

Title	On congruent holomorphic mappings into a Hermitian symmetric space
Author(s)	Taniguchi, Yoshiharu
Citation	Osaka Journal of Mathematics. 1999, 36(4), p. 793-804
Version Type	VoR
URL	https://doi.org/10.18910/11089
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

ON CONGRUENT HOLOMORPHIC MAPPINGS INTO A HERMITIAN SYMMETRIC SPACE

YOSHIHARU TANIGUCHI

(Received January 9, 1998)

1. Introduction

It is one of basic problems in differential geometry to find geometric conditions that determine the congruence class of a submanifold of a given manifold. For example, a general hypersurface of n -dimensional Euclidean space \mathbf{R}^n is determined up to congruence by its first and second fundamental forms. It is another classical result that for a certain generic hypersurfaces of \mathbf{R}^n ($n \geq 4$), the first fundamental form alone is sufficient to determine the congruence class. On the other hand, we know some remarkable facts in the complex-analytic case. First, any complex submanifold of a complex space form has the metric rigidity, i.e., it is determined up to congruence by its first fundamental form alone (the rigidity theorem of Calabi [1]). Secondly, Green [2] asserts even if the ambient space S is a general Kähler manifold with real-analytic Kähler metric, a certain generic holomorphic mapping (a "nondegenerate" mapping in his terminology) into it has the local metric rigidity: There is actually shown the existence of such a finite-dimensional family of local real-analytic hypersurfaces of S that a holomorphic mapping into S has the local metric rigidity unless its image lies in any of them. Only one cannot find any geometric conditions that assure the nondegeneracy for a mapping.

In this article, we shall give local differential-geometric conditions that determine the congruence class of a holomorphic mapping into a Hermitian symmetric space under considerably general settings. For a technical reason, we have to restrict ourselves to the case where mappings are *infinitesimally full* in our sense. In particular we cannot deal with totally geodesic complex submanifolds. But our results extend Calabi's rigidity theorem to a direction different from [2], since our "infinitesimally full" condition is equivalent to the "full" condition used in [1] in the case of complex space forms.

2. Preliminaries

Let S be a simply connected, complex N -dimensional Hermitian symmetric space with metric tensor g . Denote by TS , T^*S , and $T_g^r S$ the tangent bundle, cotangent bundle, and tensor bundle of type (r, s) of S respectively. Their complexifications will

be denoted by $\mathbf{T}S$, \mathbf{T}^*S , and $\mathbf{T}_s^r S$ respectively. As usual, $\mathbf{T}S$ splits into a direct sum of its complex vector subbundles T^+S and T^-S according to the eigenvalues $\pm\sqrt{-1}$ of the complex structure J of S respectively. The set of complex vector fields on S will be denoted by $\mathbf{Z}(S)$. Throughout this paper, we always assume that the Riemannian metric g , curvature tensor R , and the Levi-Civita connection ∇ of S are complex-linearly extended. So we define the Christoffel symbols Γ_{st}^r by

$$(2.1) \quad (\nabla_{\partial/\partial w^s} \partial/\partial w^t)_q = \sum_{r=1}^N \Gamma_{st}^r(q) (\partial/\partial w^r)_q, \quad (q \in \tilde{U}, 1 \leq s, t \leq N),$$

where $(\tilde{U}; w^1, \dots, w^N)$ is a local holomorphic coordinate system valid in an open set \tilde{U} of S .

Now let M be a complex n -dimensional connected complex manifold and f a holomorphic mapping of M into S . Let V be an open set of M . A *complex tensor field Q along f over V of type (r, s)* is a smooth cross section of the induced bundle $\pi_s^r : f|_V^* \mathbf{T}_s^r S \rightarrow V$, where $f|_V$ denotes the restriction of f to V . It may be considered in a usual way as a smooth mapping $p \mapsto Q_p$ from V into $\mathbf{T}_s^r S$ such that $\pi_s^r(Q_p) = f(p)$ for any $p \in V$. Q will be called a complex vector field (resp. complex 1-form) along f if $(r, s) = (0, 1)$ (resp. $(r, s) = (1, 0)$). The set of complex vector fields along f is denoted by $\mathbf{Z}_f(V)$, while the set of complex vector fields on V by $\mathbf{Z}(V)$. Further, we denote by $\mathbf{Z}_f^+(V)$ (resp. $\mathbf{Z}_f^-(V)$) the set of elements in $\mathbf{Z}_f(V)$ which take the values in T^+S (resp. T^-S). An element $Z \in \mathbf{Z}_f(V)$ is called holomorphic if $Z \in \mathbf{Z}_f^+(V)$ and if it is holomorphic as a mapping into T^+S . Let $(U; z^1, \dots, z^n)$ be a local holomorphic coordinate system valid in an open set U of M . Then each $f_*(\partial/\partial z^i)$ is an element of $\mathbf{Z}_f^+(U)$ and holomorphic. If $f(U) \subset \tilde{U}$, the mapping $p \mapsto (\partial/\partial w^r)_{f(p)}$ is an holomorphic element of $\mathbf{Z}_f^+(U)$, which we denote by $(\partial/\partial w^r)_f$. Similarly $(\partial/\partial \bar{w}^r)_f$ means an element of $\mathbf{Z}_f^-(U)$. Then every $Z \in \mathbf{Z}_f(U)$ can be uniquely expressed as

$$(2.2) \quad Z = \sum_{r=1}^N Z^r (\partial/\partial w^r)_f + \sum_{r=1}^N Z^{\bar{r}} (\partial/\partial \bar{w}^r)_f,$$

