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HOMOLOGY BOUNDARY LINKS AND FUSION CONSTRUCTIONS

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Introduction

It is well-known [6] that a smooth link of spheres $L^n \subset S^{n+2}$ is trivial if and only if

- i) the link group $\pi = \pi_1(S^{n+2} \setminus L)$ is free on the set of meridians of L , and
- ii) the homotopy groups $\pi_j(S^{n+2} \setminus L)$ are trivial for $2 \leq j \leq [(n+1)/2]$,

at least when $n \neq 2$. It is of considerable interest to study classes of links for which the link group satisfies weaker “freeness” conditions than i). The following three consecutively enlarged classes turn out to be of special importance:

- (I) *Boundary links.* These are links whose components bound disjoint oriented manifolds in S^{n+2} (Seifert surfaces). Equivalently, there is an epimorphism $\pi \rightarrow F$ (F =free group of rank equal to the number of components of L), which maps meridians to generators.
- (II) *Homology boundary links (HBLs).* Here we drop the condition on meridians. A geometric interpretation in terms of “singular” Seifert surfaces is known (compare [15]).
- (III) *Sublinks of homology boundary links (SHB-links).* This class arises since class (II) is not closed with respect to sublinks.

Interest in these classes comes mostly from the study of link concordance. In higher dimensions a classification of boundary links up to boundary link concordance is known (compare for example [9]). Recently T. Cochran and J. Levine proved that each *HBL* is *concordant* to a fusion of a boundary link [3]. Roughly, a *fusion* of an r -component link is an $(r-j)$ -component link, which is formed by attaching j 1-handles (bands) to the link. The question whether in the last statement *concordance* can be replaced by *isotopy* is in fact related to the Andrews Curtis Conjecture (see [3]). Cochran and Levine define an obstruction for a *HBL* to be a boundary link, the *pattern*, an isotopy invariant, which in the author’s opinion has not yet been studied adequately. In 1989, T. Cochran and K. Orr proved the result, surprising to the experts, that there are *HBLs*, which are not even *concordant* to boundary links [4]. Their examples arise from a “completed” fusion construction, which preserves the number of components:

Canonical unknotted n -spheres linking the bands are added to a fusion to form a *strong fusion link*. Finally, the class (III) naturally appears in the study of disk link concordance via J. Ledermets exact sequence and in Levine's discussion of \hat{F} -links ([11] and [10]).

In this paper we shall study (strong) fusion and the classes (I)-(III) up to isotopy.

J. Hillman gave several examples of ribbon links in S^3 to distinguish the three classes [7]. His proofs use Wirtinger calculus and the ribbon property. It is our main aim to find the actual geometric reason for why these examples work. The first step in this direction was made by T. Cochran. He proved that each fusion (resp. strong fusion) of a boundary link is a *SHB-link* (resp. *HBL*). It is easy to see that both fusion constructions preserve class (III). This suggests the following questions:

- a) When is a (strong) fusion of a *HBL* a *HBL*?
- b) When is a (strong) fusion of a boundary link a boundary link?

We will apply techniques of [2], [3] and [10] to translate both questions into pure problems in combinatorial group theory. In case b) the hard work is contained in [3], namely the reduction to a property of the pattern of the (strong) fusion. Our contribution is to actually compute the pattern. We think that it is important to recognize that the answer for strong fusion in case a) only depends on the pattern of the original link, in case b) only depends on the bands.

Our methods work in all dimensions. No reference to the fundamental group of a specific given link is needed. From our computations we deduce the following result:

Theorem 1. *A strong fusion of a boundary link along a band is a boundary link if and only if one can choose Seifert surfaces which intersect the band only in its boundary.*

Here the main point is to have sufficient control over the relation between bands and Seifert surfaces. The necessary algebraic gadget is the band word (which is well defined up to a certain action of an extension of the special automorphism group of the free group): The transverse oriented intersection of the oriented band and an arbitrary choice of Seifert surfaces (whose components are labelled by generators of the free group) represents an element of the free group in the obvious way.

Theorem 1 may look technical, but gives a very convenient method not only to construct large classes of *HBLs*, which are not boundary links, but actually to *decide* when a strong fusion of a boundary link is a boundary link.

For example we prove:

Theorem 2. *A strong fusion of a two-component link is not a boundary link if and only if the corresponding (reduced) band word contains an occurrence*

$x_2^\varepsilon x_1^\eta (\varepsilon, \eta \in \{\pm 1\})$. (Here the band is oriented from the first to the second component).

Many results (for example 2.4 and 3.4) admit obvious generalizations to “several” bands, but to reduce technical details we shall keep to the case of a single band.

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1. Basic Notions

A link L is a smooth oriented submanifold of S^{n+2} diffeomorphic to a disjoint union of n -spheres. Throughout isotopy means smooth (in particular ambient) isotopy.

Let $N(L)$ be a closed tubular neighbourhood of L and $X(L):=S^{n+2} \setminus \text{int}(N(L))$ be the exterior. For any subset $A \subset S^{n+2}$ let $\pi(A):=\pi_1(S^{n+2} \setminus A)$.

We consider pairs (L, b) , where $L=(K_1, \dots, K_r)$ is an r -component link and $b=(\beta, u)$ is a smooth arc $\beta \subset S^{n+2}$ together with a vectorfield u normal to β , such that

- a) $L \cap \beta = \partial \beta$ is contained in K_1 and K_2 ,
- b) $u|_{\partial \beta}$ is also normal to L ,
- c) the orientations of S^{n+2} induced by L , u and some orientation of β are opposite to each other in the two endpoints $\partial \beta$.

Following [3] we define:

DEFINITION 1.1. The data (L, b) determine a 1-handle, i.e. a submanifold of S^{n+2} diffeomorphic to $I \times D^n$ (unique up to isotopy moving $\partial I \times D^n$ only in L), which is used to form the connected sum $K_1 \#_b K_2$ of K_1 and K_2 . The resulting $(r-1)$ -component link $L(b)=(K_1 \#_b K_2, K_3, \dots, K_r)$ is called the *fusion of L along the band b* .

For $n=1$ any two choices of u for some arc β differ by some element of $\pi_1(SO_2) \cong \mathbb{Z}$ twisting the vectorfield. But for $n \geq 2$ any two vectorfields are homotopic rel boundary. So the isotopy class rel boundary of β determines b . This was pointed out by J. Levine.

We choose an orientation preserving embedding $\bar{b}: I \times D^{n+1} \rightarrow S^{n+2}$ (which is unique up to isotopy), such that $\bar{b}(I \times (1/2) D^n)$ is the 1-handle corresponding to b . Then $C_b := \bar{b}((1/2) \times S^n)$ is the *n-sphere linking b*, which is contained in $S^{n+2} \setminus L$ and in $S^{n+2} \setminus L(b)$ by suitable choice of \bar{b} . We shall orient β from K_1 to K_2 . Then C_b inherits a canonical orientation.

DEFINITION 1.2. The link $\hat{L}(b)=(K_1 \#_b K_2, C_b, K_3, \dots, K_r)$ is called the *strong fusion of L along b* .

DEFINITION 1.3. Pairs (L, b) and (L', b') are called *isotopic*, if there is a diffeomorphism h of S^{n+2} , isotopic to the identity, such that $h(L)=L'$ and $h(b)=b'$, i.e. $h(\beta)=\beta'$ and $u'_{h(x)}=dh_x(u_x)$ for all $x \in \beta$.

Then fusion (resp. strong fusion) is a construction from isotopy classes of pairs (L, b) , where L is an r -component link, to isotopy classes of $(r-1)$ (resp. r)-component links.

