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ON THE $MSp$ HATTORI-STONG PROBLEM

RIKIO ŌKITA

(Received September 22, 1975)

1. Introduction

In the present paper, we work in the category of $CW$-spectra due to Adams [1]. For any ring spectrum $E$, we denote by $E_*(\ )$ and $E^*(\ )$ the associated homology and cohomology functors and by $E_*$ the coefficient ring. The unit of $E$ is denoted by $u^E: S \to E$. Let $E$ be a ring spectrum and $F$ a spectrum. Consider the spectrum morphism

$$u^E \land 1: F = S \land F \to E \land F .$$

Then $u^E \land 1$ induces the generalized Hurewicz map

$$h^E = (u^E \land 1)_*: F_* \to E_*(F) .$$

For $E=H$, we denote $h^E$ simply by $h$.

Ray [7] has conjectured that the Hurewicz map

$$(1.1) \quad h^{KO}: MSp_n \to KO_n(MSp)$$

is a split monomorphism for any integer $n$ and has shown that it is a split monomorphism for $n \leq 20$. Later Segal [14] has shown that the map (1.1) is not a monomorphism for $n=31$ (since $MSp_{31} \cong \mathbb{Z}$) and that Ray's $MSp$ Hattori-Stong conjecture is false.

But still we may conjecture that the map

$$(1.2) \quad h^{KO/Tors}: MSp_*/\text{Tors} \to KO_*(MSp)/\text{Tors}$$

is a split monomorphism, where Tors denotes the torsion subgroup.

For any ring spectrum $E$, we put

$$W^E_\# = \{ x \in MSp_* \otimes \mathbb{Q}; h^E(x) \in E_*(MSp)/\text{Tors} \subset E_*(MSp) \otimes \mathbb{Q} \} .$$

Then $MSp_*/\text{Tors} \subset W^E_\#$. And the map (1.2) is a split monomorphism if and only if $MSp_*/\text{Tors} = W^{KO}_\#$.

Let $L_\#$ be a subring of $MSP_* \otimes \mathbb{Q}$. We put

$$Q(L_\#) = L_\#/(L_\# \cap D_\#) ,$$
where $D_\ast$ is the ideal of all decomposable elements in $\text{MSp}_\ast \otimes \mathbb{Q}$.

In this paper, we prove the following two theorems.

**Theorem 1.1.** The inclusion $i : \text{MSp}_\ast / \text{Tors} \to W^{\text{KO}}_\ast$ induces the isomorphism

$$i_\ast : Q(\text{MSp}_\ast / \text{Tors}) \cong Q(W^{\text{KO}}_\ast)$$

(Cf. Proposition 3.12).

**Theorem 1.2.** The Hurewicz map

$$h^{\text{KO}} : \text{MSp}_n \to \text{KO}_n(\text{MSp})$$

is a split monomorphism for $n \leq 30$. In particular, we have

$$\text{MSp}_n / \text{Tors} = W^{\text{KO}}_n \quad \text{for} \quad n < 32.$$}

The author wishes to express his hearty appreciation to Professor S. Araki.

2. Calculations in $W^*_\ast$ and $W^{\text{KO}}_\ast$

We denote by $i : \text{CP}^n \to \text{CP}^\infty$ (resp. $i : \text{HP}^n \to \text{HP}^\infty$) the inclusion map. Let $E$ be a ring spectrum having a class $x \in E^i(\text{CP}^\infty)$ (resp. $x \in E^i(\text{HP}^\infty)$) such that

$$E^*(\text{CP}^\infty) = E^*[x]/(x^{n+1})$$

for each integer $n \geq 1$ and $x \in \tilde{E}^i(\text{CP}^\infty) = \tilde{E}^i(S^i)$ (resp. $x \in \tilde{E}^i(\text{HP}^\infty) = \tilde{E}^i(S^i)$) is represented by the unit $u^E$, where $x = i^*(x)$. As is well known, $x$ determines the Thom isomorphism $\phi : E_\ast(\text{BU}) \cong E_\ast(\text{MU})$ (resp. $\phi : E_\ast(\text{BSp}) \cong E_\ast(\text{MSp})$). Let $j : \text{CP}^n \to \text{BU}$ (resp. $j : \text{HP}^n \to \text{BSp}$) be the inclusion map and $y_i \in E_\ast(\text{CP}^n)$ (resp. $y_i \in E_\ast(\text{HP}^n)$) dual to $x_i$. Put $y_i = \phi^i(x_i)$. Then we have

$$E_\ast(\text{MU}) = E_\ast[y_i, y_{i+1}, \ldots, y_i, \ldots]$$

(resp. $E_\ast(\text{MSp}) = E_\ast[y_i, y_{i+1}, \ldots, y_i, \ldots]$),

where $y_i \in E_\ast(\text{MU})$ (resp. $y_i \in E_\ast(\text{MSp})$).

In $H^i(\text{CP}^n)$, choose $x$ to be $e_i$, the first Chern class of the universal $U(1)$-bundle $\xi^1$ over $\text{CP}^n$. In this case, we denote $y_i$ by $b_i$. Then we have

$$H_\ast(\text{MU}) = \mathbb{Z}[b_1, b_2, \ldots, b_i, \ldots], \quad b_i \in H_\ast(\text{MU}).$$

In $\tilde{H}^i(\text{CP}^n)$, choose $x$ to be $c_f$, the first Conner-Floyd Chern class of $\xi^1$, represented by the homotopy equivalence $\text{CP}^n = \text{MU}(1)$.

Let $z \in K_2$ be such that $i^*(\xi^1 - 1) = 2\gamma$ in $K^0(\text{CP}^1)$, where $\gamma \in \tilde{K}^2(\text{CP}^1) = \tilde{K}^2(S^1)$ is represented by the unit $u^K$. Then we have

$$K_\ast = \mathbb{Z}[x, x^{-1}] \quad \text{and} \quad H_\ast(K) = \mathbb{Q}[t, t^{-1}].$$
where \( t = h(z) \).

In \( \tilde{K}^{*}(CP^{\infty}) \), choose \( x \) to be \( z^{-1}(\xi^{1} - 1) \). As is well known, there is a unique ring spectrum morphism \( g: MU \to K \) such that \( g_{*}(cf_{1}) = z^{-1}(t^{1} - 1) \).

In \( \tilde{H}^{*}(HP^{\infty}) \), choose \( x \) to be \( p_{1} \), the first symplectic Pontrjagin class of the universal \( Sp(1) \)-bundle \( \xi^{1} \) over \( HP^{\infty} \). In this case we denote \( y_{i} \) by \( q_{i} \). Then we have

\[
H_{*}(MSp) = Z[q_{1}, q_{2}, \ldots, q_{i}, \ldots], q_{i} \in H_{*}(MSp).
\]

In \( \tilde{MSp}^{*}(HP^{\infty}) \), choose \( x \) to be \( p_{1} \), the first Conner-Floyd symplectic Pontrjagin class of \( \xi^{1} \), represented by the homotopy equivalence \( HP^{\infty} \simeq MSp(1) \). In this case, we denote \( y_{i} \) by \( q_{i} \).

