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Osaka J. Math.
13 (1976), 547–566

ON THE MSp HATTORI-STONG PROBLEM

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(Received September 22, 1975)

1. Introduction

In the present paper, we work in the category of CW-spectra due to Adams [1]. For any ring spectrum E, we denote by $E_*(\ )$ and $E^*(\ )$ the associated homology and cohomology functors and by E_* the coefficient ring. The unit of E is denoted by $u^E: S \to E$. Let E be a ring spectrum and F a spectrum. Consider the spectrum morphism

$$u^E \wedge 1: F = S \wedge F \to E \wedge F\,.$$

Then $u^E \wedge 1$ induces the generalized Hurewicz map

$$h^E = (u^E \wedge 1)_*: F_* \to E_*(F)\,.$$

For $E=H$, we denote h^E simply by h.

Ray [7] has conjectured that the Hurewicz map

(1.1) $\qquad h^{KO}: MSp_n \to KO_n(MSp)$

is a split monomorphism for any integer n and has shown that it is a split monomorphism for $n \leqq 20$. Later Segal [14] has shown that the map (1.1) is not a monomorphism for $n=31$ (since $MSp_{31} \cong Z_2$) and that Ray's MSp Hattori-Stong conjecture is false.

But still we may conjecture that the map

(1.2) $\qquad h^{KO}/\mathrm{Tors}: MSp_*/\mathrm{Tors} \to KO_*(MSp)/\mathrm{Tors}$

is a split monomorphism, where Tors denotes the torsion subgroup.

For any ring spectrum E, we put

$$W_*^E = \{x \in MSp_* \otimes \boldsymbol{Q};\ h^E(x) \in E_*(MSp)/\mathrm{Tors} \subset E_*(MSp) \otimes \boldsymbol{Q}\}\,.$$

Then $MSp_*/\mathrm{Tors} \subset W_*^E$. And the map (1.2) is a split monomorphism if and only if $MSp_*/\mathrm{Tors} = W_*^{KO}$.

Let L_* be a subring of $MSp_* \otimes \boldsymbol{Q}$. We put

$$Q(L_*) = L_*/(L_* \cap D_*)\,,$$

where D_* is the ideal of all decomposable elements in $MSp_*\otimes \boldsymbol{Q}$.

In this paper, we prove the following two theorems.

Theorem 1.1. *The inclusion* i: $MSp_*/\mathrm{Tors}\to W_*^{KO}$ *induces the isomorphism*

$$i_*\colon Q(MSp_*/\mathrm{Tors}) \cong Q(W_*^{KO})$$

(Cf. Proposition 3.12).

Theorem 1.2. *The Hurewicz map*

$$h^{KO}\colon MSp_n \to KO_n(MSp)$$

is a split monomorphism for $n\leqq 30$. *In particular, we have*

$$MSp_n/\mathrm{Tors} = W_n^{KO} \qquad \textit{for} \quad n<32\,.$$

The author wishes to express his hearty appreciation to Professor S. Araki.

2. Calculations in W_*^K and W_*^{KO}

We denote by ${}_ni\colon CP^n\to CP^\infty$ (resp. ${}_ni\colon HP^n\to HP^\infty$) the inclusion map. Let E be a ring spectrum having a class $x\in \tilde{E}^2(CP^\infty)$ (resp. $x\in\tilde{E}^4(HP^\infty)$) such that

$$E^*(CP^n) = E_*[{}_nx]/({}_nx^{n+1}) \text{ (resp. } E^*(HP^n) = E_*[{}_nx]/({}_nx^{n+1}))$$

for each integer $n\geqq 1$ and ${}_1x\in\tilde{E}^2(CP^1)=\tilde{E}^2(S^2)$ (resp. ${}_1x\in\tilde{E}^4(HP^1)=\tilde{E}^4(S^4)$) is represented by the unit u^E, where ${}_nx={}_ni^*(x)$. As is well known, x determines the Thom isomorphism $\phi\colon E_*(BU)\cong E_*(MU)$ (resp. $\phi\colon E_*(BSp)\cong E_*(MSp)$). Let $j\colon CP^\infty\to BU$ (resp. $j\colon HP^\infty\to BSp$) be the inclusion map and $y_i'\in E_*(CP^\infty)$ (resp. $y_i'\in E_*(HP^\infty)$) dual to x^i. Put $y_i=\phi j_*(y_i')$. Then we have

$$E_*(MU) = E_*[y_1, y_2, \cdots, y_i, \cdots]$$
$$\text{(resp. } E_*(MSp) = E_*[y_1, y_2, \cdots, y_i, \cdots])\,,$$

where $y_i\in E_{2i}(MU)$ (resp. $y_i\in E_{4i}(MSp)$).

In $\tilde{H}^2(CP^\infty)$, choose x to be c_1, the first Chern class of the universal $U(1)$-bundle ζ^1 over CP^∞. In this case, we denote y_i by b_i. Then we have

$$H_*(MU) = \boldsymbol{Z}[b_1, b_2, \cdots, b_i, \cdots],\ b_i\in H_{2i}(MU)\,.$$

In $\widetilde{MU}^2(CP^\infty)$, choose x to be cf_1, the first Conner-Floyd Chern class of ζ^1, represented by the homotopy equivalence $CP^\infty\simeq MU(1)$.

Let $z\in K_2$ be such that ${}_1i^*(\zeta^1-1)=z\gamma$ in $K^0(CP^1)$, where $\gamma\in\tilde{K}^2(CP^1)=\tilde{K}^2(S^2)$ is represented by the unit u^K. Then we have

$$K_* = \boldsymbol{Z}[z, z^{-1}] \quad\text{and}\quad H_*(K) = \boldsymbol{Q}[t, t^{-1}]\,,$$

where $t=h(z)$.

In $\tilde{K}^2(CP^\infty)$, choose x to be $z^{-1}(\zeta^1-1)$. As is well known, there is a unique ring spectrum morphism $g: MU\to K$ such that $g_*(cf_1)=z^{-1}(\zeta^1-1)$.

In $\tilde{H}^4(HP^\infty)$, choose x to be p_1, the first symplectic Pontrjagin class of the universal $Sp(1)$-bundle ξ^1 over HP^∞. In this case we denote y_i by q_i. Then we have

$$H_*(MSp) = \boldsymbol{Z}[q_1, q_2, \cdots, q_i, \cdots],\ q_i\in H_{4i}(MSp)\,.$$

In $\widetilde{MSp}^4(HP^\infty)$, choose x to be pf_1, the first Conner-Floyd symplectic Pontrjagin class of ξ^1, represented by the homotopy equivalence $HP^\infty\simeq MSp(1)$. In this case, we denote y_i by qf_i.

Put $\kappa_i=(gr)_*(qf_i)\in K_*(MSp)$, where $r: MSp\to MU$ is the morphism induced by the inclution $Sp\to U$. Then we have

$$K_*(MSp) = K_*[\kappa_1, \kappa_2, \cdots, \kappa_i, \cdots],\ \kappa_i\in K_{4i}(MSp)\,.$$

Let bu denote the connective BU-spectrum and $\psi: bu\to K$ the canonical morphism. Then we have

$$\psi_*: bu_n \cong K_n \quad \text{if} \quad n\geqq 0,\ bu_n = 0 \quad \text{if} \quad n<0\,.$$

And let $\tilde{\kappa}_i\in bu_*(MSp)$ be the unique class such that $\psi_*(\tilde{\kappa}_i)=\kappa_i\in K_*(MSp)$. Then we have

$$bu_*(MSp) = bu_*[\tilde{\kappa}_1, \tilde{\kappa}_2, \cdots, \tilde{\kappa}_i, \cdots]\,.$$

Therefore $\psi_*: bu_*(MSp)\to K_*(MSp)$ is a split monomorphism, so that we have

(2.1) $\qquad W_*^{bu} = W_*^K\,.$

Similarly we have

(2.2) $\qquad W_*^{bo} = W_*^{KO}\,,$

where bo denotes the connective BO-spectrum.

