ON MAXIMAL SUBMODULES OF A FINITE DIRECT SUM OF HOLLOW MODULES II

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Introduction

We have given, in [3], the structure of right artinian rings satisfying the following conditions: i) the Jacobson radical of a ring is square zero and ii) every submodule of a direct sum of hollow (local) modules is also a direct sum of hollow modules. The latter property cited above implies that every maximal submodule of a direct sum of \(t+1\)-copies of a hollow module with length \(t\) contains a direct summand.

In this paper, we shall study this property for any right artinian ring, and reproduce, in §1, the results similar to ones in [3] without the assumption that the Jacobson radical is square zero. In §2 we shall give a characterization of some rings in terms of the property above.

1 Property (**)

Let \(R\) be a ring with identity. In this paper, every \(R\)-module is a unitary right \(R\)-module. Let \(M\) be an \(R\)-module. We shall denote the Jacobson radical of \(M\) by \(J(M)\) and the radical of \(R\) by \(J\) or \(J(R)\), respectively. Throughout this paper we assume that \(R\) is a right artinian (semi-perfect) ring and every \(R\)-module \(M\) has the finite composition length, which we denote by \(|M|\). If \(M\) has a unique maximal submodule \(J(M)\), \(M\) is called hollow (local). In this case \(M \approx eR/A\) for a primitive idempotent \(e\) and a right ideal \(A\) in \(eR\).

Given a family \(N=\{N_i\}_{i=1}^\infty\) of (hollow) modules, we denote by \(D(N)\) the direct sum \(\bigoplus_{i=1}^\infty N_i\). If \(N_i=N\) for a fixed module \(N\), we indicate this by \(N^{(n)}\).

We have studied in [3] the following property:

(**) Every maximal submodule of \(D(N)\) contains a non-zero direct summand of \(D(N)\).

Since the above property is preserved by Morita equivalence, we may assume that \(R\) is a basic ring. Hence, from now on, we assume that \(R\) is a right artinian and basic ring. Let \(N\) be a hollow module with finite length. We put \(\bar{N}=N/J(N)\), and \(S (=S_N)=\text{End}_R(N)\). Then \(\Delta=\text{End}_R(\bar{N})\) is a division
ring. We have the natural homomorphism \( \varphi \) of \( S \) into \( \Delta \). It is clear that \( \ker \varphi = J(S) \) and \( \im \varphi \) is a subdivision ring of \( \Delta \), because \( |N| < \infty \). We put \( \im \varphi = S' (= S) \). We assume \( D = D(N, j, n) = \sum \bigoplus N_{i1} \oplus \sum \bigoplus N_{i2} \oplus \cdots \bigoplus \bigoplus N_{ii} \), where \( N_{i1} \cong N_{ki} \) and \( N_{i2} \cong N_{ji} \) if \( i \neq j \). Let \( M \) be a maximal submodule of \( D \). Then \( M \supset J(D) \) and \( \bar{M} = M/J(D) \) is expressed as \( \bar{M} = \sum \bigoplus \bar{M}_i \), where \( \bar{M}_i \) is a maximal submodule of \( \bigoplus \bigoplus N_{ik} \) for some \( i \) and \( \bar{M}_i = \sum \bigoplus N_{jk} \) for \( j \neq i \). Therefore, when we study the property (**), we may assume \( \bar{N}_i \cong \bar{N}_i \) for all \( i \). We shall identify all End\(_R(N_i) \) and denote them by \( \Delta \). Then \( D/J(D) \) is a \( \Delta \)-vector space and \( M \) contains a subspace \( M' \) which is a maximal subspace of \( \sum \bigoplus N_i \) for some \( k \) (\( n \geq 3 \)), (cf. [3] §2). Hence \( M \) contains a submodule \( M' \) maximal in \( \sum \bigoplus N_i \). Thus we obtain the following:

**Lemma 1.** Let \( N = \{ N_i \}_{i=1}^\infty \) be a family of hollow modules with finite length. If \( D(N') \) satisfies (***) for a subfamily \( N' = \{ N_i \}_{i=1}^{k'} \) of \( N \) with \( k' > k \geq 2 \), so does \( D(N) \) (for the case \( k = 1 \), see Theorem 6 below).