Z^r and $Z^{\bar{r}}$ being complex-valued smooth functions on U . Obviously, Z is holomorphic if and only if $Z^{\bar{r}} = 0$ and Z^r is holomorphic in U for each r .

The holomorphic mapping f gives arise to a covariant differentiation D along f that is induced from the Levi-Civita connection of S . It is given in terms of local coordinate systems as follows: Let $(U; z^1, \dots, z^n)$ a local holomorphic coordinate system in an open set U of M such that $f(U) \subset \tilde{U}$. If $Z \in \mathbf{Z}_f^+(U)$ and $Z = \sum_{r=1}^N Z^r (\partial/\partial w^r)_f$, then

$$(D_{\partial/\partial z^j} Z)_p = \sum_{r=1}^N \left\{ \frac{\partial Z^r}{\partial z^j}(p) + \sum_{s,t=1}^N \Gamma_{st}^r(f(p)) \frac{\partial f^s}{\partial z^j}(p) Z^t(p) \right\} (\partial/\partial w^r)_{f(p)}$$

$$(D_{\partial/\partial\bar{z}^j} Z)_p = \sum_{r=1}^N \frac{\partial Z^r}{\partial \bar{z}^j}(p)(\partial/\partial w^r)_{f(p)}.$$

In the following lemma we summarize some basic properties of D that are needed in later sections.

Lemma 2.1. *Let $X, X_1, X_2 \in \mathbf{Z}(V)$ and $Z, Z_1, Z_2, Z_3 \in \mathbf{Z}_f(V)$.*

(i)

$$\begin{aligned} D_{X_1} f_* X_2 - D_{X_2} f_* X_1 - f_* [X_1, X_2] &= 0, \\ D_{X_1} D_{X_2} Z - D_{X_2} D_{X_1} Z - D_{[X_1, X_2]} Z &= R(f_* X_1, f_* X_2) Z. \end{aligned}$$

(ii) D_X commutes with the complex structure J of S . In particular, if $Z \in \mathbf{Z}_f^+(V)$ (resp. $Z \in \mathbf{Z}_f^-(V)$), then $D_X Z \in \mathbf{Z}_f^+(V)$ (resp. $D_X Z \in \mathbf{Z}_f^-(V)$).

(iii) $D_{\partial/\partial\bar{z}^j} Z$ vanishes if Z is holomorphic.

(iv) D is real: $\overline{D_X Z} = D_{\bar{X}} \bar{Z}$, where the bars over expressions denote the complex conjugation.

(v) D leaves both the metric tensor g and curvature tensor R of S invariant:

$$\begin{aligned} Xg(Z_1, Z_2) &= g(D_X Z_1, Z_2) + g(Z_1, D_X Z_2) \\ D_X R(Z_1, Z_2) Z_3 &= R(D_X Z_1, Z_2) Z_3 + R(Z_1, D_X Z_2) Z_3 + R(Z_1, Z_2) D_X Z_3. \end{aligned}$$

(vi) If ψ is a complex 1-form along f over V , then

$$X\psi(Z) = D_X\psi(Z) + \psi(D_X Z).$$

(vii) Let F be a holomorphic and isometric transformation of S . Set $f' = F \circ f$ and denote by D' the covariant differentiation along f' . Then $F_* D_X Z = D'_X F_* Z$.

Here we make a further agreement on notation: Let $(U; z^1, \dots, z^n)$ be a local holomorphic coordinate system of M and f a holomorphic mapping of M into S . We write $f_j = f_*(\partial/\partial z^j)$, $D_j = D_{\partial/\partial z^j}$, $D_{\bar{j}} = D_{\partial/\partial \bar{z}^j}$. For a positive integer a , a multi-index I of order a is an a -tuple of integers (i_1, i_2, \dots, i_a) with $1 \leq i_1, i_2, \dots, i_a \leq n$. The order of a multi-index I will be often denoted by $|I|$. We denote by \mathcal{I}^a the set of multi-indices of order less than or equal to a . If $I = (i_1, \dots, i_a)$, we write $D_I = D_{i_1} \cdots D_{i_a}$, $D_I f = f_I = D_{i_1} \cdots D_{i_{a-1}} f_{i_a}$, $\partial_I \varphi = \partial_{i_1} \cdots \partial_{i_a} \varphi = \partial^a \varphi / \partial z^{i_1} \cdots \partial z^{i_a}$, φ being a complex-valued smooth function on U .