We have a Künneth isomorphism

$$H_*(\ )\otimes H_*(MSp) \cong H_*(\ \wedge MSp)$$

since $H_*(MSp)$ is torsion free. By this isomorphism we idenify $H_*(\ )\otimes H_*(MSp)$ and $H_*(\ \wedge MSp)$.

Lemma 2.1. Consider the commutative diagram

$$\begin{tikzcd} MSp_*\otimes \boldsymbol{Q} \arrow[r, "h", "\cong"'] \arrow[d, "h^K"] & H_*(MSp)\otimes \boldsymbol{Q} \arrow[d, "j"] \\ K_*(MSp)\otimes \boldsymbol{Q} \arrow[r, "h", "\cong"'] & H_*(K)\otimes H_*(MSp)\otimes \boldsymbol{Q}\,, \end{tikzcd}$$

where $j=(u^K\wedge 1)_*\colon H_*(MSp)\to H_*(K\wedge MSp)=H_*(K)\otimes H_*(MSp)$. Then we have

$$j(x)=1\otimes x$$

for any $x\in H_*(MSp)$ and

$$\begin{aligned} h(h^K(W_*^K)) &= h(K_*(MSp))\cap j(H_*(MSp)) \\ &= h(\boldsymbol{Z}[z^2, \kappa_1, \kappa_2, \cdots, \kappa_i, \cdots])\cap j(H_*(MSp)) . \end{aligned}$$

Proof. It is proven by diagram chasing that

$$j(x)=1\otimes x$$

for any $x\in H_*(MSp)$.

We have the following commutative diagram

$$\begin{tikzcd} MSp_*\otimes\boldsymbol{Q} \arrow[r, "h", "\simeq"'] \arrow[d, "h^{bu}"] \arrow[dd, bend right=60, "h^K"'] & H_*(MSp)\otimes\boldsymbol{Q} \arrow[d, "\tilde{j}"] \arrow[dd, bend left=60, "j"] \\ bu_*(MSp)\otimes\boldsymbol{Q} \arrow[r, "h", "\simeq"'] \arrow[d, "\psi_*"] & H_*(bu)\otimes H_*(MSp)\otimes\boldsymbol{Q} \arrow[d, "\psi_*\otimes 1"] \\ K_*(MSp)\otimes\boldsymbol{Q} \arrow[r, "h", "\simeq"'] & H_*(K)\otimes H_*(MSp)\otimes\boldsymbol{Q} , \end{tikzcd}$$

where $\tilde{j}=(u^{bu}\wedge 1)_*\colon H_*(MSp)\to H_*(bu\wedge MSp)=H_*(bu)\otimes H_*(MSp)$.

Now let $x\in W_*^K=W_*^{bu}$ (Cf. (2.1)). Then there is an integer $n\neq 0$ such that $nx\in MSp_*/\mathrm{Tors}$. We have

$$\begin{aligned} nh(h^{bu}(x)) &= h(h^{bu}(nx)) \\ &= \tilde{j}(h(nx))\in \tilde{j}(H_*(MSp)) . \end{aligned}$$

Since $\tilde{j}/\mathrm{Tors}\colon H_*(MSp)\to H_*(bu)/\mathrm{Tors}\otimes H_*(MSp)$ is a split monomorphism, $h(h^{bu}(x))\in\tilde{j}(H_*(MSp))$. Therefore we obtain

$$h(h^K(x))\in j(H_*(MSp)) .$$

By (2.1) and dimensional reason, we obtain

$$\begin{aligned} & h^K(x)\in\boldsymbol{Z}[z^2, \kappa_1, \kappa_2, \cdots, \kappa_i, \cdots] , \\ & h(h^K(x))\in h(\boldsymbol{Z}[z^2, \kappa_1, \kappa_2, \cdots, \kappa_i, \cdots]) . \end{aligned}$$

Conversly let $y\in K_*(MSp)$ and $h(y)\in j(H_*(MSp))$. Then

$$h(y)\in j(h(MSp_*\otimes\boldsymbol{Q}))=h(h^K(MSp_*\otimes\boldsymbol{Q})) ,$$

so that $y\in h^K(MSp_*\otimes\boldsymbol{Q})$. Consequently we obtain

$$y \in h^K(W_*^K),\ h(y) \in h(h^K(W_*^K))\,.$$

Corollary 2.2. $h(W_*^K) \subset H_*(MSp)$.

It is well known that

$$g_*(b_i) = t^i/(i+1)!\,, \tag{2.3}$$

where $g_*\colon H_*(MU) \to H_*(K)$. And we have

Lemma 2.3.

$$(gr)_*(q_i) = 2t^{2i}/[2(i+1)]!\,,$$

where $(gr)_*\colon H_*(MSp) \to H_*(K)$.

Proof. We have

$$r_*(q_i) = 2[b_{2i} - b_1 b_{2i-1} + \cdots + (-1)^{i-1} b_{i-1} b_{i+1}] + (-1)^i b_i^2\,,$$

so that the lemma follows immediately from (2.3).

Consider the commutative diagram

$$\begin{tikzcd}
MSp_*(MSp) \arrow[r, "h"] \arrow[d, "(gr)_*"] & H_*(MSp) \otimes H_*(MSp) \arrow[d, "(gr)_* \otimes 1"] \\
K_*(MSp) \arrow[r, "h"] & H_*(K) \otimes H_*(MSp)\,.
\end{tikzcd}$$

By definition, $(gr)_*(qf_i) = \kappa_i$. Therefore we have

$$h(\kappa_i) = (gr)_* \otimes 1(h(qf_i))\,,$$

so that, by Ray [9], (5·6) and Lemma 2.3, we can calculate the Hurewicz map

$$h\colon K_*(MSp) \to H_*(K) \otimes H_*(MSp)\,.$$

Therefore, by Lemma 2.1 and the fact that $h^K\colon W_*^K \to K_*(MSp)$ is a monomorphism, we obtain

Proposition 2.4. *W_*^K is generated by elements*

$$x_i (1 \leqq i \leqq 7),\ y_4,\ y_6 \quad \textit{and} \quad y_7$$

in dimensions <32, *where* x_i $(1 \leqq i \leqq 6)$ *are defined by*

$$h^K(x_1) = z^2+12\kappa_1\,,$$
$$h^K(x_2) = z^2\kappa_1-4\kappa_1^2+10\kappa_2\,,$$
$$h^K(x_3) = z^2(-3\kappa_1^2+4\kappa_2)+12\kappa_1^3-36\kappa_1\kappa_2+28\kappa_3\,,$$
$$h^K(x_4) = z^2(\kappa_1^3-2\kappa_1\kappa_2+\kappa_3)-4\kappa_1^4+14\kappa_1^2\kappa_2-4\kappa_2^2-12\kappa_1\kappa_3+6\kappa_4\,,$$
$$\begin{aligned} h^K(x_5) = {} & z^2(-7\kappa_1^4+18\kappa_1^2\kappa_2-4\kappa_2^2-11\kappa_1\kappa_3+4\kappa_4) \\ & +28\kappa_1^5-112\kappa_1^3\kappa_2+66\kappa_1\kappa_2^2+96\kappa_1^2\kappa_3-38\kappa_2\kappa_3-62\kappa_1\kappa_4+22\kappa_5\,, \end{aligned}$$
$$\begin{aligned} h^K(x_6) = {} & z^4(-2\kappa_1^4+5\kappa_1^2\kappa_2-\kappa_2^2-3\kappa_1\kappa_3+\kappa_4) \\ & +z^2(-3\kappa_1^5-10\kappa_1^3\kappa_2+24\kappa_1\kappa_2^2+13\kappa_1^2\kappa_3-18\kappa_2\kappa_3-14\kappa_1\kappa_4+8\kappa_5) \\ & +44\kappa_1^6-150\kappa_1^4\kappa_2+15\kappa_1^2\kappa_2^2+25\kappa_2^3+140\kappa_1^3\kappa_3+36\kappa_1\kappa_2\kappa_3-12\kappa_3^2-84\kappa_1^2\kappa_4 \\ & -45\kappa_2\kappa_4+18\kappa_1\kappa_5+13\kappa_6 \end{aligned}$$

and

$$y_4 = (-x_2^2+x_1x_3)/4,\ y_6 = (-x_2x_4+x_1x_5)/2 \ \textit{and}\ y_7 = (-x_3x_4+x_2x_5)/2\,.$$

