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DIFFERENTIAL HOPF ALGEBRAS MODELLED ON K-THEORY MOD p. I

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Introduction

In the present work the authors study properties of Hopf structures modelled on complex K-theory mod p of H-spaces. Namely, let X be an H-space which has a homotopy type of finite CW-complexes, and p a prime. For each choice of the admissible external multiplication μ_p of $K^*(\ ;Z_p)$ [2], $K^*(X; Z_p)$ gains a structure of algebra as well as that of coalgebra, hence a kind of structure like Hopf algebra. We study structures modelled on these structures. Our results were already partly announced in [3] and also reported in Neuchâtel conference on H-spaces, 1970 (to appear in Springer Lecture Notes Series) by the first named author.

When we compare our structures like Hopf algebras with the classical Hopf algebras modelled on the ordinary homology and cohomology of H-spaces, we will find two significant differences. The first point of difference is that the classical Hopf algebras are non-negatively graded and can be discussed sometimes by making use of an induction argument on degrees (cf., [10] etc.), but our structures are Z_2-graded and we cannot use such arguments. We use instead sometimes two filtrations (F-filtrations by algebra structure and G-filtrations by coalgebra structure) originally due to Browder [6], or some other arguments.

The second point is that in the classical Hopf algebras the relation

$$\psi\varphi = (\varphi\otimes\varphi)(1\otimes T\otimes 1)(\psi\otimes\psi)$$

of Milnor-Moore [10] is important. But in our structures the above relation may not hold in general. The above relation in the classical case is essentially based on the *commutativity of external multiplications* of ordinary homology and cohomology. But the external multiplication μ_p of $K^*(\ ; Z_p)$ may not be commutative in general, and is never commutative in case $p=2$ [2]. Fortunately the deviation formula from the commutativity is known [2]. So we regard this non-commutativity as a kind of commutativity relation replacing the ordinary twisting morphism T by the λ-modified one T_λ, (2.17). Actually we find the above relation holds also in our structures if we replace T by a suitable T_λ. In the definition of T_λ a differential comes in. Thus we talk of the differential Hopf algebras (in a modified sense) from the beginning.

When the differential is trivial we find a Hopf algebra (in Z_2-graded but not modified sense). Thus our theory contains a theory on Hopf algebras in Z_2-graded sense. Generally we do not assume the finite dimensionality even though our main interest is to apply for $K^*(X; Z_p)$ with finite CW-H-space X.

In §1 we define algebras, coalgebras and two filtrations, and discuss basic properties of these filtrations, associated graded algebras and coalgebras, some consequences of making use of these filtrations. In §2 we first formulate some basic properties of ordinary twisting morphism or, more generally, of signed permutations by making use of a semi-simplicial terminology. Then we define λ-modified twisting morphism and λ-modified permutations, and see that they also behave the same basic properties as ordinary ones. This justifies to use λ-modified permutations instead of ordinary signed permutations. In §3 we discuss differential algebras, coalgebras and their spectral sequences associated with the basic filtrations. In §4 we define λ-modified differential Hopf algebras and discuss their spectral sequences, primitivity and coprimitivity, and some related propositions. Here we check some propositions and theorems of [10] work also for our λ-modified differential Hopf algebras. In §5 we define the notion of derived Hopf algebras of λ-modified differential Hopf algebras, which will be used in §6 in connection with the characterization of primitivity and coprimitivity. In §6 we discuss our version of Milnor-Moore criterions of primitivity and coprimitivity (cf.,[10], Proposition 4.20), Theorem (6.6), (6.6*), (6.15) and (6.15*). In case $p=2$ and $\lambda d \neq 0$ the criterions fail to be criterions since we could not prove the inverse theorem in this case. Still it says something, and we could establish the primitivity or coprimitivity of all terms of spectral sequences associated with basic filtrations, Theorem (6.17).

1. Basic filtrations

1.1. All modules will be understood to be defined over a field K throughout the present work. A *Z_2-graded* module (or *G_2-module*) M is a module over K graded with indices in Z_2, i.e.,

$$M = M_0 \oplus M_1 .$$

Elements of M_0 (or of M_1) will be called of *even* type (or of *odd* type). Morphisms of G_2-modules are understood to preserve Z_2-gradings. A G_2-module M has a canonical involution σ such that

$$\sigma | M_0 = 1 \quad \text{and} \quad \sigma | M_1 = -1 . \tag{1.1}$$

When the characteristic of K is different from 2 the involution σ characterizes the Z_2-grading conversely.

Let M and N be G_2-modules. The tensor product $M \otimes N$ is a G_2-module defined by

$$(M\otimes N)_0 = M_0\otimes N_0+M_1\otimes N_1, \quad (M\otimes N)_1 = M_0\otimes N_1+M_1\otimes N_0 .$$

The ground field K is always understood as to be Z_2-graded by $K_0=K$ and $K_1=\{0\}$. Then we have canonical isomorphisms

$$M\approx M\otimes K\approx K\otimes M \tag{1.2}$$

of G_2-modules.

By a graded G_2-module M we mean a G_2-module with a non-negative grading

$$M = \oplus_{n\geqslant 0}M^n$$

such that M is (Z, Z_2)-bigraded, i.e.,

$$M_i = \oplus_{n\geqslant 0}M_i^n, \quad M_i^n = M_i\cap M^n, \quad i\in Z_2 .$$

Similarly a bigraded G_2-module is defined.

1.2. An *algebra* A is a G_2-module equipped with morphisms of G_2-modules

φ: $A\otimes A\to A$, called by a *multiplication,*
η: $K\to A$, called by a *unit.*

and

ε: $A\to K$, called by an *augmentation*,

which satisfy relations

$$\varphi(\eta\otimes 1) = \varphi(1\otimes\eta) = 1_A$$

via the canonical identification (1.2) for $M=A$,

$$\varphi_K(\varepsilon\otimes\varepsilon) = \varepsilon\varphi \quad\text{and}\quad \varepsilon\eta = 1_K .$$

Notice that ε: $(A, \varphi, \eta)\to K$ is a morphism of G_2-modules equipped with multiplications and units.

A coalgebra A is a G_2-module equipped with morphisms of G_2-modules

ψ: $A\to A\otimes A$, called by a *comultiplication,*
ε: $A\to K$, called by a *counit,*

and

η: $K\to A$, called by an *augmentation,*

which satisfy relations

$$(\varepsilon\otimes 1)\psi = (1\otimes\varepsilon)\psi = 1_A ,$$
$$\psi\eta = (\eta\otimes\eta)\psi_K \quad\text{and}\quad \varepsilon\eta = 1_K .$$

An algebra (or a coalgebra) A is called to be *associative* when the multiplication φ (or the comultiplication ψ) satisfies

$$\varphi(\varphi\otimes 1)=\varphi(1\otimes\varphi) \qquad (\text{or } (\psi\otimes 1)\psi=(1\otimes\psi)\psi)\,.$$

An algebra (or a coalgebra) A is called to be *graded* when the underlying G_2-module is graded and all structure morphisms preserve degrees. A graded algebra (or coalgebra) A is called to be *connected* when $A^0\cong K$, as usual.

Let A be an algebra (or a coalgebra). Since $\varepsilon\eta=1_K$, we have a direct sum decomposition of G_2-modules

$$\begin{aligned} A &= \operatorname{Im}\eta\oplus\operatorname{Ker}\varepsilon \\ &= K\oplus\bar{A}\,, \end{aligned} \tag{1.3}$$

where K and $\operatorname{Im}\eta$ are identified through the isomorphism η: $K\cong\operatorname{Im}\eta$, and $\operatorname{Ker}\varepsilon$ is denoted by $\bar{A}$. We regard as $1\in A$ *via* the above identification. We can also identify $\bar{A}$ with $\operatorname{Coker}\eta$. Let

$$\iota\colon\ \bar{A}\subset A \quad \text{and} \quad \rho\colon\ A\to\bar{A}$$

be the inclusion and projection respectively. The map

$$\begin{gathered} \bar{\varphi}=\rho\varphi(\iota\otimes\iota)\colon\ \bar{A}\otimes\bar{A}\to\bar{A} \\ (\text{or } \bar{\psi}=(\rho\otimes\rho)\psi\iota\colon\ \bar{A}\to\bar{A}\otimes\bar{A}) \end{gathered} \tag{1.4}$$

is called the *reduced* multiplication (or comultiplication).

Images of morphisms are called *sub algebras* (or *sub coalgebras*). Kernels of morphisms are called *ideals* (or *coideals*). Hence every ideal (or coideal) J of A is contained in $\bar{A}$, and the quotient A/J becomes an algebra (or a coalgebra) called a *quotient algebra* (or a *quotient coalgebra*). If A is an algebra and J is an ideal of A, then $K\oplus J$ is a sub algebra. If A is a coalgebra and B is a sub coalgebra, then $\bar{B}$ is a coideal of A.

Let A be an algebra (or a coalgebra) and M be a G_2-module. When M is equipped with a morphism of G_2-modules

$$\varphi_M\colon\ A\otimes M\to M \qquad (\text{or } \psi_M\colon\ M\to A\otimes M)$$

such that

$$\varphi_M(\eta_A\otimes 1_M)=1_M \qquad (\text{or } (\varepsilon_A\otimes 1_M)\psi_M=1_M)$$

via the canonical identification $K\otimes M=M$, then M is called a left *A-module* (or *A-comodule*). Right A-modules (or A-comodules) are similarly defined. When A is associative and φ_M (or ψ_M) is associative in the sense that

$$\begin{gathered} \varphi_M(\varphi_A\otimes 1)=\varphi_M(1\otimes\varphi_M) \\ (\text{or } (\psi_A\otimes 1)\psi_M=(1\otimes\psi_M)\psi_M)\,, \end{gathered}$$

then M is called to be *associative.*

1.3. Let A be an algebra (or coalgebra) and $A^{\otimes k}=A\otimes\cdots\otimes A$ stand for the k-th power of A in the sense of tensor products. Let

$$(1.5) \qquad \varphi^{(i)}=1\otimes\cdots\otimes1\otimes\varphi\otimes1\otimes\cdots\otimes1:\ A^{\otimes k+1}\to A^{\otimes k}$$

$$(\text{or } (1.5^*) \qquad \psi^{(i)}=1\otimes\cdots\otimes1\otimes\psi\otimes1\otimes\cdots\otimes1:\ A^{\otimes k}\to A^{\otimes k+1})$$

denote a map containing φ (or ψ) in the i-th tensor factor for $1\leqslant i\leqslant k$.

Define a set W_k of k-tuples of integers by

$$(1.6) \qquad W_k=\{(i_1,\dots,i_k);\ 1\leqslant i_s\leqslant s,\ 1\leqslant s\leqslant k\},\ k\geqslant1\,,$$

and a map

$$\varphi_k^{w_k}:\ A^{\otimes k+1}\to A$$
$$(\text{or } \psi_k^{w_k}:\ A\to A^{\otimes k+1})$$

for each $w_k\in W_k$ by

$$(1.7) \qquad \varphi_k^{w_k}=\varphi^{(i_1)}\varphi^{(i_2)}\cdots\varphi^{(i_k)}$$

$$(\text{or } (1.7^*) \qquad \psi_k^{w_k}=\psi^{(i_k)}\psi^{(i_{k-1})}\cdots\psi^{(i_1)})\,.$$

Putting

$$\bar\varphi_k^{w_k}=\varphi_k^{w_k}(\iota\otimes\cdots\otimes\iota):\ \bar A^{\otimes k+1}\to A$$
$$(\text{or } \bar\psi_k^{w_k}=(\rho\otimes\cdots\otimes\rho)\psi_k^{w_k}:\ A\to\bar A^{\otimes k+1})\,,$$

we define a decreasing (or increasing) filtration $\{F^kA\}$ (or $\{G^kA\}$) of A by

$$(1.8) \qquad F^0A=A,\quad F^1A=\bar A\,,$$
$$F^{k+1}A=\textstyle\sum_{w_k\in W_k}\operatorname{Im}\bar\varphi_k^{w_k}\quad\text{for}\quad k\geqslant1$$

$$(\text{or } (1.8^*) \qquad G^0A=K\,,$$
$$G^kA=\textstyle\bigcap_{w_k\in W_k}\operatorname{Ker}\bar\psi_k^{w_k}\quad\text{for}\quad k\geqslant1)\,,$$

and call it the *F-filtration* (or *G-filtration*) of A. If A is associative then these filtrations coincide with those of [1], hence essentially to those of [6].

Sometimes we denote as

$$\varphi_0=\varphi_0^{(0)}=1_A \qquad (\text{or } \psi_0=\psi_0^{(0)}=1_A)$$

and

$$\bar\varphi_0=\varphi_0\iota=\iota:\ \bar A\to A$$
$$(\text{or } \bar\psi_0=\rho\psi_0=\rho:\ A\to\bar A)\,.$$

And putting

$$\bar{\varphi}_n = \sum\circ(\oplus_{w_n\in W_n}\bar{\varphi}_n^{w_n}) \qquad (\text{or } \bar{\psi}_n = (\oplus_{w_n\in W_n}\bar{\psi}_n^{w_n})\circ\Delta), \quad n\geqslant 1\,,$$

where $\Delta\colon A\to A\oplus\cdots\oplus A$ and $\Sigma\colon A\oplus\cdots\oplus A\to A$ are the diagonal map and the dual, we have

$$F^{n+1}A = \operatorname{Im}\bar{\varphi}_n \qquad (\text{or } G^nA = \operatorname{Ker}\bar{\psi}_n)$$

for $n\geqslant O$.

The associated graded G_2-module is

$$(1.9) \qquad E_0(A) = \textstyle\sum_{k>0}E_0^kA, \quad E_0^kA = F^kA/F^{k+1}A\,,$$

$$(\text{or } (1.9^*) \qquad {}_0E(A) = \textstyle\sum_{k>0}{}_0E^kA, \quad {}_0E^kA = G^kA/G^{k-1}A)\,.$$

We put

$$(1.10) \qquad Q^kA = \bar{A}/F^{k+1}A$$

$$(\text{or } (1.10^*) \qquad P^kA = \bar{A}\cap G^kA)$$

for $k\geqslant 1$. In the case of coalgebra we have a direct sum decomposition

$$G^kA = K\oplus P^kA, \quad k\geqslant 1\,.$$

According to a notation of [10] we write also as

$$Q(A) = Q^1A \qquad (\text{or } P(A) = P^1A)$$

and call it the *module of indecomposable elements* (or *of primitive elements*) of A. In the case of algebra, if a sub G_2-module M of $\bar{A}$ is mapped isomorphically onto $Q(A)$ through the projection $\bar{A}\to Q(A)$, then we call that *M represents the module of indecomposable elements of A.*

1.4. The above filtrations of algebras (or coalgebras) can be naturally generalized to filtrations of A-modules (or -comodules). Namely, let A be an algebra (or a coalgebra) and M a left A-module (or -comodule). We define the map $\varphi^{(i)}$ (or $\psi^{(i)}$) of (1.5) for M by replacing the last tensor factor of $A^{\otimes k}$ and $A^{\otimes k+1}$ by M. Then, define the map

$$\varphi_k^{w_k}\colon A^{\otimes k}\otimes M\to M$$

$$(\text{or } \psi_k^{w_k}\colon M\to A^{\otimes k}\otimes M)$$

by the same formula (1.7) for each $w_k\in W_k$. Finally, putting

$$\bar{\varphi}_k^{w_k} = \varphi_k^{w_k}(\iota\otimes\cdots\otimes\iota\otimes 1_M)\colon \bar{A}^{\otimes k}\otimes M\to M$$

$$(\text{or } \bar{\psi}_k^{w_k} = (\rho\otimes\cdots\otimes\rho\otimes 1_M)\psi_k^{w_k}\colon M\to\bar{A}^{\otimes k}\otimes M)\,,$$

we define a decreasing (or increasing) filtration $\{F^k M\}$ (or $\{G^k M\}$) of M by

$$(1.11)\qquad \begin{aligned} F^0M &= F^1M = M\,, \\ F^{k+1}M &= \Sigma_{w_k\in W_k} \operatorname{Im} \bar{\varphi}_k^{w_k} \qquad \text{for} \quad k\geqslant 1 \end{aligned}$$

$$(\text{or } (1.11^*)\qquad \begin{aligned} G^0M &= \{0\}\,, \\ G^kM &= \cap_{w_k\in W_k} \operatorname{Ker} \bar{\psi}_k^{w_k} \qquad \text{for} \quad k\geqslant 1)\,. \end{aligned}$$

The associated graded G_2-module is

$$(1.12)\qquad E_0(M) = \textstyle\sum_k E_0^k M, \quad E_0^k M = F^k M/F^{k+1}M$$

$$(\text{or } (1.12^*)\qquad {}_0E(M) = \textstyle\sum_k {}_0E^k M, \quad {}_0E^k M = G^k M/G^{k-1}M)\,.$$

We put

$$(1.13)\qquad Q^k M = M/F^{k+1}M \qquad (\text{or } P^k M = G^k M)$$

for $k\geqslant 1$ in analogy with (1.10). Then we have

$$(1.14)\qquad Q^1M = E_0^1 M = K\otimes_A M$$

$$(\text{or } (1.14^*)\qquad P^1M = {}_0E^1M = K\square_A M)$$

by using the notations of [10], p.215 and p.219.

Let A be an algebra (or a coalgebra). $\bar{A}$ is a left A-module (or -comodule) with the structure map

$$\varphi_{\bar{A}} = \rho\varphi_A(1\otimes\iota)\colon\ A\otimes\bar{A}\to\bar{A}$$
$$(\text{or } \psi_{\bar{A}} = (1\otimes\rho)\psi_A\iota\colon\ \bar{A}\to A\otimes\bar{A})\,.$$

Observe that

$$\varphi(\iota\otimes\iota) = \iota\rho\varphi(\iota\otimes\iota) \qquad (\text{or } (\rho\otimes\rho)\psi = (\rho\otimes\rho)\psi\iota\rho)\,,$$

then by a simple computation we see that

$$F^k\bar{A} = F^k A \qquad (\text{or } G^k\bar{A} = \bar{A}\cap G^k A)$$

for $k\geqslant 1$. Thus

$$Q^k\bar{A} = Q^k A \qquad (\text{or } P^k\bar{A} = P^k A)$$

for $k\geqslant 1$. This justifies the definition (1.13).

1.5. Let A be an algebra and M a left A-module. For each $w_{i-1}\in W_{i-1}$ and $w_{n-i-1}\in W_{n-i-1}$, the set of (1.6), such that $1\leqslant i\leqslant n$-1, putting $w_{n-1}=(1, w_{i-1}, w_{n-i-1}+i)\in W_{n-1}$, we have

$$\varphi_{n-1}^{w_{n-1}} = \varphi(\varphi_{i-1}^{w_{i-1}} \otimes \varphi_{n-i-1}^{w_{n-i-1}})$$

and
$$\bar{\varphi}_{n-1}^{w_{n-1}} = \varphi(\bar{\varphi}_{i-1}^{w_{i-1}} \otimes \bar{\varphi}_{n-i-1}^{w_{n-i-1}}) .$$

From this relation we see that

(1.15) $$\varphi_M(\textstyle\sum_{i=0}^n F^i A \otimes F^{n-i} M) \subset F^n M \quad \text{for} \quad n \geqslant 0 .$$

Applying this to $M=\bar{A}$ we see that

(1.16) $$\varphi_A(\textstyle\sum_{i=0}^n F^i A \otimes F^{n-i} A) \subset F^n A \quad \text{for} \quad n \geqslant 0 .$$

(1.16) shows that φ_A induces a multiplication

$$E_0(\varphi)\colon\ E_0(A) \otimes E_0(A) \to E_0(A)$$

of $E_0(A)$ and

(1.17) *$E_0(A)$ is a graded connected algebra.*

The reduced multiplication $\bar{\varphi}$ induces the reduced multiplication

$$E_0(\bar{\varphi})_n = \overline{E_0(\varphi)_n}\colon\ \textstyle\sum_{i=1}^{n-1} E_0^i A \otimes E_0^{n-i} A \to E_0^n A$$

of $E_0(A)$ for each $n \geqslant 2$.

(1.18) **Proposition.** *$E_0(\bar{\varphi})_n$ is surjective for all $n \geqslant 2$.*

Proof. Remark that for every $w_n \in W_n$, $n \geqslant 1$, there exist $w_k \in W_k$ and $w_{n-k-1} \in W_{n-k-1}$ for some k, $0 \leqslant k < n$, such that

$$\varphi_n^{w_n} = \varphi(\varphi_k^{w_k} \otimes \varphi_{n-k-1}^{w_{n-k-1}}) .$$

(This can be easily seen by observing a tree of w_n). Then

$$\bar{\varphi}_n^{w_n} = \varphi(\bar{\varphi}_k^{w_k} \otimes \bar{\varphi}_{n-k-1}^{w_{n-k-1}}) .$$

This shows (1.18) immediately.