Put \( \kappa_{i} = (gr)_{*}(qf_{i}) \in K_{*}(MSp) \), where \( r: MSp \to MU \) is the morphism induced by the inclusion \( Sp \to U \). Then we have

\[
K_{*}(MSp) = K_{*}[\kappa_{1}, \kappa_{2}, \ldots, \kappa_{i}, \ldots], \kappa_{i} \in K_{*}(MSp).
\]

Let \( bu \) denote the connective \( BU \)-spectrum and \( \psi: bu \to K \) the canonical morphism. Then we have

\[
\psi_{*}: bu_{*} \simeq K_{*} \quad \text{if} \quad n \geq 0, \quad bu_{*} = 0 \quad \text{if} \quad n < 0.
\]

And let \( \bar{\kappa}_{i} \in bu_{*}(MSp) \) be the unique class such that \( \psi_{*}(\bar{\kappa}_{i}) = \kappa_{i} \in K_{*}(MSp) \). Then we have

\[
bu_{*}(MSp) = bu_{*}[\bar{\kappa}_{1}, \bar{\kappa}_{2}, \ldots, \bar{\kappa}_{i}, \ldots].
\]

Therefore \( \psi_{*}: bu_{*}(MSp) \to K_{*}(MSp) \) is a split monomorphism, so that we have

\[
W_{*}^{bu} = W_{*}^{K}.
\]

Similarly we have

\[
W_{*}^{bo} = W_{*}^{K^{O}},
\]

where \( bo \) denotes the connective \( BO \)-spectrum.

We have a Künneth isomorphism

\[
H_{*}(\wedge MSp) \cong H_{*}(MSp) \wedge H_{*}(MSp)
\]

since \( H_{*}(MSp) \) is torsion free. By this isomorphism we identify \( H_{*}(\wedge MSp) \) and \( H_{*}(MSp) \).

**Lemma 2.1.** Consider the commutative diagram

\[
\begin{array}{ccc}
MSp \otimes Q & \xrightarrow{h} & H_{*}(MSp) \otimes Q \\
\downarrow h^{K} & & \downarrow j \\
K_{*}(MSp) \otimes Q & \xrightarrow{h} & H_{*}(K) \otimes H_{*}(MSp) \otimes Q,
\end{array}
\]
where \( j = (u^K \wedge 1)_* : H_*(MSp) \to H_*(K \wedge MSp) = H_*(K) \otimes H_*(MSp) \). Then we have

\[
j(x) = 1 \otimes x
\]

for any \( x \in H_*(MSp) \) and

\[
h(h^K(W_x^\Lambda)) = h(K_*(MSp)) \cap j(H_*(MSp))
\]

\[
= h(Z[z^2, \kappa_1, \kappa_2, \cdots, \kappa_i, \cdots]) \cap j(H_*(MSp)).
\]

**Proof.** It is proven by diagram chasing that

\[
j(x) = 1 \otimes x
\]

for any \( x \in H_*(MSp) \).

We have the following commutative diagram

\[
\begin{array}{ccc}
MSp \otimes Q & \overset{h}{\rightarrow} & H_*(MSp) \otimes Q \\
\downarrow h^{bu} & \approx & \downarrow j \\
bu_*(MSp) \otimes Q & \overset{h}{\rightarrow} & H_*(bu) \otimes H_*(MSp) \otimes Q \\
\downarrow \varphi_* & \approx & \downarrow j \\
K_*(MSp) \otimes Q & \overset{h}{\rightarrow} & H_*(K) \otimes H_*(MSp) \otimes Q,
\end{array}
\]

where \( j = (u^{bu} \wedge 1)_* : H_*(MSp) \to H_*(bu \wedge MSp) = H_*(bu) \otimes H_*(MSp) \).

Now let \( x \in W_x^\Lambda = W_\Lambda^u \) (Cf. (2.1)). Then there is an integer \( n \neq 0 \) such that \( nx \in MSp_\Lambda / \text{Tors} \). We have

\[
nh(h^{bu}(x)) = h(h^{bu}(nx))
\]

\[
= j(h(nx)) \in j(H_*(MSp)).
\]

Since \( j/\text{Tors} : H_*(MSp) \to H_*(bu)/\text{Tors} \otimes H_*(MSp) \) is a split monomorphism, \( h(h^{bu}(x)) \in j(H_*(MSp)) \). Therefore we obtain

\[
h(h^K(x)) \in j(H_*(MSp))
\]

By (2.1) and dimensional reason, we obtain

\[
h^K(x) \in Z[z^2, \kappa_1, \kappa_2, \cdots, \kappa_i, \cdots],
\]

\[
h(h^K(x)) \in h(Z[z^2, \kappa_1, \kappa_2, \cdots, \kappa_i, \cdots]).
\]

Conversely let \( y \in K_*(MSp) \) and \( h(y) \in j(H_*(MSp)) \). Then

\[
h(y) \in j(h(MSp_\Lambda \otimes Q)) = h(h^K(MSp_\Lambda \otimes Q)),
\]

so that \( y \in h^K(MSp_\Lambda \otimes Q) \). Consequently we obtain
Corollary 2.2. \( h(W^*_K) \subset H_\#(MSp) \).

It is well known that
\[
(2.3) \quad g_\#(b_i) = t^i/(i+1)!,
\]
where \( g_\#: H_\#(MU) \to H_\#(K) \). And we have

Lemma 2.3.
\[
(\text{gr})_\#(q_i) = 2t^i/[2(i+1)]!,
\]
where \( (\text{gr})_\#: H_\#(MSp) \to H_\#(K) \).

Proof. We have
\[
\kappa_i = 2[b_{i-1}b_{i+1} + \cdots + (-1)^{i-1}b_{i-2}b_{i-1} + (-1)^ib_i],
\]
so that the lemma follows immediately from (2.3).

Consider the commutative diagram
\[
\begin{array}{ccc}
MSP_\#(MSp) & \xrightarrow{h} & H_\#(MSp) \otimes H_\#(MSp) \\
\downarrow{\text{(gr)}}_\# & & \downarrow{\text{(gr)}}_\# \otimes 1 \\
K_\#(MSp) & \xrightarrow{h} & H_\#(K) \otimes H_\#(MSp) \\
\end{array}
\]

By definition, \( (\text{gr})_\#(qf) = \kappa_i \). Therefore we have
\[
h(\kappa_i) = (\text{gr})_\# \otimes 1(h(qf)),
\]
so that, by Ray [9], (5·6) and Lemma 2.3, we can calculate the Hurewicz map
\[
h: K_\#(MSp) \to H_\#(K) \otimes H_\#(MSp).
\]
Therefore, by Lemma 2.1 and the fact that \( h^K: W^*_K \to K_\#(MSp) \) is a monomorphism, we obtain

Proposition 2.4. \( W^*_K \) is generated by elements
\[
x_i (1 \leq i \leq 7), y_4, y_6 \quad \text{and} \quad y,
\]
in dimensions \( <32 \), where \( x_i \) \( (1 \leq i \leq 6) \) are defined by
\[ h^K(x_i) = z^2 + 12x_i, \]
\[ h^K(x_i) = z^2k_1 - 4k_1^3 + 10k_2, \]
\[ h^K(x_1) = z^2(-3k_1^2 + 4k_2) + 12k_1^3 - 36k_1k_2 + 28k_3, \]
\[ h^K(x_2) = z^2(k_1^3 - 2k_1k_2^2 + k_3) - 4k_1^4 + 14k_1k_2^2 - 4k_2^3 - 12k_1k_2 + 6k_4, \]
\[ h^K(x_3) = z^2(-7k_1^3 + 18k_1k_2^2 - 4k_2^3 - 11k_3k_3 + 4k_4) \]
\[ + 28k_1^5 - 112k_1^2k_2 + 66k_1k_2^3 + 96k_2^4 - 38k_2^3k_3 - 62k_1k_4 + 22k_5, \]
\[ h^K(x_4) = z^2(-2k_1^3 + 5k_1^2k_2^2 - k_2^3 - 3k_1k_3 + k_4) \]
\[ + 9z^2(-3k_1^2 + 10k_1k_2 + 24k_1k_3^2 + 13k_3^3 - 18k_2k_3 - 14k_1k_4 + 8k_5) \]
\[ + 44k_1^5 - 150k_1^3k_2^2 + 15k_1^2k_2^3 + 25k_2^4 + 140k_1k_2^2 + 36k_1k_2k_3 + 12k_3^2 - 84k_1k_4 \]
\[ - 45k_2k_4 + 18k_1k_5 + 13k_6, \]

and

\[ y_4 = (-x_1^2 + x_1x_2)/4, \quad y_6 = (-x_2x_3 + x_1x_4)/2 \quad \text{and} \quad y_7 = (-x_3x_4 + x_2x_5)/2. \]