And we have

Lemma 2.5. Let $x\in W_*^K$, and

$$h^K(x) = f(z, \kappa_1, \kappa_2, \cdots, \kappa_i, \cdots)\in \boldsymbol{Z}[z, \kappa_1, \kappa_2, \cdots, \kappa_i, \cdots]\,.$$

Then

$$h(x) = f(0, q_1, q_2, \cdots, q_i, \cdots)\in H_*(MSp)\,.$$

For example,

$$h(x_1) = 12q_1\,,$$
$$h(x_2) = -4q_1^2+10q_2\,,$$
$$h(x_3) = 12q_1^3-36q_1q_2+28q_3\,,$$
$$h(x_4) = -4q_1^4+14q_1^2q_2-4q_2^2-12q_1q_3+6q_4\,.$$

Proof. Notice that

$$h(\kappa_i) \equiv 1\otimes q_i \bmod t\otimes 1 \quad \text{in} \quad \boldsymbol{Q}[t]\otimes H_*(MSp)$$

where $h\colon K_*(MSp)\to H_*(K)\otimes H_*(MSp)$. Then the lemma follows from Lemma 2.1.

Let $c\colon KO\to K$ be the complexification morphism. As is well known, KO_* is generated by the classes

$$e\in KO_1,\ x\in KO_4,\ y\in KO_8 \quad \text{and} \quad y^{-1}\in KO_{-8}$$

subject to the relations

$$2e = e^3 = ex = 0,\ x^2 = 4y \quad \text{and} \quad yy^{-1} = 1$$

such that

$$c_*(x) = 2z^2 \quad \text{and} \quad c_*(y) = z^4 \quad \text{in} \quad K_* .$$

Let $\sigma_i \in KO_{4i}(MSp)$ be the unique class such that $c_*(\sigma_i) = \kappa_i \in K_{4i}(MSp)$. Then we have

$$KO_*(MSp) = KO_*[\sigma_1, \sigma_2, \cdots, \sigma_i, \cdots] ,$$

and

(2.4) $\qquad W_*^{KO} \subset W_*^K .$

As a corollary to Proposition 2.4, we obtain

Proposition 2.6. *W_{4k}^{KO} has the following generators for $k \leqq 7$.*

$k = 1$: $2x_1$.
$k = 2$: $x_1^2,\ 2x_2$.
$k = 3$: $2x_1^3,\ x_1x_2,\ 2x_3$.
$k = 4$: $x_1^4,\ 2x_1^2x_2,\ x_1x_3,\ 2y_4,\ 2x_4$.
$k = 5$: $2x_1^5,\ x_1^3x_2,\ 2x_1^2x_3,\ 2x_1y_4,\ x_2x_3,\ x_1x_4,\ 2x_5$.
$k = 6$: $x_1^6,\ 2x_1^4x_2,\ x_1^3x_3,\ 2x_1x_2x_3,\ 2x_2y_4,\ x_3^2,\ 2x_1^2x_4$,
$\quad x_1x_2x_3 + x_1^2(y_4 + x_4),\ x_2x_4,\ x_1x_5,\ 2x_6$.
$k = 7$: $2x_1^7,\ x_1^5x_2,\ 2x_1^4x_3,\ 2x_1^3y_4,\ x_1^2x_2x_3,\ 2x_1x_3^2,\ 2x_3y_4,\ x_1^3x_4$,
$\quad x_1x_3^2 + x_1x_2(y_4 + x_4),\ x_3x_4,\ 2x_1^2x_5,\ x_1y_6,\ x_2x_5,\ 2x_1x_6,\ \tilde{x}_7$.

REMARK.

$$W_*^{MSU} = W_*^{KO},\ h^{MSU}(W_*^{MSU}) = H\text{-}Sp_* ,$$

where $H\text{-}Sp_*$ is the algebra of Ray [10], (2・1), and

$$h(2x_i) = h_i \in H_*(MSp)$$

for $i \leqq 4$, where h_i are the classes in [10], (3・7) (Cf. Lemma 2.5)

3. Adams spectral sequence maps

For any connective spectrum X such that X_r is finitely generated for each r, we denote by $E_*^{**}(X)$ the mod 2 Adams spectral sequence for X_* (Cf. [3], 2.2). For an integer n, we denote by F^sX_n the s-th filtration in the mod 2 Adams spectral sequence. Then we have

$$F^sX_n/F^{s+1}X_n = E_\infty^{s,s+n}(X) = E_r^{s,s+n}(X) \quad (r \text{ large})$$

Let H be a graded vector space over $\boldsymbol{Z}_2$. We define a graded vector space H' from H by

$$H'_{2n} = H_n,\ H'_{2n+1} = 0$$

for any integer n. For any connected Hopf algebra H over $\boldsymbol{Z}_2$, we denote the augmentation ideal $\sum_{i>0} H_i$ by $\bar{H}$.

We denote the mod 2 Steenrod Algebra by A. Let A'' be endowed with structure as a graded A-module by the following A-action.

$$A \otimes A'' \xrightarrow{\beta \otimes 1} A'' \otimes A'' \xrightarrow{\mu} A''.$$

Here β: $A \to A''$ is the map such that $\beta^*(x) = x^4 \in A^*$ for any $x \in A''^*$ and μ is the product map in A. Using the notation of Milnor [4], we denote $(\zeta_{j+1}^m)^*$ by m_j for any integers $m, j \geqq 0$. For any n $(0 \leqq n \leqq \infty)$, let $B(n)$ be the Hopf subalgebra of A (multiplicatively) generated by the elements 1_0, 2_j for $j < n$. The map β induces the isomorphism

(3.1) $\quad A//B \cong A''$,

where $B = B(\infty)$.

Let R be a Hopf subalgebra of A, and (C, d_C, ε_C) a R-free resolution of $\boldsymbol{Z}_2$. As is well known, A is free as a right R-module and we have the isomorphism $A/A\bar{R} \cong A \underset{R}{\otimes} \boldsymbol{Z}_2$ of A-modules. So we obtain

Lemma 3.1. There is an A-free resolution of $A/A\bar{R}$:

$$A/A\bar{R} \xleftarrow{1 \otimes \varepsilon_C} A \underset{R}{\otimes} C_0 \xleftarrow{1 \otimes d_C} A \underset{R}{\otimes} C_1 \xleftarrow{1 \otimes d_C} \cdots \xleftarrow{1 \otimes d_C} A \underset{R}{\otimes} C_i \xleftarrow{1 \otimes d_C} \cdots.$$

The following proposition is well known.

Proposition 3.2.

(1) (Serre [15]) $(H\boldsymbol{Z}_2)^*(H) \cong A/A\overline{B(0)}$

as graded A-modules.

(2) (Cf. [1], §16) $(H\boldsymbol{Z}_2)^*(bo) \cong A/A\overline{B(1)}$

as graded A-modules.