(1.19) **Corollary.** *$E_0^1 A$ represents the module of indecomposable elements of $E_0(A)$.*

(1.15) shows that φ_M induces a module structure

$$E_0(\varphi_M)\colon\ E_0(A) \otimes E_0(M) \to E_0(M) ,$$

and we have

(1.20) *$E_0(M)$ is a graded left $E_0(A)$-module.*

By a reduced A-module structure of M we mean

$$\bar{\varphi}_M = \varphi_M(\iota\otimes 1)\colon\ \bar{A}\otimes M \to M\,.$$

Then we obtain the reduced $E_0(A)$-module structure

$$\overline{E_0(\varphi_M)} = \textstyle\sum_{n\geqslant 2}\overline{E_0(\varphi_M)_n}\,,$$
$$E_0(\bar{\varphi}_M)_n = \overline{E_0(\varphi_M)_n}\colon\ \textstyle\sum_{i=1}^{n-1} E_0^i A\otimes E_0^{n-i}M \to E_0^n M$$

of $E_0(M)$. By the same proof as (1.18) we obtain

(1. 21) **Proposition.** $E_0(\bar{\varphi}_M)_n$ *is surjective for* $n\geqslant 2$.

(1. 22) **Corollary.** $Q^1E_0(M)\cong E_0^1 M = Q^1M$.

1.6. Next, let A be a coalgebra and M a left A-comodule. By a reduced A-comodule structure of M we mean

$$\bar{\psi}_M = (\rho\otimes 1)\psi_M\colon\ M \to \bar{A}\otimes M\,.$$

As a dual of (1.15) we obtain

(1. 15*) $$\psi_M(G^nM)\subset \textstyle\sum_{r=0}^{n} G^rA\otimes G^{n-r}M \quad \text{for} \quad n\geqslant 1\,.$$

Proof. It is sufficient to show that

$$\bar{\psi}_M(P^nM)\subset \textstyle\sum_{r=1}^{n-1} P^rA\otimes P^{n-r}M \quad \text{for} \quad n\geqslant 2\,.$$

For each $w_i\in W_i$ and $w_j\in W_j$, the set of (1.6), such that $i+j=n$-1, $0\leqslant i<n$, putting $w_n=(1,\ w_i,\ w_j+i+1)\in W_n$, we have

$$(\bar{\psi}_i^{w_i}\otimes\bar{\psi}_j^{w_j})\bar{\psi}_M(P^nM) = \bar{\psi}_n^{w_n}(P^nM) = 0\,.$$

Thus

$$\begin{aligned}\bar{\psi}_M(P^nM) &\subset \textstyle\bigcap_{i+j=n-1,w_i,w_j}\operatorname{Ker}(\bar{\psi}_i^{w_i}\iota\otimes\bar{\psi}_j^{w_j})\\ &= \textstyle\bigcap_{i+j=n-1}\operatorname{Ker}(\bar{\psi}_i\iota\otimes\bar{\psi}_j)\\ &= \textstyle\bigcap_{i+j=n-1}(P^iA\otimes M+\bar{A}\otimes P^jM)\,.\end{aligned}$$

In particular

$$\begin{aligned}\bar{\psi}_M(P^nM) &\subset (\bar{A}\otimes P^{n-1}M)\cap(P^{n-1}A\otimes M)\\ &\subset P^{n-1}A\otimes P^{n-1}M\,.\end{aligned}$$

Then, by an induction on t we obtain

$$\bar{\psi}_M(P^nM)\subset P^{n-t}A\otimes P^{n-1}M + \textstyle\sum_{k=1}^{t-1} P^{n-k}A\otimes P^kM$$

for $t\geqslant 1$. Finally, putting $t=n$-1 we complete the proof.

Applying (1.15*) to $M=\bar{A}$ we see that

(1. 16*) $$\psi_A(G^nA)\subset\sum_{r=0}^{n}G^rA\otimes G^{n-r}A \quad \text{for} \quad n\geqq 0\,.$$

(1.16*) shows that ψ_A induces a comultiplication

$${}_0E(\psi)\colon {}_0E(A)\to{}_0E(A)\otimes{}_0E(A)$$

of ${}_0E(A)$ and

(1. 17*) ${}_0E(A)$ *is a graded connected coalgebra.*

The reduced comultiplication $\bar{\psi}_A$ induces the reduced comultiplication

$${}_0E(\bar{\psi})_n=\overline{{}_0E(\psi)_n}\colon {}_0E^nA\to\sum_{r=1}^{n-1}{}_0E^rA\otimes{}_0E^{n-r}A$$

of ${}_0E(A)$ for each $n\geqq 2$.

(1. 18*) **Proposition.** ${}_0E(\bar{\psi})_n$ *is injective for all* $n\geqq 2$.

Proof. Let $x\in\bar{A}$ be such that

$$\bar{\psi}(x)\in\sum_{i=1}^{n-2}P^iA\otimes P^{n-i-1}A, \quad n\geqq 2.$$

Since for every $w_{n-1}\in W_{n-1}$ there exist $w_{k-1}\in W_{k-1}$ and $w_{n-k-1}\in W_{n-k-1}$ such that

$$\psi_{n-1}^{w_{n-1}}=(\psi_{k-1}^{w_{k-1}}\otimes\psi_{n-k-1}^{w_{n-k-1}})\psi\,,$$

we have

$$\begin{aligned}\bar{\psi}_{n-1}^{w_{n-1}}(x)&=(\bar{\psi}_{k-1}^{w_{k-1}}\otimes\bar{\psi}_{n-k-1}^{w_{n-k-1}})\bar{\psi}(x)\\&\in\sum_{i=1}^{n-2}(\bar{\psi}_{k-1}^{w_{k-1}}\otimes\bar{\psi}_{n-k-1}^{w_{n-k-1}})(P^iA\otimes P^{n-i-1}A)=\{0\}\,.\end{aligned}$$

Thus

$$x\in P^{n-1}A\,.$$ q.e.d.

(1. 19*) **Corollary.** ${}_0E^1A=P({}_0E(A))$.

(1. 15*) shows that ψ_M induces a comodule structure

$${}_0E(\psi_M)\colon {}_0E(M)\to{}_0E(A)\otimes{}_0E(M)\,.$$

and we have

(1. 20*) ${}_0E(M)$ *is a graded left* ${}_0E(A)$*-comodule.*

The reduced A-comodule structure $\bar{\psi}_M$ of M induces the reduced ${}_0E(A)$-comodule structure of ${}_0E(M)$:

$$\overline{{}_0E(\psi_M)} = \sum_{n\geqslant 1} \overline{{}_0E(\psi_M)_n}\,,$$
$${}_0E(\bar{\psi}_M) = {}_0\overline{E(\psi_M)_n}\colon\ {}_0E^nM \to \sum_{i=1}^{n-1} {}_0E^iA \otimes {}_0E^{n-i}M\,.$$

By the same proof as (1.18*) we obtain

(1. 21*) **Proposition.** *${}_0E(\bar{\psi}_M)_n$ is injective for $n \geqslant 2$.*

(1. 22*) **Corollary.** $P^1{}_0E(M) = {}_0E^1(M) \cong P^1M$.

1.7. Let $f\colon A \to B$ be a morphism of algebras (or coalgebras). By definition clearly f preserves F-filtrations (or G-filtrations). The induced map

$$E_0(f)\colon E_0(A) \to E_0(B)$$
$$(\text{or } {}_0E(f)\colon {}_0E(A) \to {}_0E(B))$$

is a morphism of graded algebras (or coalgebras).

Let A be an algebra (or a coalgebra), M and N be A-modules (or -comodules). If $g\colon M \to N$ is a morphism of A-modules (or -comodules), g also preserves F-filtrations (or G-filtrations) and the induced map

$$E_0(g)\colon E_0(M) \to E_0(N)$$
$$(\text{or } {}_0E(g)\colon {}_0E(M) \to {}_0E(N))$$

is a morphism of graded $E_0(A)$-modules (or ${}_0E(A)$-comodules).

In the following subsections we will show that some basic propositions of Milnor-Moore [10] (Props. 1.4, 1.5, 1.6, 2.4, 2.5) hold also in our case under suitable formulations.

1.8. For the purposes to prove some properties of A using F- (or G-) filtrations we need sometimes their completeness conditions. Let A be an algebra (or coalgebra). When the F- (or G-) filtration of A is complete, i.e.,

$$\cap_{k\geqslant 0} F^kA = \{0\} \qquad (\text{or } \cup_{k\geqslant 0} G^kA = A)\,,$$

we call A to be *semi-connected.* Remark that a graded connected algebra (or coalgebra) is semi-connected. If A is semi-connected and of finite dimension over K, then $E_0(A)$ (or ${}_0E(A)$) is isomorphic to A as G_2-modules.

Similarly we call a left A-module (or -comodule) M to be *semi-connected* if its F- (or G-) filtration is complete. Remark also that a graded left A-module (or (-module) is semi-connected if A is graded and connected.

Usually a decreasing filtration of a module topologizes it. For an algebra A or a left A-module M we topologize it by F-filtration. Then A or M is a Hausdorff space if semi-connected.

(1.23) **Proposition.** *Let A be a semi-connected algebra. Then $\bar{A} = \{0\}$ if and only if $Q(A) = \{0\}$.*

Proof. The "only if" part is evident.

Suppose that $Q(A)=\{0\}$, i.e., $\bar{A}=F^2A$. We shall prove that $\bar{A}=F^nA$ for all $n\geqslant 1$ by an induction on n. If we assume that $\bar{A}=F^2A=\cdots=F^nA$, $n\geqslant 2$, then $E_0^iA=0$ for $1\leqslant i\leqslant n\text{-}1$, and by (1.18) we obtain

$$E_0^nA=\{0\}, \quad \text{i.e.,} \quad \bar{A}=F^nA=F^{n+1}A\,.$$

Now

$$\bar{A}=\cap_{n\geqslant 1}F^nA=\{0\}$$

since A is semi-connected. q.e.d.

By exactly the same proof we obtain the following

(1.24) **Proposition.** *Let A be an algebra and M a semi-connected left A-module. Then $M=\{0\}$ if and only if $Q^1M=K\otimes_A M=\{0\}$.*

(1.23*) **Proposition.** *Let A be a semi-connected coalgebra. Then $\bar{A}=\{0\}$ if and only if $P(A)=\{0\}$.*

Proof. The "only if" part is evident.

Suppose that $P(A)=\{0\}$. By (1.18*) and an induction on n, we see that

$$P^nA=\{0\} \qquad \text{for all} \quad n\geqslant 1\,.$$

Then, since A is semi-connected

$$\bar{A}=\cup P^nA=\{0\}\,. \qquad \text{q.e.d.}$$

In the same way we obtain the following

(1.24*) **Proposition.** *Let A be a coalgebra and M a semi-connected left A-comodule. Then $M=\{0\}$ if and only if $P^1M=K\square_A M=\{0\}$.*

1.9. Let f: $A\to B$ be a morphism of algebras. If $f(A)$ is dense in B (topologized by F-filtration) then we call f to be *almost surjective*. Similarly we define the almost surjectivity of a morphism of A-modules. Denote by $\hat{A}$ the completion of A by the topology of F-filtration, i.e.,

$$\hat{A}=K\oplus \varprojlim Q^nA\,.$$

Then f: $A\to B$ is almost surjective if and only if $\hat{f}$: $\hat{A}\to\hat{B}$ is surjective.

(1.25) **Proposition.** *Let $f: A\to B$ be a morphism of algebras. The following four conditions are equivalent:*

i) *$f: A\to B$ is almost surjective,*

ii) *$Q(f): Q(A)\to Q(B)$ is surjective,*

iii) $Q^n f: Q^n A \to Q^n B$ *is surjective for all* $n \geqslant 1$,
iv) $E_0^n f: E_0^n A \to E_0^n B$ *is surjective for all* $n \geqslant 1$.

Proof. We prove in the order: i)→ii)→iv)→iii)→i). To say that f is almost surjective means that for any $b \in B$ and any $n \geqslant 1$ there exists an element $a_n \in A$ such that $b\text{-}f(a_n) \in F^n B$. Thus "i)→ii)" and "iii)→i)" follow immediately.

ii)→iv): We prove by an induction on n. ii) is equivalent to say that $E_0^1 f$ is surjective. In the following commutative diagram

$$\begin{array}{ccc}\sum_{i=1}^{n-1} E_0^i A \otimes E_0^{n-i} A & \xrightarrow{E_0^n \bar{\varphi}} & E_0^n A \\ \Big\downarrow {\scriptstyle \sum E_0^i f \otimes E_0^{n-i} f} & & \Big\downarrow {\scriptstyle E_0^n f} \\ \sum_{i=1}^{n-1} E_0^i B \otimes E_0^{n-i} B & \xrightarrow[E_0^n \bar{\varphi}]{} & E_0^n B\end{array}$$

$E_0^n \bar{\varphi}$ is surjective for $n \geqslant 2$ by (1.18) and $\sum_{i=1}^{n-1} E_0^i f \otimes E_0^{n-i} f$ is surjective by the induction hypothesis. Thus $E_0^n f$ is surjective.

iv)→iii): By the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & E_0^n A & \to & Q^n A & \to & Q^{n-1} A & \to & 0 \\ & & \downarrow E_0^n f & & \downarrow Q^n f & & \downarrow Q^{n-1} f & & \\ 0 & \to & E_0^n B & \to & Q^n B & \to & Q^{n-1} B & \to & 0 \end{array}$$

and an induction on n we get the proof. q.e.d.

By a parallel argument to the above with some minor changes we get the following

(1.26) **Proposition.** *Let* $f: M \to N$ *be a morphism of left* A*-modules. The following four conditions are equivalent:*

i) $f: M \to N$ *is almost surjective,*
ii) $1 \otimes_A f: K \otimes_A M \to K \otimes_A N$ *is surjective,*
iii) $Q^n f: Q^n M \to Q^n N$ *is surjective for all* $n \geqslant 1$,
iv) $E_0^n f: E_0^n M \to E_0^n N$ *is surjective for all* $n \geqslant 1$.

Let A be an associative algebra, M an associative left A-module and C a G_2-module. Define the left A-module structures on $A \otimes C$ and $A \otimes M$ by $\varphi_A \otimes 1$. Let $f: C \to M$ be a morphism of G_2-modules. Then

$$1_A \otimes f: A \otimes C \to A \otimes M \quad \text{and} \quad \varphi_M: A \otimes M \to M$$

become morphisms of left A-modules.

(1.27) **Proposition.** *Under the above situation the composition* $\varphi_M(1 \otimes f)$: $A \otimes C$

$\to M$ is almost surjective if and only if the composition πf: $C\to K\otimes_A M$ is a surjection of G_2-modules, where π: $M\to K\otimes_A M$ is the canonical projection.

Proof. Since

$$K\otimes_A(A\otimes C)\approx C \quad \text{and} \quad K\otimes_A(A\otimes M)\approx M$$

canonically, we see that

$$1\otimes_A(\varphi_M(1\otimes f)) = \pi f$$

via canonical identifications. Then the proposition follows from (1.26). q.e.d.

1.10. By parallel discussions to **1.9.** we obtain the following propositions. The details are left to readers.

(1.25*) **Proposition.** *Let f: $A\to B$ be a morphism of coalgebras and A be semi-connected. The following four conditions are equivalent:*

i) *f: $A\to B$ is injective,*
ii) *$P(f)$: $P(A)\to P(B)$ is injective,*
iii) *$P^n f$: $P^n A\to P^n B$ is injective for all $n\geqslant 1$,*
iv) *${}_0E^n f$: ${}_0E^n A\to {}_0E^n B$ is injective for all $n\geqslant 1$.*

(1.26*) **Proposition.** *Let f: $M\to N$ be a morphism of left A-comodules and M be semi-connected. The following four conditions are equivalent:*

i) *f: $M\to N$ is injective,*
ii) *$1\square_A f$: $K\square_A M\to K\square_A N$ is injective,*
iii) *$P^n f$: $P^n M\to P^n N$ is injective for all $n\geqslant 1$,*
iv) *${}_0E^n f$: ${}_0E^n M\to {}_0E^n N$ is injective for all $n\geqslant 1$.*

Let A be an associative coalgebra, M an associative left A-comodule and C a G_2-module. Define the left A-comodule structures on $A\otimes C$ and $A\otimes M$ by $\psi_A\otimes 1$. Let f: $M\to C$ be a morphism of G_2-modules. Then

$$\psi_M:\ M\to A\otimes M \quad \text{and} \quad 1_A\otimes f:\ A\otimes M\to A\otimes C$$

become morphisms of left A-comodules.

(1.27*) **Proposition.** *Under the above situation assume that M is semi-connected. Then the composition $(1\otimes f)\psi_M$: $M\to A\otimes C$ is injective if and only if the composition fi: $K\square_A M\to C$ is injective, where i: $K\square_A M\to M$ is the canonical injection.*

1.11. For the sake of our later references we list the following easily proved propositions, which form some counterpart of Propositions (1.26) and (1.26*).

(1.28) **Proposition.** *Let f: $M\to N$ be a morphism of left A-modules. $Q^n f$: $Q^n M$*

$\to Q^n N$ is injective for all $n \geqslant 1$ if and only if $E_0^n f$: $E_0^n M \to E_0^n N$ is injective for all $n \geqslant 1$. Next assume that M is semi-connected, then the above equivalent conditions imply that f: $M \to N$ is injective.

(1. 28*) **Proposition.** *Let f: $M \to N$ be a morphism of left A-comodules. $P^n f$: $P^n M \to P^n N$ is surjective for all $n \geqslant 1$ if and only if ${}_0E^n f$: ${}_0E^n M \to {}_0E^n N$ is surjective for all $n \geqslant 1$. Next suppose that N is semi-connected, then the above equivalent conditions imply that f: $M \to N$ is surjective.*

2. λ-modified permutations

2.1. A *differential* G_2-module M is a G_2-module equipped with a differential d: $M \to M$, i.e.

$$dM_i \subset M_{i+1}, \quad i \in Z_2, \quad \text{and} \quad d^2 = 0 . \tag{2. 1}$$

We regard K as a differential G_2-module endowed with a trivial differential.

Let M and N be differential G_2-modules. $M \otimes N$ is a differential G_2-module by

$$d_{M \otimes N} = d_M \otimes 1 + \sigma \otimes d_N ,$$

where σ is the involution (1.1) of M.

More generally: let $M_1, \dots, M_n$ be n differential G_2-modules. We define the i-th *partial* differential d_i of $M_1 \otimes \cdots \otimes M_n$ by

$$d_i = \sigma \otimes \cdots \otimes \sigma \otimes d \otimes 1 \otimes \cdots \otimes 1 \tag{2. 2}$$

with d in the i-th tensor factor for $1 \leqslant i \leqslant n$, where σ is the canonical involution (1.1). Obviously we have

$$d_i^2 = 0 \quad \text{and} \quad d_i d_j + d_j d_i = 0$$

for all i and j. The canonical differential G_2-module structure of $M_1 \otimes \cdots \otimes M_n$ is given with the *total* differential

$$d = d_1 + \cdots + d_n .$$

2.2. Let $M_1, \dots, M_n$ be n differential G_2-modules. As usual, for every permutation $s \in \mathfrak{S}_n$ we can associate a morphism of differential G_2-modules

$$T_s: M_1 \otimes \cdots \otimes M_n \to M_{\bar{s}(1)} \otimes \cdots \otimes M_{\bar{s}(n)}$$

($\bar{s} = s^{-1}$), the so called signed permutation of tensor factors, i.e.,

$$T_s(x_1 \otimes \cdots \otimes x_n) = \varepsilon_s(x_1, \dots, x_n) x_{\bar{s}(1)} \otimes \cdots \otimes x_{\bar{s}(n)} ,$$

$x_i \in M_i$, $1 \leqslant i \leqslant n$, and $\varepsilon_s(x_1, \dots, x_n)$ is a sign such as, if $s=(i, i+1)$, a transposition, $\varepsilon_s(x_1, \dots, x_n)=(-1)^{pq}$ for $p=$type x_i and $q=$type x_{i+1}.