3. A Congruence Theorem for Holomorphic Mappings

Let o be a point of M and $(U; z^1, \dots, z^n)$ a local holomorphic coordinate system around o . We set $f(o) = \tilde{o}$. Let $T_{\tilde{o}}^+ S = (\pi^+)^{-1}(\tilde{o})$, where π^+ is the projection $T^+ S \rightarrow S$. For a positive integer a , we define the a -th complex osculating space $O_f^a(o)$ to f at o as the complex linear subspace of $T_{\tilde{o}}^+ S$ spanned by all the $(D_I f)_o$ with $I \in \mathcal{I}^a$. We set $O_f^0(o) = 0$. If $O_f^d(o) = T_{\tilde{o}}^+ S$ and $O_f^{d-1}(o) \neq T_{\tilde{o}}^+ S$ for a positive integer d , f is said to be *infinitesimally full of order d at o* . Further, f is said to be *infinitesimally full* if there exists a positive integer d and a point $o \in M$ such that f is infinitesimally full of order d at o .

The following theorem is due to E. Calabi.

Theorem 3.1 (Calabi). *Let f and f' be holomorphic imbeddings of a connected complex manifold M into a complex space form S with metric tensor g . If f is full and if $f^*g = f'^*g$ on M , then f' is also full and there exists uniquely a holomorphic and isometric transformation F of S such that $F \circ f = f'$.*

The following theorem is our congruence theorem for holomorphic mappings into a simply connected Hermitian symmetric space.

Theorem 3.2. *Let f and f' be holomorphic mappings of a connected complex manifold M into a simply connected Hermitian symmetric space S with metric tensor g . Let R be the Riemannian curvature of S . Denote by D and D' the covariant differentiations along f and f' respectively. Let $(U; z^1, \dots, z^n)$ be a local holomorphic coordinate system around a point $o \in M$. Suppose that f is infinitesimally full of order d at o . Moreover, suppose that (i) $f^*g = f'^*g$ on U , (ii) $R(D_i f, \overline{D_{I_1} f}, D_{I_2} f, \overline{D_{I_3} f}) = R(D'_i f', \overline{D'_{I_1} f'}, D'_{I_2} f', \overline{D'_{I_3} f'})$ on U ($1 \leq i \leq n, |I_1| \leq d-1, |I_2| \leq d, |I_3| \leq d-1$), and (iii) $R(D_{I_1} f, \overline{D_{I_2} f}, D_{I_3} f, \overline{D_{I_4} f})_o = R(D'_{I_1} f', \overline{D'_{I_2} f'}, D'_{I_3} f', \overline{D'_{I_4} f'})_o$ ($|I_\nu| \leq d, 1 \leq \nu \leq 4$). Then f' is also infinitesimally full of order d at o and there exists uniquely a holomorphic and isometric transformation F such that $F \circ f = f'$.*

REMARK 3.1. When S is a complex space form, an imbedding f of M into S is said to be full if there exists no proper, totally geodesic complex submanifold of S including $f(M)$ (cf. [1]). In Theorem 3.1, one can replace the condition that f is full by the one that it is infinitesimally full because both conditions are equivalent in this case, as will be shown in the last section.

Now let f and f' be holomorphic mapping of M into S . In order to prove Theorem 3.2, we may assume that $f(o) = f'(o)$, because the group of holomorphic and isometric transformations of S is transitive and the condition of the theorem does not change under the group. Let $(\tilde{U}; w^1, \dots, w^N)$ be a local holomorphic coordinate system around $f(o) = \tilde{o}$. We may assume that $f(U) \subset \tilde{U}$ and $f'(U) \subset \tilde{U}$, shrinking U if it is necessary. Set $f^r = w^r \circ f$ and $f'^r = w^r \circ f'$ for $r = 1, 2, \dots, N$.

Proposition 3.1. *The holomorphic mapping f coincides with f' on M if and only if $f(o) = f'(o)$ and $(D_I f)_o = (D'_I f')_o$ for all the multi-indices I .*

For the proof, we need lemmas.

Lemma 3.1. *Let c be a positive integer. If $f(o) = f'(o)$ and $\partial_I f^r(o) = \partial_I f'^r(o)$ for any $r = 1, \dots, N$ and $I \in \mathcal{I}^c$, then $(D_I(\partial/\partial w^r))_f)_o = (D'_I(\partial/\partial w^r))_{f'}_o$ for any $r = 1, \dots, N$ and $I \in \mathcal{I}^c$.*