(3) (Cf. [3], THEOREM II. 4) $(H\boldsymbol{Z}_2)^*(MSp) \cong A'' \otimes S''$

as graded coalgebra and A-modules (A oprating on S'' trivially), where S is the graded coalgebra over $\boldsymbol{Z}_2$ such that

$$S^* \cong \boldsymbol{Z}_2[V_2, V_4, V_5, \cdots, V_i, \cdots],\ i \neq 2^a - 1,\ \deg V_i = i.$$

As a result of Proposition 3.2, the following proposition is obtained by (3.1) and Lemma 3.1.

Proposition 3.3.

(1) $E_2(H) \cong Ext_{B(0)}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$.

(2) $E_2(bo) \cong Ext_{B(1)}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$.

(3) $E_2(MSp) \cong Ext_B(\boldsymbol{Z}_2, \boldsymbol{Z}_2) \otimes \boldsymbol{Z}_2[v_2, v_4, v_5, \cdots, v_i, \cdots]$,
$$i \neq 2^a-1,\ v_i=[V_i] \in E_2^{0,4i}(MSp).$$

A $B(n)$-free resolution of $\boldsymbol{Z}_2$ has been constructed by Liulevicius [3]. Let $Y(n)$ be the $\boldsymbol{Z}_2$-vector space with basis

$$\left\{ I\otimes J;\ \begin{matrix} I=(i_0, i_1, \cdots, i_{n-1}),\ J=(j_0, j_1, \cdots, j_n), \text{ where } I, J \text{ are} \\ \text{sequences of non-negative, finitely non-zero integers.} \end{matrix} \right\}.$$

Let

$$\deg I\otimes J = (\textstyle\sum (i_r+j_r),\ \sum [i_r(2^{r+2}-2)+j_r(2^{r+1}-1)]).$$

We define a $B(n)$-homomorphism $d(n)$: $B(n)\otimes Y(n) \to B(n)\otimes Y(n)$ by

$$\begin{aligned} d(n)(I\otimes J) = \sum_k [& 1_k I\otimes(J-\Delta_k)+2_k(I-\Delta_k)\otimes J \\ & +(j_{k+1}+1)(I-\Delta_k)\otimes(J-\Delta_0+\Delta_{k+1}) \\ & +(j_{k+1}+1)1_0(I-\Delta_0-\Delta_k)\otimes(J+\Delta_{k+1}) \\ & +\binom{j_{k+1}+2}{2}(I-\Delta_0-2\Delta_k)\otimes(J+2\Delta_{k+1})] \\ + \sum_{k<t} & (j_{k+1}+1)(j_{t+1}+1)(I-\Delta_0-\Delta_k-\Delta_t)\otimes(J+\Delta_{k+1}+\Delta_{t+1}). \end{aligned}$$

Here we set $I-\Delta_r=0$ if $i_r=0$ and $J-\Delta_r=0$ if $j_r=0$. Then

$$B(n)\otimes Y(n) = (B(n)\otimes Y(n),\ d(n),\ \varepsilon(n))$$

is the $B(n)$-free resolution of $\boldsymbol{Z}_2$ constructed by him, where $\varepsilon(n)$: $B(n)\otimes Y(n)_0 \to \boldsymbol{Z}_2$ is the unique $B(n)$-homomorphism. Put

$$\langle J\rangle = (0)\otimes J.$$

Then we have

$$d(n)\langle J\rangle = \textstyle\sum 1_k\langle J-\Delta_k\rangle.$$

Using the notation of [3] for $Hom_{B(n)}(B(n)\otimes Y(n), \boldsymbol{Z}_2) = Y(n)^*$, let

$$\begin{aligned} k_j &= [x_j] \in Ext_{B(n)}^{1,2^{j+2}-2}(\boldsymbol{Z}_2, \boldsymbol{Z}_2), \\ q_0 &= [y_0] \in Ext_{B(n)}^{1,1}(\boldsymbol{Z}_2, \boldsymbol{Z}_2), \\ \tau_j &= [y_0 y_{j+1}^2 + x_0 x_j y_{j+1}] \in Ext_{B(n)}^{3,2^{j+3}-1}(\boldsymbol{Z}_2, \boldsymbol{Z}_2), \\ \omega_0 &= [y_1^4] \in Ext_{B(n)}^{4,12}(\boldsymbol{Z}_2, \boldsymbol{Z}_2). \end{aligned}$$

Proposition 3.4. (Liulevicius [3])

(1) $Ext_{B(0)}(Z_2, Z_2) = Z_2[q_0]$.

(2) $Ext_{B(1)}(Z_2, Z_2)$ *has multiplicative generators* q_0, k_0, τ_0 *and* ω_0 *with bidegrees* (1,1), (1,2), (3,7) *and* (4,12) *respectively subject to the relations*

$$q_0k_0 = 0,\ k_0^3 = 0,\ k_0\tau_0 = 0 \quad \text{and} \quad \tau_0^2 = q_0^2\omega_0 .$$

Corollary 3.5.

(1) $E_\infty(H) = E_2(H)$.

(2) $E_\infty(bo) = E_2(bo)$.

Lemma 3.6. For any integer n, there is an integer $s_0=s_0(n)$ such that

$$Ext_B^{s,s+n}(Z_2, Z_2) = (Z_2[q_0, \{\tau_j\}])^{s,s+n} \qquad \text{if} \quad s \geqq s_0 .$$

Proof. Let $\tilde{B}(m)$ be the Hopf subalgebra of B (multiplicatively) generated by $B(m)$, 1_{m+1} $(0 \leqq m < \infty)$. By Segal [12], PROPOSITION 2.3, there is a spectral sequence ${}_mE_*^{***}$ such that

$$ {}_mE_1 = Ext_{\tilde{B}(m)}(Z_2, Z_2) \otimes F(\Omega^*) \ (\Omega = B(m+1)//\tilde{B}(m)) ,$$
$$ ({}_mE_\infty)^{s,t} \cong Ext^{s,t}_{B(m+1)}(Z_2, Z_2) .$$

Since $\Omega = E_{Z_2}[k_m']$, $k_m' = [2_m]$, we have $F(\Omega^*) = Z_2[k_m]$, $\deg k_m = (1, 2^{m+2}-2)$. And $Ext_{\tilde{B}(m)}(Z_2, Z_2) = Ext_{B(m)}(Z_2, Z_2) \otimes Z_2[q_{m+1}]$, $\deg q_{m+1} = (1, 2^{m+2}-1)$. Therefore

$$ {}_mE_1 = Ext_{B(m)}(Z_2, Z_2) \otimes Z_2[k_m] \otimes Z_2[q_{m+1}] .$$

Then we have

$$ d_1(q_{m+1}) = q_0 k_m $$

and all d_r in ${}_mE$ are trivial on $Ext_{B(m)}(Z_2, Z_2) \otimes Z_2[k_m]$ (Cf. [12]).

Now we prove by induction on m that there is an integer $s_0 = s_0(n,m)$ such that

$$Ext^{s,s+n}_{B(m)}(Z_2, Z_2) = (Z_2[q_0, \{\tau_j ; j \leqq m-1\}])^{s,s+n} \qquad \text{if} \quad s \geqq s_0 .$$

For $m=0$, it is true by Proposition 3.4, (1). Assume that it is true for m. Since $\deg q_{m+1} = (1, 1+(2^{m+2}-2))$, $2^{m+2}-2 \geqq 1$ and $\deg k_m = (1, 1+(2^{m+2}-3))$, $2^{m+2}-3 \geqq 1$, there is an integer $s_0' = s_0'(n, m)$ such that

$$({}_mE_2)^{s,s+n} = (Z_2[q_0, \{\tau_j ; j \leqq m-1\}, q_{m+1}^2])^{s,s+n} \qquad \text{if} \quad s \geqq s_0' .$$

Clearly there is an integer $s_0'' = s_0''(n, m) \geqq s_0'$ such that

$$({}_mE_2)^{s,s+n} = (Z_2[q_0, \{\tau_j ; j \leqq m-1\}, q_0 q_{m+1}^2])^{s,s+n} \qquad \text{if} \quad s \geqq s_0'' .$$

$q_0 q_{m+1}^2$ is a permanent cycle and τ_m is represented by $q_0 q_{m+1}^2$. Put $s_0(n, m+1)= s_0''(n, m)$ then

$$Ext_{B(m+1)}^{s,s+n}(Z_2, Z_2) = (Z_2[q_0, \{\tau_j; j\leqq m\}])^{s,s+n} \qquad \text{if} \quad s\geqq s_0(n, m+1)\,.$$

From the fact that $Ext_B^{s,s+n}(Z_2, Z_2)\cong Ext_{B(m)}^{s,s+n}(Z_2, Z_2)$ if $2^{m+2}-3>n$, the lemma follows.