Since \( R \) is semi-perfect, \( N \cong eR/A \) for a primitive idempotent \( e \) and a right ideal \( A \) in \( eR \). Then \( \Delta = eRe/eFe \) and \( S_N = \{ x \in eRe \mid xA \subset A \} \). We sometimes denote \( S_N \) by \( \Delta(A) \).

We have defined a max. quasiprojective module in [2]. This is nothing but \( \Delta = S_N \) in our case.

**Theorem 1.** Let \( N \) be a hollow module with \( |N| < \infty \). Then the following conditions are equivalent:

1) \( N \) is a max. quasiprojective.
2) \( N^{(2)} \) has the property of simple modules modulo the radical (see [1]).
3) \( N^{(n)} \) has the above property for \( n \geq 2 \).
4) \( N^{(k)} \) satisfies (***)

Proof. It is clear from [1], [2], except 4).

1) \( \leftrightarrow \) 4). This is clear from Theorem 2 below.

From Theorem 1 we are interested in case where \( \Delta \cong S_N = S \). We may assume that \( \Delta \) is a right \( S \)-vector space and we denote the dimension of \( \Delta \) by \( [\Delta : S] \).

**Theorem 2 ([3], Lemma 5).** Let \( N, \Delta, \) and \( S \) be as above. Then \( [\Delta : S] = k < \infty \) if and only if \( N^{(k+1)} \) satisfies (**), but \( N^{(k)} \) does not.

We shall give a more general result than Theorem 2. Let \( N_1 \) and \( N_2 \) be hollow modules with \( |N_1| \leq |N_2| < \infty \). We assume \( \bar{N}_1 \cong \bar{N}_2 \). We shall identify
\( \bar{N}_1 \) and \( \bar{N}_2 \) and denote \( \text{End}_E(\bar{N}_1) \) by \( \Delta \). Then we have the natural mapping \( \varphi \) of \( \text{Hom}_E(N_2, N_1) \) into \( \Delta \). Put \( \text{im } \varphi = \Delta(N_2, N_1) \) which is a right \( \bar{S}_{N_2} \)-subspace of \( \Delta \). We can express \( N_i = eR/A_i \), \( i = 1, 2 \). Then \( |A_1| \geq |A_2| \) and \( \text{Hom}_E(N_2, N_1) = \{ x \in eRe | xA_2 \subseteq A_1 \} \).

**Theorem 2'.** Let \( N_1 \) and \( N_2 \) be hollow modules with finite length \( (\bar{N}_1 \cong \bar{N}_2) \).

If \( \Delta(\Delta(N_2, N_1)); S_{N_2} \leq k \), \( D = N_2^{(k+1)} \oplus N_1 \) satisfies \( (**) \). Conversely, if \( D \) satisfies \( (**) \) and \( |N_2| \geq |N_1| \) then \( \Delta(\Delta(N_2, N_1)); S_{N_2} \leq k \).

Proof. We assume first \( |N_2| \geq |N_1| \). We may assume \( N_1 = eR/A_i \) for \( i = 1, 2 \). Put \( D = eR/A_2 \oplus \cdots \oplus eR/A_2 \oplus eR/A_1 \). Assume \( D \) satisfies \( (**) \). Let \( \{ \delta_1, \delta_2, \ldots, \delta_{k+1} \} \) be any set of elements in \( \Delta \). We shall express every element in \( D \) as \( (a_1, a_2, \ldots, a_{k+2}) \), where the \( a_i \) are in \( eR \) and \( \delta_i \) is the residue class of \( a_i \) in \( eR/A \).

Take \( \alpha_1 = (\delta, \delta, \ldots, \delta, \delta) \), \( \alpha_2 = (\delta, \delta, \ldots, \delta, \delta_2) \), \ldots, \( \alpha_{k+1} = (\delta, \ldots, \delta, \delta, \delta_1) \). Let \( M \) be the submodule of \( D \) generated by \( \{ \alpha_j \} \) and the elements in \( J(D) \). Then \( M \) is a maximal submodule of \( D \). Put \( \bar{D} = \bar{D}[J(D)] \) \( \Rightarrow \bar{M} = M/J(D) \). \( M \) contains a non-zero direct summand \( M_1 \) of \( D \) by \( (**) \).