Proof. By successively differentiating both sides of

$$(D_i(\partial/\partial w^r))_f)_p = \sum_{s,t} \Gamma_{sr}^t(f(p)) \frac{\partial f^s}{\partial z^t}(p) (\partial/\partial w^t)_f)_p,$$

we see that $(D_I(\partial/\partial w^r))_f)_o$ is uniquely determined by the values

$$\frac{\partial^a \Gamma_{sr}^t}{\partial w^{s_1} \dots \partial w^{s_a}}(f(o)), \quad \frac{\partial^b f^r}{\partial z^{j_1} \dots \partial z^{j_b}}(o), \quad (\partial/\partial w^r)_f)_o$$

such that $0 \leq a \leq |I| - 1$ and $0 \leq b \leq |I|$. By the condition, all the values above are common for f' . This means $(D_I(\partial/\partial w^r))_f)_o = (D'_I(\partial/\partial w^r))_{f'}_o$. □

Lemma 3.2. *Let c be a positive integer. Suppose that $f(o) = f'(o)$ and $(D_I f)_o = (D'_I f')_o$ for any $I \in \mathcal{I}^c$. Then $\partial_I f^r(o) = \partial_I f'^r(o)$ for any $r = 1, \dots, N$ and $I \in \mathcal{I}^c$.*

Proof. We proceed by induction on c . The assertion is obvious if $c = 1$. Assume that we have shown our assertion for positive integer less than c . If $I = (i_1, \dots, i_c)$, we have

$$\begin{aligned} (D_I f)_o &= (D_{i_1} \cdots D_{i_{c-1}} \sum_r f_{i_c}^r(\partial/\partial w^r))_f)_o \\ (3.1) \quad &= \sum_r \partial_{i_1} \cdots \partial_{i_{c-1}} f_{i_c}^r(o) (\partial/\partial w^r)_f)_o \\ &\quad + \sum_r \sum^* \partial_{i_{\sigma(1)}} \cdots \partial_{i_{\sigma(a)}} f_{i_c}^r(o) (D_{i_{\tau(1)}} \cdots D_{i_{\tau(b)}} (\partial/\partial w^r))_f)_o, \end{aligned}$$

where the summation \sum^* is taken over certain $\sigma(1), \dots, \sigma(a)$ and $\tau(1), \dots, \tau(b)$ such that $0 \leq a \leq c - 2$, $1 \leq b \leq c - 1$, and $a + b = c - 1$. Equation (3.1) is also valid for f' and D' if we replace f by f' and D by D' .

Now suppose that $f(o) = f'(o)$ and $(D_I f)_o = (D'_I f')_o$ for any $I \in \mathcal{I}^c$. By the assumption of induction, we have in particular

$$\partial_K f^r(o) = \partial_K f'^r(o) \quad (1 \leq r \leq N, K \in \mathcal{I}^{c-1}),$$

which implies by Lemma 3.1 $(D_K(\partial/\partial w^r)_f)_o = (D'_K(\partial/\partial w^r)_{f'})_o$ ($K \in \mathcal{I}^{c-1}$). Comparing the above identity (3.1) for f with the corresponding one for f' , we have

$$\partial_{i_1} \cdots \partial_{i_{c-1}} f_{i_c}^r(o) = \partial_{i_1} \cdots \partial_{i_{c-1}} f_{i_c}'(o),$$

completing the induction. □

Proposition 3.1 is now obvious by the above two lemmas.

Lemma 3.3. (i) For any multi-index I and integers j, k ($1 \leq j, k \leq n$),

$$\partial/\partial z^k g(\overline{f_I}, f_j) = g(\overline{f_{k,I}}, f_j).$$

(ii) For any k ($1 \leq k \leq n$) and multi-index $I = (i_1, \dots, i_c)$ ($c \geq 2$),

$$D_{\overline{k}} f_I = \sum_I^* R(\overline{f_k}, f_{I'}) f_{I''},$$

where the summation \sum_I^* is taken over all $I' = (i_{\sigma(1)}, \dots, i_{\sigma(a-1)}, i_{\sigma(a)})$ and $I'' = (i_{\tau(1)}, \dots, i_{\tau(b)}, i_c)$ such that

$$\begin{aligned} 1 \leq a, \quad 0 \leq b, \quad a + b = c - 1 \\ \sigma(1) < \dots < \sigma(a) \leq c - 1, \quad \tau(1) < \dots < \tau(b) \leq c - 1 \\ \{\sigma(1), \dots, \sigma(a), \tau(1), \dots, \tau(b)\} = \{1, \dots, c - 1\}. \end{aligned}$$

(iii) For any k ($1 \leq k \leq n$), multi-indices I ($|I| \geq 2$) and K ,

$$\partial/\partial z^k g(\overline{f_I}, f_K) = g(\overline{f_I}, f_{k,K}) + \sum_I^* g(R(f_k, \overline{f_{I'}}) \overline{f_{I''}}, f_K).$$