To formulate the basic properties of signed permutations we shall use certain semi-simplicial structure of categories. Let $\mathfrak{G}=\mathfrak{D}\mathfrak{G}_2$ be the category of differential G_2-modules over a field K. We regard the associativity

$$(L\otimes M)\otimes N = L\otimes(M\otimes N)$$

and the obvious isomorphism

$$K\otimes M = M\otimes K = M$$

as identities in $\mathfrak{G}$. For each integer $n \geqslant 1$ let

$$\mathfrak{G}^{(n)} = \mathfrak{G}\times\cdots\times\mathfrak{G} \qquad (n \text{ fold})$$

be the n fold product category of $\mathfrak{G}$. Objects and morphisms of $\mathfrak{G}^{(n)}$ are respectively $(M_1, \dots, M_n)$, $M_i \in \text{obj }\mathfrak{G}$, and $(f_1, \dots, f_n)$, $f_i \in \text{Morph }\mathfrak{G}$, $1 \leqslant i \leqslant n$. Let

$$D_i\colon \mathfrak{G}^{(n+1)} \to \mathfrak{G}^{(n)}, \tag{2.3}$$

$1 \leqslant i \leqslant n$, be a functor given by

$$\begin{aligned} D_i(M_1, \dots, M_{n+1}) &= (M_1, \dots, M_{i-1}, M_i\otimes M_{i+1}, M_{i+2}, \dots, M_{n+1}), \\ D_i(f_1, \dots, f_{n+1}) &= (f_1, \dots, f_{i-1}, f_i\otimes f_{i+1}, f_{i+2}, \dots, f_{n+1}). \end{aligned}$$

Let

$$F_i\colon \mathfrak{G}^{(n)} \to \mathfrak{G}^{(n+1)}, \tag{2.4}$$

$1 \leqslant i \leqslant n+1$, be a functor given by

$$\begin{aligned} F_i(M_1, \dots, M_n) &= (M_1, \dots, M_{i-1}, K, M_i, \dots, M_n), \\ F_i(f_1, \dots, f_n) &= (f_1, \dots, f_{i-1}, 1_K, f_i, \dots, f_n). \end{aligned}$$

As is easily seen we obtain relations:

$$\begin{aligned} D_iF_i &= D_{i-1}F_i = 1, \\ D_jF_i &= F_iD_{j-1}, \quad i<j, \\ D_jF_i &= F_{i-1}D_j, \quad j<i\text{-}1. \end{aligned}$$

Thus

$$\mathfrak{G}^* = \{\mathfrak{G}^{(1)}, \mathfrak{G}^{(2)}, \dots, \mathfrak{G}^{(n)}, \dots\}$$

is equipped with a semi-simplicial structure (in a dual sense) with D_i's as *degeneracy* operators and F_i's as *face* operators.

The n-th symmetric group $\mathfrak{S}_n$ operates on $\mathfrak{G}^{(n)}$ as a group of automorphisms. Namely every $s\in\mathfrak{S}_n$ determines a functor

$$s:\ \mathfrak{G}^{(n)}\to\mathfrak{G}^{(n)}$$

by

$$s(M_1, \dots, M_n) = (M_{\bar{s}(1)}, \dots, M_{\bar{s}(n)})\,,$$
$$s(f_1, \dots, f_n) = (f_{\bar{s}(1)}, \dots, f_{\bar{s}(n)})\,,$$

$\bar{s}=s^{-1}$. These functors s form a group of automorphisms of $\mathfrak{G}^{(n)}$. There hold the following relations between these automorphisms and degeneracy or face operators: for any $s\in\mathfrak{S}_n$ and any D_i: $\mathfrak{G}^{(n+1)}\to\mathfrak{G}^{(n)}$, $1\leqslant i\leqslant n$, there exists $D_i^* s\in\mathfrak{S}_{n+1}$ uniquely such that

$$s\circ D_i = D_j\circ D_i^* s,\quad j = s(i); \tag{2.5}$$

for any $s\in\mathfrak{S}_{n+1}$ and any F_i: $\mathfrak{G}^{(n)}\to\mathfrak{G}^{(n+1)}$, $1\leqslant i\leqslant n+1$, there exists $F_i^* s\in\mathfrak{S}_n$ uniquely such that

$$s\circ F_i = F_j\circ F_i^* s,\quad j = s(i)\,. \tag{2.6}$$

And we have following relations with respect to compositions:

$$D_j^* s'\circ D_i^* s = D_i^*(s'\circ s),\quad j = s(i)\,, \tag{2.7}$$
$$F_j^* s'\circ F_i^* s = F_i^*(s'\circ s),\quad j = s(i)\,. \tag{2.8}$$

We define a *canonical* functor

$$T = T^{(n)}:\ \mathfrak{G}^{(n)}\to\mathfrak{G} \tag{2.9}$$

by

$$T(M_1, \dots, M_n) = M_1\otimes\cdots\otimes M_n\,,$$
$$T(f_1, \dots, f_n) = f_1\otimes\cdots\otimes f_n\,,$$

for each $n\geqslant 1$. We have the following obvious relations:

$$T^{(n)}\circ D_i = T^{(n+1)}\quad\text{for all}\quad i,\ 1\leqslant i\leqslant n,\ \text{and} \tag{2.10}$$
$$T^{(n+1)}\circ F_i = T^{(n)}\quad\text{for all}\quad i,\ 1\leqslant i\leqslant n+1\,. \tag{2.11}$$

Now we can regard the signed permutations as natural transformations

$$T_s:\ T^{(n)}\to T^{(n)}\circ s,\quad s\in\mathfrak{S}_n,\quad n\geqslant 1\,, \tag{2.12}$$

which satisfy the relations:

$$T_{s'}T_s = T_{s's}\,,\quad T_{id} = id.\,, \tag{2.13}$$
$$T_s\circ D_i = T_{D_i^* s}\qquad (s\in\mathfrak{S}_n,\quad 1\leqslant i\leqslant n)\,, \tag{2.14}$$
$$T_s\circ F_i = T_{F_i^* s}\qquad (s\in\mathfrak{S}_{n+1},\quad 1\leqslant i\leqslant n+1)\,. \tag{2.15}$$

2.3. Take an element $\lambda\in K$ and fix it. Let $M_1, \dots, M_n$ be differential G_2-modules over K. Using the partial differentials d_i of (2.2) we define *λ-modified switching maps*

$$T_{i,\lambda}\colon M_1\otimes\cdots\otimes M_n\to M_1\otimes\cdots\otimes M_{i-1}\otimes M_{i+1}\otimes M_i\otimes M_{i+2}\otimes\cdots\otimes M_n$$

by

$$T_{i,\lambda}=(1+\lambda d_i d_{i+1})T_i \tag{2.16}$$

for $1\leqslant i\leqslant n-1$. As is easily seen we have

$$T_{i,\lambda}d_i=d_{i+1}T_{i,\lambda},\quad T_{i,\lambda}d_{i+1}=d_iT_{i,\lambda},\quad 1\leqslant i\leqslant n-1\,,$$
$$T_{i,\lambda}d_j=d_jT_{i,\lambda}\quad\text{for}\quad j\neq i,\ i+1\,.$$

In particular $T_{i,\lambda}$'s are morphisms of differential G_2-modules. When $n=2$ we put

$$T_{1,\lambda}=T_\lambda\,,$$

i.e.,

$$T_\lambda=(1+\lambda(d\sigma)\otimes d)T\,, \tag{2.17}$$

called the *λ-modified twisting morphism*, then we have

$$T_{i,\lambda}=1\otimes\cdots\otimes 1\otimes T_\lambda\otimes 1\otimes\cdots\otimes 1 \tag{2.18}$$

with T_λ in the i-th place for $1\leqslant i\leqslant n$-1. Obviously the ordinary switching maps and twisting morphism are the 0-modified ones.

A routine computation shows that

$$\begin{aligned} &T_{i,\lambda}^2=1\quad\text{for}\quad 1\leqslant i\leqslant n\text{-}1\,,\\ &T_{i,\lambda}T_{i+1,\lambda}T_{i,\lambda}=T_{i+1,\lambda}T_{i,\lambda}T_{i+1,\lambda}\quad\text{for}\quad 1\leqslant i\leqslant n\text{-}2\,,\\ &T_{i,\lambda}T_{j,\lambda}=T_{j,\lambda}T_{i,\lambda}\quad\text{for}\quad i+1<j\,. \end{aligned} \tag{2.19}$$

Since the corresponding relations for transpositions $t_i=(i,\ i+1)$ are fundamental relations of $\mathfrak{S}_n$ [7], p. 287, if we define for each $s\in\mathfrak{S}_n$ a map

$$T_{s,\lambda}\colon M_1\otimes\cdots\otimes M_n\to M_{\bar{s}(1)}\otimes\cdots\otimes M_{\bar{s}(n)}$$

($\bar{s}=s^{-1}$) by expressing s as a product of t_i's, replacing each t_k in this expression by the corresponding $T_{k,\lambda}$ and putting $T_{s,\lambda}$ to be equal to thus obtained composition of $T_{i,\lambda}$'s, then $T_{s,\lambda}$ is a uniquely determined map regardless of choices of the expression of s. So defined, the map $T_{s,\lambda}$ is called the *λ-modified permutation* (corresponding to $s\in\mathfrak{S}_n$). $T_{s,\lambda}$ is a morphism of differential G_2-modules since $T_{i,\lambda}$'s are so. Obviously $T_s=T_{s,0}$ for each $s\in\mathfrak{S}_n$.

Now we can regard the λ-modified permutations as natural transformations

$$(2.12)_\lambda \qquad T_{s,\lambda}\colon T^{(n)} \to T^{(n)}\circ s, \quad s\in\mathfrak{S}_n, \quad n\geqslant 1\,.$$

(2. 20) **Proposition.** *λ-modified permutations satisfy the relations:*

$$(2.13)_\lambda \qquad T_{s',\lambda}T_{s,\lambda} = T_{s's,\lambda}, \quad T_{id,\lambda} = id.,$$

$$(2.14)_\lambda \qquad T_{s,\lambda}\circ D_i = T_{D_i^* s,\lambda} \qquad (s\in\mathfrak{S}_n, \quad 1\leqslant i\leqslant n),$$

$$(2.15)_\lambda \qquad T_{s,\lambda}\circ F_i = T_{F_i^* s,\lambda} \qquad (s\in\mathfrak{S}_{n+1}, \quad 1\leqslant i\leqslant n+1)\,.$$

Proof. $(2.13)_\lambda$ is obvious by the definition of $T_{s,\lambda}$. To prove $(2.14)_\lambda$ it is sufficient to prove for $s=t_k$, a transposition, by the relation (2.7); then by (2.18) it is sufficient to prove the case $n=2$. In the latter case the relations are easily obtained by routine calculations.

The proof of $(2.15)_\lambda$ is also reduced by (2.8) and (2.18) to the case $n=1$. This case is obvious because K has a trivial differential. q.e.d.

(2. 21) **Corollary.** *Suppose a proposition P on differential G_2-modules involving signed permutations T_s is true and proven only using properties* (2.13), (2.14) *and* (2.15) *as those of T_s's. Then a proposition P_λ, obtained by replacing T_s by $T_{s,\lambda}$ for each s ($\lambda\in K$ is fixed), is also true.*

An iterated application of $(2.14)_\lambda$ implies

(2.22) **Proposition.** *Let $M_1, \dots, M_n$ be differential G_2-modules. Suppose each M_i is a tensor product $M_i=N_{k_{i-1}+1}\otimes\cdots\otimes N_{k_i}$ of differential G_2-modules, $1\leqslant i\leqslant n$, $0=k_0<k_1<k_2<\cdots<k_n=r$. A permutation s of $\{1, \dots, n\}$ induces naturally a (blockwise) permutation τ of $\{1, \dots, r\}$. Then, for $\lambda\in K$,*

$$T_{s,\lambda} = T_{\tau,\lambda}$$

regarded as a morphism

$$N_1\otimes\cdots\otimes N_r \to N_{\bar\tau(1)}\otimes\cdots\otimes N_{\bar\tau(r)} \qquad (\bar\tau = \tau^{-1})\,.$$

An iterated application of $(2.15)_\lambda$ implies

(2.23) **Proposition.** *Let $M_1, \dots, M_n$ be differential G_2-modules. Suppose $\{M_1, \dots, M_n\}$ contains some number of K's. Deleting some K's from $\{M_1, \dots, M_n\}$ we obtain $\{N_1, \dots, N_r\}$. A permutation s of $\{1, \dots, n\}$ induces a permutation τ of $\{1, \dots, r\}$. Then, for $\lambda\in K$,*

$$T_{s,\lambda} = T_{\tau,\lambda}$$

regarded as a morphism

$$N_1\otimes\cdots\otimes N_r \to N_{\bar\tau(1)}\otimes\cdots\otimes N_{\bar\tau(r)} \qquad (\bar\tau = \tau^{-1})\,.$$

3. Differential algebras and coalgebras

3.1. Let A be an algebra (or a coalgebra) endowed with a differential d as a G_2-module. When d commutes with the structurure morphisms φ, η and ε (or (ψ, ε and η) then we call A to be a *differential* algebra (or coalgebra). The ground field K is a differential algebra as well as coalgebra endowed with a trivial differential.

Remark that the assumption that d commutes with the multiplication φ (or comultiplication ψ) implies that d commutes with the unit η (or ε), [1], p.512. By our present definition we further assume that d commutes with the augmentation ε (or η). Thus it is stronger than the definition of [1], p.512.

Let A and B be differential algebras (or coalgebras). Putting

$$(3.1) \qquad \varphi_\lambda = (\varphi\otimes\varphi)(1\otimes T_\lambda\otimes 1)\colon\ A\otimes B\otimes A\otimes B\to A\otimes B$$
$$(\text{or } \psi_\lambda = (1\otimes T_\lambda\otimes 1)(\psi\otimes\psi)\colon\ A\otimes B\to A\otimes B\otimes A\otimes B)$$

and

$$\eta = \eta_A\otimes\eta_B, \quad \varepsilon = \varepsilon_A\otimes\varepsilon_B, \quad d = d_1+d_2$$

as usual, $A\otimes B$ becomes a differential algebra (or coalgebra), which has a multiplication φ_λ (or comultiplication ψ_λ) different from the ordinary tensor product and will be called the λ-*modified tensor product* of A and B, denoted by $(A\otimes B)_\lambda$. Thus $(A\otimes B)_0=A\otimes B$, the ordinary tensor product.

Every algebra (or coalgebra) can be regarded as a differential one by putting $d=0$ (trivial differential). In this case we have

$$(A\otimes B)_\lambda = A\otimes B$$

for any $\lambda\in K$.

Let A, B and C be differential algebras (or coalgebras) and $\lambda\in K$. We have the associativity

$$(3.2) \qquad ((A\otimes B)_\lambda\otimes C)_\lambda = (A\otimes(B\otimes C)_\lambda)_\lambda\,.$$

We denote this by $(A\otimes B\otimes C)_\lambda$. More generally we put

$$(A_1\otimes\cdots\otimes A_n)_\lambda = ((A_1\otimes\cdots\otimes A_{n-1})_\lambda\otimes A_n)_\lambda$$

inductively for n differential algebras (or coalgebras) A_i, $1\leqslant i\leqslant n$. Because of the above associativity (3.2) it is equal to $(A_1\otimes(A_2\otimes\cdots\otimes A_n)_\lambda)_\lambda$ and so on.

If a differential algebra (or coalgebra) A satisfies the relation

$$(3.3) \qquad \varphi T_\lambda = \varphi \qquad (\text{or } T_\lambda\psi = \psi), \quad \lambda\in K\,,$$

then we call A to be λ-*commutative.*

(3.4) *If A and B are associative or λ-commutative differential algebras (or coalgebras), then $(A\otimes B)_\lambda$ is also associative or λ-commutative.*
Proofs of (3.2) and (3.4) are of course not hard by direct computations. But we would rather regard them as cases to apply the principle (2.21) because the case $\lambda=0$ is obvious and classical.

3.2. To apply the principle (2.21) for differential algebras and coalgebras the following observations would be useful. Let A be a differential algebra (or coalgebra), $M_1, \dots, M_n$ be differential G_2-modules and $\lambda\in K$. Replacing M_i by A and $A\otimes A$ we consider the following morphism of differential G_2-modules

$$(3.5)\qquad \begin{aligned} &\varphi^{(i)} = 1\otimes\cdots\otimes 1\otimes\varphi\otimes 1\otimes\cdots\otimes 1:\\ &\quad M_1\otimes\cdots\otimes M_{i-1}\otimes A\otimes A\otimes M_{i+1}\otimes\cdots\otimes M_n\\ &\quad \to M_1\otimes\cdots\otimes M_{i-1}\otimes A\otimes M_{i+1}\otimes\cdots\otimes M_n\\ &(\text{or } \psi^{(i)} = 1\otimes\cdots\otimes 1\otimes\psi\otimes 1\otimes\cdots\otimes 1:\\ &\quad M_1\otimes\cdots\otimes M_{i-1}\otimes A\otimes M_{i+1}\otimes\cdots\otimes M_n\\ &\quad \to M_1\otimes\cdots\otimes M_{i-1}\otimes A\otimes A\otimes M_{i+1}\otimes\cdots\otimes M_n) \end{aligned}$$

with φ (or ψ) in the i-th tensor factor, $1\leqslant i\leqslant n$. Since φ (or ψ) is a morphism of differential G_2-modules, the naturality of λ-modified permutations and $(2.14)_\lambda$ imply the relation

$$(3.6)\qquad \begin{gathered} T_{s,\lambda}\varphi^{(i)} = \varphi^{(j)}T_{D_i^* s,\lambda}\\ (\text{or } T_{D_i^* s,\lambda}\psi^{(i)} = \psi^{(j)}T_{s,\lambda}) \end{gathered}$$

for each $s\in\mathfrak{S}_n$, $j=s(i)$. Similarly, replacing φ(or ψ) by η or ε, we have relations

$$(3.7)\qquad T_{s,\lambda}\eta^{(i)} = \eta^{(j)}T_{F_i^* s,\lambda} \quad\text{and}\quad \varepsilon^{(j)}T_{s,\lambda} = T_{F_i^* s,\lambda}\varepsilon^{(i)}$$

regarded as

$$\begin{aligned} \eta^{(i)}:\ & M_1\otimes\cdots\otimes M_{i-1}\otimes M_{i+1}\otimes\cdots\otimes M_n\\ &\to M_1\otimes\cdots\otimes M_{i-1}\otimes A\otimes M_{i+1}\otimes\cdots\otimes M_n \end{aligned}$$

and

$$\begin{aligned} \varepsilon^{(i)}:\ & M_1\otimes\cdots\otimes M_{i-1}\otimes A\otimes M_{i+1}\otimes\cdots\otimes M_n\\ &\to M_1\otimes\cdots\otimes M_{i-1}\otimes M_{i+1}\otimes\cdots\otimes M_n, \end{aligned}$$

where $j=s(i)$.

3.3. Let A and B be differential algebras (or coalgebras). We have the relation

$$(3.8)\qquad \begin{gathered} T_\lambda\varphi_\lambda = \varphi_\lambda(T_\lambda\otimes T_\lambda)\\ (\text{or } \psi_\lambda T_\lambda = (T_\lambda\otimes T_\lambda)\psi_\lambda) \end{gathered}$$

for $\lambda \in K$, where φ_λ (or ψ_λ) denotes the multiplications (or comultiplications) of $(A\otimes B)_\lambda$ and of $(B\otimes A)_\lambda$ simultaneously, and T_λ: $(A\otimes B)_\lambda \to (B\otimes A)_\lambda$.

This relation is also an immediate consequence of (2.22) and the naturality of T_λ.

Next, let $A_1, \dots, A_n$ be n differential algebras (or coalgebras) and $\lambda \in K$. Let $c(n)=(1, 2, \dots, n)$, a cyclic permutation of order n. Put

$$T_{c(n),\lambda} = C_{n,\lambda}, \quad \lambda\text{-}modified\ cyclic\ permutation.$$

Putting $(A_1\otimes\cdots\otimes A_{n-1})_\lambda = B$, by (2.22) we see that $C_{n,\lambda}=T_\lambda$ as a map: $B\otimes A_n \to A_n\otimes B$. Then, since $(A_1\otimes\cdots\otimes A_n)_\lambda=(B\otimes A_n)_\lambda$, by (3.8) we obtain

$$\begin{aligned} C_{n,\lambda}\varphi_\lambda &= \varphi_\lambda(C_{n,\lambda}\otimes C_{n,\lambda}) \\ (\text{or } \psi_\lambda C_{n,\lambda} &= (C_{n,\lambda}\otimes C_{n,\lambda})\psi_\lambda), \end{aligned} \tag{3.9}$$

where φ_λ (or ψ_λ) denotes the multiplications (or comultiplications) of $(A_1\otimes\cdots\otimes A_n)_\lambda$ and of $(A_n\otimes A_1\otimes\cdots\otimes A_{n-1})_\lambda$ simultaneously.

3.4. Let A be a differential algebra (or coalgebra) and M a left A-module (or -comodule) which is a differential G_2-module. When the structure map φ_M (or ψ_M) commutes with differentials on M and $A\otimes M$, then we call M to be a *differential* left A-module (or -comodule).

