Title	On maximal submodules of a finite direct sum of hollow modules. II
Author(s)	Harada, Manabu
Citation	Osaka Journal of Mathematics. 1984, 21(3), p. 671-677
Version Type	VoR
URL	https://doi.org/10.18910/11086
rights	
Note	

Harada, M.
Osaka J. Math.
21 (1984), 671–677

ON MAXIMAL SUBMODULES OF A FINITE DIRECT SUM OF HOLLOW MODULES II

MANABU HARADA

(Received April 26, 1983)

Introduction

We have given, in [3], the structure of right artinian rings satisfying the following conditions: i) the Jacobson radical of a ring is square zero and ii) every submodule of a direct sum of hollow (local) modules is also a direct sum of hollow modules. The latter property cited above implies that every maximal submodule of a direct sum of $t+1$-copies of a hollow module with length t contains a direct summand.

In this paper, we shall study this property for any right artinian ring, and reproduce, in §1, the results similar to ones in [3] without the assumption that the Jacobson radical is square zero. In §2 we shall give a characterization of some rings in terms of the property above.

1 Property (**)

Let R be a ring with identity. In this paper, every R-module is a unitary right R-module. Let M be an R-module. We shall denote the Jacobson radical of M by $J(M)$ and the radical of R by J or $J(R)$, respectively. Throughout this paper we assume that R is a right artinian (semi-perfect) ring and every R-module M has the finite composition length, which we denote by $|M|$. If M has a unique maximal submodule $J(M)$, M is called *hollow* (*local*). In this case $M\approx eR/A$ for a primitive idempotent e and a right ideal A in eR.

Given a family $\boldsymbol{N}=\{N_i\}_{i=1}^t$ of (hollow) modules, we denote by $\mathrm{D}(\boldsymbol{N})$ the direct sum $\sum_{i=1}^t \oplus N_i$. If $N_i=N$ for a fixed module N, we indicate this by $N^{(t)}$. We have studied in [3] the following property:

(**) *Every maximal submodule of $D(\boldsymbol{N})$ contains a non-zero direct summand of $D(\boldsymbol{N})$.*

Since the above property is preserved by Morita equivalence, we may assume that R is a basic ring. Hence, from now on, we assume that R is a right artinian and basic ring. Let N be a hollow module with finite length. We put $\bar{N}=N/J(N)$, and $S\ (=S_N)=\mathrm{End}_R(N)$. Then $\Delta=\mathrm{End}_R(\bar{N})$ is a division ring. We have the natural homomorphism φ of S into Δ. It is clear that $\ker\varphi=J(S)$ and im φ is a subdivision ring of Δ, because $|N|<\infty$. We put im $\varphi=\bar{S}_N(=\bar{S})$. We assume $D=D(N_j, n)=\sum_i\oplus N_{1i}\oplus\sum_i\oplus N_{2i}\oplus\cdots\oplus\sum_i\oplus N_{ti}$, where $\bar{N}_{k1}\approx\bar{N}_{ki}$ and $\bar{N}_{i1}\not\approx\bar{N}_{j1}$ if $i\neq j$. Let M be a maximal submodule of D. Then $M\supset J(D)$ and $\bar{M}=M/J(D)$ is expressed as $\bar{M}=\sum_j\oplus\bar{M}_j$, where $\bar{M}_i$ is a maximal submodule of $\sum_k\oplus\bar{N}_{ik}$ for some i and $\bar{M}_j=\sum_k\oplus\bar{N}_{jk}$ for $j\neq i$. Therefore, when we study the property (**), we may assume $\bar{N}_i\approx\bar{N}_1$ for all i. We shall identify all $\mathrm{End}_R(\bar{N}_i)$ and denote them by Δ. Then $D=\bar{D}/J(D)$ is a Δ-vector space and $\bar{M}$ contains a subspace $\bar{M}'$ which is a maximal subspace of $\sum_{i\neq k}\oplus\bar{N}_i$ for some k $(n\geqslant 3)$, (cf. [3] §2). Hence M contains a submodule M' maximal in $\sum_{i\neq k}\oplus N_i$. Thus we obtain the following:

Lemma 1. *Let $N=\{N_i\}_{i=1}^{k'}$ be a family of hollow modules with finite length. If $D(N')$ satisfies (**) for a subfamily $N'=\{N_i\}_{i=1}^{k}$ of N with $k'>k\geqslant 2$, so does* $D(N)$ (for the case $k=1$, see Theorem 6 below).