The following two results are essential to the understanding of (strong) fusion, the second one being an easy improvement of [2, 2.3].

Proposition 1.4. *The diffeomorphism type of the $(n+2)$ -manifold, which is the result of surgery on the longitudes of $\hat{L}(b)$ (0-framing on C_b for $n=1$) does not depend on the band b .*

This is folklore (see for example [18]). ■

Proposition 1.5. *Each fusion or strong fusion of a SHB-link is a SHB-link.*

Proof. Recall that the *HBL* condition only depends on the $(n+2)$ -manifold, which is the result of (0-framed for $n=1$) surgery on the longitudes of the link in consideration. Let L be a sublink of the *HBL* L' , b a band for L . We can arrange that $\bar{b}(I \times D^{n+1})$ does not intersect $L' \setminus L$. The link $L' \cup C_b$ is also a *HBL*. We fuse a parallel copy of K_2 to K_1 to get the link L'' . L'' is a *HBL*, since the fusion corresponds to a handle-slide on the surgery manifold corresponding to $L' \cup C_b$. But $L(b)$ and $\hat{L}(b)$ are both sublinks of L'' . ■

REMARK 1.6. By 1.4 all properties of $\hat{L}(b)$, which only depend on the surgery-manifold, do not depend on the bands. This applies for instance in those cases, where the existence of a homomorphism from the link group into a certain group is considered, which *kills longitudes*. Examples: *HBLs*, *E-links*, \hat{F} -links, vanishing $\overline{\mu}$ -invariants (compare [2], [3]).

Now we study the way in which basings can be used as a coordinate system for bands. This will be more general than actually needed for the purposes of this paper, but can be applied in many situations.

For a given link L choose basepoints $*_i \in \partial N(L)$ in the components T_i of $\partial N(L)$, $1 \leq i \leq r$, $* \in X(L)$. A *basing* of the i -th component is a choice of proper arc $\gamma_i \subset X(L)$ with $\gamma_i(0)=*$, $\gamma_i(1)=*_i$. Note that each basing of the i -th component determines a *meridian* μ_i and a *longitude* λ_i in the usual way, i.e. $\mu_i = [\gamma_i, m_i, \gamma_i^{-1}]$ and $\lambda_i = [\gamma_i, l_i, \gamma_i^{-1}]$ in $\pi_1(L)$, where m_i resp. l_i are meridional resp. longitudinal curves on T_i .

There is an obvious notion of (ambient) isotopy of triples (L, γ, b) , where $\gamma = (\gamma_1, \gamma_2)$ is a basing of the first two components of L and b is a band joining

these components.

Let $\pi_1(T_1)$ and $\pi_1(T_2)$ be considered as subgroups of $\pi(L)$ via γ . The arc $\tilde{\beta} := \beta \cap X(L)$ has its two boundary points in $T_1 \cup T_2$.

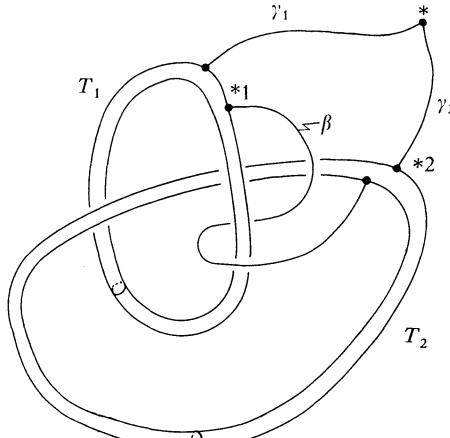


Figure 1

Up to isotopy fixing (L, γ) these can be assumed to be $*_1, *_2$. Let $\bar{\beta}$ be the double coset $\pi_1(T_1) [\gamma_1 \tilde{\beta} \gamma_2^{-1}] \pi_1(T_2)$. We define $\rho(L, \gamma, b) = (\pi(L), \bar{\beta})$. Then ρ is well defined from isotopy classes of triples (L, γ, b) to isomorphism classes of pairs $(\pi, \bar{\beta})$. Here an isomorphism of $(\pi, \bar{\beta})$ and $(\pi', \bar{\beta}')$ is an isomorphism of groups $\bar{h}: \pi \rightarrow \pi'$, such that $\bar{h}(\bar{\beta}) = \bar{\beta}'$.

In higher dimensions we can reconstruct b from $\bar{\beta}$ as follows:

Proposition 1.7. *If $n \geq 2$ and for a given pair (L, γ) and bands b, b' we know that $\bar{\beta} = \bar{\beta}' \subset \pi(L)$, then there exists an isotopy h of S^{n+2} fixing L , which maps b to b' .*

Proof. Because of Levine's observation we don't have to worry about normal vectorfields. We know that $\bar{\beta} = \bar{\beta}'$, so $a_1 [\gamma_1 \tilde{\beta} \gamma_2^{-1}] a_2 = [\gamma_1 \tilde{\beta}' \gamma_2^{-1}]$ for $a_1 \in \pi_1(T_1)$, $a_2 \in \pi_1(T_2)$. Then it is easy to see that $a'_1 \tilde{\beta} a'_2 \simeq \bar{\beta}'$ rel boundary for certain loops $a'_i \in \pi_1(T_i, *_i)$, $i=1, 2$ (\simeq means homotopic). Now isotope β fixing L by moving $\tilde{\beta}(0)$ in T_1 and $\tilde{\beta}(1)$ in T_2 until $\tilde{\beta} \simeq \bar{\beta}'$ rel boundary. Since homotopy implies isotopy for $n \geq 2$ the result follows. ■

If $n=1$, then bands may link and a single band may knot. In this case $\bar{\beta}$ does not measure "band information" completely, even if we forget twisting of bands.

EXAMPLE 1.8. The Whitehead double of a knot $K \subset S^3$ is the fusion of the Hopf-link H in S^3 along a band, which follows the knot. The $\bar{\beta}$ for the corresponding pairs (H, b) are all the same.

2. Bands for Boundary Links

We shall introduce an invariant of (L, b) for boundary links L , which is derived from ρ in section 1.

Throughout F will denote the free group of rank r with basis $\{x_1, \dots, x_r\}$, where r is the number of components of L . Recall that (via Pontryagin Thom construction) a choice $\{V_1, \dots, V_r\}$ of bordism class rel boundaries of Seifert surfaces for L corresponds to a pair (ϕ, μ) , where $\mu: F \rightarrow \pi(L)$ is a meridian map (homotopy basing), and $\phi: \pi(L) \rightarrow F$ is a splitting homomorphism [5], [9]. Such a pair (ϕ, μ) will be called *special*.

Let $\gamma = (\gamma_1, \dots, \gamma_r)$ be a basing, μ_i be associated to γ_i , and (ϕ, μ) be a special pair with $\mu(x_i) = \mu_i$.

DEFINITION 2.1. The *band word* corresponding to (ϕ, μ) is the double-coset mod $\langle x_1 \rangle$ from the left and $\langle x_2 \rangle$ from the right of the word $w := \phi[\gamma_1 \tilde{\beta} \gamma_2^{-1}] \in F(\langle \rangle)$ denotes the subgroup generated by).

This is of course just $\phi(\tilde{\beta})$ with $\tilde{\beta}$ from section 1.

There is the following geometric way to derive the *band word* w of (L, b) from the Seifert surfaces $\{V_i\}$ corresponding to (ϕ, μ) : Go along β (from K_1 to K_2), which we assume transversal to $\cup V_i$. Each time β intersects V_i with positive resp. negative orientation write down x_i resp. x_i^{-1} . Note that each occurrence $x_j x_i^{-1}$ in the not reduced word w can be eliminated geometrically by attaching a 1-handle to V_j , thus performing surgery rel boundaries. If $w=1$, then $L(b)$ and $\hat{L}(b)$ are boundary links in the obvious way.