Let A and B be differential algebras (or coalgebras), and M and N be differential left A- and B-modules (or -comodules) respectively. Putting

$$\begin{aligned} \varphi_\lambda &= (\varphi_M\otimes\varphi_N)(1\otimes T_\lambda\otimes 1)\colon (A\otimes B)_\lambda\otimes M\otimes N \to M\otimes N \\ (\text{or } \psi_\lambda &= (1\otimes T_\lambda\otimes 1)(\psi_M\otimes\psi_N)\colon M\otimes N \to (A\otimes B)_\lambda\otimes M\otimes N), \end{aligned} \tag{3.10}$$

we define a $(A\otimes B)_\lambda$-module (or -comodule) structure on $M\otimes N$, denoted by $(M\otimes N)_\lambda$ and called by λ-*modified tensor product* of M and N. $(M\otimes N)_\lambda$ becomes a differential left $(A\otimes B)_\lambda$-module (or -comodule). When M and N are associative then $(M\otimes N)_\lambda$ is also an associative $(A\otimes B)_\lambda$-module (or -comodule).

3.5. Let A be a differential algebra (or coalgebra). Since the structure morphisms commute with the differential, $H(A)$ gains the induced multiplication (or comultiplication), unit (or counit) and augmentation. Thus $H(A)$ is an algebra (or coalgebra). If A is associative or λ-commutative for some $\lambda\in K$, then $H(A)$ is associative or commutative. If B is another differential algebra (or coalgebra) then

$$H((A\otimes B)_\lambda) = H(A)\otimes H(B) \tag{3.11}$$

as an algebra (or a coalgebra) for any $\lambda\in K$.

Since the differential commutes with η and ε, it preserves the direct sum decomposition (1.3). In particular

(3. 12) $$d(\bar{A})\subset\bar{A}.$$

This implies that

(3. 13) $$d(F^nA)\subset F^nA \qquad (\text{or } d(G^nA)\subset G^nA)$$

for each $n\geqslant 0$. Thus the F- (or G-) filtration determines the spectral sequence

(3. 14) $$E_r(A)=\textstyle\sum_{n\geqslant 0}E_r^nA \qquad (\text{or } {}_rE(A)=\textstyle\sum_{n\geqslant 0}{}_rE^nA),$$

$r\geqslant 0$, as usual. (1.16) (or (1.16*)) and the standard arguments about spectral sequences imply

(3.15) **Proposition.** *If A is a differential (associative) algebra (or coalgebra), then $E_r(A)$ (or ${}_rE(A)$) is a differential graded and connected (associative) algebra (or coalgebra) for each $r\geqslant 0$, of which the differential d_r has degree r (or $-r$). Furthermore, if A is λ-commutative for some $\lambda\in K$ then $E_0(A)$ (or ${}_0E(A)$) is also λ-commutative for the same λ and $E_r(A)$ (or ${}_rE(A)$) are commutative for all $r\geqslant 1$.*

If f: $A\to B$ is a morphism of differential algebras (or coalgebras) then f induces morphism

$$E_r(f)\colon E_r(A)\to E_r(B) \qquad (\text{or } {}_rE(f)\colon {}_rE(A)\to {}_rE(B)),$$

$r\geqslant 0$, of spectral sequences of graded algebras (or coalgebras).

Let A be a differential algebra (or coalgebra) and M a differential left A-module (or -comodule). Then we can see that $H(M)$ is a left $H(A)$-module (or -comodule), and also that the differential d: $M\to M$ preserves F- (or G-) filtrations of M and hence we get a spectral sequence $E_r(M)$ (or ${}_rE(M)$), $r\geqslant 0$, of which the r-th term is a differential left $E_r(A)$-module (or ${}_rE(A)$-comodule) for each $r\geqslant 0$.

3.6. Let A and B be differential algebras (or coalgebras) and $\lambda\in K$. Putting $(A\otimes B)_\lambda=C$ and denoting the multiplications (or comultiplications) of A, B and C by ${}_A\varphi$, ${}_B\varphi$ and ${}_C\varphi$ (or ${}_A\psi$, ${}_B\psi$ and ${}_C\psi$) respectively, we shall express ${}_C\varphi_n^{w_n}$ (or ${}_C\psi_n^{w_n}$) in terms of ${}_A\varphi_n^{w_n}$ and ${}_B\varphi_n^{w_n}$ (or ${}_A\psi_n^{w_n}$ and ${}_B\psi_n^{w_n}$) for each $w_n\in W_n$, the sets of (1.6).

Let $u_k\in\mathfrak{S}_{2k}$ be a permutation such that

$$u_k(2i-1)=i \quad\text{and}\quad u_k(2i)=k+i, \quad 1\leqslant i\leqslant k.$$

We denote as

$$U_k=T_{u_k} \quad\text{and}\quad U_{k,\lambda}=T_{u_k,\lambda},$$

By an induction on n we can easily show that

$$\begin{gathered}{}_{A\otimes B}\varphi_n^{w_n}=({}_A\varphi_n^{w_n}\otimes{}_B\varphi_n^{w_n})U_{n+1}\\ (\text{or } U_{n+1}\,{}_{A\otimes B}\psi_n^{w_n}={}_A\psi_n^{w_n}\otimes{}_B\psi_n^{w_n})\end{gathered}$$

for each $w_n \in W_n$, $n \geqslant 1$. Remark that this is the case to apply the principle (2.21) making use of (3.6). Hence we obtain the relation

$$ {}_C\varphi_n^{w_n} = ({}_A\varphi_n^{w_n} \otimes {}_B\varphi_n^{w_n}) U_{n+1,\lambda} \tag{3. 16}$$
$$(\text{or } U_{n+1,\lambda}\, {}_C\psi_n^{w_n} = {}_A\psi_n^{w_n} \otimes {}_B\psi_n^{w_n})$$

for $w_n \in W_n$, $n \geqslant 1$.

3.7. Let A and B be differential algebras and $\lambda \in K$. We have

(3.17) **Proposition.** $\sum_{r=0}^{n} F^r A \otimes F^{n-r} B = F^n(A \otimes B)_\lambda$ *for* $n \geqslant 0$.

Proof. Put $(A\otimes B)_\lambda = C$ and denote the multiplication of C by ${}_C\varphi$. For $w_{r-1} \in W_{r-1}$ and $w_{s-1} \in W_{s-1}$, $r+s=n$, we have

$$\operatorname{Im} {}_A\bar{\varphi}_{r-1}^{w_r-1} \otimes \operatorname{Im} {}_B\bar{\varphi}_{s-1}^{w_s-1} = {}_C\varphi((\operatorname{Im} {}_A\bar{\varphi}_{r-1}^{w_r-1} \otimes K) \otimes (K \otimes \operatorname{Im} {}_B\bar{\varphi}_{s-1}^{w_s-1}))$$
$$\subset {}_C\varphi(\operatorname{Im} {}_C\bar{\varphi}_{r-1}^{w_r-1} \otimes \operatorname{Im} {}_C\bar{\varphi}_{s-1}^{w_s-1}) = \operatorname{Im} {}_C\varphi_{n-1}^{w_n-1}$$

where $w_{n-1} = (1, w_{r-1}, w_{s-1}+r) \in W_{n-1}$. Thus

$$\sum_{r=0}^{n} F^r A \otimes F^{n-r} B \subset F^n C .$$

Next, for each $w_{n-1} \in W_{r-1}$

$${}_C\bar{\varphi}_{n-1}^{w_n-1} = ({}_A\varphi_{n-1}^{w_n-1} \otimes {}_B\varphi_{n-1}^{w_n-1}) U_{n,\lambda}(\iota_C \otimes \cdots \otimes \iota_C)$$

by (3.16) since

$$\iota_C = \eta_A \otimes \iota_B + \iota_A \otimes \eta_B + \iota_A \otimes \iota_B$$

and since d commutes with η and ι,

$$\operatorname{Im} {}_C\bar{\varphi}_{n-1}^{w_n-1} \subset \sum \operatorname{Im} {}_A\varphi_{n-1}^{w_n-1}(\pi_{i_1} \otimes \cdots \otimes \pi_{i_n}) \otimes \operatorname{Im} {}_B\varphi_{n-1}^{w_n-1}(\pi_{j_1} \otimes \cdots \otimes \pi_{j_n})$$

where π_i and π_j are η or ι of A and B respectively, and the summation runs over $\{i_1, \dots, i_n, j_1, \dots, j_n\}$ having no pairs (i_s, j_s) such that $\pi_{i_s} = \pi_{j_s} = \eta$. Since

$$\varphi_{n-1}^{w_n-1}(\pi_{i_1} \otimes \cdots \otimes \pi_{i_n}) = \bar{\varphi}_{n-i-1}^{w_{n-i}-1}$$

for a suitable $w_{n-i-1} \in W_{n-i-1}$ if $\{\pi_{i_1}, \dots, \pi_{i_n}\}$ contains exactly i η's, we have

$$\operatorname{Im} {}_C\bar{\varphi}_{n-1}^{w_n-1} \subset \sum_{i+j\leqslant n, w_{n-i-1}, w_{n-j-1}} \operatorname{Im} {}_A\bar{\varphi}_{n-i-1}^{w_{n-i}-1} \otimes \operatorname{Im} {}_B\bar{\varphi}_{n-j-1}^{w_{n-j}-1}$$
$$\subset \sum_{i+j\leqslant n} F^{n-i} A \otimes F^{n-j} B .$$

Thus

$$F^n C \subset \sum_{r+s\leqslant n} F^r A \otimes F^s B = \sum_{r+s=n} F^r A \otimes F^s B . \qquad \text{q.e.d.}$$

By (3.17) we obtain immediately the following

(3.18) **Proposition.** *If A and B are differential algebras then for $\lambda\in K$ we have*

i) $E_0((A\otimes B)_\lambda)\cong(E_0(A)\otimes E_0(B))_\lambda$

and

ii) $E_r((A\otimes B)_\lambda)\cong E_r(A)\otimes E_r(B)$

for $r\geqslant 1$ as graded differential algebras.

3.8. Let A and B be differential coalgebras and $\lambda\in K$. As a dual of (3.17) we have

(3. 17*) **Proposition.** $\sum_{r=0}^{n} G^rA\otimes G^{n-r}B=G^n(A\otimes B)_\lambda$ *for $n\geqq 0$.*

Proof. Denoting comultiplications of A, B and $C=(A\otimes B)_\lambda$ by ${}_A\psi$, ${}_B\psi$ and ${}_C\psi$ respectively, (1.16*) and (3.16) imply that

$$ {}_C\psi_n^{w_n}(G^rA\otimes G^{n-r}B)\subset\sum_{\substack{i_1+\cdots+i_{n+1}=r\\ j_1+\cdots+j_{n+1}=n-r}} U_{n+1,\lambda}^{-1}(G^{i_1}A\otimes\cdots\otimes G^{i_{n+1}}A \otimes G^{j_1}B\otimes\cdots\otimes G^{j_{n+1}}B) $$

for each $w_n\in W_n$. In each summand of the right hand side above, there exists t, $1\leqslant t\leqslant n+1$, such that $i_t=j_t=0$. Hence

$$ {}_C\bar{\psi}_n^{w_n}(G^rA\otimes G^{n-r}B)=0 \qquad \text{for each} \quad w_n\in W_n $$

and we obtain

$$ \sum_{r=0}^{n} G^rA\otimes G^{n-r}B\subset G^nC\,. $$

Conversely, let $x\in G^nC$. Since

$$ \rho_C=\varepsilon_A\otimes\rho_B+\rho_A\otimes\varepsilon_B+\rho_A\otimes\rho_B $$

and the mixed tensor products of ε_A, ρ_A, ε_B and ρ_B are projectors in a direct sum decomposition of $(A\otimes B)^{\otimes k}$ derived from the decomposition (1.3) of A and B, letting π^i denote ε or ρ we have

$$ (*) \qquad (\pi_A^{i_1}\otimes\pi_B^{j_1}\otimes\cdots\otimes\pi_A^{i_{n+1}}\otimes\pi_B^{j_{n+1}})\,{}_C\psi_n^{w_n}(x)=0 $$

for all $w_n\in W_n$ unless $\pi^{i_s}=\pi^{j_s}=\varepsilon$ for some s, $1\leqslant s\leqslant n+1$.

Now, for each $w_i\in W_i$ and $w_j\in W_j$ such that $i+j=n-1$, $0\leqslant i\leqslant n-1$, we put $w_n=(1, w_i, w_j+i+1)$. Then $\psi_n^{w_n}=(\psi_i^{w_i}\otimes\psi_j^{w_j})\psi$ and we have

$$ (\varepsilon_A\otimes\rho_B)^{\otimes i+1}\otimes(\rho_A\otimes\varepsilon_B)^{\otimes j+1}\,{}_C\psi_n^{w_n}(x)=0 $$

from (*) on one hand. By an easy calculation it is equal to

$$ U_{n+1,\lambda}^{-1}((1)^{\otimes i+1}\otimes({}_A\bar{\psi}_j^{w_j}\otimes{}_B\bar{\psi}_i^{w_i})(x)\otimes(1)^{\otimes j+1}) $$

on the other hand, where $1=\eta(1)$ in A and B respectively. Thus

$$({}_A\bar{\psi}_j^{w_j} \otimes {}_B\bar{\psi}_i^{w_i})(G^nC) = 0$$

for each $w_i \in W_i$ and $w_j \in W_j$ such that $i+j=n-1$ and $0 \leqslant i \leqslant n-1$. Hence

$$G^nC \subset \cap_{i=0}^{n-1}(G^iA \otimes B + A \otimes G^{n-i-1}B).$$

In (*), putting $\pi^{i_1} = \cdots = \pi^{i_n} = \varepsilon$, $\pi^{i_{n+1}} = \varepsilon + \rho$ and $\pi^{j_1} = \cdots = \pi^{j_{n+1}} = \rho$, we obtain

$$G^nC \subset A \otimes G^nB,$$

and also

$$G^nC \subset G^nA \otimes B.$$

Hence

$$G^nC \subset G^nA \otimes G^nB.$$

Finally using an induction similar as in the proof of (1.15*) we obtain

$$G^nC \subset \sum_{r=0}^{n} G^rA \otimes G^{n-r}B. \qquad \text{q.e.d.}$$

From (3.17*) we obtain the following

(3.18*) **Proposition.** *Let A and B be differential coalgebras and $\lambda \in K$, then*

i) ${}_0E((A \otimes B)_\lambda) \cong ({}_0E(A) \otimes {}_0E(B))_\lambda$

and

ii) ${}_rE((A \otimes B)_\lambda) \cong {}_rE(A) \otimes {}_rE(B)$

for $r \geqq 1$ as graded differential coalgebras.

REMARK. Applying similar arguments as in **3.7** and **3.8** to $(M \otimes N)_\lambda$, where M and N are differential left A- and B-modules (or comodules), we can obtain a similar Künneth formula for the spectral sequence of $(M \otimes N)_\lambda$. But it is somewhat tedious and unnecessary in our later discussions, so we omit the details of them.

3.9. Let A be a differential algebra (or coalgebra). The filtration induced from F- (or G-) filtration of A is defined by

$$\bar{F}^nH(A) = \mathrm{Im}\,[H(F^nA) \to H(A)]$$
$$(\text{or } \bar{G}^nH(A) = \mathrm{Im}\,[H(G^nA) \to H(A)]).$$

This filtration is of course not the same as the F- (or G-) filtration of $H(A)$. Nevertheless we can easily see that

(3.19) $\qquad \bar{F}^nH(A) \supset F^nH(A) \qquad (\text{or } \bar{G}^nH(A) \subset G^nH(A))$

for all $n \geqq 0$.

In coalgebra case we can easily deduce from (3.19) the following

(3.20) **Proposition.** *Let A be a differential and semi-connected coalgebra. Then $H(A)$ is also semi-connected.*

In algebra case we can not prove the corresponding proposition. To this end we need much stronger condition than the "semi-connected".

Let A be an algebra (or a coalgebra). We call A to be *finitely semi-connected* if there exists $n>0$ such that $F^nA=\{0\}$ (or $G^nA=A$). Thus, in case of an associative algebra A, to say that A is finitely semi-connected is equivalent to say that the unique maximal ideal $\bar{A}$ is nilpotent. When A is of finite dimension as a module over K, "semi-connected" is equivalent to "finitely semi-connected".

Now from (3.19) we can easily see the following

(3.21) **Proposition.** *Let A be a differential algebra (or coalgebra). If A is finitely semi-connected, then $H(A)$ is also finitely semi-connected.*

4. λ-modified differential Hopf algebras

4.1. Let a G_2-module A be endowed with a structure of algebra as well as that of coalgebra. When the algebra unit and augmentation coincide with the coalbegra augmentation and counit respectively, then we call A a *quasi pre Hopf algebra*. Furthermore, when the multiplication and the comultiplication is associative then we call A a *pre Hopf algebra.*

When a (quasi) pre Hopf algebra A is equipped with a differential so that A is a differential algebra as well as a differential coalgebra, then we call A a *differential* (quasi) pre Hopf algebra. In this case by (3.15) we have two spectral sequences

$$\{E_r(A),\quad r\geqq 0\}\quad\text{and}\quad \{{}_rE(A),\quad r\geqq 0\}$$

of graded algebras and coalgebras respectively.

If a differential (quasi) pre Hopf algebra A satisfies

$$\psi\varphi=(\varphi\otimes\varphi)(1\otimes T_\lambda\otimes 1)(\psi\otimes\psi) \tag{4. 1}$$

for some $\lambda\in K$, then we call A a λ-*modified differential* (*quasi*) *Hopf algebra,* or simply a (*quasi*) $(d,\ \lambda)$ *-Hopf algebra.* Thus, to say that a differential (quasi) pre Hopf algebra A is a (quasi) $(d,\ \lambda)$-Hopf algebra is equivalent to say that

$$\psi\colon\ A\to(A\otimes A)_\lambda$$

is a morphism of differential algebras or that

$$\varphi\colon\ (A\otimes A)_\lambda\to A$$

is a morphism of differential coalgebras.

A(quasi) $(d,\ 0)$-Hopf algebra is simply called a *differential* (*quasi*) *Hopf alge-*

bra; furthermore, when the differential is trivial or ignored, it is a *(quasi) Hopf algebra*. Any (quasi) Hopf algebra can be regarded as a (quasi) (d, λ)-Hopf algebra with $d=0$ for any $\lambda \in K$.

We obtain easily

(4.2) **Proposition.** *If A is a (quasi) (d, λ)-Hopf algebra, then $H(A)$ is a (quasi) Hopf algebra with the induced structure morphisms.* (Cf., [1], proposition 2.4).

Examples of (d, λ)-Hopf algebras are rich in mod p K-theory of H-spaces. Differential near Hopf algebras of [1] are $(d,1)$-Hopf algebras over $K=Z_2$. If we use a non-commutative external multiplication in $K^*(\ ; Z_p)$ [2], then $K^*(X; Z_p)$ of a finite CW-H-space X becomes a quasi (d, λ)-Hopf algebra with $\lambda \neq 0$. (Cf., [1], §5).

4.2. Let A be a (quasi) (d, λ)-Hopf algebra. Morphisms ψ and φ induce a morphism of differential algebras

$$E_r(\psi): E_r(A) \to E_r((A \otimes A)_\lambda)$$

and of differential coalgebras

$${}_rE(\varphi): {}_rE((A \otimes A)_\lambda) \to {}_rE(A)$$

for $r \geqslant 0$. (3.18) and (3.18*) imply that $E_r(\psi)$ (or ${}_rE(\varphi)$) defines a comultiplication (or multiplication) of $E_r(A)$ (or ${}_rE(A)$). Thus we obtain Hopf structures in $E_r(A)$ (or ${}_rE(A)$) and we can easily see the following

(4.3) **Proposition.** *Let $\lambda \in K$ and A be a (quasi) (d, λ)-Hopf algebra. In the spectral sequence $\{E_r(A), r \geqslant 0\}$ (or $\{{}_rE(A), r \geqslant 0\}$) associated with the F- (or G-) filtration,*

i) *the term $E_0(A)$ (or ${}_0E(A)$) is a graded connected (quasi) (d, λ)-Hopf algebra with the differential d_0 of degree 0,*

ii) *the term $E_r(A)$ (or ${}_rE(A)$) is a graded connected differential (quasi) Hopf algebra with the differential d_r of degree r (or $-r$) for $r \geqslant 1$, and*

iii) *$E_{r+1}(A) \cong H(E_r(A))$ (or ${}_{r+1}E(A) \cong H({}_rE(A))$ as (quasi) Hopf algebras for $r \geqslant 0$. (Cf., [1], proposition 2.9).*

4.3. Let A and B be differential (quasi) pre Hopf algebra. For $\lambda \in K$ we denote by $(A \otimes B)_\lambda$ a differential (quasi) pre Hopf algebra $A \otimes B$ with the λ-modified multiplication and λ-modified comultiplication.