Since R is semi-perfect, $N\approx eR/A$ for a primitive idempotent e and a right ideal A in eR. Then $\Delta=eRe/eJe$ and $S_N=\{x\in eRe\,|\,xA\subset A\}$. We sometimes denote $\bar{S}_N$ by $\Delta(A)$.

We have defined a max. quasiprojective module in [2]. This is nothing but $\Delta=\bar{S}_N$ in our case.

Theorem 1. *Let N be a hollow module with $|N|<\infty$. Then the following conditions are equivalent:*

1) *N is a max. quasiprojective.*
2) *$N^{(2)}$ has the lifting property of simple modules modulo the radical* (see [1]).
3) *$N^{(n)}$ has the above property for $n\geqslant 2$.*
4) *$N^{(2)}$ satisfies* (**).

Proof. It is clear from [1], [2], except 4).

1)$\leftrightarrow$4). This is clear from Theorem 2 below.

From Theorem 1 we are interested in case where $\Delta\supsetneqq\bar{S}_N=\bar{S}$. We may assume that Δ is a right $\bar{S}$-vector space and we denote the dimension of Δ by $[\Delta:\bar{S}]$.

Theorem 2 ([3], Lemma 5). *Let N, Δ, and $\bar{S}$ be as above. Then* $[\Delta:\bar{S}]$ $=k<\infty$ *if and only if $N^{(k+1)}$ satisfies* (**), *but $N^{(k)}$ does not.*

We shall give a more general result than Theorem 2. Let N_1 and N_2 be hollow modules with $|N_1|\leqslant|N_2|<\infty$. We assume $\bar{N}_1\approx\bar{N}_2$. We shall identify

$\bar{N}_1$ and $\bar{N}_2$ and denote $\mathrm{End}_R(\bar{N}_1)$ by Δ. Then we have the natural mapping φ of $\mathrm{Hom}_R(N_2, N_1)$ into Δ. Put im $\varphi=\Delta(N_2\ N_1)$ which is a right $\bar{S}_{N_2}$-subspace of Δ. We can express $N_i=eR/A_i$ $i=1, 2$. Then $|A_1|\geqslant|A_2|$ and $\mathrm{Hom}_R(N_2, N_1)$ $=\{x\in eRe\,|\,xA_2\subset A_1\}$.

Theorem 2′. *Let N_1 and N_2 be hollow modules with finite length ($\bar{N}_1\approx\bar{N}_2$). If $[\Delta/\Delta(N_2, N_1)\colon S_{N_2}]\leq k$, $D=N_2^{(k+1)}\oplus N_1$ satisfies (**). Conversely, if D satisfies (**) and $|N_2|\geqslant|N_1|$ then $[\Delta/\Delta(N_2\ N_1)\colon \bar{S}_{N_2}]\leqslant k$.*