And we have

**Lemma 2.5.** Let \( x \in W^*_K \), and

\[ h^K(x) = f(z, k_1, k_2, \ldots, k_i, \ldots) \in \mathbb{Z}[z, k_1, k_2, \ldots, k_i, \ldots]. \]

Then

\[ h(x) = f(0, q_1, q_2, \ldots, q_i, \ldots) \in H_*(MSP). \]

For example,

\[ h(x_1) = 12q_1, \]
\[ h(x_2) = -4q_1^2 + 10q_2, \]
\[ h(x_3) = 12q_1^3 - 36q_1q_2 + 28q_3, \]
\[ h(x_4) = -4q_1^4 + 14q_1^2q_2 - 4q_2^2 - 12q_1q_3 + 6q_4. \]

**Proof.** Notice that

\[ h(k_i) \equiv 1 \otimes q_i \mod t \otimes 1 \quad \text{in} \quad Q[t] \otimes H_*(MSP) \]

where \( h : K_*(MSP) \rightarrow H_*(K) \otimes H_*(MSP) \). Then the lemma follows from Lemma 2.1.

Let \( c : KO \rightarrow K \) be the complexification morphism. As is well known, \( KO_* \) is generated by the classes

\[ e \in KO_1, \quad x \in KO_4, \quad y \in KO_8 \quad \text{and} \quad y^{-1} \in KO_{-8} \]

subject to the relations

\[ 2e = e^2 = ex = 0, \quad x^2 = 4y \quad \text{and} \quad yy^{-1} = 1 \]
such that
\[ c_*(x) = 2z^2 \quad \text{and} \quad c_*(y) = z^4 \quad \text{in} \quad K_* . \]

Let \( \sigma_i \in KO_*^{d}(MSp) \) be the unique class such that \( c_*(\sigma_i) = \kappa_i \in K_*^{d}(MSp) \).
Then we have
\[ KO_*(MSp) = KO_*[\sigma_1, \sigma_2, \ldots, \sigma_i, \ldots] , \]
and

\[ (2.4) \quad W_0^{KO} \subset W_*^{K} . \]

As a corollary to Proposition 2.4, we obtain

**Proposition 2.6.** \( W_*^{KO} \) has the following generators for \( k \leq 7 \).

\[
\begin{align*}
  k = 1: & \quad 2x_1, \\
  k = 2: & \quad x_1^2, 2x_2, \\
  k = 3: & \quad 2x_1^3, x_1x_2, 2x_3, \\
  k = 4: & \quad x_1^4, 2x_1^3x_2, x_2x_3, 2y_4, 2x_4, \\
  k = 5: & \quad 2x_1^5, x_1^4x_2, 2x_1^3x_3, 2x_1^2x_4, x_2x_5, x_3x_4, 2x_6, \\
  k = 6: & \quad x_1^6, 2x_1^5x_2, x_1^4x_3, 2x_1^3x_4, 2x_1^2x_5, x_2x_6, x_3x_7, 2x_8, \\
        & \quad x_1x_2^2 + x_1x_2(y_4 + x_5), x_2x_3, x_4y_5, x_5x_6, 2x_7x_8, x_9. \\
  k = 7: & \quad 2x_1^7, x_1^6x_2, 2x_1^5x_3, 2x_1^4x_4, x_1^3x_5, 2x_1^2x_6, 2x_1x_7, x_8y_9, 2x_9x_{10}, 2x_8x_9, x_{11}. 
\end{align*}
\]

**Remark.**
\[ W_*^{MSU} = W_*^{KO}, \quad h_*^{MSU}(W_*^{MSU}) = H-Sp_* , \]

where \( H-Sp_* \) is the algebra of Ray [10], (2.1), and
\[ h(2x_i) = h_i \in H_*^{d}(MSp) \]
for \( i \leq 4 \), where \( h_i \) are the classes in [10], (3.7) (Cf. Lemma 2.5)

### 3. Adams spectral sequence maps

For any connective spectrum \( X \) such that \( X_r \) is finitely generated for each \( r \), we denote by \( E_*^{**}(X) \) the mod 2 Adams spectral sequence for \( X_* \) (Cf. [3], 2.2). For an integer \( n \), we denote by \( F^sX_* \) the \( s \)-th filtration in the mod 2 Adams spectral sequence. Then we have
\[ F^sX_n/F^{s+1}X_n = E_*^{s,s+n}(X) = E_*^{s,s+n}(X) \quad (r \text{ large}) \]

Let \( H \) be a graded vector space over \( Z_2 \). We define a graded vector space \( H' \) from \( H \) by
for any integer \(n\). For any connected Hopf algebra \(H\) over \(\mathbb{Z}_2\), we denote the augmentation ideal \(\bigoplus_{i>0} H_i\) by \(\mathcal{H}\).

We denote the mod 2 Steenrod Algebra by \(A\). Let \(A''\) be endowed with structure as a graded \(A\)-module by the following \(A\)-action.

\[
A \otimes A'' \xrightarrow{\beta \otimes 1} A'' \otimes A'' \xrightarrow{\mu} A''
\]

Here \(\beta: A \to A''\) is the map such that \(\beta^*(x) = x^2 \in A^*\) for any \(x \in A''^*\) and \(\mu\) is the product map in \(A\). Using the notation of Milnor [4], we denote \((\xi_{j+i}^m)^*\) by \(m_j\) for any integers \(m, j \geq 0\). For any \(n (0 \leq n \leq \infty)\), let \(B(n)\) be the Hopf subalgebra of \(A\) (multiplicatively) generated by the elements \(1, 2_j\) for \(j < n\). The map \(\beta\) induces the isomorphism

\[
A/B \approx A''
\]

where \(B = B(\infty)\).

Let \(R\) be a Hopf subalgebra of \(A\), and \((C, d_C, \epsilon_C)\) a \(R\)-free resolution of \(\mathbb{Z}_2\). As is well known, \(A\) is free as a right \(R\)-module and we have the isomorphism \(A/R \approx A \otimes \mathbb{Z}_2\) of \(A\)-modules. So we obtain

**Lemma 3.1.** There is an \(A\)-free resolution of \(A/R\):

\[
A/R \xrightarrow{\otimes \epsilon_C} A \otimes C \xrightarrow{1 \otimes d_C} A \otimes C \xrightarrow{1 \otimes d_C} \cdots \xrightarrow{1 \otimes d_C} A \otimes C \xrightarrow{1 \otimes d_C} \cdots
\]

The following proposition is well known.

**Proposition 3.2.**

1. (Serre [15]) \((HZ_2)^*(H) \approx A/AB(0)\)

   as graded \(A\)-modules.

2. (Cf. [1], §16) \((HZ_2)^*(bo) \approx A/AB(1)\)

   as graded \(A\)-modules.