Proof. Since f_j is holomorphic, we have (i). We shall show (ii) by induction on $|I|$. If $I = (i_1, i_2)$, we have

$$D_{\overline{k}} f_{i_1, i_2} = D_{\overline{k}} D_{i_1} f_{i_2} = R(\overline{f_k}, f_{i_1}) f_{i_2},$$

showing (ii) in the case $|I| = 2$. Assume that we have shown (ii) for any I such that $|I| \leq c$. Let $I = (i_1, \dots, i_c)$. Then for any i_0 , we have

$$\begin{aligned} D_{\overline{k}} f_{i_0, I} &= D_{\overline{k}} D_{i_0} f_{i_1, \dots, i_c} \\ &= R(\overline{f_k}, f_{i_0}) f_{i_1, \dots, i_c} + D_{i_0} D_{\overline{k}} f_{i_1, \dots, i_c} \\ &= R(\overline{f_k}, f_{i_0}) f_{i_1, \dots, i_c} + \sum_I^* \{ R(\overline{f_k}, f_{i_0, i_{\sigma(1)}, \dots, i_{\sigma(a)}}) f_{i_{\tau(1)}, \dots, i_{\tau(b)}} \\ &\quad + R(\overline{f_k}, f_{i_{\sigma(1)}, \dots, i_{\sigma(a)}}) f_{i_0, i_{\tau(1)}, \dots, i_{\tau(b)}} \}, \end{aligned}$$

completing the induction. (iii) follows from (ii). □

Lemma 3.4. For any multi-indices I and K , $g(\overline{f_I}, f_K)$ is uniquely determined by the functions $g(\overline{f_i}, f_j)$ and $g(R(\overline{f_i}, \overline{f_{I'}})\overline{f_{I''}}, f_{K'})$ on U such that $1 \leq i, j \leq n$, $|I'|, |I''| < |I|$, and $|K'| < |K|$.

Proof. If $I = (i_1, i_2, \dots, i_a)$, Lemma 3.3 (i) implies that $g(\overline{f_I}, f_k)$ is uniquely determined by $g(\overline{f_{i_a}}, f_k)$. Then from Lemma 3.3 (iii), we obtain our assertion for $g(\overline{f_I}, f_K)$ by induction on $|K|$. □

For a positive integer a , we denote by $G(f, a)$ the set of functions $g(\overline{f_i}, f_j)$ and $g(R(\overline{f_i}, \overline{f_{I_1}})\overline{f_{I_2}}, f_K)$ such that $1 \leq i, j \leq n$, $I_1, I_2 \in \mathcal{I}^{a-1}$, and $K \in \mathcal{I}^a$.

Lemma 3.5. Suppose that there exist multi-indices L_1, L_2, \dots, L_N such that the set $\{f_{L_1}, f_{L_2}, \dots, f_{L_N}\}$ forms a basis of $T_{f(p)}^+ S$ for every $p \in U$. Set $d = \max_{1 \leq r \leq N} |L_r|$. Then for any multi-indices I and K , $g(\overline{f_I}, f_K)$ is uniquely determined by $G(f, d)$.

Proof. By Lemma 3.4, the assertion is obvious if $|I| \leq d$ and $|K| \leq d$. In particular, $g(\overline{f_{L_r}}, f_{L_s})$ is uniquely determined by $G(f, d)$. We next show our assertion is true in the case where $I = L_r$ and K is an arbitrary multi-index. We proceed by induction on $|K|$. Assume $g(\overline{f_{L_r}}, f_K)$ is determined by $G(f, d)$ if $|K| \leq c$. Let $|K| = c$. If we write $f_K = \sum_s A_K^s f_{L_s}$ with certain functions A_K^s on U , then every A_K^s is determined by $g(\overline{f_{L_r}}, f_K)$ ($1 \leq r \leq N$) and hence by $G(f, d)$ by the assumption of induction. If $|L_r| \geq 2$, then by Lemma 3.3(iii),

$$\begin{aligned} g(\overline{f_{L_r}}, f_{k,K}) &= \partial_k g(\overline{f_{L_r}}, f_K) - \sum_{L_r}^* g(R(\overline{f_k}, \overline{f_{L'}})\overline{f_{L''}}, f_K) \\ &= \partial_k g(\overline{f_{L_r}}, f_K) - \sum_{L_r}^* \sum_s A_K^s g(R(\overline{f_k}, \overline{f_{L'}})\overline{f_{L''}}, f_{L_s}), \end{aligned}$$

which shows that $g(\overline{f_{L_r}}, f_{k,K})$ is determined by $G(f, d)$. When $|L_r| = 1$, the same conclusion follows from Lemma 3.3 (i). Thus our assertion is true for $I = L_r$ and arbitrary K .