Lemma 2.2. A bordism class rel boundaries of Seifert surfaces for L determines $w \in F$ up to the $\mathbb{Z} \times \mathbb{Z}$ -action above. Moreover, each such change of w by this action is induced by a bordism rel boundaries.

Proof. Let (\tilde{V}_i) be a bordism rel boundaries between (V_i) and (V'_i) in $X(L) \times I$. By transversality $\beta \times I$ intersects $\cup \tilde{V}_i$ in a collection of arcs. The boundary points of these arcs are contained in $V_1, \dots, V_r, V'_1, \dots, V'_r$ or in $\partial V_i \times I$ for $i=1, 2$. This proves the first assertion. Since we can isotope β to change w by multiplication with powers of x_1 from the left and x_2 from the right arbitrarily, the result is proved. ■

Now we consider a variation of the special pair (ϕ, μ) . Let CA denote the group of special automorphisms of F , i.e. automorphisms which map each generator to a conjugate of itself (F free on $\{x_1, \dots, x_r\}$). It is known that CA acts transitively on special pairs (ϕ, μ) by $\alpha \cdot (\phi, \mu) = (\alpha \circ \phi, \phi \circ \mu^{-1})$ for $\alpha \in CA$ [5]. Moreover, CA is generated by special automorphisms $\overline{\alpha_{ij}} (i \neq j)$, where $\overline{\alpha_{ij}}(x_j) = x_i x_j x_i^{-1}$ and $\overline{\alpha_{ij}}(x_l) = x_l$ for $l \neq j$ [9].

It is easy to see how w changes, when (ϕ, μ) is replaced by $\overline{\alpha_{ij}} \cdot (\phi, \mu)$. Let

$\alpha_{ij} \cdot w$ denote the word corresponding to Seifert surfaces determined by $\overline{\alpha_{ij}} \cdot (\phi, \mu)$:

$$(*) \quad \begin{cases} \alpha_{ij} \cdot w = \overline{\alpha_{ij}}(w), & j=1, 2 \\ \alpha_{i1} \cdot w = x_i^{-1} \overline{\alpha_{i1}}(w), & i=1 \\ \alpha_{i2} \cdot w = \overline{\alpha_{i2}}(w) x_i, & i=2 \end{cases}$$

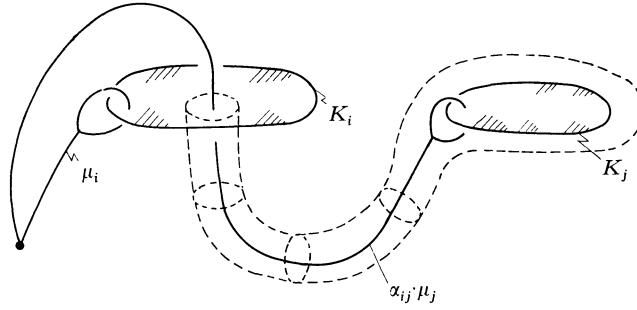


Figure 2

This suggests to introduce an extension of CA as follows:

Let $F^r = F \times F \times \cdots \times F$, and for each $\alpha = (g_1, \dots, g_r) \in F^r$ let $\bar{\alpha}$ denote the endomorphism of F given by $\bar{\alpha}(x_i) = g_i x_i g_i^{-1}$ for $1 \leq i \leq r$. Let \tilde{CA} denote the group of all $\alpha \in F^r$, such that $\bar{\alpha}$ is an automorphism of F , i.e. $\bar{\alpha} \in CA$. Then there is the exact sequence:

$$1 \rightarrow \mathbb{Z}^r \rightarrow CA \rightarrow \tilde{CA} \rightarrow 1$$

where $(n_1, \dots, n_r) \in \mathbb{Z}^r$ is mapped to $(x_1^{n_1}, \dots, x_r^{n_r}) \in \tilde{CA}$.

The group structure on \tilde{CA} is explicitly given by

$$\begin{aligned} (g_1, \dots, g_r) (h_1, \dots, h_r) &= (\bar{\alpha}(h_1) g_1, \dots, \bar{\alpha}(h_r) g_r) \\ (g_1, \dots, g_r)^{-1} &= (\bar{\alpha}^{-1}(g_1^{-1}), \dots, \bar{\alpha}^{-1}(g_r^{-1})). \end{aligned}$$

Note that \tilde{CA} is generated by elements

$$\alpha_{ij} = (1, \dots, 1, x_i, 1, \dots, 1)$$

↑
j

for $1 \leq i, j \leq r$. From $(*)$ we deduce:

Lemma 2.3. \tilde{CA} acts on F by

$$\alpha \cdot w = g_1^{-1} \bar{\alpha}(w) g_2$$

for $\alpha = (g_1, \dots, g_r) \in \tilde{CA}$.

Proposition 2.4. *Let L be a boundary link with band b . Then the \tilde{CA} -orbit of a band word w is an isotopy invariant of (L, b) . Moreover, each (even not reduced) word in this orbit of w is realized as band word for some choice of Seifert surfaces.*

Proof. This follows from 2.2 and 2.3, [9] and [5]. If we change (L, b) in its isotopy class, then Seifert surfaces can be chosen, such that the band word is unchanged. Finally note that any occurrence $x_j x_j^{-1}$ can also be generated geometrically by an obvious finger-move. ■

Corollary 2.5. *Let w be a band word for (L, b) , L be a boundary link. Then there are Seifert surfaces for L , which intersect β only in $\partial\beta$, if and only if 1 is in the \tilde{CA} -orbit of w .*

DEFINITION 2.6. A band for L is called *essential* if 1 is not in the \tilde{CA} -orbit of w .

EXAMPLES 2.7. Let (L, b) be a given pair.

- a) If w does not involve x_1 or x_2 , then b is not essential, since $\overline{\alpha_{i1}}(w)=w$ or $\overline{\alpha_{i2}}(w)=w$ (compare (*)).
- b) If $r=2$, then the usual action by automorphisms of CA on F is by inner automorphisms, i.e. $\alpha_{12}(w)=x_1 w x_1^{-1}$ and $\alpha_{21}(w)=x_2 w x_2^{-1}$, so $\alpha_{12} \cdot w = x_1 w$ and $\alpha_{21} \cdot w = w x_2^{-1}$. Thus the \tilde{CA} -orbit is the \mathbb{Z}^2 -orbit.

It is in general difficult to decide, when 1 is in the \tilde{CA} -orbit of a given word w . The orbit of 1 is the set of all $g_1 g_2^{-1}$, for which there are $g_3, \dots, g_r \in F$, such that $\alpha=(g_1, \dots, g_r) \in \tilde{CA}$. Of course $\alpha \in \tilde{CA}$ if and only if $\{g_i x_i g_i^{-1}\}$ is a basis of F . This is equivalent to the existence of $h_3, \dots, h_r \in F$, such that

$$(x_1, w x_2 w^{-1}, h_3 x_3 h_3^{-1}, \dots, h_r x_r h_r^{-1})$$

is a basis of F . This suggests the following notion:

DEFINITION 2.8. Let $\{x_i\}$ be a fixed set of generators of F . Let $Y \subset F$ be a set of conjugates of x_i 's. Y is called *special primitive* if there is a supplementary set Z of conjugates of x_i 's, such that $Y \cup Z$ is a basis of F .