3. (Cf. [3], THEOREM II. 4) \((HZ_2)^*(MSp) \approx A'' \otimes S''\)

   as graded coalgebra and \(A\)-modules (\(A\) operating on \(S''\) trivially), where \(S\) is the graded coalgebra over \(\mathbb{Z}_2\) such that

\[
S^* \approx \mathbb{Z}_2[V_2, V_4, V_5, \ldots, V_i, \ldots], \quad i \equiv 2^a - 1, \quad \deg V_i = i.
\]

As a result of Proposition 3.2, the following proposition is obtained by (3.1) and Lemma 3.1.
Proposition 3.3.

(1) \( E_2(H) \cong \Ext_{B(0)}(Z_2, Z_2) \).

(2) \( E_2(ho) \cong \Ext_{B(1)}(Z_2, Z_2) \).

(3) \( E_2(MSp) \cong \Ext_{B}(Z_2, Z_2) \otimes Z[Z_2, v_0, v_2, \ldots, v_i, \ldots] \),

\[ i \neq 2^a - 1, \quad v_i = [V_i] \in E_2^{2^a}(MSp) \).

A \( B(n) \)-free resolution of \( \mathbb{Z}_2 \) has been constructed by Liulevicius [3]. Let \( Y(n) \) be the \( \mathbb{Z}_2 \)-vector space with basis

\[ \{ I \otimes J; \begin{cases} I = (i_0, i_1, \ldots, i_{n-1}), \quad J = (j_0, j_1, \ldots, j_n), \text{ where } I, J \text{ are } \text{ sequences of non-negative, finitely non-zero integers.} \end{cases} \} . \]

Let

\[ \deg I \otimes J = \left( \sum (i_r + j_r), \sum \left[ i_r (2^{r+2} - 2) + j_r (2^{r+1} - 1) \right] \right) . \]

We define a \( B(n) \)-homomorphism \( d(n): B(n) \otimes Y(n) \rightarrow B(n) \otimes Y(n) \) by

\[ d(n)(I \otimes J) = \sum_k [1_k I \otimes (J - \Delta_k) + 2_k (I - \Delta_k) \otimes J \]

\[ + (j_{k+1} + 1)(I - \Delta_k) \otimes (J - \Delta_0 + \Delta_{k+1}) \]

\[ + (j_{k+1} + 1) \cdot (I - \Delta_0 - \Delta_k) \otimes (J + \Delta_{k+1}) \]

\[ + \left( \frac{j_{k+1} + 2}{2} \right) (I - \Delta_0 - 2 \Delta_k) \otimes (J + 2 \Delta_{k+1}) \]

\[ + \sum_{k < k'} (j_{k+1} + 1)(j_{k'} + 1)(I - \Delta_0 - \Delta_k - \Delta_{k'}) \otimes (J + \Delta_{k+1} + \Delta_{k'+1}) \].

Here we set \( I - \Delta_r = 0 \) if \( i_r = 0 \) and \( J - \Delta_r = 0 \) if \( j_r = 0 \). Then

\( B(n) \otimes Y(n) = (B(n) \otimes Y(n), d(n), \varepsilon(n)) \)

is the \( B(n) \)-free resolution of \( \mathbb{Z}_2 \) constructed by him, where \( \varepsilon(n): B(n) \otimes Y(n) \rightarrow \mathbb{Z}_2 \)

is the unique \( B(n) \)-homomorphism. Put

\[ \langle J \rangle = (0) \otimes J \].

Then we have

\[ d(n)\langle J \rangle = \sum_k \langle J - \Delta_k \rangle . \]

Using the notation of [3] for \( \text{Hom}_{B(n)}(B(n) \otimes Y(n), \mathbb{Z}_2) = Y(n)^* \), let

\[ k_j = [x_j] \in \Ext_{B(n)}^{2^j+2-2}(Z_2, Z_2) , \]

\[ q_j = [y_i] \in \Ext_{B(n)}^{2^j}(Z_2, Z_2) , \]

\[ \tau_j = [y_j] \in \Ext_{B(n)}^{2^j+3-1}(Z_2, Z_2) , \]

\[ \omega_0 = [y_i] \in \Ext_{B(n)}^{12}(Z_2, Z_2) . \]
**Proposition 3.4.** (Liulevicius [3])

1. \( \text{Ext}_{B(0)}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[q_0] \).

2. \( \text{Ext}_{B(13)}(\mathbb{Z}_2, \mathbb{Z}_2) \) has multiplicative generators \( q_0, k_0, \tau_0 \) and \( \omega_0 \) with bidegrees \((1,1), (1,2), (3,7) \) and \((4,12) \) respectively subject to the relations

\[
q_0k_0 = 0, \ k_0^2 = 0, \ k_0\tau_0 = 0 \quad \text{and} \quad \tau_0^2 = q_0^2\omega_0.
\]

**Corollary 3.5.**

1. \( E_\infty(H) = E_4(H) \).

2. \( E_\infty(b_0) = E_4(b_0) \).

**Lemma 3.6.** For any integer \( n \), there is an integer \( s_0 = s_0(n) \) such that

\[
\text{Ext}_B^{s+n}(\mathbb{Z}_2, \mathbb{Z}_2) = (\mathbb{Z}_2[q_{s}], \{\tau_j\})^{s+n} \quad \text{if} \quad s \geq s_0.
\]

Proof. Let \( \tilde{B}(m) \) be the Hopf subalgebra of \( B \) (multiplicatively) generated by \( B(m), 1_{m+1} \) \((0 \leq m < \infty) \). By Segal [12], PROPOSITION 2.3, there is a spectral sequence \( mE_{***} \) such that

\[
mE_1 = \text{Ext}_{B(m)}(\mathbb{Z}_2, \mathbb{Z}_2) \otimes F(\Omega^*) \quad (\Omega = B(m+1)/\tilde{B}(m)),
\]

\[
(mE_\infty)^{s+n} \cong \text{Ext}_{B(m+1)}^{s+n}(\mathbb{Z}_2, \mathbb{Z}_2).
\]

Since \( \Omega = E_{2k}[k_m], k_m' = [2_m] \), we have \( F(\Omega^*) = \mathbb{Z}_2[k_m], \deg k_m = (1, 2^{m+2} - 2) \). And \( \text{Ext}_{B(m)}(\mathbb{Z}_2, \mathbb{Z}_2) = \text{Ext}_{B(m)}(\mathbb{Z}_2, \mathbb{Z}_2) \otimes \mathbb{Z}_2[q_{m+1}], \deg q_{m+1} = (1, 2^{m+2} - 1) \). Therefore

\[
mE_1 = \text{Ext}_{B(m)}(\mathbb{Z}_2, \mathbb{Z}_2) \otimes \mathbb{Z}_2[k_m] \otimes \mathbb{Z}_2[q_{m+1}].
\]

Then we have

\[
d_*(q_{m+1}) = q_{k_m}
\]

and all \( d_r \) in \( mE \) are trivial on \( \text{Ext}_{B(m)}(\mathbb{Z}_2, \mathbb{Z}_2) \otimes \mathbb{Z}_2[k_m] \) (Cf. [12]).