Let

$$G = {}_mG = A/\overline{AB(m)}\otimes(HZ_2)^*(MSp) = (HZ_2)^*({}_mM)\otimes(HZ_2)^*(MSp)$$

(A operating on $(HZ_2)^*(MSp)$ trivially), where $m=0$ or 1 and ${}_0M=H$, ${}_1M=bo$. And we define a map

$$\Phi = {}_m\Phi\colon G\to(HZ_2)^*(M\wedge MSp) \quad (M = {}_mM)$$

by $\Phi([a]\otimes u)=\sum [a_i']\cdot a_i''u$ for $a\in A$, $u\in(HZ_2)^*(MSp)$, where $\psi(a)= \sum a_i'\otimes a_i''$. Then we have

Lemma 3.7. (Cf. [1], §16) Φ is an isomorphism of graded coalgebras and A-modules.

We identify G and $(HZ_2)^*(M\wedge MSp)$ by Φ.

Corollary 3.8.

(1) $E_2(H\wedge MSp)=Z_2[q_0, v_1, v_2, \cdots, v_i, \cdots]$.

(2) $E_2(bo\wedge MSp)=E_2(bo)\otimes Z_2[v_1, v_2, \cdots, v_i, \cdots]$.

Here $v_i\in E_2^{0,4i}({}_mM \wedge MSp)$, where

$$v_i = [\zeta_j] \quad \text{if} \quad i = 2^j-1,\ v_i = [V_i] \quad \text{if} \quad i\neq 2^a-1$$

$((HZ_2)_*(MSp) = A''^*\otimes S''^*)$.

Corollary 3.9.

(1) $E_\infty(H\wedge MSp)=E_2(H\wedge MSp)$.

Therefore we have

$$F^sH_n(MSp) = \{x\in H_n(MSp);\ 2^s\,|\,x\}\,.$$

(2) $E_\infty(bo\wedge MSp)=E_2(bo\wedge MSp)$.

Lemma 3.10. For any $u\in(HZ_2)^*(MSp)$, we have

$$(u^M\wedge 1)^*(1\otimes u) = u\,,$$

where $(u^M\wedge 1)^*\colon G\to(HZ_2)^*(MSp)$.

Proof. For any $v\in(HZ_2)_*(MSp)$, we can prove by diagram chaising that

$$(u^M\wedge 1)_*(v)=1\cdot v\in(HZ_2)_*(M\wedge MSp)\,.$$

Therefore we have

$$(u^M\wedge 1)^*(1\cdot u)=u$$

for any $u\in(HZ_2)^*(MSp)$, where $(u^M\wedge 1)^*\colon (HZ_2)^*(M\wedge MSp)\to(HZ_2)^*(MSp)$. Since $\Phi^{-1}(1\cdot u)=1\otimes u$, the lemma follows.

For any ring spectrum X and any spectrum Y, $u^X\wedge 1\colon Y\to X\wedge Y$ induces the spectral sequence map

$$h^X\colon E_*^{**}(Y)\to E_*^{**}(X\wedge Y)\,.$$

For $X=H$, we denote h^X simply by h.

Lemma 3.11.

(1-a) $h(v_i)=v_i$ if $i\neq 2^a-1$.

(1-b) $h(Ext_B(Z_2, Z_2))$ is contained in the ring

$${}_0R=Z_2[q_0, v_1, v_3, \cdots, v_{2^a-1},\cdots]\,.$$

(1-c) $h(\tau_j)=q_0^3(v_{2^{j+1}-1}+$demcoposables in $Z_2[v_1, v_3, \cdots, v_{2^a-1}, \cdots])\in{}_0R$.

(2-a) $h^{bo}(v_i)=v_i$ if $i\neq 2^a-1$.

(2-b) $h^{bo}(Ext_B(Z_2, Z_2))$ is contained in the ring

$${}_1R=Ext_{B(1)}(Z_2, Z_2)\otimes Z_2[v_1, v_3, \cdots, v_{2^a-1}, \cdots]\,.$$

(2-c) $h^{bo}(\tau_j)=\tau_0(v_{2^j-1}^2+$other terms in $Z_2[v_1, v_3, \cdots, v_{2^j-1}])+q_0^3(v_{2^{j+1}-1}+$decomposables in $Z_2[v_1, v_3, \cdots, v_{2^a-1}, \cdots])\in{}_1R$,
where $v_0=1$.

(2-c′) Let $u\in(Z_2[q_0, \{\tau_a\}])^{s,t}\subset Ext_B^{s,t}(Z_2, Z_2)$. Then we have

$$h^{bo}(u)\in Z_2[q_0, \tau_0, \{v_{2^a-1}\}]$$

and

$$h^{bo}(u)\notin Z_2[q_0, \{v_{2^a-1}\}] \quad \text{if} \quad u\notin Z_2[q_0]\,.$$

(2-d) $h^{bo}(k_j)=k_0(v_{2^j-1}+$decomposables in $Z_2[v_1, v_3, \cdots, v_{2^a-1}, \cdots])\in{}_1R$.

Proof. We porve only (2). We can prove (1) in the same way. Applying Lemma 3.1 to the resolution $B(n)\otimes Y(n)$, we obtain an A-free resolurion of $A/A\overline{B(n)}$:

$$A/A\overline{B(n)}\xleftarrow{\varepsilon}A\xleftarrow{d}A\otimes Y(n)_1\xleftarrow{d}\cdots\xleftarrow{d}A\otimes Y(n)_s\xleftarrow{d}\cdots\,.$$

Then $(A\otimes Y(1)\otimes A''\otimes S'', d\otimes 1\otimes 1, \varepsilon\otimes 1\otimes 1)$ is an A-free resolution of

$$ {}_1G = A/A\overline{B(1)}\otimes A''\otimes S'' $$

and $(A\otimes Y(\infty)\otimes S'', d\otimes 1, \beta\otimes 1)$ an A-free resolution of

$$ (H\boldsymbol{Z}_2)^*(MSp) = A''\otimes S''\,. $$

We can define an A-homomorphism $f_s\colon A\otimes Y(1)_s\otimes A''\to A\otimes Y(\infty)_s$ for each $s\geqq 0$ such that

$$ \{f_s\otimes 1;\ A\otimes Y(1)_s\otimes A''\otimes S''\to A\otimes Y(\infty)_s\otimes S''\} $$

is a homomorphism of A-free resolutions, that is,

$$ (u^{bo}\wedge 1)^*(\varepsilon\otimes 1\otimes 1) = (\beta\otimes 1)(f_0\otimes 1) $$

and $\qquad (f_s\otimes 1)(d\otimes 1\otimes 1) = (d\otimes 1)(f_{s+1}\otimes 1) \qquad$ for any $s\geqq 0$,

where $(u^{bo}\wedge 1)^*\colon A/A\overline{B(1)}\otimes A''\otimes S''\to A''\otimes S''$ (Cf. Lemma 3.10). Partial construction of $\{f_s\}$ is given as the following ((∘)~(iii), (i′)).