Of course special primitive implies primitive, but the converse does not hold:

EXAMPLE 2.9. Let $w=x_3 x_2 x_1$, $r=3$. Then $\{x_1, w x_2 w^{-1}\}$ is primitive, since $\{x_1, w x_2 w^{-1}\}$ is a basis of F . Assume that $(x_1, w x_2 w^{-1}, h x_3 h^{-1})$ is a basis for some $h \in F$. In particular x_2 is a word in these three elements. For $x \in F(x_1, x_2, x_3)$ let \bar{x} denote the image in $F(x_1, x_2)$ under the canonical projection. Then x_2 is a word in x_1 and $\bar{w} x_2 \bar{w}^{-1}$ since $\bar{h} x_3 \bar{h}^{-1}=1$. But it is easy to see that this is not

possible either by a length argument or by Whitehead's algorithm [19].

For $w \in F$ let $|w|$ denote the length of the reduced word w .

QUESTION 2.10. Given $w \in F$ of minimal length under the \mathbb{Z}^2 -action. Assume that there exists an $\alpha \in \widetilde{CA}$, such that $|\alpha \cdot w| < |w|$. Does there exist an elementary generator α_{ij} , such that $|\alpha_{ij} \cdot w| < |w|$?

3. Detection of (Homology) Boundary Links

We recall results from [3]: Let L be a HBL with epimorphism $\phi: \pi(L) \rightarrow F$ and meridians $\{\mu_i\}$. Let $\text{Out}(F)$ be the group of outer automorphisms of F and $\theta: \text{Out}(F) \rightarrow \text{Aut}(F')$ be defined by $\theta(\alpha)(g_1, \dots, g_r) = (\alpha(g_1), \dots, \alpha(g_r))$. Let the group structure on the semidirect product $F' \times_{\theta} \text{Out}(F)$ be given by

$$((g_1, \dots, g_r), \alpha)((g'_1, \dots, g'_r), \alpha') = ((g_1 \alpha(g'_1), \dots, g_r \alpha(g'_r), \alpha \alpha')).$$

Then $F' \times_{\theta} \text{Out}(F)$ acts on F' by

$$((g_1, \dots, g_r), \alpha) \cdot (h_1, \dots, h_r) = (g_1 \alpha(h_1) g_1^{-1}, \dots, g_r \alpha(h_r) g_r^{-1})$$

The following results are proved in [3]:

- i) The orbit of $(\phi(\mu_1), \dots, \phi(\mu_r))$ under this action, denoted by $P(L) = [\phi(\mu_i)]$, is called the *pattern* of L , and is an invariant of L .
- ii) Each r -tuple (u_1, \dots, u_r) of elements, which normally generate F , is realized as $(\phi(\mu_i))$ for some ribbon link.

Theorem 3.1. (Smythe [15], Cochran-Levine [3]). *A HBL L is a boundary link if and only if $(x_1, \dots, x_r) \in P(L)$ for generators x_1, \dots, x_r of F .*

Note that ϕ can always be changed, such that $\phi(\mu_i) \equiv x_i \pmod{[F, F]}$. But this condition is *not* preserved by the $(F' \times_{\theta} \text{Out}(F))$ -action.

The next result provides, similar to 3.1, a method to decide, when a SHB-link is a HBL.

It is convenient to introduce the following notion:

DEFINITION 3.2. Let G be a group which abelianizes to \mathbb{Z}' . Then G is called *onto-free* if there exists an epimorphism from G onto the free group of rank r .

Proposition 3.3. *Let L' be an $(r+l)$ -component HBL with meridians $\{\mu'_i\}$, and let $\phi': \pi(L') \rightarrow F' = F(x_1, \dots, x_{r+l})$ be an epimorphism. Let $L' = (L, L'')$ for some r -component link L . Then L is a HBL if and only if the group $G = G(r, \phi', \{\mu'_i\}) = (x_1, \dots, x_{r+l} \mid \phi'(\mu'_{r+1}), \dots, \phi'(\mu'_{r+l}))$ is onto-free.*

Proof. We have the commutative diagram:

$$\begin{array}{ccc} \pi(L') & \xrightarrow{\phi'} & F' \\ pr \downarrow & & \downarrow pr' \\ \pi(L) & \xrightarrow{\tilde{\phi}'} & G \end{array}$$

where $\tilde{\phi}'$ is induced by ϕ' .

Assume that $\psi: G \rightarrow F$ is an epimorphism. Then $\psi\tilde{\phi}': \pi(L) \rightarrow F$ is an epimorphism, and L is a HBL. Conversely suppose that $\vartheta: \pi(L) \rightarrow F$ is onto. Let $\mu': F' \rightarrow \pi(L')$ be a splitting homomorphism for ϕ' . We want to show that $\vartheta \circ pr \circ \mu': F' \rightarrow F$ factors through a homomorphism $G \rightarrow F$ (which will be onto, since $F' \cong \pi(L')/\pi(L')_\omega \rightarrow \pi(L)/\pi(L)_\omega \cong F$ is onto). But $\mu' \phi'(\mu'_j) = \mu'_j c_j$ for some $c_j \in \pi(L')_\omega$ for $1 \leq j \leq r+l$, since the kernel of ϕ' is $\pi(L')_\omega$. If $j > r$ then $pr(\mu'_j) = 1 \in \pi(L)$ and $pr(c_j) \in \pi(L)_\omega$, so $\vartheta \circ pr(c_j) = 1 \in F$. ■

Theorem 3.4. *Let (L, b) be a pair, L a HBL with meridians $\{\mu_i\}$ and epimorphism ϕ . Then $L(b)$ resp. $\hat{L}(b)$ are HBLs if and only if the groups*

$$(x_1, \dots, x_r | \phi(\mu_2)^{-1} \phi(B)^{-1} \phi(\mu_1) \phi(B))$$

resp.

$$(x_1, \dots, x_r, z | \phi(\mu_2)^{-1} z^{-1} \phi(\mu_1) z))$$

are onto-free. Here $B = [\gamma_1 \tilde{\beta} \gamma_2^{-1}] \in \pi(L)$ for basings γ_1, γ_2 of K_1, K_2 , which induce the meridians μ_1, μ_2 , and $\tilde{\beta} = X(L) \cap \beta$.

Proof. We know that $L(b)$ (resp. $\hat{L}(b)$) are sublinks of the HBL $L(b)' := (L(b), K'_2)$ (resp. $(\hat{L}(b))' := (\hat{L}(b), K'_2)$), where K'_2 is a parallel pushout of K_2 . It follows from the proof of [3, 3.1] that the pair $(\phi, \{\mu_i\})$ induces a pair $(\phi', \{\mu'_i\})$ for $L(b)'$, such that $\phi'(\mu'_i) = \phi(B^{-1} \mu_1^{-1} B \mu_2)$. This proves the assertion about $L(b)$. In case of strong fusion we first add C_b to L and extend ϕ to the epimorphism $\pi(L \cup C_b) = \pi(L) * \mathbb{Z} \rightarrow F(x_1, \dots, x_r, z)$ in the obvious way (* means free product). We choose a meridian of C_b , which maps to z by this extension. Then we follow [3, 3.1] again. It turns out that always $G(r, \hat{\phi}', \{\hat{\mu}'_i\}) \cong (x_1, \dots, z, z | z^{-1} \phi(\mu_1)^{-1} z \phi(\mu_2))$ for each $\hat{\phi}': \pi(\hat{L}(b)') \rightarrow F(x_1, \dots, x_r, z)$ and meridians $\{\hat{\mu}'_i\}$ of $\hat{L}(b)'$. ■

REMARK 3.5. Since the isomorphism classes of $G(r-1, \phi', \{\mu'_i\})$ and $G(r, \hat{\phi}', \{\hat{\mu}'_i\})$ do not depend on input data $(\phi, \{\mu_i\})$, we will write $G_{L,b} \cong G(r-1, \phi', \{\mu'_i\})$ resp. $G_L \cong G(r, \hat{\phi}', \{\hat{\mu}'_i\})$.