Now we prove by induction on \( m \) that there is an integer \( s_0 = s_0(n, m) \) such that

\[
\text{Ext}_B^{s+n}(\mathbb{Z}_2, \mathbb{Z}_2) = (\mathbb{Z}_2[q_{s}], \{\tau_j; j \leq m-1\})^{s+n} \quad \text{if} \quad s \geq s_0.
\]

For \( m = 0 \), it is true by Proposition 3.4, (1). Assume that it is true for \( m \). Since \( \deg q_{m+1} = (1, 1+(2^{m+2} - 2)), 2^{m+2} - 2 \geq 1 \) and \( \deg k_m = (1, 1+(2^{m+2} - 3)), 2^{m+2} - 3 \geq 1 \), there is an integer \( s_0 = s_0(n, m) \) such that

\[
(mE_2)^{s+n} = (\mathbb{Z}_2[q_{s}], \{\tau_j; j \leq m-1\}, q_{m+1})^{s+n} \quad \text{if} \quad s \geq s_0'.
\]

Clearly there is an integer \( s_0'' = s_0''(n, m) \geq s_0' \) such that

\[
(mE_2)^{s+n} = (\mathbb{Z}_2[q_{s}], \{\tau_j; j \leq m-1\}, q_{m+1})^{s+n} \quad \text{if} \quad s \geq s_0''.
\]
\[ q_{\delta_{m+1}} \] is a permanent cycle and \( \tau_{m} \) is represented by \( q_{\delta_{m+1}} \). Put \( s_{0}(n, m+1) = s_{0}''(n, m) \) then

\[
\text{Ext}_{B_{m+1}}^{s_{0}^{2}+s_{0}^{n}}(Z_{2}, Z_{2}) = (Z_{2}[q_{0}, \{\tau_{j}; j \leq m\}])^{s_{0}^{2}+s_{0}^{n}} \quad \text{if} \quad s_{0} \leq s_{0}(n, m+1).
\]

From the fact that \( \text{Ext}_{B_{m+1}}^{s_{0}^{2}+s_{0}^{n}}(Z_{2}, Z_{2}) \cong \text{Ext}_{B_{m}}^{s_{0}^{2}+s_{0}^{n}}(Z_{2}, Z_{2}) \) if \( 2m+2-3 > n \), the lemma follows.

Let

\[ G = mG = A|AB(m) \otimes (HZ_{2})^{*}(MSp) = (HZ_{2})^{*}(mM) \otimes (HZ_{2})^{*}(MSp) \]

(\( A \) operating on \( (HZ_{2})^{*}(MSp) \) trivially), where \( m = 0 \) or \( 1 \) and \( \cdot M = H, \cdot M = bo. \)

And we define a map

\[ \Phi = m\Phi: G \rightarrow (HZ_{2})^{*}(M \wedge MSp) \quad (M = mM) \]

by \( \Phi([a] \otimes u) = \sum [a_{i}] \cdot a_{i}'' u \) for \( a \in A, u \in (HZ_{2})^{*}(MSp), \) where \( \psi(a) = \sum a_{i} \otimes a_{i}'' \). Then we have

**Lemma 3.7.** (Cf. [1], §16) \( \Phi \) is an isomorphism of graded coalgebras and \( A \)-modules.

We identify \( G \) and \( (HZ_{2})^{*}(M \wedge MSp) \) by \( \Phi \).

**Corollary 3.8.**

1. \( E_{n}(H \wedge MSp) = Z_{2}[q_{0}, v_{1}, v_{2}, \ldots, v_{i}, \ldots]. \)
2. \( E_{n}(bo \wedge MSp) = E_{n}(bo) \otimes Z_{2}[v_{1}, v_{2}, \ldots, v_{i}, \ldots]. \)

Here \( v_{i} \in E_{n}^{v_{i} + 1}(M \wedge MSp), \) where

\[ v_{i} = [V_{i}] \quad \text{if} \quad i = 2^{j} - 1, \quad v_{i} = [V_{i}] \quad \text{if} \quad i = 2^{n} - 1 \]

\((HZ_{2})^{*}(MSp) = A^{*} \otimes S^{*} \).

**Corollary 3.9.**

1. \( E_{n}(H \wedge MSp) = E_{n}(H \wedge MSp) \).

Therefore we have

\[ F^{*}H_{n}(MSp) = \{ x \in H_{n}(MSp); 2^{e} \mid x \} . \]

2. \( E_{n}(bo \wedge MSp) = E_{n}(bo \wedge MSp). \)

**Lemma 3.10.** For any \( u \in (HZ_{2})^{*}(MSp), \) we have

\( (u^{M} \wedge 1)^{*}(1 \otimes u) = u \),

where \( (u^{M} \wedge 1)^{*}: G \rightarrow (HZ_{2})^{*}(MSp). \)
Proof. For any $v \in (HZ_2)_*(MSp)$, we can prove by diagram chasing that
\[(u^M \wedge 1)_*(v) = 1 \cdot v \in (HZ_2)_*(M \wedge MSp).\]
Therefore we have
\[(u^M \wedge 1)^*(1 \cdot u) = u\]
for any $u \in (HZ_2)_*(MSp)$, where $(u^M \wedge 1)^*: (HZ_2)_*(M \wedge MSp) \to (HZ_2)_*(MSp)$. Since $\Phi^-(1 \cdot u) = 1 \otimes u$, the lemma follows.

For any ring spectrum $X$ and any spectrum $Y$, $u^X \wedge 1: Y \to X \wedge Y$ induces the spectral sequence map
\[h^X: E^{**}(Y) \to E^{**}(X \wedge Y).\]
For $X = H$, we denote $h^X$ simply by $h$.

**Lemma 3.11.**

(1-a) $h(v_i) = v_i$ if $i \neq 2^a - 1$.
(1-b) $h(\text{Ext}_B(Z_2, Z_2))$ is contained in the ring
\[R = \mathbb{Z}_2[q_0, v_1, v_3, \ldots, v_{2^a-1}, \ldots].\]
(1-c) $h(\tau_j) = q_0^j(v_{2^j+1-1} + \text{decomposables in } Z_2[v_1, v_3, \ldots, v_{2^a-1}, \ldots]) \in R$.
(2-a) $h^{bo}(v_i) = v_i$ if $i \neq 2^a - 1$.
(2-b) $h^{bo}(\text{Ext}_B(Z_2, Z_2))$ is contained in the ring
\[R = \text{Ext}_B(Z_2, Z_2) \otimes \mathbb{Z}_2[v_1, v_3, \ldots, v_{2^a-1}, \ldots].\]
(2-c) $h^{bo}(\tau_j) = \tau_j(v_{2^j+1-1} + \text{other terms in } Z_2[v_1, v_3, \ldots, v_{2^j-1}, \ldots] + q_0^j(v_{2^j+1-1} + \text{decomposables in } Z_2[v_1, v_3, \ldots, v_{2^a-1}, \ldots]) \in R$,
where $v_0 = 1$.

(2-c') Let $u \in (Z_2[q_0, \{\tau_j\}])^{*,*} \subset \text{Ext}_B^{*,*}(Z_2, Z_2)$. Then we have
\[h^{bo}(u) \in Z_2[q_0, \tau_0, \{v_{2^a-1}\}]\]
and
\[h^{bo}(u) \in Z_2[q_0, \{v_{2^a-1}\}] \quad \text{if } u \in Z_2[q_0].\]

(2-d) $h^{bo}(k_j) = k_j(v_{2^j-1} + \text{decomposables in } Z_2[v_1, v_3, \ldots, v_{2^a-1}, \ldots]) \in R$.

Proof. We prove only (2). We can prove (1) in the same way. Applying Lemma 3.1 to the resolution $B(n) \otimes Y(n)$, we obtain an $A$-free resolution of $A/AB(n)$:

\[\begin{array}{ccccccccc}
\vdots & d & d & d & d & \cdots & d & A \otimes Y(n) & d & \cdots & d & A/AB(n) & \epsilon & A/AB(n) & A \otimes Y(n) & A \otimes Y(n) \end{array}\]
Then \((A \otimes Y(1) \otimes A'' \otimes S'')\), \(d \otimes 1 \otimes 1, \delta \otimes 1 \otimes 1\) is an \(A\)-free resolution of \(G = A | AB(1) \otimes A'' \otimes S''\) and \((A \otimes Y(\infty) \otimes S'')\), \(d \otimes 1, \beta \otimes 1\) an \(A\)-free resolution of \((HZ_2)^*(MSp) = A'' \otimes S''\).