(4.4) **Proposition.** *Let $\lambda \in K$. If A and B are (quasi) (d, λ)-Hopf algebras then $(A \otimes B)_\lambda$ is also a (quasi) (d, λ)-Hopf algebra.*

Proof. The proposition is classical for $\lambda=0$. By the principle (2.21) using (3.6) we conclude (4.4) for general $\lambda \in K$. q.e.d.

Now (3.18), (3.18*) and (4.4) show that

(4.5) **Proposition.** *Let $\lambda\in K$, and A and B be (quasi) (d, λ)-Hopf algebras.*

i) $E_0((A\otimes B)_\lambda)\cong(E_0(A)\otimes E_0(B))_\lambda$

and

$$ {}_0E((A\otimes B)_\lambda)\cong({}_0E(A)\otimes{}_0E(B))_\lambda$$

as graded (quasi) (d, λ)-Hopf algebras;

ii) $E_r((A\otimes B)_\lambda)\cong E_r(A)\otimes E_r(B)$

and

$$ {}_rE((A\otimes B)_\lambda)\cong{}_rE(A)\otimes{}_rE(B)$$

as graded (quasi) differential Hopf algebras for $r\geqslant 1$:

iii) *any morphism $f: A\to B$ of (quasi) (d, λ)-Hopf algebras induces morphisms $E_r(f): E_r(A)\to E_r(B)$ and ${}_rE(f): {}_rE(A)\to{}_rE(B)$ of graded (quasi) (d, λ)-Hopf algebras for $r=0$ and of graded differential (quasi) Hopf algebras for $r\geqslant 1$.*

4.4. Let A be a quasi (d, λ)-Hopf algebra. Since

$$\psi: A\to(A\otimes A)_\lambda \quad\text{and}\quad \varphi: (A\otimes A)_\lambda\to A$$

are morphisms of differential algebras and coalgebras respectively, for each $w_n\in W_n$, the set of (1.6),

$$\psi_n^{w_n}: A\to(A^{\otimes n+1})_\lambda \quad\text{and}\quad \varphi_n^{w_n}: (A^{\otimes n+1})_\lambda\to A$$

are morphisms of differential algebras and coalgebras respectively, and preserve F-filtrations and G-filtrations respectively. Then, by (3.17) and (3.17*) we see that

$$\psi_n^{w_n}(F^m A)\subset\textstyle\sum_{i_1+\cdots+i_{n+1}=m}F^{i_1}A\otimes\cdots\otimes F^{i_{n+1}}A$$

and

$$\varphi_n^{w_n}(\textstyle\sum_{i_1+\cdots+i_{n+1}=m}G^{i_1}A\otimes\cdots\otimes G^{i_{n+1}}A)\subset G^m A$$

for each $w_n\in W_n$, $n\geqslant 1$ and $m\geqslant 0$. In particular $\bar{\psi}_{n-1}^{w_{n-1}}$ and $\bar{\varphi}_{n-1}^{w_{n-1}}$ induces

$$Q^n(\bar{\psi}_{n-1}^{w_{n-1}}): Q_n A\to(Q^1A)^{\otimes n} \tag{4.6}$$

and

$$P^n(\bar{\varphi}_{n-1}^{w_{n-1}}): (P^1A)^{\otimes n}\to P^n A \tag{4.6*}$$

for each $w_{n-1}\in W_{n-1}$, $n\geqslant 2$.

4.5. Let A be (quasi) pre Hopf algebra. For each $n\geqslant 1$ we define a map

$$\nu_n: P^n A\to Q^n A$$

as a composition of the inculsion $P^nA \to \bar{A}$ and the projection $\bar{A} \to Q^nA$. particularly important is the map

$$\nu = \nu_1 \colon P(A) \to Q(A).$$

According to [6] we call A to be *primitive, coprimitive* or *biprimitive* if ν is surjective, injective or bijective respectively.

(4.7) **Proposition.** *Let A be (quasi) (d, λ)-Hopf algebra. If A is primitive, coprimitive or biprimitive then ν_n is surjective, injective or bijective respectively for every $n \geqslant 1$.*

Proof. Suppose A is primitive. Then ν_1 is surjective. We prove the surjectivity of ν_n by an induction on n. Let $n \geqslant 2$ and consider the following commutative diagram:

$$\begin{tikzcd}
\oplus(P^1A)^{\otimes n} \arrow[d, "\oplus(\nu_1)^{\otimes n}"] \arrow[rr, "\oplus P^n(\bar{\varphi}^{w_{n-1}}_{n-1})"] & & P^nA \arrow[dd, "\nu_n"] & P^{n-1}A \arrow[l] \arrow[dd, "\nu_{n-1}"] \\
\oplus(Q^1A)^{\otimes n} \arrow[d, equal] & & & \\
\oplus(E^1_0A)^{\otimes n} \arrow[r, "\oplus E^n_0(\bar{\varphi}^{w_{n-1}}_{n-1})"] & E^n_0A \arrow[r] & Q^nA \arrow[r] & Q^{n-1}A,
\end{tikzcd}$$

where the direct sum $\oplus$ runs over all $w_{n-1} \in W_{n-1}$, $P^n(\bar{\varphi}^{w_{n-1}}_{n-1})$ is the map of (4.6*) and $E^n_0(\bar{\varphi}^{w_{n-1}}_{n-1})$: $(E^0_1A)^{\otimes n} \to E^n_0A$ coincides with $E_0(\bar{\varphi})^{w_{n-1}}_{n-1}$. (1.18) implies that $\oplus E^n_0(\bar{\varphi}^{w_{n-1}}_{n-1})$ is surjective; $\oplus(\nu_1)^{\otimes n}$ is surjective since ν_1 is so; ν_{n-1} is surjective by an assumption of the induction. Then, since $E^n_0A \to Q^nA \to Q^{n-1}A$ is exact, by chasing the above diagram we see easily the surjectivity of ν_n.

Next, suppose A is coprimitive. ν_1 is injective. We prove the injectivity of ν_n by an induction on n. Let $n \geqslant 2$ and consider the following commutative diagram:

$$\begin{tikzcd}
\oplus({}_0E^1A)^{\otimes n} \arrow[d, equal] & {}_0E^n \arrow[l, "\oplus_0 E^n(\psi^{w_{n-1}}_{n-1})"'] & P^nA \arrow[l] \arrow[dd, "\nu_n"] & P^{n-1}A \arrow[l] \arrow[dd, "\nu_{n-1}"] \\
\oplus(P^1A)^{\otimes n} \arrow[d, "\oplus(\nu_1)^{\otimes n}"] & & & \\
\oplus(Q^1A)^{\otimes n} & & Q^nA \arrow[ll, "\oplus Q^n(\bar{\psi}^{w_{n-1}}_{n-1})"] \arrow[r] & Q^{n-1}A,
\end{tikzcd}$$

where the direct sum $\oplus$ runs over all $w_{n-1} \in W_{n-1}$. This diagram is dual to the previous one. Then, by a dual argument to the above we see the injectivity of ν_n.

q.e.d.

4.6. Let A be a quasi (d, λ)-Hopf algebra. Sometimes the semi-connectivities of A as algebra and as coalgebra are not independent to each other.

(4.8) **Proposition.** *Let A be a quasi (d, λ)-Hopf algebra. If A is coprimitive and* *semi-connected as a coalgebra, then A is semi-connected as an algebra.*

Proof. Take any $x \in \bigcap_{k \geqslant 1} F^k A$. Since A is semi-connected as a coalgebra there exists an integer n such that $x \in P^n A$. But $\nu_n(x)=0$ since $x \in F^{n+1}A$. Now A is coprimitive, hence (4.7) implies that $x=0$. q.e.d.

We can not prove the dual statement of (4.8) by a similar reason as (3.20). Again, under the stronger condition of "finitely semi-connected" the dual statement is true.

(4.9) **Proposition.** *Let A be a quasi (d, λ)-Hopf algebra. If A is coprimitive and finitely semi-connected as a coalgebra, then A is finitely semi-connected as an algebra.*

(4.9*) **Proposition.** *Let A be a quasi (d, λ)-Hopf algebra. If A is primitive and finitely semi-connected as an algebra, then A is finitely semi-connected as a coalgebra.*

Proofs of these propositions are easy.

4.7. By definitions, (1.19) and (1.19*) we see

(4.10) **Proposition.** *Let A be a quasi (d, λ)-Hopf algebra. $E_0(A)$ is primitive and ${}_0E(A)$ is coprimitive.* (Cf., [6], proposition 1.3).

Next we prove

(4.11) **Proposition.** *Let A be a quasi (d, λ)-Hopf algebra which is semi-connected as an algebra. If $E_0(A)$ is coprimitve then A is coprimitive.*

Proof. We show that $P^1A \cap F^2A = \{0\}$. Take any element $x \in P^1A \cap F^nA$ for any $n \geqslant 2$. Then

$$\{x\} \in P^1(E_0(A)) \cap E_0^n A, \quad n \geqslant 2,$$

where $\{x\}$ denotes the element of $E_0^n A$ represented by x. By the assumption and (4.10) $E_0(A)$ is biprimitive. Therefore

$$\{x\} = 0, \quad \text{i.e.,} \quad x \in P^1A \cap F^{n+1}A.$$

Thus

$$P^1A \cap F^nA = P^1A \cap F^{n+1}A \qquad \text{for all} \quad n \geqslant 2,$$

i.e.,

$$P^1A \cap F^2A = P^1A \cap F^nA \qquad \text{for all} \quad n \geqslant 2.$$

Now

$$P^1A \cap F^2A = \bigcap_{n \geqslant 2} (P^1A \cap F^nA) = \{0\}$$

since A is semi-connected as an algebra. q.e.d.

As a dual of (4.11) we obtain

(4.11*) **Proposition.** *Let A be a quasi (d, λ)-Hopf algebra which is semi-connected as a coalgebra. If ${}_0E(A)$ is primitive then A is primitive.*

Proof. Take any element $x \in \bar{A}$. We show that x is congruent to an element of P^1A modulo F^2A. Since A is semi-connected as a coalgebra, there exists $n>0$ such that $x \in P^nA$. Suppose that $n \geqslant 2$. Since ${}_0E(A)$ is biprimitive by the assumption and (4.10), $\{x\}$ is decomposable, where $\{x\}$ denotes the class in ${}_0E^nA$ represented by x. Thus, there exists $u \in P^{n-1}A$ such that

$$x \equiv u \bmod F^2A \cap P^nA ,$$

whence, by an induction on n in the descending order, we see that x is congruent to an element of P^1A mod F^2A q.e.d.

4.8. From now up to the end of this section we will see how some propositions and theorems of Milnor-Moore [10] work also for our (d, λ)-Hopf algebras.

Let A and B be associative algebras (or coalgebras) and f: $A \rightarrow B$ (or f: $B \rightarrow A$) a morphism of algebras (or coalgebras) which is left normal in the sense of [10], Definitions 3.3 and 3.5. We regard B as a left A-module (or -comodule) as usual. Put $C = K \otimes_A B$ (or $C = K \square_A B$). Then C is a G_2-module and, as is easily seen, has an induced structure of an algebra (or a coalgebra) by that of B. Further suppose that A and B are differential algebras (or coalgebras) and f is a morphism of differential algebras (or coalgebras). Then C obtains a differential induced by that of B so that C becomes a differential algebra (or coalgebra). Finally suppose that A and B are quasi (d, λ)-Hopf algebras for $\lambda \in K$ and f a morphism of quasi (d, λ)-Hopf algebras, then the relation (4.1) of B shows that C obtains also a comultiplication (or multiplication) induced by that of B so that C becomes a quasi (d, λ)-Hopf algebra for the same $\lambda \in K$. Thus we obtain the following two propositions.

(4.12) **Proposition.** *Let $\lambda \in K$ and f: $A \rightarrow B$ be a morphism of quasi (d, λ)-Hopf algebras which is left normal as a morphism of algebras. Let $\pi: B \rightarrow C = K \otimes_A B$ be the natural projection, and assume that the multiplications of A and B are associative. Then C has a unique structure of quasi (d, λ)-Hopf algebra such that π is a morphism of quasi (d, λ)-Hopf algebras.*

(4.12*) **Proposition.** *Let $\lambda \in K$ and f: $B \rightarrow A$ be a morphism of quasi (d, λ)-Hopf algebras which is left normal as a morphism of coalgebras. Let $i: C = K \square_A B \rightarrow B$ be the natural injection, and assume that the comultiplications of A and B are associative. Then C has a unique structure of quasi (d, λ)-Hopf algebra such that i is a morphism of quasi (d, λ)-Hopf algebras.*

4.9. A morphism of algebras or A-modules, where A is an algebra, is called to be an *almost isomorphism* or *almost bijective* if it is injective and almost surjective.

Let A and B be algebras (or coalgebras) and f: $A \to B$ (or f: $B \to A$) a morphism of algebras (or coalgebras). Put $C = K \otimes_A B$ (or $C = K \square_A B$) and π: $B \to C$ (or i: $C \to B$) be the canonical projection (or injection). By the definitions of C we have the following relation:

$$\text{(4.13)} \qquad \pi\varphi_B(f \otimes 1) = \pi(\varepsilon_A \otimes 1) \colon A \otimes B \to C$$

$$\text{(or (4.13*)} \qquad (f \otimes 1)\psi_B i = (\eta_A \otimes 1)i \colon C \to A \otimes B)\,,$$

where we used the identification $K \otimes B = B$. These relations will be used frequently in the following discussions.

(4.14) **Proposition.** *Let i: $A \to B$ be an injection of quasi (d, λ)-Hopf algebras which is left normal as a morphism of algebras, and π: $B \to C = K \otimes_A B$ be the canonical projection. Assume that B is semi-connected and associative as an algebra. then there exists an almost isomorphism*

$$h \colon A \otimes C \to B$$

of left A-modules such that $i = h(1 \otimes \eta_C)$ *and* $\pi h = \varepsilon_A \otimes 1$. *If we assume furthermore that B is finitely semi-connected as an algebra or that A and B are graded connected, and i is degree-preserving, then h is an isomorphism.*

Proof. Choose a morphism j: $C \to B$ of G_2-modules such that $\pi j = 1_C$, $j\eta_C = \eta_B$ and $\varepsilon_B j = \varepsilon_C$ (but not required to commute with differentials). Define

$$h \colon A \otimes C \to B$$

as the composition

$$A \otimes C \xrightarrow{1 \otimes j} A \otimes B \xrightarrow{i \otimes 1} B \otimes B \xrightarrow{\varphi} B$$

of morphisms of left A-modules, where $B \otimes B$ is an A-module by the A-module structure of the left tensor factor. The relations $i = h(1 \otimes \eta_C)$ and $\pi h = \varepsilon_A \otimes 1$ are easily obtained. We have

$$1_K \otimes_A h = \pi j = 1_C\,.$$

Thus by (1.27) we see that h is almost surjective.

Next we show the injectivity of h. Define a map

$$g \colon B \to B \otimes C$$

as the composition

$$B \xrightarrow{\psi} B \otimes B \xrightarrow{1 \otimes \pi} B \otimes C\,.$$

Using (4.13) we can easily see that g is a morphism of left A-modules, and we have

$$gh = \alpha(i\otimes 1)\,,$$

where $\alpha=(\varphi_B\otimes\pi)(1\otimes\psi_B)(1\otimes j)$: $B\otimes C\to B\otimes C$, is a morphism of left B-modules. Now, since $i\otimes 1$ is injective, to show the injectivity of h it is sufficient to show that α is injective. Since α preserves F-filtrations of $B\otimes C$, which are given by $F^kB\otimes C$, by restricting α we have a map

$$F^k\alpha\colon F^kB\otimes C\to F^kB\otimes C$$

for each $k\geqslant 0$. Here

$$\begin{aligned} F^k\alpha \bmod F^{k+1}B\otimes C &\equiv (\varphi\otimes\pi)(1\otimes\eta\varepsilon\otimes 1)(1\otimes\psi)(1\otimes j) \\ &= (1\otimes\pi)(1\otimes j) = id. \end{aligned}$$

This shows that the map

$$E_0^k\alpha\colon E_0^kB\otimes C\to E_0^kB\otimes C$$

induced by α is an identity map for each $k\geqslant 0$. Thus α is injective by (1.28).

Finally suppose that B is finitely semi-connected as an algebra (or A and B are graded connected, and i is degree-preserving), then F-topology of B (or of B^n for each degree n) is discrete. Thus almost surjectivity of h implies the surjectivity of h. q.e.d.

(4.14*) **Proposition.** *Let π: $B\to A$ be a surjection of quasi (d, λ)-Hopf algebras which is left normal as a morphism of coalgebras, and i: $C=K\square_A B\to B$ be the canonical injection. Assume that B is semi-connected and associative as a coalgebra, then there exists an isomorphism h: $B\to A\otimes C$ of left A-comodules such that $hi=\eta_A\otimes 1$ and $\pi=(1\otimes\varepsilon_C)h$.*

Proof. Choose a morphism j: $B\to C$ of G_2-modules such that $ji=1_C$, $\varepsilon_C j=\varepsilon_B$ and $j\eta_B=\eta_C$. Define

$$h\colon B\to A\otimes C$$

as the composition

$$B \xrightarrow{\psi} B\otimes B \xrightarrow{\pi\otimes 1} A\otimes B \xrightarrow{1\otimes j} A\otimes C$$

of morphisms of left A-comodules. h is injective by (1.27*) and satisfies the relations: $hi=\eta_A\otimes 1$ and $\pi=(1\otimes\varepsilon_C)h$, as is easily seen. Define a map

$$g\colon B\otimes C\to B$$

as the composition

$$B\otimes C \xrightarrow{1\otimes i} B\otimes B \xrightarrow{\varphi} B\,.$$

Using (4.13*) we can see that g is a morphism of left A-comodules, and we have

$$hg = (\pi\otimes 1)\alpha\,,$$

where $\alpha=(1\otimes j)(1\otimes\varphi_B)(\psi_B\otimes i)$: $B\otimes C\to B\otimes C$, is a morphism of left B-comodules. Then, discussing in a similar way as in (4.14), we see that ${}_0E^k\alpha=1$ for all $k\geqslant 0$. Thus α is surjective by (1.28*). Hence h is surjective. q.e.d.

4.10. The following two propositions correspond to Proposition 4.9 of [10].

(4.15) **Proposition.** *Let i: $A\to B$ be an injection of quasi (d, λ)-Hopf algebras which is left normal as a morphism of algebras, and π: $B\to C=K\otimes_A B$ be the canonical projection. Assume that the multiplication of B is associative, and that B is finitely semi-connected as an algebra or that A and B are graded connected, and i is degree-preserving, then i gives an isomorphism*

$$A\cong B\square_C K$$

of quasi (d, λ)-Hopf algebras by restricting range.

Proof. We use the maps and notations given in the proof of (4.14). It is sufficient to show that the sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\bar g} B\otimes\bar C$$

is exact, where $\bar g=(1\otimes\rho_C)g$.

$$\begin{aligned} gi &= (1\otimes\pi)\psi_B i = (1\otimes\pi)(i\otimes i)\psi_A \\ &= (i\otimes(\eta_C\varepsilon_A))\psi_A = i\otimes\eta_C\,, \end{aligned}$$

thus

$$\bar g i = 0\,.$$

Next assume that $\bar g(b)=0$ for $b\in B$, then

$$\begin{aligned} g(b) &= (1\otimes(\eta_C\varepsilon_C))g(b) = (1\otimes\eta_C)(1\otimes\varepsilon_C)(1\otimes\pi)\psi_B(b) \\ &= (1\otimes\eta_C)(1\otimes\varepsilon_B)\psi_B(b) = (1\otimes\eta_C)(b)\,. \end{aligned}$$

By the proof of (4.14), h: $A\otimes C\to B$ and α: $B\otimes C\to B\otimes C$ are isomorphisms such that $gh=\alpha(i\otimes 1)$ and $\alpha(1\otimes\eta_C)=1\otimes\eta_C$. Then we have

$$\begin{aligned} i(1\otimes\varepsilon_C)h^{-1}(b) &= (1\otimes\varepsilon_C)(i\otimes 1)h^{-1}(b) \\ &= (1\otimes\varepsilon_C)\alpha^{-1}g(b) \\ &= (1\otimes\varepsilon_C)\alpha^{-1}(1\otimes\eta_C)(b) = b\,, \end{aligned}$$

i.e.,

$$b\in \operatorname{Im} i\,.$$

q.e.d.