The results 3.4 and 3.1 translate the geometric questions a) and b) of the introduction into algebra. For the rest of this paragraph we shall deduce several sufficient conditions from this and carry out computations, which are needed later on. We recall the following result characterizing freeness of 1-

relator groups:

Lemma 3.6. (Rappaport [14], Whitehead [19]). *The 1-relator group $(x_1, \dots, x_r | R)$ is free if and only if the relator R is a primitive element of $F = F(x_1, \dots, x_r)$, i.e. R is member of a basis of F .*

Corollary 3.7. *Let L be a HBL admitting an epimorphism $\phi: \pi(L) \rightarrow F$, such that $\phi(\mu_1) = x_1$ and $\phi(\mu_2) = x_2$ (we call such links partial boundary). Let b be a band for L . Then $L(b)$ is a HBL. Moreover*

$$G_{L,b} \cong G_w := (x_1, \dots, x_r | x_2 = w^{-1} x_1 w),$$

where $w = \phi(B)$, $B = [\gamma_1 \tilde{\beta} \gamma_2^{-1}]$. Here $(\phi, \{\mu_i\})$ is chosen such that $\phi(\mu_1) = x_1$, $\phi(\mu_2) = x_2$, and γ_1, γ_2 are basings of K_1, K_2 inducing μ_1, μ_2 . If L is a boundary link, then w is a band word for b .

Proof. This follows from 3.4 and 3.6. ■

Of course each boundary link is partial boundary. Moreover, if $L = (K_1, \dots, K_r)$ is partial boundary, then the sublink (K_1, K_2) is a boundary link. But the following example shows that this is not sufficient.

EXAMPLE 3.8. Let $L = (K_1, K_2, K_3)$ be a HBL with $(\phi, \{\mu_i\})$, such that

$$(\phi(\mu_1), \phi(\mu_2), \phi(\mu_3)) = (x_1, x_3^{-1} x_2^{-1} x_1 x_2 x_1^{-1} x_3 x_2, x_3)$$

If L would be partial boundary, then there would exist an automorphism ψ of F , such that $\psi(x_1) = x_1$ and $\psi(x_2) = g\phi(\mu_2)g^{-1}$ for some $g \in F$. In particular $\{x_1, g\phi(\mu_2)g^{-1}\}$ would be primitive for some $g \in F$. By [19] it is easily proved that this is not true. Thus L is not partial boundary. On the other hand $[x_1, x_2^{-1} x_1 x_2 x_1 x_2]$ is the pattern of a 2-component boundary link.

From 3.7 we deduce:

Corollary 3.9. *Let L be a partial boundary link and $w = \phi(B)$ as in 3.7. Then $L(b)$ is a HBL, if*

- (i) $x_2^{-1} w^{-1} x_1 w$ is primitive (this is equivalent to G_w free) or
- (ii) w maps to $\langle t \rangle \subset F' := F(t, x_3, \dots, x_r)$ under the epimorphism $F \rightarrow F'$ given by $(x_1 \mapsto t, x_2 \mapsto t, x_i \mapsto x_i \text{ for } 3 \leq i \leq r)$.

Proof. (i) follows from 3.7 and 3.6. The epimorphism in (ii) induces an epimorphism $\chi_w: G_w \rightarrow F'$, so G_w is onto-free. ■

Corollary 3.10. *Let L be a boundary link with band b and band word w satisfying (ii) in 3.10. Then $L(b)$ is a boundary link.*

Proof. We choose a special pair (ϕ, μ) for L . Then there is a special pair

$(\phi', \{\mu'_i\})$ for $L(b)'$ (compare [CL, 3.1]), such that the diagram:

$$\begin{array}{ccc}
 \pi(L(b)') & \xrightarrow{\phi'} & F \\
 pr \downarrow & & \downarrow pr' \\
 \pi(L(b)) & \xrightarrow{\tilde{\phi}'} & G_w \\
 & & \downarrow \chi_w \\
 & & F(t, x_3, \dots, x_r)
 \end{array}$$

commutes. Thus we see that $[t, x_3, \dots, x_r]$ is the pattern of $L(b)$. ■

The following question is related to the notion of partial boundary links:
For which r there are HBLs L , which are not boundary links, but for which $\hat{L}(b)$ is a HBL?

A positive answer can be given for $r \geq 3$:

EXAMPLE 3.11. Let L be partial boundary with pattern $[x_1, x_2, x_3 x_2 x_3^{-1} x_1 x_2 x_3]$. Using 3.1 and [19] it is easy to prove that L is not a boundary link. But $\hat{L}(b)$ is a HBL by 3.7.

For $r=2$ the following two conjectures are equivalent by 3.4, stating the problem in the geometric and a purely algebraic way:

CONJECTURE (H). Let L be 2-component HBL. If $\hat{L}(b)$ is a HBL for some (and hence any) band b , then L is a boundary link.

CONJECTURE (H'). Let $u_1, u_2 \in F(x_1, x_2) = F$ be elements, which normally generate F , but $(x_1, x_2) \notin [u_1, u_2]$. Then the HNN-extension $G_u := (x_1, x_2, z | u_2 = z^{-1} u_1 z)$ is not onto-free.

Now we give a convenient sufficient condition for when a fusion of a boundary link is a HBL and compute the resulting pattern. We begin with an example, which shows that it is not necessary that the band is inessential to L .

EXAMPLE 2.9. (continued). Let L be a 3-component boundary link and b a band for L with band-word $w = x_3 x_2 x_1$. We have seen in 2.9 that $\{x_1, ww_2w^{-1}\}$ is primitive, but not special primitive. So $L(b)$ is a HBL for an essential band b .

DEFINITION 3.12. Let $r \geq 2$ and L be an $(r+1)$ -component boundary link. A band b for L is called *tame* if for some choice of Seifert surfaces the band word w has the form $w = w_1 x_{r+1}^{-1} w_2 \in F(x_1, \dots, x_{r+1})$ (or $w = w_1 x_{r+1} w_2$) for words $w_1, w_2 \in F(x_1, \dots, x_r) = F$. A pair (L, b) as above is called *tame*. □

REMARKS 3.13.

- a) A strong fusion of an r -component link L along a band b is a fusion of the $(r+1)$ -component link (L, C_b) along a tame band b' with band-word $w' = x_{r+1}^{-1} w$, where w is a band word corresponding to b . Fusion of a tame pair is a more general “algebraic” version of strong fusion. Figure 3 shows a tame band for the trivial 3-component link.

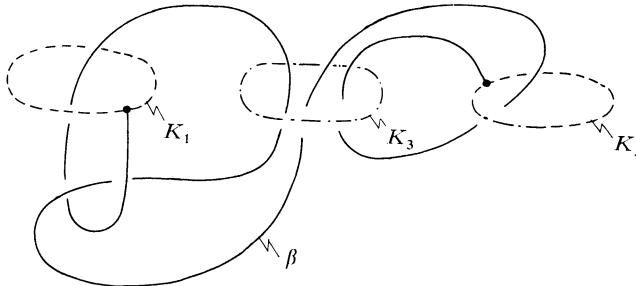


Figure 3

- b) Tameness of b is equivalent to the existence of Seifert surfaces, such that the band word has the form $x_{r+1}^{-1} w$. This follows from 1.4.