We may assume that \( M_1 \) is indecomposable and hence cyclic. Let \( \beta \) be its generator. Then \( \beta = \beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_{k+1} y_{k+1} + j \), where the \( y_i \) are in \( eR \) and \( j \) is in \( J(D) \). Since \( \beta \in J(D) \), we may assume that the \( y_i \) are in \( eRe \) and \( \delta_1 = 0 \) (\( R \) is basic). Consider an epimorphism \( \psi \) of \( eR \) onto \( eR \) given by setting \( \psi(r) = \beta r \); \( r \in eR \). \( \beta \) satisfies \( (**) \).

Put \( \beta = (\delta y_1 + j_1, \delta y_2 + j_2, \ldots, \delta y_{k+1} + j_{k+2}, \delta y_1 + \delta y_2 + \ldots + \delta_{k+1} y_{k+1} + j_{k+2}) \), where the \( j_i \) are in \( eR \). Let \( x \in \ker \psi \). Then \( x = x e \xi \in A_2 \). Hence \( x \in eR(\bar{z})^{-1} A_2 \) and so \( |M_1| > |\beta eR| = |eR/\ker \psi| = |eR| (\bar{z})^{-1} A_2 \). Since \( |eR/A_2| > |eR/A_1| \) and \( M_1 \) is an indecomposable direct summand of \( D \), \( |M_1| \leq |eR/A_2| \). Hence \( |M_1| = |eR/A_2| \), which implies \( \ker \psi = (\bar{z})^{-1} A_2 \). Therefore \( (ey_i + j_i)(\bar{z})^{-1} A_2 \subseteq A_2 \) for \( i = 2, \ldots, k+1 \) and \( (\delta y_1 + \ldots + \delta_{k+1} y_{k+1} + j_{k+2})(\bar{z})^{-1} A_2 \subseteq A_2 \). Accordingly, \( \varphi(ey_i + j_i)(\bar{z})^{-1} = \bar{y}_i \bar{z}^{-1} \in \Delta(A_2) \) and \( \varphi((\delta y_1 + \ldots + \delta_{k+1} y_{k+1} + j_{k+2})(\bar{z})^{-1}) = \bar{\delta}_1 + \bar{\delta}_2 y_2 + \ldots + \bar{\delta}_{k+1} y_{k+1} + \bar{\delta}_1 \). Hence \( \Delta(\Delta((2, A_1); \Delta(A_2)) \leq k \).

Conversely, we assume that \( \Delta(\Delta(A_2, A_1); \Delta(A_2)) \leq k \) and \( M \) is a maximal submodule of \( D \). Then \( M \supseteq J(D) \). Let \( \pi_i \) be the projection of \( D \) onto the \( i \)-th component. If \( \pi_i(M) = 0 \) for some \( j, M = \sum_{i=1}^{\infty} \oplus N_i \oplus J(N_i) \). Hence we may assume \( \pi_i(M) \neq 0 \) for all \( i \). Then \( M \) contains a basis \( \{ \alpha_1, \alpha_2, \ldots, \alpha_{k+1} \} \) as above. Since \( \Delta(\Delta(A_2, A_1); \Delta(A_2)) \leq k \), there exists a set \( \{ \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{k+1} \} \) in \( \Delta(A_2) \) such that \( \sum \bar{y}_i y_i \in \Delta(A_2, A_1) \). Hence \( M \) contains an element \( \beta = \sum \alpha_i y_i = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{k+1}, \sum \bar{y}_i y_i) \), and so \( M \) contains a direct summand of \( D \) by [3], Lemma 17. If we put \( N_2 = N_2 \) in the theorem, then we have Theorem 2. Finally we assume \( |N_2| > |N_1| \).

Then there are no epimorphisms of \( N_2 \) onto \( N_1 \), and so \( \Delta(N_2, N_1) = 0 \). Hence \( \Delta(\Delta(N_2, N_1)); S_{N_2} = \Delta; S_{N_2} \leq k \). Therefore \( D(k+2) \) satisfies \( (**) \) by Theorem 2 and Lemma 1.
The argument given in [3], §3 shows that the converse part in Theorem 2' does not hold without the assumption \( |N_2| \geq |N_1| \).