(4.15*) **Proposition.** *Let π: $B \to A$ be a surjection of quasi (d, λ)-Hopf algebras which is left normal as a morphism of coalgebras, and i: $C = K \square_A B \to B$ be the canonical injection. Assume that B is semi-connected and associative as a coalgebra, then π induces an isomorphism*

$$B \otimes_C K \cong A$$

of quasi (d, λ)-Hopf algebras by passing to quotients.

Proof. We use the maps and notations appeared in the proof of (4.14*). It is sufficient to show that the sequence

$$B \otimes \bar{C} \xrightarrow{\bar{g}} B \xrightarrow{\pi} A \longrightarrow 0$$

is exact, where $\bar{g} = g(1 \otimes \iota_C)$.

$$\begin{aligned} \pi g &= \pi \varphi_B(1 \otimes i) = \varphi_A(\pi \otimes \pi)(1 \otimes i) \\ &= \varphi_A(\pi \otimes (\eta_A \varepsilon_C)) = \pi \otimes \varepsilon_C , \end{aligned}$$

thus

$$\pi \bar{g} = 0 .$$

Next assume that $\pi(b) = 0$ for $b \in B$. By the proof of (4.14*) h: $B \to A \otimes C$ and α: $B \otimes C \to B \otimes C$ are isomorphisms satisfying $\pi = (1 \otimes \varepsilon_C)h$, $(1 \otimes \varepsilon_C)\alpha = 1 \otimes \varepsilon_C$ and $hg = (\pi \otimes 1)\alpha$. Therefore

$$(1 \otimes \varepsilon_C)h(b) = \pi(b) = 0 ,$$

i.e.,

$$h(b) \in A \otimes \bar{C} .$$

Hence, choosing a morphism k: $A \to B$ of G_2-modules such that $\pi k = 1_A$, we have

$$\alpha^{-1}(k \otimes 1)h(b) \in B \otimes \bar{C},$$

and

$$\begin{aligned} g\alpha^{-1}(k \otimes 1)h(b) &= g\alpha^{-1}(k \otimes 1)h(b) \\ &= h^{-1}(\pi \otimes 1)(k \otimes 1)h(b) = b , \end{aligned}$$

i.e.,

$$b \in \operatorname{Im} \bar{g} .$$

q.e.d.

By the above two propositions and Propositions 3.11 and 3.12 of [10] we obtain

(4.16) **Proposition.** *Under the assumptions of Proposition* (4.15) *or* (4.15*) (*in this case, exchanging the notations between A and C*) *there is a commutative diagram*

$$\begin{array}{ccccccc} 0 \to & P(A) & \to & P(B) & \to & P(C) & \\ & \downarrow & & \downarrow & & \downarrow & \\ & Q(A) & \to & Q(B) & \to & Q(C) & \to 0 \end{array}$$

with exact rows.

4.11. The following two propositions correspond to Proposition 4.11 of [10]. The proofs are almost parallel to that of [10] if we use our (4.15) and (4.15*) instead of Proposition 4.9 of [10]. So we omit the proofs.

(4.17) **Proposition.** *Suppose i: $A \to B$ and j: $B \to C$ are injections of quasi (d, λ)-Hopf algebras which are left normal as morphisms of algebras. Assume that the multiplication of C is associative, and that C is finitely semi-connected as an algebra or that A, B and C are graded connected, and i, j are degree preserving. Let $B' = K \otimes_A B$, $C' = K \otimes_A C$ and j': $B' \to C'$ be the morphism of quasi (d, λ)-Hopf algebras induced by j. Then j' is injective and left normal as a morphism of algebras, and*

$$K \otimes_B C = K \otimes_{B'} C'.$$

(4.17*) **Proposition.** *Suppose τ: $B \to A$ and π: $C \to B$ are surjections of quasi (d, λ)-Hopf algebras which are left normal as morphisms of coalgebras. Assume that C is semi-connected and associative as a coalgebra. Let $B' = K \square_A B$, $C' = K \square_A C$ and π': $C' \to B'$ be the morphism of quasi (d, λ)-Hopf algebras induced by π. Then π' is surjective and left normal as a morphism of coalgebras, and*

$$K \square_{B'} C' = K \square_B C\,.$$

4.12. Let A be a semi-connected coalgebra. We define an integer valued function ω on A by $\omega(x)$="the least integer n such that $x \in G^n A$". We call ω the *primitivity function* on A. The following proposition corresponds to Proposition 4.13 of [10].

(4.18) **Proposition.** *Let A be a (quasi) (d, λ)-Hopf algebra which is semi-connected as a coalgebra. Then A is a direct limit of sub (quasi) (d, λ)-Hopf algebras which are finitely generated as algebra.*

The proof is the same as [10], Proposition 4.13, if we replace the choice of $x \in A - B$ with least degree by a choice of such an element with least value $\omega(x)$, and the use of degree property of $\bar{\psi}(x)$ by the use of (1.16*). So we omit the details of the proof.

Let A be a (quasi) (d, λ)-Hopf algebra such that $A = \bigcup_{i \in I} A_i$, a direct limit of sub (quasi) (d, λ)-Hopf algebras, where I is directed by the inclusion of A_i. Since the direct limit is an exact functor for modules over a field, the functors P^i, Q^i, E_0, ${}_0E$, and H commute with the direct limit, i.e.,

$$\begin{aligned} &P^k A = \cup_{i \in I} P^k A_i\,, \quad F^k A = \cup_{i \in I} F^k A_i\,, \\ &Q^k A = \varinjlim_{i \in I} Q^k A_i\,, \quad {}_0E(A) = \varinjlim_{i \in I} {}_0E(A_i)\,, \\ &E_0(A) = \varinjlim_{i \in I} E_0(A_i) \quad \text{and} \quad H(A) = \varinjlim_{i \in I} H(A_i)\,. \end{aligned}$$

5. Derived Hopf algebras

5.1. In this section all modules, algebras and so on, are understood over a field K of characteristic $p \neq 0$. Let Π be a multiplicative cyclic group of order p and M a Π-module, i.e., Π operates on M. Suppose M admits a basis $X=\{x_\iota\}$ which is Π-invariant. Then X is decomposed uniquely as a disjoint union $X=Y \cup Z$ of Π-invariant subsets such that Π fixes every element of Y and operates freely on Z. We call Y and Z respectively the Π-*fixed* and Π-*free* part of X.

Let π be a generator of Π and put

$$\Delta = 1-\pi \quad \text{and} \quad \sum = 1+\pi+\cdots+\pi^{p-1}\,.$$

Since $\Delta\sum=\sum\Delta=0$, for a Π-module M we can define the quotients

$$\Phi(M) = \operatorname{Ker} \Delta/\operatorname{Im} \sum \quad \text{and} \quad \Psi(M) = \operatorname{Ker} \sum/\operatorname{Im} \Delta\,. \tag{5.1}$$

Since $\sum=\Delta^{p-1}$, we have inclusions

$$\operatorname{Ker} \Delta \subset \operatorname{Ker} \sum \quad \text{and} \quad \operatorname{Im} \sum \subset \operatorname{Im} \Delta$$

and a canonical map

$$\kappa\colon \Phi(M) \to \Psi(M)\,. \tag{5.2}$$

Following lemmas are obvious.

(5.3) **Lemma.** *Let M and N be Π-modules, then*

$$\Phi(M \oplus N) \cong \Phi(M) \oplus \Phi(N)$$

and

$$\Psi(M \oplus N) \cong \Psi(M) \oplus \Psi(N)$$

canonically.

(5.4) **Lemma.** *Let M be a Π-module admitting a Π-invariant basis X, and Y be the Π-fixed part of X. Then the canonical map κ induces isomorphisms*

$$\Phi(M) \cong \Psi(M) \cong K\{Y\}\,.$$

the submodule of M generated by Y.

5.2. Let M be a differential G_2-module. For $\lambda \in K$, the λ-modified cyclic permutation $C_{p,\lambda}$ makes $M^{\otimes p}$ a Π-module. In this case Δ and $\sum$ are denoted by Δ_λ and $\sum_\lambda$ respectively. We denote also as

$$\Phi(M^{\otimes p}, C_{p,\lambda}) = \Phi_\lambda M \quad \text{and} \quad \Psi(M^{\otimes p}, C_{p,\lambda}) = \Psi_\lambda M \tag{5.5}$$

for simplicity. Since $C_{p,\lambda}$ commutes with the differentials, $\Phi_\lambda M$ and $\Psi_\lambda M$

obtain the induced structures of differential G_2-modules.

(5.6) **Lemma.** *Let M and N be differential G_2 modules. We have canonical isomorphisms*

$$\Phi_\lambda(M \oplus N) \cong \Phi_\lambda M \oplus \Phi_\lambda N$$

and

$$\Psi_\lambda(M \oplus N) \cong \Psi_\lambda M \oplus \Psi_\lambda N$$

of differential G_2-modules.

Proof. We have a direct sum decompsoition

$$(M \oplus N)^{\otimes p} = M^{\otimes p} \oplus N^{\otimes p} \oplus B$$

of Π-modules, where $B = M \otimes N^{\otimes p-1} \oplus \cdots$. Π operates freely on the set of direct summands $\{M \otimes N^{\otimes p-1}, \dots\}$ of B. Choosing a set of representatives of Π-orbits in this set, $M \otimes N^{\otimes p-1}$, $M^{\otimes 2} \otimes N^{\otimes p-2}$ and so on, and choosing a d-stable homogeneous basis of M and of N respectively, make a basis for each representative of the above Π-orbits as usual. The union of these bases and their successive $C_{p,\lambda}$-transforms makes a Π-free basis of B. Thus Lemmas (5.3) and (5.4) conclude Lemma (5.6). q.e.d.

5.3. Let M be a differential G_2-module. Choosing a d-stable homogeneous basis $\{x_\iota, dx_\iota, y_\kappa\}$ of M, where $dy_\kappa = 0$, we can decompose M as a direct sum of differential sub G_2-modules:

$$M = \oplus_\iota B_\iota \oplus_\kappa C_\kappa\,,$$

where $B_\iota = K\{x_\iota, dx_\iota\}$ and $C_\kappa = K\{y_\kappa\}$. Thus

$$\Phi_\lambda M \cong \oplus_\iota \Phi_\lambda B_\iota \oplus_\kappa \Phi_\lambda C_\kappa \tag{5.7}$$

and

$$\Psi_\lambda M \cong \oplus_\iota \Psi_\lambda B_\iota \oplus_\kappa \Psi_\lambda C_\kappa \tag{5.7'}$$

by (5.6) for $\lambda \in K$. We can now reduce the discussions of $\Phi_\lambda M$ and $\Psi_\lambda M$ to those of $\Phi_\lambda B$, $\Psi_\lambda B$, $\Phi_\lambda C$ and $\Psi_\lambda C$, where $B = K\{x, dx\}$ and $C = K\{y\}$ with $dy = 0$.

$\Phi_\lambda C$ and $\Psi_\lambda C$ are easily discussed.

$$C^{\otimes p} = K\{y^{\otimes p}\} \cong K$$

and $y^{\otimes p}$ is fixed by $C_{p,\lambda}$ regardless of type y. Thus, by (5.4), we obtain

$$\Phi_\lambda C \cong \Psi_\lambda C \cong K \tag{5.8}$$

generated by $y^{\otimes p}$.

Next we discuss $\Phi_\lambda B$ and $\Psi_\lambda B$.

Case i): $p=2$ *and* $\lambda=0$. In this case $\{x\otimes x,\ dx\otimes dx,\ x\otimes dx,\ dx\otimes x\}$ forms a Π-invariant basis of $B\otimes B$ and its Π-fixed part is $\{x\otimes x,\ dx\otimes dx\}$. Thus

$$\Phi_\lambda B \cong \Psi_\lambda B \cong K\{x\otimes x,\ dx\otimes dx\}\ . \tag{5.9.1}$$

Case ii): $p=2$ *and* $\lambda\neq 0$. In this case $\{x\otimes x,\ x\otimes x+\lambda dx\otimes dx,\ x\otimes dx,\ dx\otimes x\}$ forms a Π-invariant Π-free basis of $B\otimes B$. Hence

$$\Phi_\lambda B \cong \Psi_\lambda B \cong \{0\}\ . \tag{5.9.2}$$

Case iii) *p odd.* By a monomial in $B^{\otimes p}$ we mean a mixed tensor product of x and dx. A monomial in $B^{\otimes p}$ is called of height k if it contains exactly k dx's in its tensor factors. Let B_r denote the submodule of $B^{\otimes p}$ generated by monomials of height $\geqslant r$, which is clearly $C_{p,\lambda}$-stable. B_r defines a decreasing filtration of $B^{\otimes p}$:

$$B^{\otimes p} = B_0 \supset B_1 \supset \cdots \supset B_p = K\{(dx)^{\otimes p}\} \supset B_{p+1} = \{0\}\ .$$

The operation on B_r/B_{r+1} induced by $C_{p,\lambda}$ coincides with C_p, and as is easily seen B_r/B_{r+1} admits a Π-free basis for $0<r<p$ and a Π-fixed basis for $r=0$ and p, namely, $\{x^{\otimes p}\}$ for $r=0$ and $\{(dx)^{\otimes p}\}$ for $r=p$. Choosing a set of representatives of Π-orbits of this Π-free basis for $0<r<p$, by an induction on descending order of r we see easily that B_r admits a Π-invariant basis of which Π-fixed part consists only of $(dx)^{\otimes p}$ for $1\leqslant r\leqslant p$.

Let B^{ev} and B^{od} denote the submodule of $B^{\otimes p}$ generated by monomials of height even and odd respectively. By definition of λ-modified switching maps we see that B^{ev} and B^{od} are Π-stable, and we obtain direct sum decompositions

$$B^{\otimes p} = B^{ev}\otimes B^{od}$$

and

$$B_1 = B^{ev}\cap B_1 \oplus B^{od}\cap B_1$$

into Π-stable submodules. Furthermore, a simple check of the above choice of Π-invariant basis of B_1 shows that this basis can be decomposed as a union of Π-invariant bases of $B^{ev}\cap B_1$ and of $B^{od}\cap B_1$. Since $B^{ev}\cap B_1$ does not contain $(dx)^{\otimes p}$, finally we see that $B^{ev}\cap B_1$ admits a Π-free basis. Thus

$$\Phi(B^{ev}\cap B_1) = \Psi(B^{ev}\cap B_1) = \{0\}$$

by (5.4).

(5.10) **Lemma.** *There exists an element*

$$b_{p,\lambda}(x) \in B^{ev}\cap B_1$$

such that $x^{\otimes p}+b_{p,\lambda}(x)$ *is* Π*-fixed.* $b_{p,\lambda}(x)$ *is unique modulo Im* $\sum_\lambda$.

Using this lemma we see easily that the union of $\{x^{\otimes p}+b_{p,\lambda}(x)\}$ with the Π-free basis of $B^{ev}\cap B_1$ forms a Π-invariant basis of B^{ev} of which the Π-fixed part consists only of $x^{\otimes p}+b_{p,\lambda}(x)$. Since $B^{od}=B^{od}\cap B_1$, B^{od} admits a Π-invariant basis of which the Π-fixed part consists only of $(dx)^{\otimes p}$. Thus $B^{\otimes p}$ admits a Π-invariant basis of which the Π-fixed part is $\{x^{\otimes p}+b_{p,\lambda}(x), (dx)^{\otimes p}\}$. Therefore,

$$\Phi_\lambda B\cong\Psi_\lambda B\cong K\{x^{\otimes p}+b_{p,\lambda}(x),\ (dx)^{\otimes p}\}\,. \tag{5.9.3}$$

Proof of Lemma 5.10. Observe that

$$\Delta_\lambda(x^{\otimes p})\in B^{ev}\cap B_1\,.$$

Since $\Delta_\lambda(x^{\otimes p})\in\operatorname{Ker}\sum_\lambda$ and $\Psi(B^{ev}\cap B_1)=\{0\}$ we see that

$$\Delta_\lambda(x^{\otimes p})\in\Delta_\lambda(B^{ev}\cap B_1)\,,$$

i.e., there exists an element $b_{p,\lambda}(x)\in B^{ev}\cap B_1$ such that

$$\Delta_\lambda(x^{\otimes p}+b_{p,\lambda}(x))=0\,.$$

Let $b'\in B^{ev}\cap B_1$ be another element such that $x^{\otimes p}+b'$ is $C_{p,\lambda}$-invariant. Then

$$\Delta_\lambda(b_{p,\lambda}(x)-b')=0\,.$$

Since $\Phi(B^{ev}\cap B_1)=\{0\}$ we see that

$$b_{p,\lambda}(x)-b'\in\operatorname{Im}\textstyle\sum_\lambda\,.$$ q.e.d.

5.4. By (5.7), (5.7'), (5.8) and (5.9.1-3) we obtain

(5.11) **Proposition.** *Let M be a differential G_2-module and $\lambda\in K$. The canonical map* (5.2) *induces an isomorphism*

$$\Phi_\lambda M\cong\Psi_\lambda M$$

of differential G_2-modules.

Next we show

(5.12) **Proposition.** *Let M be a differential G_2-module and $\lambda\in K$. The induced differential on $\Phi_\lambda M\cong\Psi_\lambda M$ is trivial.*

Proof. Since $1\mid\operatorname{Ker}\Delta_\lambda=C_{p,\lambda}|\operatorname{Ker}\Delta_\lambda$ we have

$$\begin{aligned} d|\operatorname{Ker}\Delta_\lambda &= \textstyle\sum_{1\leqslant k\leqslant p}d_k|\operatorname{Ker}\Delta_\lambda = \sum_{1\leqslant k\leqslant p}d_kC_{p,\lambda}^{k-1}|\operatorname{Ker}\Delta_\lambda \\ &= \textstyle\sum_{1\leqslant k\leqslant p}C_{p,\lambda}^{k-1}d_1|\operatorname{Ker}\Delta_\lambda = \sum_\lambda d_1|\operatorname{Ker}\Delta_\lambda\subset\operatorname{Im}\sum_\lambda\,. \end{aligned}$$

Thus

$$d=0 \quad \text{on} \quad \Phi_\lambda M. \qquad\qquad \text{q.e.d.}$$

Φ_λ (or Ψ_λ) is clearly functorial. Let

$$0 \to L \to M \to N \to 0$$

be an exact sequence of differential G_2-modules. Choosing a canonical basis of L and then extending it to a canonical basis of M, we see that the above exact sequence is a direct sum of the following types of exact sequences:

$$0 \longrightarrow 0 \longrightarrow B \xrightarrow{1_B} B \longrightarrow 0,$$
$$0 \longrightarrow 0 \longrightarrow C \xrightarrow{1_C} C \longrightarrow 0,$$
$$0 \longrightarrow C \longrightarrow B \longrightarrow C \longrightarrow 0,$$
$$0 \longrightarrow B \xrightarrow{1_B} B \longrightarrow 0 \longrightarrow 0,$$
$$0 \longrightarrow C \xrightarrow{1_C} C \longrightarrow 0 \longrightarrow 0,$$

where B and C are elementary differential G_2-modules discussed in **5.3**. Apply Φ_λ (or Ψ_λ) on these sequences and use (5.8) and (5.9.1-3), then we see easily that

$$\Phi_\lambda L \to \Phi_\lambda M \to \Phi_\lambda N$$

is exact, and in case p odd or $p=2$ and $\lambda d=0$

$$0 \to \Phi_\lambda L \to \Phi_\lambda M \to \Phi_\lambda N \to 0$$

is exact. Thus we obtain

(5.13) **Proposition.** *Let $\lambda \in K$ and assume that p is odd or that $p=2$ and $\lambda d=0$. Then Φ_λ (or Ψ_λ) is an exact functor defined on the category of differential G_2-modules.*

5.5. Let A be a differential algebra (or coalgebra) and $\lambda \in K$. By (3.9) $C_{p,\lambda}\colon (A^{\otimes p})_\lambda \to (A^{\otimes p})_\lambda$ is a morphism of differential algebras (or coalgebras). Thus we obtain

$$\Delta_\lambda \varphi_\lambda = \varphi_\lambda(\Delta_\lambda \otimes 1 + C_{p,\lambda} \otimes \Delta_\lambda)$$
$$(\text{or } \psi_\lambda \Delta_\lambda = (\Delta_\lambda \otimes 1 + C_{p,\lambda} \otimes \Delta_\lambda)\psi_\lambda)$$

where φ_λ (or ψ_λ denotes the multiplication (or comultiplication) of $(A^{\otimes p})_\lambda$. Therefore we obtain

(5.14) **Lemma.** i) *When A is a differential algebra then Ker Δ_λ is a differential*

sub algebra of $(A^{\otimes p})_\lambda$.
ii) *When* A *is a differential coalgebra then Im* Δ_λ *is a differential coideal of* $(A^{\otimes p})_\lambda$.