(∘) For $(\zeta_1^{n_1}\zeta_2^{n_2}\cdots\zeta_j^{n_j}\cdots)^*\in A'' = Y(1)_0\otimes A''$,

$$ f_0[(\zeta_1^{n_1}\zeta_2^{n_2}\cdots\zeta_j^{n_j}\cdots)^*] = (\zeta_1^{4n_1}\zeta_2^{4n_2}\cdots\zeta_j^{4n_j}\cdots)^*\in A = A\otimes Y(\infty)_0\,. $$

(i) $f_1(\langle\Delta_0\rangle\otimes 2_{j-1}) = 8_{j-1}\langle\Delta_0\rangle + 6_{j-1}\langle\Delta_j\rangle$ for $j\geqq 2$,
$f_1(\langle\Delta_0\rangle\otimes 2_0) = 8_0\langle\Delta_0\rangle + 6_0\langle\Delta_1\rangle + 2_0\langle\Delta_2\rangle$,
$f_1(\langle\Delta_1\rangle\otimes 2_{j-1}) = 8_{j-1}\langle\Delta_1\rangle + 4_{j-1}\langle\Delta_{j+1}\rangle$,
$f_1(\langle\Delta_0\rangle\otimes 1_j) = 4_j\langle\Delta_0\rangle + 2_j\langle\Delta_{j+1}\rangle$.

(ii) $f_2(\langle\Delta_0+\Delta_1\rangle\otimes 2_{j-1}) = 8_{j-1}\langle\Delta_0+\Delta_1\rangle + 6_{j-1}\langle\Delta_1+\Delta_j\rangle + 4_{j-1}\langle\Delta_0+\Delta_{j+1}\rangle$ $+ 2_{j-1}\langle\Delta_j+\Delta_{j+1}\rangle$ for $j\geqq 2$,
$f_2(\langle\Delta_0+\Delta_1\rangle\otimes 2_0) = 8_0\langle\Delta_0+\Delta_1\rangle + 4_0\langle\Delta_0+\Delta_2\rangle$,
$f_2(\langle 2\Delta_1\rangle\otimes 2_{j-1}) = 8_{j-1}\langle 2\Delta_1\rangle + 4_{j-1}\langle\Delta_1+\Delta_{j+1}\rangle + \langle 2\Delta_{j+1}\rangle$,
$f_2(\langle 2\Delta_0\rangle\otimes 1_j) = 4_j\langle 2\Delta_0\rangle + 2_j\langle\Delta_0+\Delta_{j+1}\rangle + \langle 2\Delta_{j+1}\rangle$.

(iii) $f_3(\langle\Delta_0+2\Delta_1\rangle\otimes 2_{j-1}) = 8_{j-1}\langle\Delta_0+2\Delta_1\rangle + 6_{j-1}\langle 2\Delta_1+\Delta_j\rangle$ $+ 4_{j-1}\langle\Delta_0+\Delta_1+\Delta_{j+1}\rangle + 2_{j-1}\langle\Delta_1+\Delta_j+\Delta_{j+1}\rangle + \langle\Delta_0+2\Delta_{j+1}\rangle$,
$f_3(\langle 3\Delta_0\rangle\otimes 1_j) = 4_j\langle 3\Delta_0\rangle + 2_j\langle 2\Delta_0+\Delta_{j+1}\rangle + \langle\Delta_0+2\Delta_{j+1}\rangle$.

(i′) $f_1([\Delta_0\otimes(0)]\otimes 1_{j-1}) = 4_{j-1}\Delta_0\otimes(0) + 1_0 1_j\Delta_{j-1}\otimes(0) + \Delta_j\otimes(0)$ for $j\geqq 2$,
$f_1([\Delta_0\otimes(0)]\otimes 1_0) = 4_0\Delta_0\otimes(0) + \Delta_1\otimes(0)$.

We have

$$Hom_A(f_s\otimes 1,\ 1)=f_s^*\otimes 1\colon\ Y(\infty)_s^*\otimes S''\to Y(1)_s^*\otimes A''\otimes S'',$$

where $f_s^*\colon\ Y(\infty)_s^*\to Y(1)_s^*\otimes A''$ and $1\colon\ S''^*\to S''^*$. So we obtain (2-a) and (2-b).

By (iii), we obtain

$$f_3^*(y_0y_{j+1}^2+x_0x_jy_{j+1})=y_0y_1^2\otimes\zeta_j^2+\text{other terms in } Y(1)^*\otimes A'' \qquad \text{for } j\geqq 1$$

and

$$f_3^*(y_0y_{j+1}^2+x_0x_jy_{j+1})=y_0^3\otimes\zeta_{j+1}+\text{other terms in } Y(1)^*\otimes A''.$$

Obviously we have $f_3^*(y_0y_1^2+x_0^2y_1)=y_0y_1^2\otimes 1+$ other terms, so that

$$f_3^*(y_0y_{j+0}^2+x_0x_jy_{j+1})=y_0y_1^2\otimes\zeta_j^2+y_0^3\otimes\zeta_{j+1}+\text{other terms in } Y(1)^*\otimes A'' \qquad \text{for } j\geqq 0,$$

where $\zeta_0=1$. No $y_0y_1^2+$other terms in $(Y(1)_3^*)^7$ is coboundary and $Ext_{B(1)}^{3,7}(Z_2, Z_2)=\{0,\ \tau_0\}$. $(Y(1)_3^*)^3=\{0,\ y_0^3\}$. Therefore we have

$$h^{bo}(\tau_j)=\tau_0v_{2^j-1}^2+q_0^3v_{2^{j+1}-1}+\text{other terms in } {}_1R.$$

From the dimensional reason, (2-c) follows.

(2-d) can be proven by (i′).

Now we prove (2-c′). We define a ring homomorphism

$$\gamma\colon\ \boldsymbol{Z}_2[q_0,\ \tau_0,\ \{v_{2^a-1}\}]\to\boldsymbol{Z}_2[\tau_0,\ \{v_{2^a-1}\}]$$

by $\gamma(q_0)=0$, $\gamma(\tau_0)=\tau_0$, $\gamma(v_{2^a-1})=v_{2^a-1}$. And we define a decreasing flitration $\{F^s\}$ in $\boldsymbol{Z}_2[\tau_0,\ \{v_{2^a-1}\}]$ by

$$F^0=\boldsymbol{Z}_2[\tau_0,\ \{v_{2^a-1}\}],$$
$$F^{s+1}=(\text{the ideal of } F^0 \text{ generated by } \{v_{2^a-1};\ a\geqq 1\}F^s)$$

Then $F^sF^t\subset F^{s+t}$ and $\gamma h^{bo}(\tau_j)\equiv\tau_0v_{2^j-1}^2$ mod higher filtration.

Let

$$u=q_0^{s'}u',\ s'\geqq 0,\ u'\in\boldsymbol{Z}_2[q_0,\ \{\tau_a\}],$$
$$u' \text{ is not divisible by } q_0 \text{ in } \boldsymbol{Z}_2[q_0,\ \{\tau_a\}].$$

If $u\notin\boldsymbol{Z}_2[q_0]$ then u' has the form

$$u'=\sum_{\substack{0\leqq s_0\leqq m,\\ 0\leqq s_1,\cdots,s_j,\cdots,\\ (s_0,s_1,\cdots,s_j,\cdots)\neq(0)}} b^{(s_0,s_1,\cdots,s_j,\cdots)}\tau_0^{s_0}\tau_1^{s_1}\cdots\tau_j^{s_j}\cdots+q_0u'',$$
$$b^{(s_0,s_1,\cdots,s_j,\cdots)}\in\boldsymbol{Z}_2,\ u''\in\boldsymbol{Z}_2[q_0,\ \{\tau_a\}]$$

($m\geqq 0$ and there is $(s_1,\cdots,s_j,\cdots)$ such that $b^{(m,s_1,\cdots,s_j,\cdots)}\neq 0$). We have

$$s_0+s_1+\cdots+s_j+\cdots=(s-s')/3 \qquad \text{if} \quad b^{(s_0,s_1,\cdots,s_j,\cdots)}\neq 0\,.$$

Therefore we obtain

$$\gamma h^{bo}(u') \equiv \sum_{\substack{0\leqq s_1,\cdots,s_j,\cdots,\\ (m,s_1,\cdots,s_j,\cdots)\neq(0)}} b^{(m,s_1,\cdots s_j,\cdots)}\tau_0^m(v_0v_1^2)^{s_1}\cdots(\tau_0v_{2^j-1}^2)^{s_j}\cdots \quad \text{mod higher filtration,}$$

so that (2-c′) is proven.