We can define an \(A\)-homomorphism \(f_s: A \otimes Y(1)_s \otimes A'' \rightarrow A \otimes Y(\infty)_s \otimes S''\) for each \(s \geq 0\) such that
\[
\{f_s \otimes 1; A \otimes Y(1)_s \otimes A'' \otimes S'' \rightarrow A \otimes Y(\infty)_s \otimes S''\}
\]
is a homomorphism of \(A\)-free resolutions, that is,
\[
(\mu^b \wedge 1)*(\delta \otimes 1 \otimes 1) = (\beta \otimes 1)(f_s \otimes 1)
\]
and
\[
(f_s \otimes 1)(d \otimes 1 \otimes 1) = (d \otimes 1)(f_{s+1} \otimes 1) \quad \text{for any } s \geq 0,
\]
where \((\mu^b \wedge 1)*: A | AB(1) \otimes A'' \otimes S'' \rightarrow A'' \otimes S''\) (Cf. Lemma 3.10). Partial construction of \(\{f_s\}\) is given as the following ((i)~(iii), (i')).

(i) For \((\xi^n; \xi^n, \ldots \xi^n, \ldots) \in A'' = Y(1)_o \otimes A''\),
\[
f_o[(\xi^n; \xi^n, \ldots \xi^n, \ldots)] = (\xi^n; \xi^n, \ldots \xi^n, \ldots) \in A = A \otimes Y(\infty)_o.
\]

(ii) \(f_s(\langle \Delta_0 + \Delta_i \rangle \otimes 2j-1) = 8_{j-1}\langle \Delta_0 + \Delta_i \rangle + 6_{j-1}\langle \Delta_j \rangle \) for \(j \geq 2\),
\[
f_s(\langle \Delta_0 + \Delta_i \rangle \otimes 2) = 8_{0}\langle \Delta_0 + \Delta_i \rangle + 6_{0}\langle \Delta_0 + \Delta_i \rangle + 2_{0}\langle \Delta_i \rangle ,
\]
\[
f_s(\langle \Delta_0 + \Delta_i \rangle \otimes 2j-1) = 8_{j-1}\langle \Delta_0 + \Delta_i \rangle + 4_{j-1}\langle \Delta_{j+1} \rangle ,
\]
\[
f_s(\langle \Delta_0 + \Delta_i \rangle \otimes 1j) = 4_{j}\langle \Delta_0 \rangle + 2_{j}\langle \Delta_i \rangle + 2_{j}\langle \Delta_{j+1} \rangle.
\]

(iii) \(f_s(\langle \Delta_0 + 2\Delta_i \rangle \otimes 2j-1) = 8_{j-1}\langle \Delta_0 + 2\Delta_i \rangle + 6_{j-1}\langle 2\Delta_i + \Delta_j \rangle + 4_{j-1}\langle \Delta_0 + \Delta_i + \Delta_{j+1} \rangle + 2_{j-1}\langle \Delta_0 + \Delta_i + \Delta_{j+1} \rangle \) for \(j \geq 2\),
\[
f_s(\langle \Delta_0 + 2\Delta_i \rangle \otimes 2) = 8_{0}\langle \Delta_0 + 2\Delta_i \rangle + 4_{0}\langle \Delta_0 + \Delta_i \rangle ,
\]
\[
f_s(\langle 2\Delta_i \rangle \otimes 2j-1) = 8_{j-1}\langle 2\Delta_i \rangle + 4_{j-1}\langle \Delta_0 + \Delta_i + \Delta_{j+1} \rangle + \langle 2\Delta_{j+1} \rangle ,
\]
\[
f_s(\langle 2\Delta_i \rangle \otimes 1j) = 4_{j}\langle 2\Delta_0 \rangle + 2_{j}\langle \Delta_0 + \Delta_{j+1} \rangle + \langle 2\Delta_{j+1} \rangle.
\]

We have
\[ \text{Hom}_A(f_s \otimes 1, 1) = f_\ast^\ast \otimes 1 : Y(\infty)^\ast \otimes S'' \to Y(1)^\ast \otimes A'' \otimes S'', \]

where \( f_\ast^\ast : Y(\infty)^\ast \to Y(1)^\ast \otimes A'' \) and \( 1 : S'' \to S''^* \). So we obtain (2-a) and (2-b).

By (iii), we obtain
\[ f_\ast^\ast(y_0 y_j^2 \cdots x_0 x_j^2 y_{j+1}) = y_0 y_j^2 \otimes \xi_j^2 + \text{other terms in } Y(1)^\ast \otimes A'' \quad \text{for } j \geq 1 \]
and
\[ f_\ast^\ast(y_0 y_j^2 \cdots x_0 x_j^2 y_{j+1}) = y_0 \otimes \xi_{j+1} + \text{other terms in } Y(1)^\ast \otimes A''. \]

Obviously we have \( f_\ast^\ast(y_0 y_j^2 \cdots x_0 x_j^2 y_{j+1}) = y_0 y_j^2 \otimes 1 + \text{other terms, so that} \)
\[ f_\ast^\ast(y_0 y_j^2 \cdots x_0 x_j^2 y_{j+1}) = y_0 y_j^2 \otimes \xi_j^2 + y_0 \otimes \xi_{j+1} + \text{other terms in } Y(1)^\ast \otimes A'' \quad \text{for } j \geq 0, \]
where \( \xi_0 = 1 \). No \( y_0 y_j^2 + \text{other terms in } (Y(1)^\ast)' \) is coboundary and
\[ \text{Ext}_{B_0}(Z_2, Z_2) = \{0, \tau_0\}. \]
\( Y(1)^\ast = \{0, y_0^\ast\} \). Therefore we have
\[ h^{bo}(\tau_j) = \tau_{v_j^2 j+1} + \text{higher filtration of terms in } R. \]

From the dimensional reason, (2-c) follows.

(2-d) can be proven by (i').

Now we prove (2-c'). We define a ring homomorphism
\[ \gamma : Z_2[q_0, \tau_0, \{v_s^*\}] \to Z_2[\tau_0, \{v_s^*\}] \]
by \( \gamma(q_0) = 0, \gamma(\tau_0) = \tau_0, \gamma(v_s^*) = v_s^* - 1 \). And we define a decreasing filtration \( \{F^a\} \)
in \( Z_2[\tau_0, \{v_s^*\}] \) by
\[ F^0 = Z_2[\tau_0, \{v_s^*\}], \]
\[ F^{a+1} = (\text{the ideal of } F^a \text{ generated by } \{v_s^* - 1, a \geq 1\} F^a) \]

Then \( F^a F^b \subset F^{a+b} \) and \( \gamma h^{bo}(\tau_j) \equiv v_j^2 j^2 \mod \text{higher filtration}. \)

Let
\[ u = g_0^s u', s' \geq 0, u' \in Z_2[q_0, \{\tau_0\}], \]
\[ u' \text{ is not divisible by } q_0 \text{ in } Z_2[q_0, \{\tau_0\}]. \]

If \( u \neq Z_2[q_0] \) then \( u' \) has the form
\[ u' = \sum_{0 \leq s_0 \leq m, 0 \leq s_1 \leq \cdots, s_j \leq \cdots} b^{(s_0, s_1, \cdots, s_j, \cdots)} \tau_0^s \tau_1^{s_1} \cdots \tau_j^{s_j} \cdots + q_0 u'' \]

\( (m \geq 0 \text{ and there is } (s_1, \cdots, s_j, \cdots) \text{ such that } b^{(s_1, \cdots, s_j, \cdots)} = 0). \)
We have
Therefore we obtain
\[\gamma h^0(u') \equiv \sum_{0 \leq s_1, \ldots, s_i, \ldots, (m, s_1, \ldots, s_i, \ldots) \neq (0)} b^{(m, s_1, \ldots, s_i, \ldots)} \tau_1^{(v_0 v_1)^{s_1}} \tau_i^{(v_i v_{i+1})^{s_i}} \cdots (v_i^2)^{s_i}] \cdots \mod \text{higher filtration},\]
so that (2-c') is proven.