Next we prove

(5.15) **Lemma.** i) *When* A *is a differential algebra then Im* $\sum_\lambda$ *is a differential ideal of Ker* Δ_λ.
ii) *When* A *is a differential coalgebra then* $\Psi_\lambda A$ *is a differential sub coalgebra of Coker* Δ_λ.

Proof. Remark that

$$1|\text{Ker } \Delta_\lambda = C_{p,\lambda}|\text{Ker } \Delta_\lambda .$$

Hence

$$\begin{aligned}&\varphi_\lambda(\textstyle\sum_\lambda \otimes 1)|B\otimes \text{Ker } \Delta_\lambda \\ &\quad = \varphi_\lambda(1\otimes 1 + C_{p,\lambda}\otimes C_{p,\lambda} + \cdots + C_{p,\lambda}^{p-1}\otimes C_{p,\lambda}^{p-1})|B\otimes \text{Ker } \Delta_\lambda \\ &\quad = \textstyle\sum_\lambda \varphi_\lambda |B\otimes \text{Ker } \Delta_\lambda ,\end{aligned}$$

where $B=(A^{\otimes p})_\lambda$. Thus

$$\varphi_\lambda((\text{Im } \textstyle\sum_\lambda)\otimes(\text{Ker } \Delta_\lambda))\subset \text{Im } \textstyle\sum_\lambda .$$

Similarly,

$$\varphi_\lambda((\text{Ker } \Delta_\lambda)\otimes(\text{Im } \textstyle\sum_\lambda))\subset \text{Im } \textstyle\sum_\lambda .$$

That is, i) is proved.

Next, for $x\in B=(A^{\otimes p})_\lambda$,

$$\begin{aligned}&(\textstyle\sum_\lambda\otimes 1)\psi_\lambda(x) \\ &\quad \equiv (1\otimes 1 + C_{p,\lambda}\otimes C_{p,\lambda} + \cdots + C_{p,\lambda}^{p-1}\otimes C_{p,\lambda}^{p-1})\psi_\lambda(x) \\ &\quad \equiv \psi_\lambda \textstyle\sum_\lambda(x) \qquad \text{mod} \quad B\otimes(\text{Im } \Delta_\lambda) .\end{aligned}$$

Thus

$$\psi_\lambda(\text{Ker } \textstyle\sum_\lambda)\subset(\text{Ker } \textstyle\sum_\lambda)\otimes B + B\otimes(\text{Im } \Delta_\lambda) .$$

Similarly

$$\psi_\lambda(\text{Ker } \textstyle\sum_\lambda)\subset B\otimes(\text{Ker } \textstyle\sum_\lambda) + (\text{Im } \Delta_\lambda)\otimes B .$$

Hence

$$\psi_\lambda(\text{Ker } \textstyle\sum_\lambda)\subset B\otimes(\text{Im } \Delta_\lambda) + (\text{Im } \Delta_\lambda)\otimes B + (\text{Ker } \textstyle\sum_\lambda)\otimes(\text{Ker } \textstyle\sum_\lambda) ,$$

which proves ii). q.e.d.

5.6. Now let A be a (quasi) (d, λ)-Hopf algebra for $\lambda\in K$. By (4.4) $(A^{\otimes p})_\lambda$ is also a (quasi) (d, λ)-Hopf algebra. By (5.15) $\Phi_\lambda A$ and $\Psi_\lambda A$ are differential algebra and coalgebra respectively with induced structures from $(A^{\otimes n})_\lambda$. (5.12) says that these differentials are trivial, and (5.11) says that there exists a canonical isomorphism $\Phi_\lambda A \cong \Psi_\lambda A$ of G_2-modules. So, if we identify $\Phi_\lambda A$ and $\Psi_\lambda A$ by this canonical isomorphism, then $\Phi_\lambda A$ gains structures of an algebra and a coalgebra. Since these structures of $\Phi_\lambda A$ are induced from the corresponding ones of $(A^{\otimes p})_\lambda$ it is a (quasi) (d, λ)-Hopf algebra with the trivial differential, i.e., we have

(5.16) **Proposition.** *Let A be a (quasi) (d, λ)-Hopf algebra. $\Phi_\lambda A = \Psi_\lambda A$ is a (quasi) Hopf algebra.*

We call $\Phi_\lambda A$ the *derived Hopf algebra* of A. By (5.6) we see also that

$$\overline{\Phi_\lambda A} = \Phi_\lambda \bar{A} \quad \text{and} \quad \overline{\Psi_\lambda A} = \Psi_\lambda \bar{A}\,. \tag{5.17}$$

6. Primitivity and coprimitivity

6.1. Let A be a quasi (d, λ)-Hopf algebra for $\lambda \in K$. By *ideals, sub algebras* and *quotient algebras* (or *coideals*, *sub coalgebras* and *quotient coalgebras*) of A we mean those of the *underlying* algebra (or coalgebra) of A.

The maps

$$\begin{aligned}
\varphi_a &= \varphi(\varphi\otimes 1 - 1\otimes\varphi)\colon\ (A\otimes A\otimes A)_\lambda \to A\,,\\
\varphi_c &= \varphi(T_\lambda - 1)\colon\ (A\otimes A)_\lambda \to A\,,\\
\psi_a &= (\psi\otimes 1 - 1\otimes\psi)\psi\colon\ A \to (A\otimes A\otimes A)_\lambda
\end{aligned}$$

and

$$\psi_c = (T_\lambda - 1)\psi\colon\ A \to (A\otimes A)_\lambda$$

measure deviations from associativities and λ-commutativities.

(6.1) **Lemma.** i) *$Im\ \varphi_a$ and $Im\ \varphi_c$ are differential coideals of A;*
ii) *$Ker\ \psi_a$ and $Ker\ \psi_c$ are differential sub algebras of A.*

Proof. Since $\varphi(\varphi\otimes 1)$, $\varphi(1\otimes\varphi)$, φT_λ and φ are morphisms of coalgebras by (4.1) and (3.8), putting $B=(A\otimes A)_\lambda$ and $C=(A\otimes A\otimes A)_\lambda$ we have

$$\psi\varphi_a = (\varphi_a\otimes(\varphi(\varphi\otimes 1)) + (\varphi(1\otimes\varphi))\otimes\varphi_a)\psi_c$$

and

$$\psi\varphi_c = (\varphi_c\otimes\varphi T_\lambda + \varphi\otimes\varphi_c)\psi_B\,,$$

which prove i).

Similarly we have

$$\psi_a\varphi = \varphi_c(\psi_a\otimes((\psi\otimes 1)\psi) + ((1\otimes\psi)\psi)\otimes\psi_a)$$

and

$$\psi_c\varphi = \varphi_B(\psi_c\otimes T_\lambda\psi+\psi\otimes\psi_c)\,,$$

which show ii). q.e.d.

6.2. As the first step to discuss our version of Milnor-Moore criterion of coprimitivity [10], Proposition 4.20, we have

(6.2) **Proposition.** *Let A be a quasi (d, λ)-Hopf algebra which is semi-connected as a coalgebra. If A is coprimitive then the multiplication is associative and λ-commutative.*

Proof. It is sufficient to show that φ_a and φ_c are zero maps.

By (6.1) Coker φ_a is a quotient coalgebra of A. Since Im $\varphi_a\subset F^2$A we have a map f: $\overline{\text{Coker } \varphi_a}\to Q(A)$ such that the following diagram

$$\begin{tikzcd} P(A) \arrow[r, "i"] \arrow[d, "P(\pi)"] & \bar{A} \arrow[r, "j"] \arrow[d, "\bar{\pi}"] & Q(A) \\ P(\text{Coker } \varphi_a) \arrow[r] & \overline{\text{Coker } \varphi_a} \arrow[ur, "f"'] & \end{tikzcd}$$

is commutative, where π: $A\to$Coker φ_a is the canonical projection, a morphism of coalgebras, and $ji=\nu$: $P(A)\to Q(A)$. As ν is injective by assumption the above diagram shows that $P(\pi)$ is injective. Then π is injective by (1.25*). Hence

$$\text{Ker } \pi = \text{Im } \varphi_a = \{0\}\,.$$

A parallel discussion shows that φ_c is also a zero map. q.e.d.

(6. 2*) **Proposition.** *Let A be a quasi (d, λ)-Hopf algebra which is semi-connected as an algebra. If A is primitive then the comultiplication is associative and λ-commutative.*

Proof. We show that ψ_a and ψ_c are zero maps.

By (6.1) Ker ψ_a is a sub algebra of A. Evidently $\psi_a(P(A))=0$. Thus we have a map g: $P(A)\to\overline{\text{Ker } \psi_a}$ such that the following diagram

$$\begin{tikzcd} P(A) \arrow[r, "i"] \arrow[dr, "g"'] & \bar{A} \arrow[r, "j"] & Q(A) \\ & \overline{\text{Ker } \psi_a} \arrow[u, "\bar{k}"] \arrow[r] & Q(\text{Ker } \psi_a) \arrow[u, "Q(k)"'] \end{tikzcd}$$

is commutative, where k: Ker $\psi_a\to A$ is the inclusion. As $\nu=ji$ is surjective by assumption, we see that $Q(k)$ is surjective. Then Ker ψ_a is dense in A by (1.25). On the other hand, ψ_a preserves F-filtrations since $(\psi\otimes 1)\psi$ and $(1\otimes\psi)\psi$ are so, which means that ψ_a is continuous. Thus Ker ψ_a is closed in A since $(A\otimes A\otimes A)_\lambda$ is semi-connected as an algebra, i.e., Hausdorff. Hence

$$\mathrm{Ker}\,\psi_a = A\,.$$

A parallel discussion shows also that

$$\mathrm{Ker}\,\psi_c = A\,. \qquad \text{q.e.d.}$$

6.3. Let A be a differential algebra (or coalgebra). If the multiplication φ (or comultiplication ψ is associative, then, for each $n\geqslant 1$, $\varphi_n^{w_n}$ (or $\psi_n^{w_n}$) are the same map for all choices of $w_n\in W_n$, the set of (1.6). In this case we denote it simply by φ_n (or ψ_n) (which coincides with the notation of [6]).

Now assume that the characteristic of K, denoted by p, is non-zero and the *multiplication* φ (*or comultiplication* ψ) *of* A *is associative and* λ*-commutative for some* $\lambda\in K$. Using the notations of §5 we define a map

$$\xi'_\lambda\colon\ \mathrm{Ker}\,\textstyle\sum_\lambda\to A \quad (\text{or } \eta'_\lambda\colon\ A\to\mathrm{Coker}\,\textstyle\sum_\lambda)$$

by

$$\xi'_\lambda = \varphi_{p-1}i \quad (\text{or } \eta'_\lambda = \pi\psi_{p-1})\,,$$

where i: Ker $\sum_\lambda\to(A^{\otimes p})_\lambda$ and π: $(A^{\otimes p})_\lambda\to$Coker $\sum_\lambda$ are the canonical inclusion and projection respectively. Since φ (or ψ) is λ-commutative we have

$$\varphi_{p-1}\Delta_\lambda = 0 \quad (\text{or } \Delta_\lambda\psi_{p-1} = 0)$$

by (3.3) and $(2.13)_\lambda$. And, passing to quotient (or restricting range) we have the induced map

$$\xi''_\lambda\colon\ \Psi_\lambda A\to A \quad (\text{or } \eta''_\lambda\colon\ A\to\Phi_\lambda A)\,.$$

Define

$$\xi_\lambda\colon\ \Phi_\lambda A\to A \quad (\text{or } \eta_\lambda\colon\ A\to\Psi_\lambda A)$$

as the composition

$$\xi_\lambda = \xi''_\lambda\circ\kappa \quad (\text{or } \eta_\lambda = \kappa\circ\eta''_\lambda)$$

where κ is the canonical map (5.2). Since φ (or ψ) is λ-commutative, φ_{p-1}: $(A^{\otimes p})_\lambda\to A$ (or ψ_{p-1}: $A\to(A^{\otimes p})_\lambda$) is a morphism of differential algebras (or coalgebras). Thus the induced morphism ξ_λ (or η_λ) is a morphism of algebras (or coalgebras).

Now suppose A is a quasi (d, λ)-Hopf algebra, char $K=p\neq 0$, and the multiplication φ (or comultiplication ψ) is associative and λ-commutative. φ_{p-1} (or ψ_{p-1}) is a morphism of coalgebras (or algebras) by (4.1). Thus the induced one ξ''_λ (or η''_λ) is also so. Thus we obtain

(6.3) **Proposition.** *Let A be a quasi (d, λ)-Hopf algebra over a field K of characteristic $p \neq 0$. If the multiplication φ (or comultiplication ψ) is associative and λ-commutative then φ_{p-1} (or ψ_{p-1}) induces a morphism of (d, λ)-Hopf algebras*

$$\xi_\lambda\colon \Phi_\lambda A \to A \quad (\textit{or } \eta_\lambda\colon A \to \Psi_\lambda A)\,. \tag{6.4}$$

Since the differential of $\Phi_\lambda A$ is trivial by (5.12),

$$\mathrm{Im}\,\xi_\lambda \quad (\text{or Coim}\ \eta_\lambda) \tag{6.5}$$

is a quasi Hopf algebra with an associative and commutative multiplication (or comultiplication) which is a sub (or quotient) Hopf algebra of A.

6.4. We use the following notations:

$$\bar{\xi}_\lambda = \xi_\lambda | \overline{\Phi_\lambda A} \quad \text{and} \quad \bar{\eta}_\lambda = \eta_\lambda | \bar{A}\,.$$

(6.6) **Theorem.** *Let $\lambda \in K$ and A be a quasi (d, λ)-Hopf algebra which is semi-connceted as a coalgebra. If A is coprimitive then the multiplication is associative, λ-commutative and, when the characteristic of K is non-zero, $\bar{\xi}_\lambda$ is a zero map.*

Proof. The first half is only a repetition of (6.2).

By (6.3) $\xi_\lambda(\Phi_\lambda A)$ is a sub coalgebra of A, hence $\xi_\lambda(\overline{\Phi_\lambda A})$ is a coideal of A. Since ξ_λ is induced by φ_{p-1} and every element of $\overline{\Phi_\lambda A}$ has a representative in $\bar{A}^{\otimes p}$ by (5.17), we see easily that $\xi_\lambda(\overline{\Phi_\lambda A}) \subset F^2 A$ and obtain an induced map f: $\bar{A}/\xi_\lambda(\overline{\Phi_\lambda A}) \to Q(A)$ such that the following diagram

$$\begin{tikzcd}
P(A) \arrow[r, "i"] \arrow[d, "P(\pi)"] & \bar{A} \arrow[r, "j"] \arrow[d, "\bar{\pi}"] & Q(A) \\
P(A/\xi_\lambda(\overline{\Phi_\lambda A})) \arrow[r] & \bar{A}/\xi_\lambda(\overline{\Phi_\lambda A}) \arrow[ur, "f"'] &
\end{tikzcd}$$

is commutative, similar to that of (6.2). Then a parallel argument to (6.2) shows

$$\xi_\lambda(\overline{\Phi_\lambda A}) = \{0\}\,. \qquad \text{q.e.d.}$$

As a dual of (6.6) we obtain

(6.6*) **Theorem.** *Let $\lambda \in K$ and A be a quasi (d, λ)-Hopf algebra which is semi-connected as an algebra. If A is primitive then the comultiplication is associative, λ-commutative and, when the characteristic of K is non-zero and $\Psi_\lambda A$ is semi-connected as an algebra, $\bar{\eta}_\lambda$ is a zero map.*

Proof. Again the first part is only a repetition of (6.2*).

By (6.3) Ker η_λ is an ideal of A. Hence, putting $B = K \oplus \mathrm{Ker}\,\eta_\lambda$, B is a sub algebra of A. An easy computation shows that

$$\psi_{p-1}(P(A)) \equiv 0 \quad \text{mod} \quad \text{Im} \textstyle\sum_\lambda .$$

Thus $P(A) \subset \text{Ker}\, \eta_\lambda = \bar{B}$ and we have a commutative diagram

$$\begin{tikzcd} P(A) \arrow[r, "i"] \arrow[dr] & \bar{A} \arrow[r, "j"] & Q(A) \\ & \bar{B} \arrow[u, "\bar{k}"] \arrow[r] & Q(B) \arrow[u, "Q(k)"'] \end{tikzcd},$$

similar to that of (6.2*). Hence Ker η_λ is dense in $\bar{A}$ with respect to F-topology of A as in (6.2*). Now Ker η_λ is closed in $\bar{A}$ since η_λ is continuous as a morphism of algebras and $\Psi_\lambda A$ is Hausdorff by assumption. Thus

$$\text{Ker}\, \eta_\lambda = \bar{A} . \qquad \text{q.e.d.}$$

REMARK. The additional assumption that $\Psi_\lambda \bar{A}$ is semi-connected as an algebra is perhaps awkward in the above Theorem. Yosimura [11] has proved recently the following fact: *Let A be a differential algebra over a field K, char $K=p \neq 0$, and $\lambda \in K$; when p is odd or $p=2$ and $\lambda d=0$, A is semi-connected if and only if the algebra $\Psi_\lambda A$ is so; when $p=2$ and $\lambda d \neq 0$, $H(A)$ is semi-connected if and only if $\Psi_\lambda A$ is so.* Thus this awkward assumption about $\Psi_\lambda A$ can be eliminated in major case, or replaced by another assumption that $H(A)$ is semi-connected as an algebra in the other case. Because of (3.21), when A is finite dimensional over K we can eliminate the assumption about $\Psi_\lambda A$ completely.

6.5. Let A be a quasi (d, λ)-Hopf algebra over a field K, char $K=p \neq 0$. The above two Theorems (6.6) and (6.6*) give necessary conditions for coprimitivity and primitivity of A. We show later that, in the major case, i.e., p odd or $p=2$ and $\lambda d=0$, these are also sufficient conditions. Here we denote these conditions by

$<$CP$>$ *the multiplication φ is associative, λ-commutative and ξ_λ is a zero map,* and

$<$P$>$ *the comultiplication ψ is associative, λ-commutative and η_λ is a zero map.*

Here we have

(6.7) **Proposition.** *The properties $<$CP$>$ and $<$P$>$ of A are respectively hereditary to $H(A)$.*

Proof. Clearly the associativity and λ-commutativity is hereditary to $H(A)$. Now in the exact sequence

$$0 \to d\bar{A} \to Z(\bar{A}) \to H(\bar{A}) \to 0$$

each term has trivial differential. Hence

$$0 \to \Phi_\lambda(d\bar{A}) \to \Phi_\lambda(Z(\bar{A})) \to \Phi_\lambda(H(\bar{A})) \to 0$$

is exact by (5.13). Observe the following commutative diagram:

$$\begin{array}{ccccc} \Phi_\lambda(\bar{A}) & \longleftarrow & \Phi_\lambda(Z(\bar{A})) & \xrightarrow{\Phi_\lambda(\pi)} & \Phi_\lambda(H(\bar{A})) \\ \downarrow{\scriptstyle \xi_\lambda} & & \downarrow{\scriptstyle \xi_\lambda(Z(\bar{A}))} & & \downarrow{\scriptstyle \xi_\lambda(H(A))} \\ \bar{A} & \xleftarrow{i} & Z(\bar{A}) & \longrightarrow & H(\bar{A}) \end{array}$$

i is injective and $\Phi_\lambda(\pi)$ is surjective. Hence $\xi_\lambda=0$ implies that $\xi_\lambda(H(A))=0$, i.e., $<CP>$ is hereditary to $H(A)$.

The case of $<P>$ can be proved by a parallel discussion. q.e.d.

6.6. One of the main steps in proving the sufficiency of $<CP>$ and $<P>$ for coprimitivity and primitivity in major cases, is to reduce the problem to some graded cases and come back to A by making use of (4.11) and (4.11*).

(6.8) **Proposition.** *Let $\lambda\in K$, char $K=p$, and A be a quasi (d, λ)-Hopf algebra. Assume that p is odd or that $p=2$ and $\lambda d=0$. If A satisfies $<CP>$ then $E_0(A)$ also satisfies $<CP>$.*

Proof. The associativity and λ-commutativity of $E_0(\varphi)$ is clear. We prove only that ξ_λ is a zero map for $E_0(A)$.