By Lemma 3.11, (2-a) and (2-d), we obtain

Proposition 3.12. *Let k be an integer, and $x\in MSp_{4k+1}$ represented by an element $\neq 0$ of $E_\infty^{1,1+(4k+1)}(MSp)$. Then $h^{KO}(x)\neq 0$ in $KO_{4k+1}(MSp)$.*

Proof. $Ext_B^1(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$ is a $\boldsymbol{Z}_2$-vector space generated by $\{q_0, k_0, k_1, \cdots, k_j, \cdots\}$.

By Lemma 3.6 and Lemma 3.11, (1), we obtain

Lemma 3.13. Let s, t be integers, and $u\in E_\infty^{s,t}(MSp)$ such that $q_0^n u\neq 0$ for any integer $n\geqq 0$. Then $h(u)\neq 0$ in $E_\infty^{s,t}(H\wedge MSp)$.

REMARK. Lemma 3.13 follows also from [12], PROPOSITION 3.2.

4. Proof of Thorem 1.1

For any integer k, we denote by g_k the composition of the following sequence of homomorphism

$$MSp_{4k}\otimes\boldsymbol{Q} \xrightarrow{h} H_{4k}(MSp)\otimes\boldsymbol{Q} \xrightarrow{p_k} \boldsymbol{Q}\,,$$

where $p_k(x)$ is the coefficient of q_k in x for any $x\in H_{4k}(MSp)\otimes\boldsymbol{Q}$. We have the commutative diagram

$$\begin{tikzcd}
MSp_{4k}/\mathrm{Tors} \arrow[r, "q"] \arrow[d, "i"] & Q_{4k}(MSp_*/\mathrm{Tors}) \arrow[r, "u_1"] \arrow[d, "i_*"] & \boldsymbol{Z} \\
W_{4k}^{KO} \arrow[r, "q"] & Q_{4k}(W_*^{KO}) \arrow[ur, "u_2"'] &
\end{tikzcd}$$

Here q denotes the quotient map, and u_1, u_2 are the maps such that

$$g_k\,|\,MSp_{4k}/\mathrm{Tors}=u_1\circ q,\ g_k\,|\,W_{4k}^{KO}=u_2\circ q$$

(Cf. Corollary 2.2 and (2.4)). Since u_2 is a monomorphism, Theorem 1.1 is equivalent to

(4.1) $\quad g_k(MSp_{4k}/\mathrm{Tors})\supset g_k(W_{4k}^{KO})$ for any integer k.

By [9], (6·4), we have

(4.2) $\quad MSp_*/\mathrm{Tors}\otimes \boldsymbol{Z}[\frac{1}{2}] = W_*^{KO}\otimes \boldsymbol{Z}[\frac{1}{2}]$.

Therefore (4.1) is equivalennt to

(4.3) $\quad 2^s|g_k(MSp_{4k}/\mathrm{Tors})\Rightarrow 2^s|g_k(W_{4k}^{KO})$ for any integer k.

Let E be a ring spectrum. Then, obviously, we have

(4.4) $\quad f_*(W_*^E)\subset W_{*-n}^E$.

for any morphism $f\colon MSp\to MSp$ of degree $-n$, where $f_*\colon MSp_*\otimes \boldsymbol{Q}\to MSp_{*-n}\otimes \boldsymbol{Q}$.

Making use of Proposition 2.6, Lemma 2.5 and (4.4), we can prove the following proposition in the same way as that of Segal [13].

Proposition 4.1.

(1) *For any integer k, $g_k(W_{4k}^{KO})$ is divisible by 2. If k is a power of 2 then it is divisible by 4.*

(2) *Let k be an odd integer. Then $h(W_{4k}^{KO})$ is divisible by 4 in $H_{4k}(MSp)$. In particular, $g_k(W_{4k}^{KO})$ is divisible by 4.*

And further, making use of some results in §3, we obtain

Proposition 4.2. *If $k=2^j-1$, j an integer >0, then $g_k(W_{4k}^{KO})$ is divisible by 8.*

Proof. Let $x\in W_{4k}^{KO}$ and $x\neq 0$. By Lemma 3.6, there is an integer $n\geqq 0$ such that $2^n x\in MSp_{4k}/\mathrm{Tors}\subset MSp_{4k}$ is represented by an element of $\boldsymbol{Z}_2[q_0, \{\tau_m\}, \{v_i; i\neq 2^a-1\}]\cap E_\infty(MSp)$. Let $2^n x$ be represneted by $u\in E_\infty^s(MSp)$, $s\geqq 0$.

(i) In case $u\in \boldsymbol{Z}_2[q_0, \{v_i; i\neq 2^a-1\}]$: There is a decomposable element $y\in H_{4k}(MSp)$ such that

$$h(2^n x)\equiv y \bmod F^{s+1}H_{4k}(MSp)\,.$$

Therefore, by Corollary 3.9, (1), $g_k(2^n x)$ is divisible by 2^{s+1}, so that $g_k(x)$ is divisible by 2^{s+1-n}. By Lemma 3.13, $h(2^n x)$ is not divisible by 2^{s+1}, so that $h(x)$ is not divisible by 2^{s+1-n}. By Proposition 4.1, (2), we have $s+1-n\geqq 3$. Consequently $g_k(x)$ is divisible by 8.

(ii) In case $u\notin \boldsymbol{Z}_2[q_0, \{v_i; i\neq 2^a-1\}]$: By Lemma 3.11, (2-a) and (2-c′), we have

$$h^{bo}(u)\notin \boldsymbol{Z}_2[q_0, \{v_i\}]\subset E_\infty(bo\wedge MSp)\,.$$

By (2.2), $h^{bo}(x)\in bo_{4k}(MSp)$. Let $h^{bo}(x)$ be represented by $w\in E_\infty^{*,*+4k}(bo\wedge MSp)$. By Proposition 3.4, (2), $h^{bo}(2^n x)=2^n h^{bo}(x)$ is represented by $q_0^n w$, so that

$$h^{bo}(u) = q_0^n w,\ w \in E_*^{s-n}(bo \wedge MSp)\,.$$

Then $w \notin \boldsymbol{Z}_2[q_0, \{v_i\}]$, so that $s-n \geqq 3$. $h(2^n x)$ is divisible by 2^s, so that $h(x)$ is divisible by 2^{s-n}. Consequently $h(x)$ is divisible by 8.

Let $n_j(n_1, n_2, \cdots, n_r) \in MSp_{2N-4j}$ be the Stong-Ray classes in [11], where $N = \sum_{i=1}^{r} (2n_i - 1)$.

Proposition 4.3.