Proposition 3.14. *Let (L, b) be a tame pair. Then $L(b)$ is a HBL.*

Proof. We know that

$$G_w = (x_1, \dots, x_r, x_{r+1} | x_2 = w^{-1} x_1 w)$$

with $w = w_1 x_{r+1}^{-1} w_2$.

Then

$$(x_1, w x_2 w^{-1}, x_3, \dots, x_r, w)$$

is a basis of $F(x_1, \dots, x_{r+1})$ and 3.6 applies. ■

Theorem 3.15. *If (L, b) is tame, then the pattern of $L(b)$ is given by*

$$P = [x_1, x_3, \dots, x_r, \bar{w}_2 x_2 \bar{w}_1],$$

where for each $u \in F$ let \bar{u} denote the image of u under the homomorphism $x_i \mapsto x_i$ for $i \neq 2$ and $x_2 \mapsto x_2 x_1 x_2^{-1}$.

Proof.

For a tame band (L, b) we shall construct an epimorphism

$$\varepsilon: \pi(L(b)) \rightarrow F(x_1, \dots, x_r)$$

as follows:

There is the canonical map

$$\tilde{\phi}' : \pi(L(b)) \rightarrow G_w$$

induced by

$$\phi' : \pi(L(b)') \rightarrow F(x_1, \dots, x_{r+1})$$

(compare [3, 3.1]). Recall that $L(b)' = (L(b), K'_2)$, where K'_2 is a parallel copy of K_2 . So for a choice of meridians $\{\mu'_i\}$ of $L(b) = (K_1 \#_b K_2, K_3, \dots, K_{r+1})$ we can assume that $\tilde{\phi}'(\mu'_i) = x_i$, $i = 1, 3, \dots, (r+1)$. Here μ'_1 is a meridian of $K_1 \#_b K_2$ and μ'_i is a meridian of K_i for $3 \leq i \leq r+1$.

Our assumptions imply the existence of a canonical epimorphism

$$\bar{\varepsilon} : G_w \rightarrow F(x_1, \dots, x_r) = F$$

given by

$$\begin{cases} \bar{\varepsilon}(x_1) = x_1 \\ \bar{\varepsilon}(wx_2w^{-1}) = x_1 \\ \bar{\varepsilon}(w) = x_2^{-1} \\ \bar{\varepsilon}(x_i) = x_i, \quad \text{for } 3 \leq i \leq r. \end{cases}$$

So $\bar{\varepsilon}(x_2) = x_2 x_1 x_2^{-1}$, and $\bar{\varepsilon}(w_1 x_{r+1}^{-1} w_2) = x_2^{-1}$ implies $\bar{\varepsilon}(x_{r+1}) = \bar{\varepsilon}(w_2) x_2 \bar{\varepsilon}(w_1)$. Now we define $\varepsilon := \bar{\varepsilon} \circ \tilde{\phi}' : \pi(L(b)) \rightarrow F$. ■

4. More Examples

Now we construct concrete families of links to distinguish the classes (I)-(III). We need the following result:

Lemma 4.1. (Hillman [7, V, Thm.1]). *Let G be a group which abelianizes to \mathbb{Z}^r and with a presentation of deficiency r . If G is onto-free, then the Alexander ideal $E_r(G)$ is principal.*

For definitions and further details we refer to [7].

EXAMPLE 4.2. Let L be a 3-component boundary link and b a band for L with band-word $w = x_3^{-1} x_1 x_2 x_3$. We compute $E_2(G)$ for $G = (x_1, x_2, x_3 | w^{-1} x_1 w x_2^{-1})$. The Jacobi-matrix of this presentation is

$$||t^{-2} + t^{-1}s^{-1} - t^{-2}s^{-1}, s^{-1} - 1 - s^{-1}t^{-1}, s^{-1}(t - t^{-1} + t^{-2} - 1)||,$$

where $x_1, x_2 \mapsto t$ and $x_3 \mapsto s$ under abelianization.

So $E_2(G)$ is the ideal

$$(1 - t - s, 1 - t + ts, 1 - t - t^2 + t^3) = (1 + t, 1 - t - s, (1 - t)^2(1 + t)) = (1 + t, 2 - s),$$

which is certainly not principal. It follows by 3.4. that the 2-component link $L(b)$ is not a *HBL*.

For $n=1$ and L the trivial link this is the example given by Hillman [7, V, Figure 1].

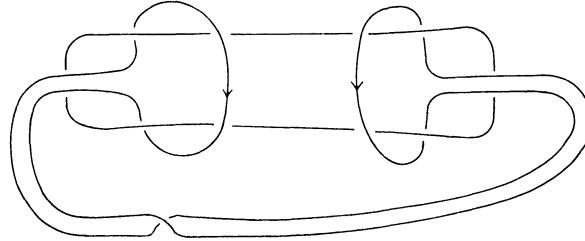


Figure 4

The next two examples show how to construct *SHB*-links, which are not *HBLs*.

EXAMPLE 4.3. Let L be a 2-component boundary link and b_i (resp. b'_i) be bands with words $w_i = x_2^i x_1$ (resp. $x_2 x_1^i$), $i \in \mathbb{Z} \setminus \{0\}$. Then the pattern of $L_i = L(b_i)$ (resp. L'_i) is given by $(x_1, x_2(x_2 x_1^i x_2^{-1}) x_1)$ (resp. $(x_1, x_1(x_2^{-1} x_1^i x_2) x_2)$).

It is easy to prove that

$$G_{L_i} \cong G_{-i,1} \cong G_{L'_i},$$

where $G_{i,j}$ is the Baumslag-group

$$(a, b, u | u[a^i, u] [a^j, b]),$$

which is known to be parafree not free (compare [1]).

Thus L_i resp. L'_i are *SHB*-links, which are not *HBL*.

For $n=1$ and $i=1$ this is Example VI, Figure 1 in [7].

EXAMPLE 4.4. Let $r=3$ and V, W be elements of the commutator subgroup of $F(x_1, x_3) \subset F(x_1, x_2, x_3) = F$, $V \neq 1$. Let L be a *HBL* with $\phi(\mu_1) = W x_1$ and $\phi(\mu_2) = W x_2 [V, x_1^{-1} x_2]$. For given (V, W) we can construct a (ribbon) link, such that $(w_1, w_2, w_3) = (W x_1, W x_2 [V, x_1^{-1} x_2], x_3)$ are the images of the meridians. In fact, by [3] we only have to check that the normal closure of $\{w_1, w_2, w_3\}$ is F . But $x_1 = W^{-1} w_1$, and W is a commutator in $F(x_1, x_3)$, thus is contained in the normal closure of x_3 ($x_3 = 1$ implies $W = 1$). Also: $x_2 = W^{-1} w_2 [x_1^{-1} x_2, V] = W^{-1} w_2 (x_1^{-1} x_2 V x_2^{-1} x_1) V^{-1}$ is in the normal closure, since $V \in F(x_1, x_3)$ and x_1 is in the normal closure. By 3.4. we know that

$$\begin{aligned} G_L &\cong (x_1, x_2, x_3, z | W x_2 [V, x_1^{-1} x_2] = z^{-1} W x_1 z) \\ &\cong (x_1, x_2, x_3, z | x_2 [V, x_2] = [(W x_1)^{-1}, z]). \end{aligned}$$

This group is parafree not free by [17, 5.1, Thm. C]. So G_L is not onto-free and $\hat{L}(b)$ is not a *HBL*. For $W=1$ we can construct the link L and thus