**Theorem 3.** Let \( \{N_i\}_{i=1}^t \) be a set of hollow modules. Assume \( |N_i| = |N_1| \), \( N_i \cong N_1 \) and \( [\Delta: S_{ni}] = k < \infty \) for all \( i \). Put \( D = N_1^{(s)} \oplus N_2^{(s)} \oplus \cdots \oplus N_i^{(s)} \), where \( k + 1 = \sum s_i \) and \( s_i \geq 1 \). Then \( D \) satisfies \( (**\) \) if and only if \( N_i \cong N_1 \) for all \( i \).

Proof. If \( N_i \cong N_1 \) for all \( i \), then \( D(N_i, k+1) \) satisfies \( (**\) \) by Theorem 2. Conversely, assume the property above. Since \( t \geq 2 \) and \( \sum s_i = k+1 \), \( s_i \leq k \).

We shall first show that some two of \( \{N_i\}_{i=1}^t \) are isomorphic to each other. According to Theorem 2 there exists a maximal submodule \( M_0 \) of \( N_i^{(s)} \), which contains no non-zero direct summands of \( N_i^{(s)} \). It is clear that \( M_0 \) is generated by \( f_i(N_i^{(s)}) \) and the set of elements \( \{\theta_i = (\delta, \cdots, \delta, \delta, \cdots, \delta_{ii}) \in N^{(s)}\} \), where \( \delta_{ii} \) are elements of \( eRe \). Let \( \{\tilde{\delta}_{ii}, \tilde{\delta}_{ii2}, \cdots, \tilde{\delta}_{iit}\} \) be a set of independent elements of \( \Delta \) over \( S_{ni} \) for \( i \leq t-1 \). We can assume \( N_i = eR/A_i \). Let \( M \) be the submodule of \( D \) generated by \( \{a_{ij} = (\delta, \cdots, \delta, \delta, \cdots, \delta_{ii})\}_{i=1, j=1} \), where \( k_{ij} = s_i + \cdots + s_{i-1} + j \) and \( J(D) \). As in the proof of Theorem 2', put \( \beta = (\tilde{\delta}_{ii} y_{11} + \tilde{\delta}_{iit} y_{1t} + \cdots, \tilde{\delta}_{iit} y_{t-1} + \tilde{\delta}_{iit} y_{tt} + \cdots + \tilde{\delta}_{iit} y_{tt-1} + \delta_{tt}) \) and assume that the direct summand \( M_i \) of \( D \), and hence of \( M \), is generated by \( \beta \). Then \( M_i = \beta R = \beta eR + (M_i \cap \Delta) = \beta R \). Since \( \beta \notin J(D) \), some \( y_{ij} \) is not in \( efe \). Assume first that \( y_{ij} = 0 \) for all \( i \leq t-1 \). Then \( M_i \subseteq N_i^{(s)} \). Let \( \pi \) and \( \pi_{ij} \) be the projections of \( D \) onto \( M_i \) and the \( j \)th component of \( N_i^{(s)} \), respectively. Since \( y_{ij} = 0 \) for some \( i, \pi_{ij}(M_i) \neq 0 \). Hence, \( M_i \) being isomorphic to some \( N_{i_p} \), \( M_i \cong N_i \). Since \( \tilde{y}_{ij} = 0 \) for \( i \leq t-1, \pi_{jj} = 0 \), where \( j \in J(\sum_{i=1}^t \oplus N_i^{(s)}), \theta = \sum \theta_{ij} y_{ij} + (0, \cdots, 0, \tilde{y}_{it}, \cdots, \tilde{y}_{tt}) \in M_i \subseteq N_i^{(s)} \). Hence \( M_i = \beta eR \) is epimorphic to \( M_i^s = \theta eR \), and so \( |M_i| \geq |M_i^s| \). Noting that \( \pi(M_i^s) = \pi(M_i) \), we know that \( \pi | M_i^s \) is an epimorphism, and hence \( \pi | M_i^s \) is an isomorphism. Therefore \( D = M_i^s \oplus ker \pi \), and so \( M_i^s \) \( (\subseteq M_i) \) is a direct summand of \( N_i^{(s)} \), which is a contradiction. Accordingly, \( y_{ij} = 0 \) for some \( i \leq t-1 \), say \( i = j = 1 \). If \( y_{ip} = 0 \) for \( p \neq 1 \), \( \pi_{ip}(M_i) = 0 \). Hence \( N_i \cong M_i \cong N_p \). Assume \( y_{pq} = 0 \) for all \( p \neq 1 \) and all \( q \). Then we have the situation similar to the proof of Theorem 2', and obtain \( \tilde{y}_{it} y_{it} \in \Delta(A_i) \). Therefore \( \delta_{ii} y_{11} + \delta_{i2} y_{12} + \cdots + \delta_{it} y_{tt} = 0 \), and so \( \pi_{it}(M_i) = 0 \), which means \( N_i \cong M_i \cong N_i \). Thus we have shown that some two of \( \{N_i\}_{i=1}^t \) are isomorphic to each other. Hence we can show the theorem by induction on \( t \).