Now let I and K be arbitrary multi-indices. Then we have immediately

$$g(\overline{f_I}, f_K) = \sum_{r,s} \overline{A_I^r} A_K^s g(\overline{f_{L_r}}, f_{L_s}),$$

completing the proof. □

Proof of Theorem 3.2. Let L_1, L_2, \dots, L_N be multi-indices such that the set $\{f_{L_1}, f_{L_2}, \dots, f_{L_N}\}$ forms a basis of $T_{f(p)}^+ S$ for any $p \in U'$, U' being an open neighborhood of o included in U . By the conditions (i), (ii), and Lemma 3.4, we have $g(\overline{f_I}, f_K) = g(\overline{f'_I}, f'_K)$ for any I and $K \in \mathcal{I}^d$. In particular, $g(\overline{f_{L_r}}, f_{L_s}) = g(\overline{f'_{L_r}}, f'_{L_s})$ for any r and s . Then the Gramian $\det(g(\overline{f'_{L_r}}, f'_{L_s}))_{r,s=1,\dots,n}$ does not vanish on U' as

well as the corresponding one for f . This means that the set $\{f'_{L_1}, f'_{L_2}, \dots, f'_{L_N}\}$ also forms a basis of $T_{f'(p)}^+ S$ for any $p \in U'$. So f' is infinitesimally full of order d at o .

Next we shall show the existence of F such that $F \circ f = f'$. By the conditions (i) and (ii) together with Lemma 3.5, we have

$$(3.2) \quad g(\overline{f_I}, f_K) = g(\overline{f'_I}, f'_K) \quad \text{on } U' \text{ for all } I \text{ and } K.$$

Let Φ be the unique unitary transformation of $T_{\tilde{o}}^+ S$ such that $\Phi(f_{L_r}(o)) = f'_{L_r}(o)$. Φ satisfies $\Phi((f_I)_o) = (f'_I)_o$ for any I , because the coefficients of $(f_I)_o$ with respect to the $(f_{L_r})_o$ coincide with those of $(f'_I)_o$ with respect to the $(f'_{L_r})_o$ by (3.2). Moreover by the condition (iii),

$$R(Z_1, \overline{Z_2}, Z_3, \overline{Z_4}) = R(\Phi(Z_1), \overline{\Phi(Z_2)}, \Phi(Z_3), \overline{\Phi(Z_4)})$$

for any $Z_1, Z_2, Z_3, Z_4 \in T_{\tilde{o}}^+ S$. Now let $\Phi^{\mathbf{R}}$ be the linear transformation of $T_{\tilde{o}} S$ induced by Φ via the natural isomorphism $\iota : T_{\tilde{o}} S \rightarrow T_{\tilde{o}}^+ S$, $\iota(X) = \frac{1}{2}(X - \sqrt{-1}JX)$. It is an orthogonal transformation commuting with $J_{\tilde{o}}$ and leaving $R_{\tilde{o}}$ invariant. There exists uniquely an isometric transformation F of S such that $F(\tilde{o}) = \tilde{o}$ and $(F_*)_{\tilde{o}} = \Phi^{\mathbf{R}}$ (cf. [3], Chapter III, Lemmas 1.2 and 1.4). Appropriately modifying the lemma last cited, we see easily that F is holomorphic. Then from Lemma 2.1 (vii), it follows $((F \circ f)_I)_o = F_*(f_I)_o = (f'_I)_o$ for any I . Hence we have $F \circ f = f'$ by Proposition 3.1.

The uniqueness of F is obvious. Thus we have finished the proof of the theorem. □

4. Complex submanifolds that are infinitesimally full of order two

In this section, we shall give a more concrete expression of Theorem 3.2 for a Kähler submanifold of S that is infinitesimally full of order two. Let M be a connected Kähler manifold. Let ∇^M be the Levi-Civita connection of M . We assume that f is a holomorphic and isometric immersion of M into S . Let α be the second fundamental form of f :

$$\alpha(X_1, X_2) = D_{X_1} f_* X_2 - f_* \nabla_{X_1}^M X_2$$

for any vector fields X_1 and X_2 on M . When we say that *the normal space to f at o is spanned by the second fundamental form*, we mean by definition that the real tangent space $T_{\tilde{o}} S$ at \tilde{o} is spanned by $f_*(X)$ and $\alpha(X, X')$ ($X, X' \in T_o M$). Note that the normal space to f at o is spanned by the second fundamental form if and only if f is infinitesimally full of order two at o . In particular, when f is a complex hypersurface, f is not totally geodesic if and only if it is infinitesimally full of order two at a point.

Theorem 4.1. *Let S be a simply connected Hermitian symmetric space S and M a connected Kähler manifold. Denote by R the Riemannian curvature of S . Let f and f' be holomorphic and isometric immersions of M into S . Let α and α' be the second fundamental forms of f and f' respectively. Suppose*