$\hat{L}(b)$ explicitly. Note that the pattern $(x_1, x_2[V, x_1^{-1}x_2], x_3)$ is equivalent to $(x_1, wx_2^{-1}w^{-1}x_1, x_3)$, where $w=x_2^{-1}x_1Vx_1^{-1}x_2$. This follows from

$$\begin{aligned}\phi(\mu_2) &= (x_2 V x_2^{-1}) x_2 x_1^{-1} (x_2 V^{-1} x_2^{-1}) x_1 \\ &= (x_2 x_1^{-1}) (x_1 V x_1^{-1}) (x_1 x_2^{-1}) (x_2 x_1^{-1}) (x_2 x_1^{-1}) (x_1 V^{-1} x_1^{-1}) (x_1 x_2^{-1}) x_1\end{aligned}$$

and the substitution $X_2^{-1}=x_2 x_1^{-1}$, i.e. $X_2=x_1 x_2^{-1}$, so

$$\phi(\mu_2) = (X_2^{-1} x_1 V x_1^{-1} X_2) X_2^{-1} (X_2^{-1} x_1 V^{-1} x_1^{-1} X_2) x_1$$

which corresponds to a change of ϕ induced by composition with the automorphism of F : $(x_1 \mapsto x_1, x_2 \mapsto x_2^{-1}x_1, x_3 \mapsto x_3)$.

The resulting pattern is the pattern of a strong fusion of a boundary link along a band with band word w . The resulting link $\hat{L}(b)$ for $V=[x_1, x_3]$ is shown in Figure 5 (where the the boundary link is chosen to be trivial).

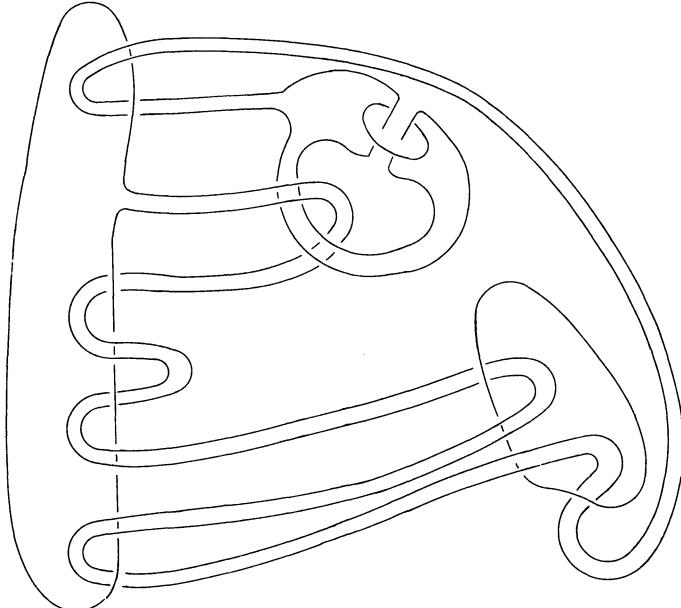


Figure 5

Question 4.5. For $n=1$, what is the relation between G_w (or G_L or $G_{L,b}$) and the ribbon group $H(R)$ of $L(b)$ (or $\hat{L}(b)$) defined by Hillman [7]?

5. Proof of Theorems 1 and 2

In this paragraph we prove Theorems 1 and 2 by solving the equivalent algebraic problem. By 2.5, 3.1 and 3.15 we know that Theorem 1 is equivalent to

Theorem 5.1. *For $w \in F$ the following two assertions are equivalent:*

- i) *The trivial word 1 is in the \widetilde{CA} -orbit of w .*
- ii) *The pattern $[x_1, \bar{w}x_2, x_3, \dots, x_r]$ contains (x_1, \dots, x_r)*

Then Theorem 2 follows from Theorem 1 and 2.7.b.

Proof of 5.1. The claim that i) implies ii) is obvious from the geometric picture (compare the remark following 2.1)

For the converse we have to recall Whitehead's algorithm [19] to decide when a pattern contains (x_1, x_2, \dots, x_r) . To each collection of words (w_1, w_2, \dots, w_r) in F there is associated a graph $\Gamma = \Gamma_w$ as follows: The graph has $2r$ vertices labelled by the generators x_1, \dots, x_r and their inverses $x_1^{-1}, \dots, x_r^{-1}$. The edges of Γ are given by the cyclic words w_k (compare [12]): For each occurrence $x_i x_j$ in some word w_k , we have an edge joining x_i to x_j^{-1} in Γ . Moreover, if some w_k begins with x_j and ends with x_i , we also have an edge joining x_i to x_j^{-1} .

A vertex v of Γ is called a cut vertex, if $\Gamma = \Gamma_1 \cup v \cup \Gamma_2$, where $\Gamma_1 \cup v \neq \Gamma_1$ and $\Gamma_2 \cup v \neq \Gamma_2$ are subgraphs, v is not isolated in $\Gamma_1 \cup v$ or $\Gamma_2 \cup v$, and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let Γ_1 be chosen, such that $v^{-1} \notin \Gamma_1$.

Lemma 5.2. (Whitehead [W]). *Given $\{w_1, \dots, w_r\} \neq \{x_1, \dots, x_r\} \subset F$. If $[w_1, \dots, w_r]$ contains (x_1, \dots, x_r) , then the graph Γ_w contains a cut vertex.*

In fact, for each cut vertex of $\Gamma = \Gamma_w$ Whitehead describes an automorphism of F (called a *simple* automorphism), which reduces the length of (w_1, \dots, w_r) (which is defined by the sum of the lengths of the cyclic words w_i) as follows: Each generating element $w \in \{x_i, x_i^{-1}\} \cap \Gamma_1$, such that $w^{-1} \notin \Gamma_1$, is mapped to wv (and w^{-1} is mapped to $v^{-1}w^{-1}$). If w and w^{-1} are vertices of Γ_1 , then w is mapped to $v^{-1}wv$.

Now we assume that $[x_1, \bar{w}x_2, x_3, \dots, x_r]$ contains a basis, but 1 is not in the \widetilde{CA} -orbit of w . We can choose w of minimal length $|w| > 1$ under the \widetilde{CA} -action. In particular, by 2.7.a there is an occurrence of $x_1^{\pm 1}$ and an occurrence of $x_2^{\pm 1}$ in w . Also, w does not begin with $x_1^{\pm 1}$ and does not end with $x_2^{\pm 1}$. So the graph Γ associated to $(x_1, \bar{w}x_2, x_3, \dots, x_r)$ has to contain at least the following edges:

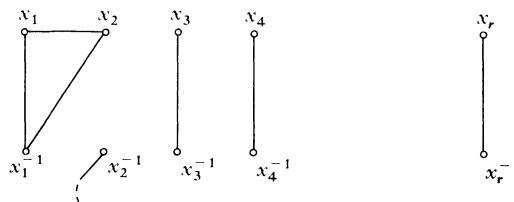


Figure 6

By 5.2 there has to be a cut vertex and a corresponding simple automorphism

determined by that vertex. There are essentially five different possible cases for v , namely $v=x_1$, $v=x_1^{-1}$, $v=x_2$, $v=x_2^{-1}$ or $v=x_j^{\pm 1}$ for some $j \geq 2$. In each case we will find a sequence of special automorphisms, whose action will strictly reduce the length of w in contradiction to the minimal choice.