Since each $E_0^n A$ is a d_0-stable submodule of $E_0(A)$, we have

$$\Phi_\lambda E_0(A) \simeq \bigoplus_{n\geq 0} \Phi_\lambda E_0^n A$$

by (5.6). Now we see that

$$\xi_\lambda\colon \Phi_\lambda E_0(A)\to E_0(A)$$

is a direct sum of maps

$$\xi_\lambda^n\colon \Phi_\lambda E_0^n A\to E_0^{np} A$$

(by the reason of degrees), $n\geqslant 0$. Since ξ_λ (of A) is induced by φ_{p-1} and φ_{p-1} preserves F-filtrations by (1.16) and (3.17), it induces a map

$$\xi_\lambda\colon \Phi_\lambda F^n A\to F^{np} A$$

for each $n\geqslant 0$. Now consider the following commutative diagram

$$\begin{array}{ccccc} \Phi_\lambda \bar{A} & \xrightarrow{\Phi_\lambda(i)} & \Phi_\lambda F^n A & \xrightarrow{\Phi_\lambda(\pi)} & \Phi_\lambda E_0^n A \\ \downarrow{\scriptstyle \xi_\lambda} & & \downarrow{\scriptstyle \xi_\lambda} & & \downarrow{\scriptstyle \xi_\lambda^n} \\ \bar{A} & \xleftarrow{i} & F^n A & \xrightarrow{\pi} & E_0^{np} A \end{array}$$

for $n>0$, where i and π are canonical inclusions and projections. By (5.13) $\Phi_\lambda(\pi)$ is surjective. Then, by a simple chasing of the above diagram we see that the assumption implies that

$$\xi_\lambda^n:\ \Phi_\lambda E_0^n A \to E_0^{np} A$$

is a zero map for each $n>0$. q.e.d.

(6.8*) **Proposition.** *Let $\lambda \in K$, char $K=p$, and A be a quasi (d, λ)-Hopf algebra. Assume that p is odd or that $p=2$ and $\lambda d=0$. If A satisfies $<P>$ then* ${}_0E(A)$ also satisfies $<P>$.

Proof. Again we prove only that $\bar{\eta}_\lambda$ is a zero map for ${}_0E(A)$. By the same reason as in the above, we have

$$\Psi_\lambda({}_0E(A)) \cong \oplus_n \Psi_\lambda({}_0E^n A)\,.$$

By the nature of the construction of Ψ_λ, $\Psi_\lambda({}_0E^n A)$ should be considered as the group of degree np, that is, if we express the graded module $\Psi_\lambda({}_0E(A))$ as

$$\Psi_\lambda({}_0E(A)) = \oplus_{m\geqslant 0} \Psi_\lambda({}_0E(A))^m,$$

then

$$\Psi_\lambda({}_0E(A))^m = \{0\} \quad \text{for} \quad m \not\equiv 0 \bmod p$$

and

$$\Psi_\lambda({}_0E(A))^{np} = \Psi_\lambda({}_0E^n A)\,.$$

Therefore, since η_λ of ${}_0E(A)$ is induced by ${}_0E(\psi_{p-1})$, it is degree preserving; putting

$$\eta_\lambda = \oplus_m \eta_\lambda^m, \quad \eta_\lambda^m = \eta_\lambda|_0E^m A\,,$$

we see that

$$\eta_\lambda^m = 0 \quad \text{for} \quad m \not\equiv 0 \bmod p$$

and

$$\eta_\lambda^{np}:\ {}_0E^{np}A \to \Psi_\lambda({}_0E^n A)$$

for each $n \geqslant 0$. Hence it is sufficient to show that $\eta_\lambda^{np}=0$ for $n>0$ under our present assumption.

Next we remark that, if we regard as $\overline{\Psi_\lambda A} = \Psi_\lambda \bar{A}$ by (5.17), then we can regard as $\bar{\eta}_\lambda$ (of A) is induced by $\bar{\psi}_{p-1}$. Now ψ_{p-1} preserves G-filtrations by (1.16*) and (3.17*), whence $\bar{\psi}_{p-1}(p^{np-1}A) \subset p^{np-1}(\bar{A}^{\otimes p})_\lambda$. Hence $\bar{\psi}_{p-1}$ induces a map

$$\bar{\psi}'_{p-1}:\ \bar{A}/p^{np-1}A \to (\bar{A}^{\otimes p})_\lambda / p^{np-1}(\bar{A}^{\otimes p})_\lambda\,.$$

$(\bar{A}^{\otimes p})_\lambda / P^{np-1}(\bar{A}^{\otimes p})_\lambda$ is a Π-module by an operation induced by $C_{p,\lambda}$. And, by exactly the same reason as in (6.3), $\bar{\psi}'_{p-1}$ induces a map

$$\bar{\eta}'_\lambda:\ \bar{A}/P^{np-1}A \to \Psi((\bar{A}^{\otimes p})_\lambda / P^{np-1}(\bar{A}^{\otimes p})_\lambda,\ C_{p,\lambda})$$

for each $n>0$. Since

$$P^{np-1}(\bar{A}^{\otimes p})_\lambda \subset \bar{A}^{\otimes p-1} \otimes P^{n-1}A + \cdots + P^{n-1}A \otimes \bar{A}^{\otimes p-1}$$

by (3.17*), we have an induced map

$$j\colon (\bar{A}^{\otimes p})_\lambda / P^{np-1}(\bar{A}^{\otimes p})_\lambda \to (\bar{A}/P^{n-1}A)^{\otimes p}$$

of Π-modules. Hence we have

$$\Psi(j)\colon \Psi((\bar{A}^{\otimes p})_\lambda / P^{np-1}(\bar{A}^{\otimes p})_\lambda, C_{p,\lambda}) \to \Psi_\lambda(\bar{A}/P^{n-1}A)\,.$$

Then we can regard the composition

$$\bar{\eta}_\lambda = \Psi(j)\eta'_\lambda\colon \bar{A}/P^{np-1}A \to \Psi_\lambda(\bar{A}/P^{n-1}A)$$

as a map induced by $\bar{\psi}_{p-1}$ for each $n>0$.

Finally consider the following commutative diagram

$$\begin{tikzcd} \bar{A} \arrow[r, "\pi"] \arrow[d, "\eta_\lambda"] & \bar{A}/P^{np-1}A \arrow[d, "\bar{\eta}_\lambda"] & {}_0E^{np}A \arrow[l, "k"'] \arrow[d, "\eta_\lambda^{np}"] \\ \Psi_\lambda \bar{A} \arrow[r, "\Psi_\lambda(\pi)"'] & \Psi_\lambda(\bar{A}/P^{n-1}A) & \Psi_\lambda({}_0E^{n}A) \arrow[l, "\Psi_\lambda(k)"] \end{tikzcd}$$

for $n>0$, where $k\colon {}_0E^mA \to \bar{A}/P^{m-1}A$ is the inclusion and $\pi\colon \bar{A} \to \bar{A}/P^{m-1}A$ is the projection. By (5.13) $\Psi_\lambda(k)$ is injective. Then, chasing the above diagram we see that the assumption implies that η_λ^{np} is a zero map for $n>0$. q.e.d.

6.7. Let A be a quasi (d, λ)-Hopf algebra, $\lambda \in K$, char $K=p\neq 0$. In this subsection we assume that *the multiplication of A is associative and λ-commutative.* We obtain the following lemmas by routine calculations involving inductions.

(6.9) **Lemma.** i) *If $x\in A$ is of even type, then*

$$\lambda(dx)^2 = 0 \quad \textit{and} \quad d(x^k) = kx^{k-1}dx;$$

ii) *if $x\in A$ is of odd type, then*

$$\lambda(dx)^2 = 2x^2,\ d(x^{2k+1}) = x^{2k}dx \quad \textit{and} \quad d(x^{2k}) = 0.$$

(6.10) **Lemma.** *Suppose that $p=2$ or that p is odd and x and y are elements of even type. Then*

$$(x+y)^k = \sum_{i=0}^k \binom{k}{i} x^i y^{k-i} + \lambda(k(k-1)/2) \sum_{i=1}^{k-1} \binom{k-2}{i-1} x^{i-1}dx \cdot y^{k-i-1}dy\,,$$
$$(xy)^k = x^k y^k + \lambda(k(k-1)/2) x^{k-1}dx \cdot y^{k-1}dy\,.$$

(6.11) **Lemma.** *Let $x\in P(A)$.* i) *Suppose that $p=2$ or that p is odd and x is* *of even type, then*

$$\psi(x^k) = \sum_{i=0}^{k}\binom{k}{i}x^i\otimes x^{k-i}+\lambda(k(k-1)/2)\sum_{i=1}^{k-1}\binom{k-2}{i-1}x^{i-1}dx\otimes x^{k-i-1}dx\,,$$

$$\psi(x^k dx) = \sum_{i=0}^{k}\binom{k}{i}(x^i dx\otimes x^{k-i}+x^i\otimes x^{k-i}dx);$$

ii) *Suppose that $p=2$ or that p is odd and x is of odd type, then*

$$\psi((dx)^k) = \sum_{i=0}^{k}\binom{k}{i}(dx)^i\otimes(dx)^{k-i},$$

$$\psi((dx)^k x) = \sum_{i=0}^{k}\binom{k}{i}((dx)^i\otimes(dx)^{k-i}x+(dx)^i x\otimes(dx)^{k-i})\,.$$

By (5.9.1-3) we see that Im ξ_λ is generated by $\{x^2;\ \lambda dx=0,\ x\in A\}$ in case $p=2$, and $\{\varphi_{p-1}(x^{\otimes p}+b_{p,\lambda}(x)),\ x\in A\}$ in case p odd. Since $b_{p,\lambda}(x)$ is a linear combination of mixed p-fold tensor products of x and dx containing dx at least two, by (6.9) we see at once

(6.12) **Lemma.** *Suppose that A satisfies $<CP>$, then*
i) *in case $p=2$,*

$$x^2 = 0 \quad \textit{if} \quad \lambda dx = 0;$$

ii) *in case p odd,*

$$x^p = 0 \quad \textit{if } x \textit{ is of even type.}$$

6.8. Let A be a graded connected quasi (d, λ)-Hopf algebra, $\lambda\in K$, char $K=p\neq 0$. A is bigraded: the one is the Z_2-grading and the other is a non-negative grading, the former is called by "type" and the latter by "degree". By a "homogeneous" element of A we mean an element with definite type and degree. In this subsection we assume that *the multiplication is associative and λ-commutative, and deg $d=0$.*

Let $x\in\bar{A}$ be a homogeneous element, then non-zero elements of $\{x^k,\ x^k dx;\ k\geqslant 0\}$ are linear independent since types or degrees are mutually different. And also non-zero elements of $\{(dx)^k,\ (dx)^k x;\ k\geqslant 0\}$ are linear independent. Therefore by (6.11) we obtain

(6.13) **Lemma.** *Let $x\in A$ be homogeneous and primitive.*
i) *Suppose that $p=2$ or that p is odd and x is of even type; if a non -zero element x^k is primitive, then $k=p^t$, $t\geqslant 0$, where $t\neq 1$ in case $p=2$ and $\lambda dx\neq 0$; if a non-zero element $x^k dx$ is primitive, then $k=0$.*
ii) *Suppose that $p=2$ or that p is odd and x is of odd type; if a non-zero element $(dx)^k$ is primitive, then $k=p^t, t\geqslant 0$; if a non-zero element $(dx)^k x$ is primitive, then $k=0$.*

6.9. Here we prove the inverse to Theorem (6.6) under very special assumptions in major cases.

(6.14) **Proposition.** *Let A be a graded connected quasi (d, λ)-Hopf algebra with deg $d=0$, whose generators as an algebra are finite and of degree 1. Assume that p is odd or that $p=2$ and $\lambda d=0$. If A satisfies $<CP>$ then A is coprimitive.*

Proof. Choose a d-stable homogeneous basis

$$\{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n, dx_1, \ldots, dx_k\},$$

$1 \leqslant k \leqslant n$, of A^1 (the module of degree 1) where $dx_j=0$ for $k+1 \leqslant j \leqslant n$. We say that A has n generators as a differential algebra. We prove the proposition by an induction on n.

First we suppose that A has a single homogeneous generator x as a differential algebra. We have two cases.

1) The case of $dx=0$; in this case we are just in the ordinary case; $A=K_p[x]/(x^p)$ when p is odd and x is of even type, or $A=\Lambda_p(x)$ otherwise by (6.9) and (6.12). Thus A is biprimitive.

2) The case of $dx \neq 0$; the elements of A are all linear combinations of elements of the form $x^k(dx)^l$. When $p=2$ or p is odd and x is of even type, we see that $(dx)^2=0$ by (6.12) for $p=2$, and by (6.9.ii) for p odd; hence the elements of A are all linear combinations of x^k, $x^k dx$, $k \geqslant 0$. When p is odd and x is of odd type, we see by (6.9.ii) that $\lambda(dx)^2=2x^2$, i.e., $x^2=\lambda/2\cdot(dx)^2$; hence the elements of A are all linear combinations of $(dx)^k$, $x(dx)^k$, $k \geqslant 0$. Now by (6.12) and (6.13) we see easily that A is biprimitive.

Next process is parallel to [10], p.233. Suppose the proposition is true for cases with less than or equal to n-1 generators as differential algebras. Let B be the sub differential algebra of A generated by $\{x_1, \ldots, x_{n-1}\}$. λ-commutativity implies that B is a normal sub algebra of A. Putting $C=K\otimes_B A$, C is a quotient differential algebra of A with one generator $\gamma=\pi(x_n)$ of differential algebra, where $\pi\colon A \to C$ is the canonical projection. Here we note that B and C are quasi (d,λ)-Hopf algebras with generators $\leqslant n-1$ as differential algebras, satisfying the assumption of the proposition. Under our assumption on p and λd, Φ_λ is an exact functor by (5.13). Thus $<CP>$ is hereditary to Sub and Quotient (d, λ)-Hopf algebras. Therefore B and C satisfy $<CP>$, hence are coprimitive by the assumption of the induction. Now we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P(B) \arrow[r] \arrow[d, "\nu_B"] & P(A) \arrow[r] \arrow[d, "\nu_A"] & P(C) \arrow[d, "\nu_C"] & \\
0 \arrow[r] & Q(B) \arrow[r] & Q(A) \arrow[r] & Q(C) \arrow[r] & 0
\end{tikzcd}$$

with exact rows by (4.15), (4.16) and our assumption on degrees of generators. It follows the injectivity of ν_A from that of ν_B and ν_C. q.e.d.

6.10. Now we prove our inverse theorem to Theorem (6.6) in cases p odd or $p=2$ and $\lambda d=0$.

(6.15) **Theorem.** *Let $\lambda \in K$ and A be a quasi (d, λ)-Hopf algebra. Suppose that A is semi-connected as an algebra, and that char $K=p$ is odd or that $p=2$ and $\lambda d=0$. If A satisfies $<CP>$ then A is coprimitive.*

Proof. By (6.8) $E_0(A)$ satisfies $<CP>$ under our assumptions. $E_0(A)$ is graded connected. Hence by (4.18) we can express as

$$E_0(A) = \cup_{i \in I} B_i$$

as a direct limit of sub quasi (d, λ)-Hopf algebras which are finitely generated as algebras. Since $E_0(A)$ is generated by elements of degree 1 as an algebra by (1.19), we can choose B_i, $i \in I$, so that they are generated by elements of degree 1. Now, since deg $d_0=0$, B_i satisfies the assumptions of (6.14) for each $i \in I$. $<CP>$ is clearly hereditary to sub (d, λ)-Hopf algebras. Thus B_i satisfies $<CP>$, hence is coprimitive by (6.14). Then, since the direct limits are exact functors.

$$\nu : P(E_0(A)) \rightarrow Q(E_0(A))$$

is injective as the limit of

$$\nu_i : P(B_i) \rightarrow Q(B_i)$$

which are injective, i.e., $E_0(A)$ is coprimitive. Finally, by (4.11) we see that A is coprimitive. q.e.d.

6.11. Let A be a graded G_2-module, i.e.,

$$\begin{aligned} A &= A_0 \oplus A_1, \quad Z_2\text{-grading} \\ &= \textstyle\sum_{n \geqslant 0} A^n, \quad \text{non-negative grading} \end{aligned}$$

such that, putting $A_t^n = A_t \cap A^n$, $A_t = \sum_{n \geqslant 0} A_t^n$ for $t \in Z_2$. Suppose A is of finite type, i.e. dim $A^n < \infty$ for each $n \geqslant 0$, then we can talk about the dual A^* of A as usual [10]. Several notions and statements about A and A^* are in duality relation. We denote here by "$X \leftrightarrow Y$" that a notion or a statement X about A is dual to Y about A^*. When A is a finite dimensional G_2-module, putting $A^0 = A$, $A^n = \{0\}$ for $n > 0$, we can regard A as a graded G_2-module of finite type, and apply the following duality relations.

The following duality are classical, cf. [10]: multiplication $\varphi \leftrightarrow$ comultiplication φ^*; φ is associative $\leftrightarrow \varphi^*$ is associative; η is a unit for $\varphi \leftrightarrow \eta^*$ is a counit for φ^*; ε is an augmentation of $\varphi \leftrightarrow \varepsilon^*$ is an augmentation of φ^*; $(A, \varphi, \eta, \varepsilon)$ is a graded algebra $\leftrightarrow (A^*, \varphi^*, \eta^*, \varepsilon^*)$ is a graded coalgebra; graded connected algebra $\leftrightarrow$ graded connected coalgebra; differential $\leftrightarrow$ differential; differential algebra $\leftrightarrow$ differential coalgebra.

Now let $\lambda \in K$. We see easily that T_λ is self-dual. Thus we obtain the following dualities: multiplication φ is λ-commutative$\leftrightarrow$comultiplication φ^* is λ-commutative; $(A, \varphi, \psi, \eta, \varepsilon, d)$ is a graded (quasi) (d, λ)-Hopf algebra$\leftrightarrow(A^*, \psi^*, \varphi^*, \varepsilon^*, \eta^*, d^*)$ is a graded (quasi) (d, λ)-Hopf algebra.

Next, let A be a graded (quasi) (d, λ)-Hopf algebra of finite type. The following duality relations are also easily seen:
$P^k A \leftrightarrow Q^k A^*$, $k \geqslant 1$; $H(A) \leftrightarrow H(A^*)$; $E_r(A) \leftrightarrow {}_rE(A^*)$, $r \geqslant 1$; A is primitive$\leftrightarrow A^*$ is coprimitive; A is semi-connected as an algebra$\leftrightarrow A^*$ is semi-connected as a coalgebra. Assume further that char $K=p \neq 0$, then $\Phi_\lambda A \leftrightarrow \Psi_\lambda A^*$: $\xi_\lambda \leftrightarrow \eta_\lambda$: A satisfies $<CP>\leftrightarrow A^*$satisfies $<P>$.

6.12. As a dual to Theorem (6.15) we obtain

(6.15*) **Theorem.** *Let $\lambda \in K$ and A be a quasi (d, λ)-Hopf algebra. Suppose that A is semi-connected as a coalgebra, and that char $K=p$ is odd or that $p=2$ and $\lambda d=0$. If A satisfies $<P>$ then A is primitive.*

Proof. By (6.8*) ${}_0E(A)$ satisfies $<P>$. ${}_0E(A)$ is graded connected. Hence by (4.18) we can express ${}_0E(A)=\bigcup_{i\in I} B_i$ as a direct limit of sub quasi (d, λ)-Hopf algebras which are finitely generated as algebras. Hence B_i is of finite type. Under our assumptions Ψ_λ is an exact functor by (5.13). Hence the property $<P>$ is hereditary to sub (d, λ)-Hopf algebras. Thus B_i satisfies $<P>$. Then the dual B_i^* satisfies $<CP>$ (cf., **6.11**). Here B_i^* is connected, hence coprimitive by (6.15). Thus B_i is primitive for each $i \in I$. Now, since the direct limits are exact functors,

$$\nu:\ P({}_0E(A)) \rightarrow Q({}_0E(A))$$

is surjective as the limit of surjective maps

$$\nu_i:\ P(B_i) \rightarrow Q(B_i)\,,$$

i.e., ${}_0E(A)$ is primitive. Finally, by (4.11*) we see that A is primitive. q.e.d.

6.13. By (6.6), (6.7) and (6.15) we obtain

(6.16) **Proposition.** *Let $\lambda \in K$, char $K \neq 0$, and A be a quasi (d, λ)-Hopf algebra. Assume that A is semi-connected as a coalgebra and $H(A)$ is semi-connected as an algebra. If A is coprimitive then $H(A)$ is coprimitive.*

Dually, by (6.6*), (6.7) and (6.15*) we obtain

(6.16*) **Proposition.** *Let $\lambda \in K$, char $K \neq 0$, and A be a quasi (d, λ)-Hopf algebra. Assume that A and $\Psi_\lambda A$ are semi-connected as an algebras and $H(A)$ is semi-connected as a coalgebra. If A is primitive then $H(A)$ is primitive.*

Now, since $E_r(A)$ and $_rE(A)$ are graded connected for $r \geqq 0$, as a corollary of (4.10), (6.16) and (6.16*) we obtain

(6.17) **Theorem.** *Let $\lambda \in K$, char $K \neq 0$, and A be a quasi (d, λ)-Hopf algebra. $E_r(A)$ is primitive and $_rE(A)$ is coprimitive for every $r \geqq 0$.*

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