From the proof above we have

**Theorem 4.** Let \( N_1 \) and \( N_2 \) be hollow modules with \( N_i \cong N_2 \). Assume \( |N_2| = |N_1| \) and \( [\Delta: S_{N_2}] = k \). Then \( N_i \cong N_2 \) if and only if \( D(k+1) = N_2^{(s)} \oplus N_1 \) satisfies \( (**\) \).
Theorem 5. Let \( \{N_i\}_{i=1}^t \) be a set of hollow modules. Assume \( |N_i| = |N_j| \) for all \( i, j \), \( \hat{N}_i \cong \hat{N}_j \), and \( [\Delta : \bar{S}_{N_i}] \geq k_i \). If \( N_i^{(k_1)} + N_i^{(k_2)} + \cdots + N_i^{(k_l)} \) satisfies (**) then some two of \( \{N_i\}_{i=1}^t \) are isomorphic to each other.

2 Direct sums of hollow modules with same length

We assume again that \( R \) is a right artinian ring.

Theorem 6. Let \( \mathcal{N} \) be a set of representatives of the isomorphism classes of hollow modules. Then there holds the following:

1) Every \( N \in \mathcal{N} \) satisfies (**), if and only if \( R \) is semi-simple.

2) Every \( N_i \oplus N_2 \) (\( N_i \in \mathcal{N} \)) satisfies (**), if and only if \( R \) is right serial.

Proof. 1) Let \( e \) be an arbitrary primitive idempotent in \( R \). If (**), then \( eR \) is hollow and hence \( ej=0 \), which proves that \( R \) is semi-simple.

2) If \( R \) is right serial then, for any \( N \in \mathcal{N}, N \cong eR/A \) with a primitive idempotent \( e \) and a characteristic submodule \( A \) of \( eR \). Hence \( \Delta(A) = \Delta \), and therefore every \( N_i \oplus N_2 \) (\( N_i \in \mathcal{N} \)) satisfies (**). Conversely, if every \( N_i \oplus N_2 \) (\( N_i \in \mathcal{N} \)) satisfies (**), then, by Theorems 2 and 4, \( \Delta = \Delta(A) \) and \( eR/A \cong eR/B \) for any primitive idempotent \( e \) and maximal submodules \( A \) and \( B \) in \( eJ \). Hence \( B=xA \) for some unit element \( x \) in \( eRe \). In view of [3], Proposition 1, we may assume that \( J^2 = 0 \). Then, since \( \Delta = \Delta(A) \), we have \( B=xA=A \). Therefore \( R \) is right serial.

Theorem 7. Let \( \mathcal{N}' \) be a set of hollow modules such that \( |N_i| = |N_j| \) and \( \hat{N}_i \cong \hat{N}_j \) for all \( i, j \), \( \hat{N}_i \in \mathcal{N}' \). Then all \( N_i \oplus N_2 \oplus N_3 \) satisfy (**), but not all \( N_i \oplus N_2 \oplus N_3 \) satisfy (**), if and only if \( \mathcal{N}' \) satisfies either

a) \( \Delta = \bar{S}_N \) for all \( N \in \mathcal{N}' \) and \( \mathcal{N}' \) contains exactly two isomorphism classes.

b) \( \Delta = \bar{S}_N \) for all \( N \in \mathcal{N}' \) and \( \mathcal{N}' \) contains exactly two isomorphism classes.

Proof. This is immediate from Lemma 1 and Theorems 3, 4 and 5.