Proof. We assume first $|N_2|\geqslant|N_1|$. We may assume $N_i=eR/A_i$ for $i=1, 2$. Put $D=eR/A_2\oplus\ \cdots\ \oplus eR/A_2\oplus eR/A_1$. Assume D satisfies (**). Let $\{\delta_1, \delta_2, \cdots, \delta_{k+1}\}$ be any set of elements in Δ. We shall express every element in D as $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_{k+1}, \tilde{a}_{k+2})$, where the a_i are in eR and $\tilde{a}_i$ is the residue class of a_i in eR/A. Take $\alpha_1=(\tilde{e}, \tilde{o}, \cdots, \tilde{o}, \tilde{\delta}_1)$, $\alpha_2=(\tilde{o}, \tilde{e}, \tilde{o}, \cdots, \tilde{o}, \tilde{\delta}_2)$, $\cdots$, α_{k+1} $=(\tilde{o}, \cdots, \tilde{o}, \tilde{e}, \tilde{\delta}_{k+1})$. Let M be the submodule of D generated by $\{\alpha_i\}_{i=1}^{k+1}$ and the elements in $J(D)$. Then M is a maximal submodule of D. Put $\bar{D}=D/J(D)$ $\supset\bar{M}=M/J(D)$. M contains a non-zero direct summand M_1 of D by (**). We may assume that M_1 is indecomposable and hence cyclic. Let β be its generator. Then $\beta=\alpha_1y_1+\alpha_2y_2+\ \cdots\ +\alpha_{k+1}y_{k+1}+j$, where the y_i are in eR and j is in $J(D)$. Since $\beta\notin J(D)$, we may assume that the y_i are in eRe and $\bar{y}_1\neq o$ (R is basic). Consider an epimorphism ψ of eR onto βeR given by setting $\psi(r)=\beta r$: $r\in eR$. Put $\beta=(\tilde{e}y_1+\tilde{j}_1, \tilde{e}y_2+\tilde{j}_2, \cdots, \tilde{e}y_{k+1}+\tilde{j}_{k+1}, \tilde{\delta}_1y_1+\tilde{\delta}_2y_2+\ \cdots$ $+\tilde{\delta}_{k+1}y_{k+1}+\tilde{j}_{k+2})$, where the j_p are in eJ, and put $z=ey_1+j_1$. Let x be in ker ψ. Then $zx=zex\in A_2$. Hence $x\in(ze)^{-1}A_2$ and so $|M_1|\geqslant|\beta eR|=|eR/\ker\psi|\geqslant$ $|eR/(ze)^{-1}A_2|=|eR/A_2|$. Since $|eR/A_2|\geqslant|eR/A_1|$ and M_1 is an indecomposable direct summand of D, $|M_1|\leqslant|eR/A_2|$. Hence $|M_1|=|eR/A_2|$, which implies ker $\psi=(ze)^{-1}A_2$. Therefore $(ey_i+j_i)(ze)^{-1}A_2\subseteq A_2$ for $i=2, \cdots, k+1$ and $(\delta_1y_1+\ \cdots\ +\delta_{k+1}y_{k+1}+j_{k+2})\ (ze)^{-1}A_2\subseteq A_1$. Accordingly, $\varphi((ey_i+j_i)\ (ze)^{-1})$ $=\bar{y}_i\bar{z}^{-1}\in\Delta(A_2)$ and $\varphi((\delta_1y_1+\ \cdots\ +\delta_{k+1}y_{k+1}+j_{k+2})(ze)^{-1})=\delta_1+\delta_2y_2z^{-1}+\ \cdots$ $+\delta_{k+1}y_{k+1}z^{-1}\in\Delta(A_2, A_1)$. Hence $[\Delta/\Delta\ (A_2, A_1)\colon \Delta(A_2)]\leqslant k$. Conversely, we assume that $[\Delta/\Delta(A_2, A_1)\colon \Delta(A_2)]\leqslant k$ and M a maximal submodule of D. Then $M\supset J(D)$. Let π_i be the projection of D onto the i-th component. If $\pi_j(\bar{M})$ $=0$ for some j, $M=\sum_{i\neq j}\oplus N_i\oplus J(N_j)$. Hence we may assume $\pi_i(\bar{M})\neq0$ for all i. Then M contains a basis $\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_{k+1}\}$ as above. Since $[\Delta/\Delta(A_2, A_1)$: $\Delta(A_2)]\leqslant k$, there exists a set $\{\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_{k+1}\}$ in $\Delta(A_2)$ such that $\sum\delta_iy_i\in\Delta(A_2,$ $A_1)$. Hence M contains an element $\beta=\sum\alpha_iy_i=(\tilde{y}_1, \tilde{y}_2, \cdots, \tilde{y}_{k+1}, \sum\tilde{\delta}_iy_i)$, and so M contains a direct summand of D by [3], Lemma 17. If we put $N_1=N_2$ in the theorem, then we have Theorem 2. Finally we assume $|N_2|<|N_1|$. Then there are no epimorphisms of N_2 onto N_1, and so $\Delta(N_2, N_1)=0$. Hence $[\Delta/\Delta(N_2, N_1)\colon \bar{S}_{N_2}]=[\Delta\colon \bar{S}_{N_2}]\leqslant k$. Therefore $D(k+2)$ satisfies (**) by Theorem 2 and Lemma 1.

The argument given in [3], §3 shows that the converse part in Theorem 2′ does not hold without the assumption $|N_2| \geqslant |N_1|$.

Theorem 3. *Let* $\{N_i\}_{i=1}^{t}$ $(t \geqslant 2)$ *be a set of hollow modules. Assume* $|N_i| = |N_1|$, $\bar{N}_i \approx \bar{N}_1$ *and* $[\Delta\colon \bar{S}_{N_i}] = k < \infty$ *for all* i. *Put* $D = N_1^{(s_1)} \oplus N_2^{(s_2)} \oplus \cdots \oplus N_t^{(s_t)}$, *where* $k+1 = \sum s_i$, *and* $s_i \geqslant 1$. Then D *satisfies* (**) *if and only if* $N_i \approx N_1$ *for all* i.