By Lemma 3.11, (2-a) and (2-d), we obtain

**Proposition 3.12.** Let \( k \) be an integer, and \( x \in \text{MSP}_{4k+1} \) represented by an element \( \Phi 0 \) of \( E_{4k+1}^1(MSP) \). Then \( h^{KO}(x) \neq 0 \) in \( KO_{4k+1}(MSP) \).

**Proof.** \( \text{Ext}^1_B(Z_2, Z_2) \) is a \( Z_2 \)-vector space generated by \( \{q_0, k_0, k_1, \ldots, k_j, \ldots\} \).

By Lemma 3.6 and Lemma 3.11, (1), we obtain

**Lemma 3.13.** Let \( s, t \) be integers, and \( u \in E_{4s+t}^1(MSP) \) such that \( q^n u = 0 \) for any integer \( n \geq 0 \). Then \( h(u) \neq 0 \) in \( E_{4s+t}(H \wedge MSP) \).

**Remark.** Lemma 3.13 follows also from [12], PROPOSITION 3.2.

4. Proof of Theorem 1.1

For any integer \( k \), we denote by \( g_k \) the composition of the following sequence of homomorphism

\[\text{MSP}_{4k} \otimes Q \rightarrow H_{4k}(MSP) \otimes Q \xrightarrow{p_k} Q,\]

where \( p_k(x) \) is the coefficient of \( q_k \) in \( x \) for any \( x \in H_{4k}(MSP) \otimes Q \). We have the commutative diagram

Here \( q \) denotes the quotient map, and \( u_1, u_2 \) are the maps such that

\[g_k |_{\text{MSP}_{4k}/\text{Tors}} = u_1 \circ q, \quad g_k |_{W_{4k}^{KO}} = u_2 \circ q\]

(Cf. Corollary 2.2 and (2.4)). Since \( u_2 \) is a monomorphism, Theorem 1.1 is equivalent to
(4.1) \( g_k(MSp_{ib}/\text{Tors}) \supset \tilde{g}_k(W^{K_0}) \) for any integer \( k \).

By [9], (6), we have

(4.2) \( MSp_{ib}/\text{Tors} \otimes Z[\frac{1}{2}] = W^{K_0}_{ib} \otimes Z[\frac{1}{2}] \).

Therefore (4.1) is equivalent to

(4.3) \( 2^k | g_k(MSp_{ib}/\text{Tors}) \Rightarrow 2^k | \tilde{g}_k(W^{K_0}) \) for any integer \( k \).

Let \( E \) be a ring spectrum. Then, obviously, we have

(4.4) \( f_*(W^Z_\mathbb{E}) \subset W^Z_{\mathbb{E} - n} \).

for any morphism \( f: MSp \rightarrow MSp \) of degree \( -n \), where \( f_*: MSp_\mathbb{E} \otimes \mathbb{Q} \rightarrow MSp_{\mathbb{E} - n} \otimes \mathbb{Q} \).

Making use of Proposition 2.6, Lemma 2.5 and (4.4), we can prove the following proposition in the same way as that of Segal [13].

**Proposition 4.1.**

1. For any integer \( k \), \( g_k(W^{K_0}) \) is divisible by \( 2 \). If \( k \) is a power of \( 2 \) then it is divisible by \( 4 \).

2. Let \( k \) be an odd integer. Then \( h(W^{K_0}) \) is divisible by \( 4 \) in \( H_{ak}(MSp) \). In particular, \( g_k(W^{K_0}) \) is divisible by \( 4 \).

And further, making use of some results in §3, we obtain

**Proposition 4.2.** If \( k = 2^j - 1, j \) an integer \( > 0 \), then \( g_k(W^{K_0}) \) is divisible by \( 8 \).

Proof. Let \( x \in W^{K_0}_\mathbb{E} \) and \( x \neq 0 \). By Lemma 3.6, there is an integer \( n \geq 0 \) such that \( 2^n x \in MSp_{ib}/\text{Tors} \subset MSp_{ib} \) is represented by an element of \( Z_\mathbb{E}[q_0, \{ \tau_m \}, \{ v_i; i = 2^n - 1 \}] \cap E_\infty(MSp) \). Let \( 2^n x \) be represented by \( u \in E_\mathbb{E}^s(MSp), s \geq 0 \).

(i) In case \( u \in Z_\mathbb{E}[q_0, \{ v_i; i = 2^n - 1 \}] \): There is a decomposable element \( y \in H_{ak}(MSp) \) such that

\[
\hat{h}(2^n x) \equiv y \mod F^{s+1}H_{ak}(MSp).
\]

Therefore, by Corollary 3.9, (1), \( g_k(2^n x) \) is divisible by \( 2^{s+1} \), so that \( g_k(x) \) is divisible by \( 2^{s+1 - n} \). By Lemma 3.13, \( \hat{h}(2^n x) \) is not divisible by \( 2^{s+1} \), so that \( h(x) \) is not divisible by \( 2^{s+1 - n} \). By Proposition 4.1, (2), we have \( s+1-n \geq 3 \). Consequently \( g_k(x) \) is divisible by \( 8 \).

(ii) In case \( u \in Z_\mathbb{E}[q_0, \{ v_i; i = 2^n - 1 \}] \): By Lemma 3.11, (2-a) and (2-c'), we have

\[
h^{bo}(u) \in Z_\mathbb{E}[q_0, \{ v_i \}] \subset E_\mathbb{E}(bo \wedge MSp).
\]

By (2.2), \( h^{bo}(u) \in bo \mathbb{E}(MSp) \). Let \( h^{bo}(u) \) be represented by \( w \in E_\mathbb{E}^{s,k+1}(bo \wedge MSp) \).

By Proposition 3.4, (2), \( h^{bo}(2^n x) = 2^n h^{bo}(x) \) is represented by \( g^*_k w \), so that
\[ h^{bo}(u) = q^u w, \quad w \in E^{s-n}_*(bo \wedge MS\ell). \]

Then \( w \in \mathbb{Z}_2[q_0, \{v_i\}] \), so that \( s-n \geq 3 \). \( h(2^n x) \) is divisible by \( 2^n \), so that \( h(x) \) is divisible by \( 2^{s-n} \). Consequently \( h(x) \) is divisible by 8.