Theorem 8. Let \( \mathcal{N}' \) be as in Theorem 7. Then all \( N_i \oplus N_2 \oplus N_3 \oplus N_4 \) satisfy (**), but not all \( N_i \oplus N_2 \oplus N_3 \) satisfy (**), if and only if \( \mathcal{N}' \) satisfies one of the following:

a) All \( N \) in \( \mathcal{N}' \) are isomorphic to each other and \( [\Delta : \bar{S}_N] = 3 \).

b) There are no \( N \) in \( \mathcal{N}' \) such that \( [\Delta : \bar{S}_N] = 3 \), and if \( l=1 \) or \( 2 \) then \( \mathcal{N}' \) contains exactly one isomorphism class of \( N \) such that \( [\Delta : \bar{S}_N] = l \).

c) \( \Delta = \bar{S}_N \) for all \( N \in \mathcal{N}' \) and \( \mathcal{N}' \) contains exactly three isomorphism classes.

Proof. This is also easy by Lemma 1 and Theorems 3, 4 and 5.

The following example will illustrate what Theorem 8 intends to expose.

Example 1. Let \( n \) be a positive integer. Let \( k \) be a field, and \( x \) an in-
determinate. Put $L=k(x)$ and $K_i=k(x^i)$. Considering $L$ as a $K_i$-vector space, for any hyper-subspaces $V$ and $V'$ in $L$ we can show directly that $\{x \in L \mid xV \subseteq V\} = K_i$ and $yV = V'$ for some $y$ in $L$. Put

$$R = \begin{pmatrix}
L & L \cdots L & L \cdots L & L \cdots L & L \\
K_1 & \ddots & K_1 & \ddots & 0 \\
& K_1 & \ddots & K_1 & \ddots \\
& & K_1 & \ddots & K_1 \\
& & & K_1 & \ddots \\
& & & & K_1 \\
& & & & & \ldots
\end{pmatrix}$$

where $i_p \neq i_q$ if $p \neq q$. Then $e_{11}J = \sum_{p=1}^{i} \oplus L_{p1}$, where $L_{p1}=(0, 0, \ldots, L, 0, \ldots)$, $i=\sum_{j=1}^{i} n_j + q + 1$, and $L_{p1} \cong L_{q1}$ if $(p, q) \neq (q', q')$. Hence, every maximal submodule in $e_{11}J$ is of the form $A_{pq}=(0, L, \ldots, L, V, L, \ldots)$, where $V$ is a hyper-subspace of $L$ over $K_{i_p}$. Further, $A_{pq}=e_{11}y_{e_{11}}A_{q'}$ for some $y$ in $L$ and $\Delta(A_{pq}) = K_{i_p}$. Therefore, for each $i$ there exist exactly $n_i$ non-isomorphic classes of maximal submodules $N_j$ in $e_{11}J$ such that $[\Delta: \Delta(\Delta(N_j))] = i_j$.

**Theorem 9.** Let $R$ be a commutative and local artinian ring and let $N$ be a set of representatives of the isomorphism classes of serial modules with length two. In case $R/J$ is infinite, if there exists a natural number $n$ such that all $N_1 \oplus N_2 \oplus \cdots \oplus N_n$ ($N_i \in N$) satisfy (**), then $R$ is a serial ring, and conversely. In case $R/J$ is finite, there exists a natural number $n$ such that all $N_1 \oplus N_2 \oplus \cdots \oplus N_n$ satisfy (**).

**Proof.** Let $K=R/J$ and $J/J^2 = \sum_{j=1}^{m} A_j$ with simple $K$-modules $A_j$. If $K$ is infinite, then $A_1 \oplus A_2$ contains infinitely many submodules isomorphic to $A_1$. Hence $N$ is infinite provided $m \geq 2$. Therefore $J/J^2 = A_1$ if and only if there exists a natural number $n$ such that all $N_1 \oplus N_2 \oplus \cdots \oplus N_n$ ($N_j \in N$) satisfy (**), and hence by [3], Proposition 1, if and only if $R$ is serial. If $K$ is finite, then $J/J^2$ is also finite. Hence $N$ contains $m$ modules, and therefore all $N_1 \oplus N_2 \oplus \cdots \oplus N_m$ satisfy (**).

Similarly, we can prove

**Theorem 10.** Let $R$ be a local algebra of finite dimension over an algebra-
icelty closed field. Let \( N \) be a representative set of the isomorphism classes of serial modules with length two. Then there exists a natural number \( n \) such that all \( N_1 \oplus N_2 \oplus \cdots \oplus N_n (N_i \in N) \) satisfy (**') if and only if \( R \) is right serial.

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References


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