Proof. If $N_i \approx N_1$ for all i, then $D(N_i, k+1)$ satisfies (**) by Theorem 2. Conversely, assume the property above. Since $t \geqslant 2$ and $\sum s_i = k+1$, $s_t \leqslant k$. We shall first show that some two of $\{N_i\}_{i=1}^{t}$ are isomorphic to each other. According to Theorem 2 there exists a maximal submodule M_0 of $N_t^{(s_t)}$, which contains no non-zero direct summands of $N_t^{(s_t)}$. It is clear that M_0 is generated by $J(N_t^{(s_t)})$ and the set of elements $\{\theta_i = (\tilde{o}, \cdots, \overset{\overset{i}{\vee}}{\tilde{e}}, \tilde{o}, \cdots, \tilde{\delta}_{ti}) \in N^{(s_t)}\}$, where the δ_{tk} are elements of eRe. Let $\{\delta_{i1}, \delta_{i2}, \cdots, \delta_{is_i}\}$ be a set of independent elements of Δ over $\bar{S}_{N_i}$ for $i \leqslant t-1$. We can assume $N_i = eR/A_i$. Let M be the submodule of D generated by $\{\alpha_{ij} = (\tilde{o}, \cdots, \overset{\overset{k_{ij}}{\vee}}{\tilde{e}}, \tilde{o}, \cdots, \tilde{\delta}_{ij})\}_{i=1, j=1}^{t}$, where $k_{ij} = s_1 + \cdots + s_{i-1} + j$ and $\mathrm{J}(D)$. As in the proof of Theorem 2′, put $\beta = (\tilde{e}y_{11} + \tilde{j}_{11}, \cdots, \tilde{e}y_{1s_1} + \tilde{j}_{1s_1}, \cdots, \tilde{e}y_{ts_t-1} + \tilde{j}_{ts_t-1}, \tilde{\delta}_{11}y_{11} + \cdots + \tilde{\delta}_{ts_t-1}y_{ts_t-1} + \tilde{j}_{k+1})$ and assume that the direct summand M_1 of D, and hence of M, is generated by β. Then $M_1 = \beta R = \beta eR + (M_1 \cap J(D)) = \beta eR + J(M_1) = \beta eR$. Since $\beta \notin \mathrm{J}(D)$, some y_{ij} is not in eJe. Assume first that $\bar{y}_{ij} = o$ for all $i \leqslant t-1$. Then $\bar{M}_1 \subseteq \bar{N}_t^{(s_t)}$. Let π and π_{ij} be the projections of D onto M_1 and the jth component of $N_i^{(s_i)}$, respectively. Since $\bar{y}_{tj} \neq o$ for some j, $\pi_{ij}(\bar{M}_1) \neq 0$. Hence, M_1 being isomorphic to some N_p, $M_1 \approx N_t$. Since $\bar{y}_{ij} = o$ for $i \leqslant t-1$, $\beta = j + \theta$, where $j \in J(\sum_{i \leqslant t-1} \oplus N_i^{(s_i)})$, $\theta = \sum \theta_i y_{ti} + (o, \cdots, o, \tilde{j}_{t1}, \cdots, \tilde{j}_{k+1}) \in M_0 \subseteq N_t^{(s_t)}$. Hence $M_1 = \beta eR$ is epimorphic to $M_0^* = \theta eR$, and so $|M_1| \geqslant |M_0^*|$. Noting that $\pi(\bar{M}_0^*) = \pi(\bar{M}_1) = \bar{M}_1$ and M_1 is hollow, we know that $\pi | M_0^*$ is an epimorphism, and hence $\pi | M_0^*$ is an isomorphism. Therefore $D = M_0^* \oplus \ker \pi$, and so M_0^* $(\subseteq M_0)$ is a direct summand of $N_t^{(s_t)}$, which is a contradiction. Accordingly, $\bar{y}_{ij} \neq o$ for some $i \leqslant t-1$, say $i = j = 1$. If $\bar{y}_{pq} \neq o$ for $p \neq 1$, $\pi_{pq}(\bar{M}_1) \neq 0$. Hence $N_1 \approx M_1 \approx N_p$. Assume $\bar{y}_{pq} = o$ for all $p \neq 1$ and all q. Then we have the situation similar to the proof of Theorem 2′, and obtain $\bar{y}_{1k}\bar{y}_{11}^{-1} \in \Delta(A_1)$. Therefore $\bar{\delta}_{11}y_{11} + \bar{\delta}_{12}y_{12} \cdots + \bar{\delta}_{1s_1}y_{1s_1} \neq o$, and so $\pi_{ts_t}(\bar{M}_1) \neq 0$, which means $N_t \approx M_1 \approx N_1$. Thus we have shown that some two of $\{N_i\}_{i=1}^{t}$ are isomorphic to each other. Hence we can show the theorem by induction on t.