Let \( n_j(n_1, n_2, \ldots, n_r) \in MS\ell_{pN-j} \) be the Stong-Ray classes in [11], where

\[ N = \sum_{i=1}^r (2n_i - 1). \]

**Proposition 4.3.**

1. (Segal [13]) For an even integer \( k > 0 \), we define integers \( s_k \) and \( t_k \) as follows. If \( k \) is not a power of 2 then we define \( s_k = 2^u + 1, 2^u \) the largest power of 2 less than \( k \), and \( t_k = k - s_k + 2 \). If \( k = 2^j \) then we define \( s_k = t_k = 2^{j-1} + 1 \). Then we have

\[ g_k(n_1(s_k, t_k)) = \begin{cases} 
2 \mod 4 & \text{if} \quad k \equiv 0 \mod 2, k \neq 2^j \\
4 \mod 8 & \text{if} \quad k = 2^j
\end{cases} \]

2. Using the notation of (1), we have

\[ g_k(n_2(s_{k+1}, t_{k+1})) = \begin{cases} 
4 \mod 8 & \text{if} \quad k \equiv 1 \mod 2, k \neq 2^j - 1 \\
8 \mod 16 & \text{if} \quad k = 2^j - 1
\end{cases} \]

(Segal [13] has proven the fact that \( g_k(MS\ell_{p^k}/\text{Tors}) \) is not divisible by 8 if \( k \equiv 1 \mod 2, k \neq 2^j - 1 \).

Now (4.3) follows from Propositions 4.1, 4.2 and 4.3, so that Theorem 1.1 is proven.

As a corollary to Proposition 4.3, we obtain

**Proposition 4.4.** \( \{n_j(n_1, n_2, \ldots, n_r) \} \) generates \( Q(MS\ell_*/\text{Tors}) \cong Q(W^K_*) \).

Proof. From Stong [17], Theorem 1, it follows that \( \{n_j(n_1, n_2, \ldots, n_r) \} \) generates \( Q(MS\ell_*/\text{Tors}) \otimes \mathbb{Z}_p \) for any odd prime \( p \).

5. **Proof of Theorem 1.2 and some remarks**

For integers \( k, s \geq 0 \), we put

\[ F^*_i = h(MS\ell_{p^k}) \cap F^*H_{ab}(MS\ell) \]

and

\[ F^*_s = h(W^{K_0}) \cap F^*H_{ab}(MS\ell). \]

The following lemma follows immediately from the definition.

**Lemma 5.1.** For \( m = 1 \) or 2, the inclusion \( F^*_m \to H_{ab}(MS\ell) \) induces the monomorphism
Lemma 5.2. \( MSp_{4k}/\text{Tors} = W^k_{14} \) if and only if
\[
F_1^{s+1}F_1^{s+1}/F_1^{s+1}F_1^{s+1} = E_{4k}^{s+4k}(H \wedge MSp)
\]
for any \( s \geq 0 \).

Proof. By (4.2), we have
\[
h(MSp_{4k}) \otimes \mathbb{Z}[\frac{1}{2}] = h(W^k_{14}) \otimes \mathbb{Z}[\frac{1}{2}].
\]
Therefore there is an integer \( s_0 = s_0(k) \) such that \( F_1^s = F_2^s \) for any \( s \geq s_0 \). Then it is easy to see that \( h(MSp_{4k}) = h(W^k_{14}) \) if and only if
\[
F_1^{s+1}F_1^{s+1} = F_1^{s+1}F_1^{s+1}
\]
for any \( s \geq 0 \).

Since \( h : MSp_{4k} \otimes \mathbb{Q} \to H_*(MSp) \otimes \mathbb{Q} \) is an isomorphism, the lemma follows.

By Theorem 1.1, Proposition 2.6, Lemmas 3.13, 5.2 and Segal [12], TABLE II, we obtain

Lemma 5.3. \( MSp_{4k}/\text{Tors} = W^k_{14} \) for \( k \leq 7 \).

By Lemma 5.3, Proposition 2.6, Lemma 3.11, (2) and [12], TABLE II, we can prove

Lemma 5.4.
\[
\text{order of } MSp_n = \text{order of } h^{\wedge 0}(MSp_n)
\]
for \( n \leq 30, n \equiv 0 \mod 4 \).

Since \( MSp_{4k} \) is torsion free for \( k \leq 7 \) by [12], Theorem 1.2 follows from Lemmas 5.3 and 5.4.

Making use of the Ray classes \( \phi \in MSp_{a_{1-1}} \) in [8], we can immediately calculate the ring structure of \( MSp_8 \) in dimensions \( \leq 30 \) except the values of \( \alpha^2 \tau \) and \( \alpha^2 \tau \), where \( \alpha \) is the generator of \( MSp_1 = \mathbb{Z}_2 \) (Cf. Ray [10], (5.25)). For example, we have

Proposition 5.5. For \( k \leq 5 \),
\[
x_1^2 MSp_{4k+1} \subset \alpha MSp_{4k+8} \quad \text{and} \quad x_7^2 MSp_{8k+2} \subset \alpha^2 MSp_{8k+8}.
\]

We can calculate the Hurewicz map (1.1) for \( n = 17 \):

Proposition 5.6. There is an indecomposable element \( \tau \in MSp_{17} \) such that
\[
h^{\wedge 0}(\tau) = e(\sigma_1^2 + y\sigma_2).
\]

Proof. Using the notation of [12], \( x_7^1 \) is represented by \( \omega_0 \) and \( 2y_1 \) by \( q_0v_2^5 \). Therefore \( 2x_7^1y_1 \) is represented by \( q_0\omega_0v_2^5 \). Since
Let \( \tau' \in \text{MSp}_{17} \) be a class represented by \( k_0 v_2^3 \). Then \( x^4 \tau' \) is represented by \( k_0 \omega v_2^3 \), so that

\[
x^4 \tau' \equiv \alpha(x_1 x_2 x_3 + x_1^2(x_2^3 + y_4 + x_i)) \mod F^4 \text{MSp}_{25}.
\]

Therefore

\[
yhK^O(\tau') = hK^O(x^4 \tau') \equiv ye(\sigma_2^2 + y \sigma_2) \mod hK^O(F^4 \text{MSp}_{25}).
\]

Since \( hK^O(F^4 \text{MSp}_{25}) = yhK^O(F^2 \text{MSp}_{17}) \), there is an element \( \lambda \in F^2 \text{MSp}_{17} \), such that

\[
yhK^O(\tau') = ye(\sigma_2^2 + y \sigma_2) + yhK^O(\lambda),
\]

\[
hK^O(\tau') = e(\sigma_2^2 + y \sigma_2) + hK^O(\lambda).
\]

We may take \( \tau = \tau' + \lambda \).

Let \( vE^{\bullet \bullet}(\text{MSp}) \) denote the Adams-Novikov spectral sequence for \( \text{MSp}_* \) (Cf. [5]). Proposition 2.6 shows us the structure of

\[
\text{MSp}_*/\text{Tors} = vE^{\bullet \bullet}(\text{MSp}) \subset vE^{\bullet \bullet}(\text{MSp})
\]

in low dimensions:

**Proposition 5.7.**

1. (Porter [6]) \( vE^{\bullet \bullet}(\text{MSp}) \cong \{ x \in \text{MSp}_* \otimes \mathbb{Q} ; r_*(x) \in MU_* \} \).
2. \( \{ x \in \text{MSp}_* \otimes \mathbb{Q} ; r_*(x) \in MU_* \} = W_*^K. \)

**Proof of (2).** Consider the commutative diagram

\[
\begin{array}{c}
\text{MSp}_* \otimes \mathbb{Q} \quad \xrightarrow{hK} \quad K_*(\text{MSp}) \otimes \mathbb{Q} \\
\downarrow r_* \quad \downarrow r_* \\
MU_* \otimes \mathbb{Q} \quad \xrightarrow{hK} \quad K_*(MU) \otimes \mathbb{Q}.
\end{array}
\]

Then \( r_*: K_*(\text{MSp}) \to K_*(MU) \) is a split monomorphism. And, by Hattori [2] or Stong [16], \( hK^*: MU_* \to K_*(MU) \) is a split monomorphism.

**OSAKA CITY UNIVERSITY**
References