- (i) *the normal space to f at a point o is spanned by the second fundamental form,*
- (ii)

$$\begin{aligned} R(f_*(X_1), f_*(X_2), f_*(X_3), f_*(X_4)) & \\ &= R(f'_*(X_1), f'_*(X_2), f'_*(X_3), f'_*(X_4)) \\ R(f_*(X_1), f_*(X_2), f_*(X_3), \alpha(X_4, X_5)) & \\ &= R(f'_*(X_1), f'_*(X_2), f'_*(X_3), \alpha'(X_4, X_5)) \end{aligned}$$

for any vector fields X_ν on M ($1 \leq \nu \leq 5$),

- (iii)

$$\begin{aligned} R(f_*(X_1), f_*(X_2), \alpha(X_3, X_4), \alpha(X_5, X_6)) & \\ &= R(f'_*(X_1), f'_*(X_2), \alpha'(X_3, X_4), \alpha'(X_5, X_6)) \\ R(f_*(X_1), \alpha(X_2, X_3), \alpha(X_4, X_5), \alpha(X_6, X_7)) & \\ &= R(f'_*(X_1), \alpha'(X_2, X_3), \alpha'(X_4, X_5), \alpha'(X_6, X_7)) \\ R(\alpha(X_1, X_2), \alpha(X_3, X_4), \alpha(X_5, X_6), \alpha(X_7, X_8)) & \\ &= R(\alpha'(X_1, X_2), \alpha'(X_3, X_4), \alpha'(X_5, X_6), \alpha'(X_7, X_8)) \end{aligned}$$

for any tangent vector X_ν to M at o ($1 \leq \nu \leq 8$).

Then the normal space to f' at o is also spanned by the second fundamental form and there exists uniquely a holomorphic and isometric transformation F of S such that $f' = F \circ f$ on M .

Proof. The assertion is immediately obtained from Theorem 3.2 and the identities

$$\begin{aligned} f_{ij} &= \alpha(f_i, f_j) + \sum_k \Lambda_{ij}^k f_k \\ f'_{ij} &= \alpha'(f'_i, f'_j) + \sum_k \Lambda'_{ij}^k f'_k, \end{aligned}$$

where Λ_{ij}^k are the Christoffel symbols of the Levi-Civita connection of M . □

5. Case where ambient space is a complex space form

A complex space form is a simply connected, complete Kähler manifold of constant holomorphic sectional curvature. According to the signature of curvature, it is

holomorphically isometric to a complex projective space, a complex vector space, or its unit disc with certain metrics. In this section we consider the case where S is a complex space form. A holomorphic mapping f of a connected complex manifold M into S is said to be *full* if its image is not included in any totally geodesic complex submanifold of S (cf. [1]). We shall show that the holomorphic mapping f is full if and only if it is infinitesimally full.

Let $O_f(o)$ be the complex linear subspace of $T_o^+ S$ spanned by all $(f_I)_o$. Note that $O_f(o)$ is independent upon the choice of local holomorphic coordinate system.

Proposition 5.1. *If $\dim_{\mathbb{C}} O_f(o) < N$, then there exists a complete, totally geodesic complex hypersurface H of S through \tilde{o} such that $f(M) \subset H$.*

Proof. By virtue of complex space form, there exists a complete, totally geodesic complex hypersurface H of S through \tilde{o} such that $O_f(o) \subset T_{\tilde{o}}^+ H$. We shall prove that $f(M) \subset H$. We may assume that the coordinate system $(\tilde{U}; w^1, \dots, w^N)$ around \tilde{o} is so chosen that $H \cap \tilde{U}$ is defined by $w^N = 0$ and that

$$(5.1) \quad \Gamma_{r,s}^N(w^1, \dots, w^{N-1}, 0) = 0 \quad (1 \leq r, s \leq N - 1).$$

We have first

$$(5.2) \quad f_I^N(o) = dw^N((f_I)_o) = 0 \quad \text{for all } I.$$

To prove $f(M) \subset H$, it will suffice to show $\partial_I f^N(o) = 0$ for every multi-indices I . We need lemmas.

Lemma 5.1. *Let a be a positive integer. Suppose that $\partial_I f^N(o) = 0$ for any $I \in \mathcal{I}^a$. Then for any smooth function Γ on \tilde{U} such that $\Gamma = 0$ on $H \cap \tilde{U}$, $\partial_I(\Gamma \circ f)$ vanishes at o for any $I \in \mathcal{I}^a$.*

Proof. For any $r = 1, 2, \dots, N - 1$, $\partial\Gamma/\partial w^r$ have the same properties as Γ . The assertion can be obtained by induction on $|I|$ from the identities

$$\begin{aligned} \partial_i(\Gamma \circ f)(p) &= \sum_{r=1}^{N-1} \frac{\partial\Gamma}{\partial w^r}(f(p)) \partial_i f^r(p) + \frac{\partial\Gamma}{\partial w^N}(f(p)) \partial_i f^N(p) \\ \partial_I \partial_i(\Gamma \circ f) &= \sum_{I', I''} \left\{ \sum_{r=1}^{N-1} \partial_{I'} \left(\frac{\partial\Gamma}{\partial w^r} \circ f \right) \partial_{I''} \partial_i f^r + \partial_{I'} \left(\frac{\partial\Gamma}{\partial w^N} \circ f \right) \partial_{I''} \partial_i f^N \right\}. \end{aligned}$$

□

Let dw_f^N be the complex 1-form along f defined by the smooth assignment $p \mapsto dw_{f(p)}^N$ from U into $\mathbb{T}^* S$.