From the proof above we have

Theorem 4. *Let* N_1 *and* N_2 *be hollow modules with* $\bar{N}_1 \approx \bar{N}_2$. *Assume* $|N_2| = |N_1|$ *and* $[\Delta\colon \bar{S}_{N_2}] = k$. *Then* $N_1 \approx N_2$ *if and only if* $D(k+1) = N_2^{(k)} \oplus N_1$ *satisfies* (**).

Theorem 5. *Let $\{N_i\}_{i=1}^{t}$ $(t\geqslant 2)$ be a set of hollow modules. Assume $|N_i| = |N_1|$, $\bar{N}_i \approx \bar{N}_1$, and $[\Delta: \bar{S}_{N_i}] \geqslant k_i < \infty$. If $N_1^{(k_1)} \oplus N_2^{(k_2)} \oplus \cdots \oplus N_t^{(k_t)}$ satisfies (**), then some two of $\{N_i\}_{i=1}^{t}$ are isomorphic to each other.*

2 Direct sums of hollow modules with same length

We assume again that R is a right artinian ring.

Theorem 6. *Let $\boldsymbol{N}$ be a set of representatives of the isomorphism classes of hollow modules. Then there holds the following:*

1) *Every $N \in \boldsymbol{N}$ satisfies (**) if and only if R is semi-simple.*
2) *Every $N_1 \oplus N_2$ $(N_i \in \boldsymbol{N})$ satisfies (**) if and only if R is right serial.*

Proof. 1) Let e be an arbitrary primitive idempotent in R. If (**) is satisfied then eR is hollow and hence $eJ=0$, which proves that R is semi-simple.

2) If R is right serial then, for any $N \in \boldsymbol{N}$, $N \approx eR/A$ with a primitive idempotent e and a characteristic submodule A of eR. Hence $\Delta(A)=\Delta$, and therefore every $N_1 \oplus N_2$ $(N_i \in \boldsymbol{N})$ satisfies (**) by Theorem 2. Conversely, if every $N_1 \oplus N_2$ $(N_i \in \boldsymbol{N})$ satisfies (**) then, by Theorems 2 and 4, $\Delta = \Delta(A)$ and $eR/A \approx eR/B$ for any primitive idempotent e and maximal submodules A and B in eJ. Hence $B = xA$ for some unit element x in eRe. In view of [3], Proposition 1, we may assume that $J^2=0$. Then, since $\Delta=\Delta(A)$, we have $B=xA=A$. Therefore R is right serial.

Theorem 7. *Let $\boldsymbol{N}'$ be a set of hollow modules such that $|N_i| = |N_j|$ and $\bar{N}_i \approx \bar{N}_j$ for all $N_i, N_j \in \boldsymbol{N}'$. Then all $N_1 \oplus N_2 \oplus N_3$ satisfy (**), but not all $N_1 \oplus N_2$ $(N_i \in \boldsymbol{N}')$, if and only if $\boldsymbol{N}'$ satisfies either*

a) *all N in $\boldsymbol{N}'$ are isomorphic to each other and $[\Delta: \bar{S}_N]=2$, or*

b) *$\Delta = \bar{S}_N$ for all $N \in \boldsymbol{N}'$ and $\boldsymbol{N}'$ contains exactly two isomorphism classes.*

Proof. This is immediate from Lemma 1 and Theorems 3, 4 and 5.

Theorem 8. *Let $\boldsymbol{N}'$ be as in Theorem 7. Then all $N_1 \oplus N_2 \oplus N_3 \oplus N_4$ satisfy (**), but not all $N_1 \oplus N_2 \oplus N_3$ $(N_i \in \boldsymbol{N}')$, if and only if $\boldsymbol{N}'$ satisfies one of the following:*

a) *All N in $\boldsymbol{N}'$ are isomorphic to each other and $[\Delta: \bar{S}_N]=3$.*

b) *There are no N in $\boldsymbol{N}'$ such that $[\Delta: \bar{S}_N]=3$, and if $l=1$ or 2 then $\boldsymbol{N}'$ contains exactly one isomorphism class of N such that $[\Delta: \bar{S}_N]=l$.*

c) *$\Delta = \bar{S}_N$ for all $N \in \boldsymbol{N}'$ and $\boldsymbol{N}'$ contains exactly three isomorphism classes.*

Proof. This is also easy by Lemma 1 and Theorems 3, 4 and 5.