(1) (Segal [13]) *For an even integer $k>0$, we define integers s_k and t_k as follows. If k is not a power of 2 then we define $s_k = 2^u + 1$, 2^u the largest power of 2 less than k, and $t_k = k - s_k + 2$. If $k = 2^j$ then we define $s_k = t_k = 2^{j-1} + 1$. Then we have*

$$g_k(n_1(s_k, t_k)) \equiv \begin{cases} 2 \bmod 4 & \text{if } k \equiv 0 \bmod 2,\ k \neq 2^j \\ 4 \bmod 8 & \text{if } k = 2^j \end{cases}$$

(2) *Using the notation of* (1), *we have*

$$g_k(n_2(s_{k+1}, t_{k+1})) \equiv \begin{cases} 4 \bmod 8 & \text{if } k \equiv 1 \bmod 2,\ k \neq 2^j - 1 \\ 8 \bmod 16 & \text{if } k = 2^j - 1 \end{cases}$$

(Segal [13] has proven the fact that $g_k(MSp_{4k}/\mathrm{Tors})$ is not divisible by 8 if $k \equiv 1 \bmod 2$, $k \neq 2^j - 1$.).

Now (4.3) follows from Propositions 4.1, 4.2 and 4.3, so that Theorem 1.1 is proven.

As a corollary to Proposition 4.3, we obtain

Proposition 4.4. *$\{n_j(n_1, n_2, \cdots, n_r) \in MSp_*\}$ generates $Q(MSp_*/\mathrm{Tors}) \cong Q(W_*^{KO})$.*

Proof. From Stong [17], Theorem 1, it follows that $\{n_1(n_1, n_2, \cdots, n_r)\}$ generates $Q(MSp_*/\mathrm{Tors}) \otimes \boldsymbol{Z}_p$ for any odd prime p.

5. Proof of Theorem 1.2 and some remarks

For integers $k, s \geqq 0$, we put

$$F_1^s = h(MSp_{4k}) \cap F^s H_{4k}(MSp)$$

and

$$F_2^s = h(W_{4k}^{KO}) \cap F^s H_{4k}(MSp)\,.$$

The following lemma follows immediately from the definition.

Lemma 5.1. For $m=1$ or 2, the inclusion $F_m^s \to H_{4k}(MSp)$ induces the monomorphism

$$F^s_m/F^{s+1}_m \to F^s H_{4k}(MSp)/F^{s+1}H_{4k}(MSp) = E^{s,s+4k}_\infty(H\wedge MSp)\,.$$

Lemma 5.2. $MSp_{4k}/\mathrm{Tors} = W^{KO}_{4k}$ if and only if

$$F^s_1/F^{s+1}_1 = F^s_2/F^{s+1}_2 \subset E^{s,s+4k}_\infty(H\wedge MSp) \qquad \text{for any } s\geqq 0\,.$$

Proof. By (4.2), we have

$$h(MSp_{4k})\otimes \boldsymbol{Z}[\tfrac{1}{2}] = h(W^{KO}_{4k})\otimes \boldsymbol{Z}[\tfrac{1}{2}]\,.$$

Therefore there is an integer $s_0=s_0(k)$ such that $F^s_1=F^s_2$ for any $s\geqq s_0$. Then it is easy to see that $h(MSp_{4k})=h(W^{KO}_{4k})$ if and only if

$$F^s_1/F^{s+1}_1 = F^s_2/F^{s+1}_2 \qquad \text{for any } s\geqq 0\,.$$

Since h: $MSp_*\otimes\boldsymbol{Q}\to H_*(MSp)\otimes\boldsymbol{Q}$ is an isomorphism, the lemma follows.

By Theorem 1.1, Proposition 2.6, Lemmas 3.13, 5.2 and Segal [12], TABLE II, we obtain

Lemma 5.3. $MSp_{4k}/\mathrm{Tors}=W^{KO}_{4k}$ for $k\leqq 7$.

By Lemma 5.3, Proposition 2.6, Lemma 3.11, (2) and [12], TABLE II, we can prove

Lemma 5.4.

$$\text{order of } MSp_n = \text{order of } h^{KO}(MSp_n)$$

for $n\leqq 30$, $n\not\equiv 0 \bmod 4$.

Since MSp_{4k} is torsion free for $k\leqq 7$ by [12], Theorem 1.2 follows from Lemmas 5.3 and 5.4.

Making use of the Ray classes $\phi_i\in MSp_{8i-3}$ in [8], we can immediately calculate the ring structure of MSp_* in dimensions $\leqq 30$ except the values of $\alpha\tilde{x}_7$ and $\alpha^2\tilde{x}_7$, where α is the generator of $MSp_1\cong \boldsymbol{Z}_2$ (Cf. Ray [10], (5·25)). For example, we have

Proposition 5.5. *For* $k\leqq 5$,

$$x_1^2 MSp_{4k+1}\subset \alpha MSp_{4k+8} \quad \text{and} \quad x_1^2 MSp_{4k+2}\subset \alpha^2 MSp_{4k+8}\,.$$

We can calculate the Hurewicz map (1.1) for $n=17$:

Proposition 5.6. *There is an indecomposable element* $\tau\in MSp_{17}$ *such that*

$$h^{KO}(\tau) = e(\sigma_2^2+y\sigma_2)\,.$$

Proof. Using the notation of [12], x_1^2 is represented by ω_0 and $2y_4$ by $q_0 v_2^2$. Therefore $2x_1^2 y_4$ is represented by $q_0\omega_0 v_2^2$. Since

$$2(x_1x_2x_3+x_1^2(x_2^2+y_4+x_4)) \equiv 2x_1^2y_4 \bmod F^6MSp_{24}\,,$$

$x_1x_2x_3+x_1^2(x_2^2+y_4+x_4)$ is represented by $\omega_0v_2^2$.

Let $\tau'\in MSp_{17}$ be a class represented by $k_0v_2^2$. Then $x_1^2\tau'$ is represented by $k_0\omega_0v_2^2$, so that

$$x_1^2\tau' \equiv \alpha(x_1x_2x_3+x_1^2(x_2^2+y_4+x_4)) \bmod F^6MSp_{25}\,.$$

Therefore

$$yh^{KO}(\tau') = h^{KO}(x_1^2\tau') \equiv ye(\sigma_2^2+y\sigma_2) \bmod h^{KO}(F^6MSp_{25})\,.$$

Since $h^{KO}(F^6MSp_{25})=yh^{KO}(F^2MSp_{17})$, there is an element $\lambda\in F^2MSp_{17}$ such that

$$yh^{KO}(\tau') = ye(\sigma_2^2+y\sigma_2)+yh^{KO}(\lambda)\,,$$
$$h^{KO}(\tau') = e(\sigma_2^2+y\sigma_2)+h^{KO}(\lambda)\,.$$

We may take $\tau=\tau'+\lambda$.

Let ${}_UE_*^{**}(MSp)$ denote the Adams-Novikov spectral sequence for MSp_* (Cf. [5]). Proposition 2.6 shows us the structure of

$$MSp_*/\mathrm{Tors} = {}_UE_\infty^{0*}(MSp)\subset {}_UE_2^{0*}(MSp)$$

in low dimensions:

Proposition 5.7.

(1) (Porter [6]) ${}_UE_2^{0*}(MSp)\cong\{x\in MSp_*\otimes \boldsymbol{Q};\ r_*(x)\in MU_*\}$.

(2) $\{x\in MSp_*\otimes \boldsymbol{Q};\ r_*(x)\in MU_*\} = W_*^K$.

Proof of (2). Consider the commutative diagram

$$\begin{array}{ccc} MSp_*\otimes \boldsymbol{Q} & \xrightarrow{h^K} & K_*(MSp)\otimes \boldsymbol{Q} \\ \downarrow{\scriptstyle r_*} & & \downarrow{\scriptstyle r_*} \\ MU_*\otimes \boldsymbol{Q} & \xrightarrow{h^K} & K_*(MU)\otimes \boldsymbol{Q}\,. \end{array}$$

Then $r_*\colon K_*(MSp)\to K_*(MU)$ is a split monomorphism. And, by Hattori [2] or Stong [16], $h^K\colon MU_*\to K_*(MU)$ is a split monomorphism.

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