Lemma 5.2. *Let a be a positive integer. If $\partial_I f^N(o) = 0$ for any $I \in \mathcal{I}^a$, then $(D_I dw_f^N)((\partial/\partial w^r)_f) = 0$ at o for every $r = 1, \dots, N - 1$ and $I \in \mathcal{I}^a$.*

Proof. We proceed by induction on a . Set $\gamma_{ir}^s(p) = \sum_{t=1}^N \Gamma_{t,r}^s(f(p)) \partial_i f^t(p)$ for $p \in U$.

The assertion is obvious for $a = 1$ from

$$(5.3) \quad D_i dw_f^N = - \sum_{r=1}^{N-1} \gamma_{ir}^N dw_f^r - \gamma_{iN}^N dw_f^N$$

and $\gamma_{ir}^N(o) = 0$. Assume that our assertion is true for a . Suppose $\partial_K f^N(o) = 0$ for any $K \in \mathcal{I}^{a+1}$, and we have in particular $D_I dw_f^N((\partial/\partial w^r)_f) = 0$ at o for any $I \in \mathcal{I}^a$. Now if $|I| \leq a$, then it follows from (5.3)

$$(5.4) \quad D_I D_i dw_f^N = - \sum_I \left\{ \sum_{r=1}^{N-1} \partial_{I'} \gamma_{ir}^N D_{I''} dw_f^r + \partial_{I'} \gamma_{iN}^N D_{I''} dw_f^N \right\}$$

where the summation \sum_I is taken over certain multi-indices I' and I'' such that $|I'| + |I''| = a$. On the other hand,

$$(5.5) \quad \partial_{I'} \gamma_{ir}^N = \sum_{I'} \left\{ \sum_{t=1}^{N-1} \partial_{I_1} (\Gamma_{tr}^N \circ f) \partial_{I_2} \partial_i f^t + \partial_{I_1} (\Gamma_{Nr}^N \circ f) \partial_{I_2} \partial_i f^N \right\},$$

where the summation $\sum_{I'}$ is taken over certain I_1 and I_2 such that $|I_1| + |I_2| = |I'|$. In (5.5), $\partial_{I_1} (\Gamma_{tr}^N \circ f)(o) = 0$ by Lemma 5.1, and $\partial_{I_2} \partial_i f^N(o) = 0$ by the assumption. We have then $\partial_{I'} \gamma_{ir}^N(o) = 0$. Hence from (5.4), $D_I D_i dw_f^N((\partial/\partial w^r)_f) = 0$ at o for any r ($1 \leq r \leq N - 1$), which completes the induction. □

Now we return to the proof of Proposition 5.1. We prove by induction on a that $\partial_I f^N(o) = 0$ for all I such that $|I| \leq a$. It is obvious for $a = 1$. We assume that the assertion is true for a . Let I be an arbitrary multi-index of order a . From $\partial_i f^N = dw_f^N(f_i)$, we have

$$\partial_I \partial_i f^N = \sum_I D_{I'} dw_f^N(f_{I''i}),$$

where the summation \sum_I is taken over certain I' and I'' such that $|I'| + |I''| = |I|$. By Lemma 5.2 together with (5.2) and the assumption of induction, each term of the right-hand side vanishes at o , which completes the induction. Thus we have finished the proof of Proposition 5.1. □

Corollary 5.1. *A holomorphic mapping of a connected complex manifold into a complex space form is full if and only if it is infinitesimally full.*

REMARK 5.1. This corollary differs from Corollary 2.1 of [5] in two points: firstly, the present "infinitesimally full" condition is somewhat different from the corresponding one in [5], and secondly only holomorphic immersions are considered in [5].

If S is a complex space form,

$$R(f_{I_1}, \overline{f_{I_2}}, f_{I_3}, \overline{f_{I_4}}) = \kappa(g(f_{I_1}, \overline{f_{I_2}})g(f_{I_3}, \overline{f_{I_4}}) + g(f_{I_1}, \overline{f_{I_4}})g(\overline{f_{I_2}}, f_{I_3}))$$

with certain constant κ . Lemma 3.4 tells us that both the conditions (ii) and (iii) of Theorem 3.2 are then consequences of (i). This means that Theorem 3.2 substantially includes Theorem 3.1.

References

- [1] E. Calabi: *Isometric imbedding of complex manifolds*, Ann. Math. **58** (1953), 1–23.
- [2] M. L. Green: *Metric rigidity of holomorphic maps into Kähler manifolds*, J. Diff. Geometry, **13** (1978), 279–286.
- [3] S. Helgason: *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.
- [4] S. Kobayashi and K. Nomizu: *Foundations of Differential Geometry, Vol. II*, Interscience, New York, 1969.
- [5] Y. Taniguchi: *The method of moving frames applied to Kähler submanifolds of complex space forms*, Osaka J. Math., **27** (1990), 593–620.

The University of Shiga Prefecture
2500 Hassaka Hikone Shiga 522–8533, Japan