The following example will illustrate what Theorem 8 intends to expose.

Example 1. Let n be a positive integer. Let k be a field, and x an indeterminate. Put $L=k(x)$ and $K_i=k(x^i)$. Considering L as a K_n-vector space, for any hyper-subspaces V and V' in L we can show directly that $\{x\in L \mid xV\subseteq V\}=K_i$ and $yV=V'$ for some y in L. Put

$$R=\begin{pmatrix} L & \overbrace{L\ \cdots\ L}^{n_1} & \overbrace{L\ \cdots\ L}^{n_2} & \overbrace{L\ \cdots\ L}^{n_3} & \cdots \\ & K_{i1} & & & \\ & \ddots & & & \\ & K_{i1} & & & 0 \\ & & K_{i2} & & \\ & & \ddots & & \\ & & K_{i2} & & \\ & & & K_{i3} & \\ & & & \ddots & \\ & & & K_{i3} & \\ 0 & & & & \ddots \end{pmatrix}$$

where $i_p\neq i_p$ if $p\neq q$. Then $e_{11}J=\sum_p\sum_{q=1}^{n_p}\oplus L_{pq}$, where $L_{pq}=(0, 0, \cdots, \overset{i}{\check{L}}, 0, \cdots)$, $i=\sum_{j=1}^{p-1}n_j+q+1$, and $L_{pq}\not\approx L_{p'q'}$ if $(p, q)\neq(p', q')$. Hence, every maximal submodule in $e_{11}J$ is of the form $A_{pq}=(0, L, \cdots, L, \overset{i}{\check{V}}, L, \cdots)$, where V is a hyper-subspace of L over K_{i_p}. Further, $A_{pq}=e_{11}ye_{11}A'_{pq}$ for some y in L and $\Delta(A_{pq})=K_{i_p}$. Therefore, for each i there exist exactly n_i non-isomorphic classes of maximal submodules N_j in $e_{11}J$ such that $[\Delta\colon \Delta(N_j)]=i_j$.

Theorem 9. *Let R be a commutative and local artinian ring and let $\boldsymbol{N}$ be a set of representatives of the isomorphism classes of serial modules with length two. In case R/J is infinite, if there exists a natural number n such that all $N_1\oplus N_2\oplus\cdots\oplus N_n$ $(N_i\in\boldsymbol{N})$ satisfy (**) then R is a serial ring, and conversely. In case R/J is finite, there exists a natural number n such that all $N_1\oplus N_2\oplus\cdots\oplus N_n$ satisfy (**).*

Proof. Let $K=R/J$ and $J/J^2=\sum_{j=1}^{m}\oplus A_j$ with simple K-modules A_j. If K is infinite, then $A_1\oplus A_2$ contains infinitely many submodules isomorphic to A_1. Hence $\boldsymbol{N}$ is infinite provided $m\geq 2$. Therefore $J/J^2=A_1$ if and only if there exists a natural number n such that all $N_1\oplus N_2\oplus\cdots\oplus N_n$ $(N_j\in\boldsymbol{N})$ satisfy (**), and hence by [3], Proposition 1, if and only if R is serial. If K is finite, then J/J^2 is also finite. Hence $\boldsymbol{N}$ contains m modules, and therefore all $N_1\oplus N_2\oplus\cdots\oplus N_{m+1}$ satisfy (**).

Similarly, we can prove

Theorem 10. *Let R be a local algebra of finite dimension over an algebra-*

ically closed field. Let $\boldsymbol{N}$ *be a representative set of the isomorphism classes of serial modules with length two. Then there exists a natural number n such that all* $N_1 \oplus N_2 \oplus \cdots \oplus N_n$ $(N_i \in \boldsymbol{N})$ *satisfy* (**) *if and only if* R *is right serial.*

The author would like to express his hearty thanks to the referee who was patient to make up the original manuscript understandable.

References

[1] M. Harada: *On lifting property on direct sums of hollow modules,* Osaka J. Math. **17** (1980), 783–791.
[2] ———: *On maximal quasiprojective modules*, J. Austral. Math. Soc. Ser. A **35** (1983), 357–368.
[3] ———: *On maximal submodules of a finite direct sum of hollow modules* I, Osaka J. Math. **21** (1984), 649–670.
[4] ———: *Serial rings and direct decompositions*, J. Pure Appl. Algebra **31** (1984), 55–61.

Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558,
Japan