(j)} \leq n \), \( 1 \leq j \leq n \).

1. Suppose that \( \deg \phi^{(j)} = n \).

By Lemma 2, there exists \( \pi \in S_{n-1}\{1, \ldots, j-1, j+1, \ldots, n\} \) such that

\[ \deg \psi^{(j)} \circ \pi \circ \phi^{(j)} < \deg \phi^{(j)} = n, \]

where

\[
\begin{align*}
\psi^{(j)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) &= (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-1}, ax_{j-1}x_{j+1} \cdots x_{n-1} - x_n).
\end{align*}
\]

From (9) and the induction hypothesis, we have

\[ \psi^{(j)} \circ \pi \circ \phi^{(j)} = (\varepsilon_1^{(j)} x_{\tau^{(j)}(1)}, \ldots, \varepsilon_{j-1}^{(j)} x_{\tau^{(j)}(j-1)}, \varepsilon_{j+1}^{(j)} x_{\tau^{(j)}(j+1)}, \ldots, \varepsilon_{n}^{(j)} x_{\tau^{(j)}(n)}), \]

where \( \tau^{(j)} \in S_{n-1}\{1, \ldots, j-1, j+1, \ldots, n\} \), \( \varepsilon_i^{(j)} = \pm 1 \).

Since \( (\psi^{(j)})^2 = \text{id} \), we have

\[ \phi^{(j)} = \pi^{-1} \circ (\varepsilon_1^{(j)} x_{\tau^{(j)}(1)}, \ldots, \varepsilon_{j-1}^{(j)} x_{\tau^{(j)}(j-1)}, \varepsilon_{j+1}^{(j)} x_{\tau^{(j)}(j+1)}, \ldots, \\
ax \varepsilon_1^{(j)} x_{\tau^{(j)}(1)} \cdots \varepsilon_{j-1}^{(j)} x_{\tau^{(j)}(j-1)} \varepsilon_{j+1}^{(j)} x_{\tau^{(j)}(j+1)} \cdots \varepsilon_{n}^{(j)} x_{\tau^{(j)}(n)} - \varepsilon_{n}^{(j)} x_{\tau^{(j)}(n)}). \]
Since \( \phi^{(j)} = (\phi_1, \ldots, \phi_{n-1})\big|_{x_j = c} \) for any \( c \) and \( \phi \) is a polynomial in \( x_1, \ldots, x_n \), it follows from (11) that

\[
\left( \phi_1, \ldots, \phi_{n-1} \right) = \pi^{-1} \circ \left( \epsilon_1^{(j)} x_{\tau(j)}(1), \ldots, \epsilon_{j-1}^{(j)} x_{\tau(j)}(j-1), \epsilon_{j+1}^{(j)} x_{\tau(j)}(j+1), \ldots, \epsilon_{n-1}^{(j)} x_{\tau(j)}(n-1) - \epsilon_n^{(j)} x_{\tau(j)}(n) \right).
\]

for some \( \pi \) and \( \tau \). Therefore,

\[
n \geq \deg \phi = 2n - 3,
\]

which contradicts with \( n \geq 4 \).

2. Now suppose that \( \deg \phi^{(j)} \leq n - 1 \) for \( j = 1, \ldots, n \).

From (9) and by the induction hypothesis, we have

\[
\phi^{(j)} = (\epsilon_1^{(j)} x_{\tau(j)}(1), \ldots, \epsilon_{j-1}^{(j)} x_{\tau(j)}(j-1), \epsilon_{j+1}^{(j)} x_{\tau(j)}(j+1), \ldots, \epsilon_{n-1}^{(j)} x_{\tau(j)}(n-1)).
\]

for any \( c \), where \( \tau^{(j)} \in S_{n-1} \{1, \ldots, j-1, j+1, \ldots, n\} \), \( \epsilon_i^{(j)} = \pm 1 \) and they may depend on \( c \). Since \( \phi^{(j)} = (\phi_1, \ldots, \phi_{n-1})\big|_{x_j = c} \) for any \( c \) and \( \phi \) is a polynomial in \( x_1, \ldots, x_n \), it follows from (12) that \( (\phi_1, \ldots, \phi_{n-1}) \) is independent of \( x_j \) for any \( j = 1, \ldots, n \). Thus, \( \deg \phi = 0 \), which is absurd since \( \phi \in G_{n, \alpha} \).

These contradictions come from the hypothesis that \( \phi_i = c \) for some \( i \), so we have \( \deg \phi_i \geq 1 \) for \( i = 1, \ldots, n \). Since \( \deg \phi \leq n \), this implies that

\[
\deg \phi_1 = \cdots = \deg \phi_n = 1,
\]

which completes the proof of Lemma 4.

\[\square\]

**Corollary 1.** Suppose that \( \phi \in G_{n, \alpha} \). Then there exists \( \varphi \in G_{n, \alpha} \), such that

\[
\varphi \circ \phi = (d_1 x_{\pi(1)}, \ldots, d_n x_{\pi(n)}).
\]

**Proof.** It follows immediately from Lemma 3 and Lemma 4.

\[\square\]

The foregoing results complete the proof of the following

**Theorem 2.** \( G_{n, \alpha} \) is a group generated by \( S_n, \rho \) and \( \psi \).
References


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