The important observation is that for each $j > 2$: $x_j \in \Gamma_1$ is equivalent to $x_j^{-1} \in \Gamma_1$. Assume that $v=x_1$. Then $x_1^{-1} \notin \Gamma_1$, $x_2 \notin \Gamma_1$. Assume that $x_2^{-1} \notin \Gamma_1$. Then the resulting simple automorphism conjugates a set of x_j 's for $j > 2$ by x_1 and leaves fixed x_2 . Since the total length of (w_1, \dots, w_r) has to be reduced, the length of $\bar{w}x_2$ has to be reduced. But the corresponding sequence of special automorphisms applied to w would have to reduce its length, which is a contradiction. Now we assume that $x_2^{-1} \in \Gamma_1$, so in addition we have to map x_2 to $x_1^{-1}x_2$. The corresponding simple automorphism applied to $\bar{w}x_2$ leads to

$$w(x_1, x_1^{-1}x_2x_1x_2^{-1}x_1, \text{conjugates of } x_3, \dots, x_r) x_1^{-1}x_2$$

which has smaller length than $\bar{w}x_2$. Here we write $w(\alpha(x_1, \dots, x_r))$ for the image of $w \in F$ under the endomorphism α of F . The substitution above can be realized by the \tilde{CA} -action as follows: We have

$$\alpha_{12}^{-1} \cdot w = \overline{\alpha_{12}}^{-1}(w) x_1^{-1},$$

and then we can apply a sequence of α_{ij} 's with $i \neq j > 2$ as before conjugating some of the x_j 's for $j > 2$. The case $v=x_1^{-1}$ is similar and will be omitted.

Assume that $v=x_2$, so $x_2^{-1} \notin \Gamma_1$. Since w has minimal length there has to be an occurrence of x_1 in w . So there are segments in w as follows:

$$x_1^\varepsilon x_{j_1}^{\varepsilon_{j_1}} x_{j_2}^{\varepsilon_{j_2}} \dots x_{j_k}^{\varepsilon_{j_k}} x_2^\delta$$

with all $j_i > 2$, $\varepsilon, \delta \neq 0$, or

$$x_1^\varepsilon x_{j_1}^{\varepsilon_{j_1}} x_{j_2}^{\varepsilon_{j_2}} \dots x_{j_k}^{\varepsilon_{j_k}}.$$

This gives rise to segments

$$x_1^\varepsilon x_{j_1}^{\varepsilon_{j_1}} \dots x_{j_k}^{\varepsilon_{j_k}} x_2 x_1^\delta$$

resp.

$$x_1^\varepsilon x_{j_1}^{\varepsilon_{j_1}} \dots x_{j_k}^{\varepsilon_{j_k}} x_2$$

in \bar{w} .

This shows in both cases that $x_1, x_1^{-1} \notin \Gamma_1$. So the corresponding automorphism just conjugates certain x_j for $j > 2$, what we know can be realized by special automorphisms on w . From the graph follows that, if $v=x_2^{-1}$, then $x_2, x_1, x_1^{-1} \notin \Gamma_1$, so again just conjugation appears.

It remains to discuss the case $v=x_j$ for $j>2$.

There are four cases to consider:

- (i) $x_2, x_2^{-1} \notin \Gamma_1$ (this is easy)
- (ii) $x_2 \in \Gamma_1, x_2^{-1} \notin \Gamma_1$,
- (iii) $x_2 \notin \Gamma_1, x_2^{-1} \in \Gamma_1$
- (iv) $x_2, x_2^{-1} \in \Gamma_1$.

We only discuss (ii), the other cases are similar to prove. We know that $x_1, x_1^{-1} \in \Gamma_1$, so the corresponding simple automorphism maps x_2 to $x_2 x_j$ and x_1 to $x_j^{-1} x_1 x_j$, thus maps $\bar{w}x_2$ to

$$w(x_j^{-1} x_1 x_j, x_2 x_1 x_2^{-1}, \text{conjugates of } x_3, \dots, x_r) x_2 x_j$$

Now $\alpha_{1j}^{-1} \cdot w = x_j \overline{\alpha_{1j}}^{-1}(w)$. So, after having applied a sequence of α_{ij} , $i>2$, we have changed w to

$$w' = x_j w(x_j^{-1} x_1 x_j, x_2, \text{conjugates of } x_3, \dots, x_r)$$

But $\bar{w}'x_2$ is the same cyclic word as the image of $\bar{w}x_2$ under the simple automorphism above. Note that the simple automorphism applied to $\bar{w}x_2$ reduces the length by the number of occurrences of the form $x_j x_1 x_j^{-1}$. Even when w begins with x_j^{-1} , this would reduce the length of $\bar{w}x_2$. But then also w begins with x_j . So we can reduce the length of w by the \widetilde{CA} -action, in contradiction to our assumption. This proves 5.1. ■

We conclude with a discussion of the following question, which is “parallel” to Theorem 1:

Let L be an r -component unlink (or completely split link) and b a band for L . When is $\hat{L}(b)$ the unlink (or completely split)?

Recall that $L=(K_1, \dots, K_r)$ is *completely split* if there are disjoint $(n+2)$ -balls $B_i \subset S^{n+2}$, such that $B_i \supset K_i$ for $1 \leq i \leq r$.

Of course, if $\hat{L}(b)$ is not a boundary link, then $\hat{L}(b)$ cannot be completely split.

Corollary 5.3. *Let L be an r -component completely split link. If b is essential to L , then $\hat{L}(b)$ is not completely split.*

For $r=2$ and $n>2$ more can be said:

Proposition 5.4. *Let L be the 2-component unlink in S^{n+2} , $n>2$, and b a band for L . Then $\hat{L}(b)$ is the unlink if and only if b is inessential.*

Proof. It is sufficient to prove that $\hat{L}(b)$ is the unlink, when b is inessential. Let $L=(K_1, K_2)$ and B_1, B_2 be the canonical disks bounding K_1, K_2 . The band word with respect to this choice of Seifert surfaces is $x_1^r x_2^s$ for integers r, s . We can isotope the band to make the band word actually trivial. This means that

the intersections of β with B_1 resp. B_2 cancel algebraically taking account the order of the intersections with respect to B_1, B_2 . Let (P, Q) be a pair of points in B_1 (or B_2), such that $\beta(t_1)=P, \beta(t_2)=Q$ and $\beta(t_1, t_2) \cap (B_1 \cup B_2) = \emptyset$. We join P and Q in B_1 by some arc α . The embedded loop $\alpha \cup \beta[t_1, t_2]$ bounds a disk $D \subset S^{n+2}$, such that $D \cap (B_1 \cup B_2) = \alpha$. So the Whitney-trick can be applied to eliminate $\{P, Q\}$ and by induction all intersections $\beta \cap \text{int}(B_1 \cup B_2)$. Finally, we can isotope β into standard position. In particular $\beta \cap S$ is a single point for a separating $(n+1)$ -sphere S for K_1 and K_2 . Thus $\hat{L}(b)$ is the unlink. ■

For $n=1$ self-knotting and section-linking of β leads to additional phenomena as was pointed out in section 1. The easiest example of a nontrivial strong fusion of a 2-component unlink with inessential band is shown in Figure 7. In fact, a tedious but straightforward computation of the HOMFLY-polynomial shows that both $\hat{L}(b)$ and $L(b)$ are nontrivial (compare [8]).

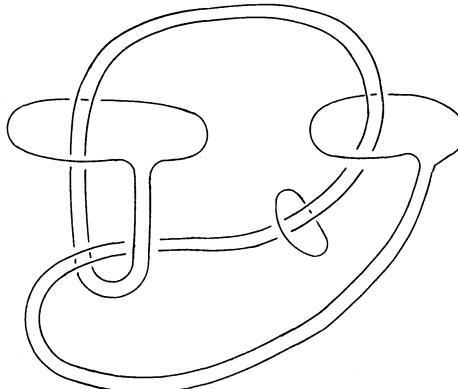


Figure 7

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