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Author(s)	Ōkita, Rikio
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ON THE MSp HATTORI-STONG PROBLEM

RIKIO ŌKITA

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1. Introduction

In the present paper, we work in the category of CW -spectra due to Adams [1]. For any ring spectrum E , we denote by $E_*(\)$ and $E^*(\)$ the associated homology and cohomology functors and by E_* the coefficient ring. The unit of E is denoted by $u^E: S \rightarrow E$. Let E be a ring spectrum and F a spectrum. Consider the spectrum morphism

$$u^E \wedge 1: F = S \wedge F \rightarrow E \wedge F.$$

Then $u^E \wedge 1$ induces the generalized Hurewicz map

$$h^E = (u^E \wedge 1)_*: F_* \rightarrow E_*(F).$$

For $E=H$, we denote h^E simply by h .

Ray [7] has conjectured that the Hurewicz map

$$(1.1) \quad h^{KO}: MSp_n \rightarrow KO_n(MSp)$$

is a split monomorphism for any integer n and has shown that it is a split monomorphism for $n \leq 20$. Later Segal [14] has shown that the map (1.1) is not a monomorphism for $n=31$ (since $MSp_{31} \cong \mathbb{Z}_2$) and that Ray's MSp Hattori-Stong conjecture is false.

But still we may conjecture that the map

$$(1.2) \quad h^{KO}/\text{Tors}: MSp_*/\text{Tors} \rightarrow KO_*(MSp)/\text{Tors}$$

is a split monomorphism, where Tors denotes the torsion subgroup.

For any ring spectrum E , we put

$$W_*^E = \{x \in MSp_* \otimes \mathbb{Q}; h^E(x) \in E_*(MSp)/\text{Tors} \subset E_*(MSp) \otimes \mathbb{Q}\}.$$

Then $MSp_*/\text{Tors} \subset W_*^E$. And the map (1.2) is a split monomorphism if and only if $MSp_*/\text{Tors} = W_*^{KO}$.

Let L_* be a subring of $MSp_* \otimes \mathbb{Q}$. We put

$$Q(L_*) = L_*/(L_* \cap D_*),$$

where D_* is the ideal of all decomposable elements in $MSp_*/\text{Tors} \otimes \mathbb{Q}$.

In this paper, we prove the following two theorems.

Theorem 1.1. *The inclusion $i: MSp_*/\text{Tors} \rightarrow W_*^{KO}$ induces the isomorphism*

$$i_*: Q(MSp_*/\text{Tors}) \cong Q(W_*^{KO})$$

(Cf. Proposition 3.12).

Theorem 1.2. *The Hurewicz map*

$$h^{KO}: MSp_n \rightarrow KO_n(MSp)$$

is a split monomorphism for $n \leq 30$. In particular, we have

$$MSp_n/\text{Tors} = W_n^{KO} \quad \text{for } n < 32.$$

The author wishes to express his hearty appreciation to Professor S. Araki.

2. Calculations in W_*^K and W_*^{KO}

We denote by ${}_n i: CP^n \rightarrow CP^\infty$ (resp. ${}_n i: HP^n \rightarrow HP^\infty$) the inclusion map. Let E be a ring spectrum having a class $x \in \tilde{E}^2(CP^\infty)$ (resp. $x \in \tilde{E}^4(HP^\infty)$) such that

$$E^*(CP^n) = E_*[{}_n x]/({}_n x^{n+1}) \quad (\text{resp. } E^*(HP^n) = E_*[{}_n x]/({}_n x^{n+1}))$$

for each integer $n \geq 1$ and ${}_1 x \in \tilde{E}^2(CP^1) = \tilde{E}^2(S^2)$ (resp. ${}_1 x \in \tilde{E}^4(HP^1) = \tilde{E}^4(S^4)$) is represented by the unit u^E , where ${}_n x = {}_n i^*(x)$. As is well known, x determines the Thom isomorphism $\phi: E_*(BU) \cong E_*(MU)$ (resp. $\phi: E_*(BSp) \cong E_*(MSp)$). Let $j: CP^\infty \rightarrow BU$ (resp. $j: HP^\infty \rightarrow BSp$) be the inclusion map and $y_i \in E_*(CP^\infty)$ (resp. $y_i' \in E_*(HP^\infty)$) dual to x^i . Put $y_i = \phi j_*(y_i')$. Then we have

$$\begin{aligned} E_*(MU) &= E_*[y_1, y_2, \dots, y_i, \dots] \\ (\text{resp. } E_*(MSp) &= E_*[y_1, y_2, \dots, y_i, \dots]), \end{aligned}$$

where $y_i \in E_{2i}(MU)$ (resp. $y_i \in E_{4i}(MSp)$).

In $\widetilde{H}^2(CP^\infty)$, choose x to be c_1 , the first Chern class of the universal $U(1)$ -bundle ζ^1 over CP^∞ . In this case, we denote y_i by b_i . Then we have

$$H_*(MU) = \mathbb{Z}[b_1, b_2, \dots, b_i, \dots], \quad b_i \in H_{2i}(MU).$$

In $\widetilde{MU}^2(CP^\infty)$, choose x to be cf_1 , the first Conner-Floyd Chern class of ζ^1 , represented by the homotopy equivalence $CP^\infty \simeq MU(1)$.

Let $z \in K_2$ be such that ${}_1 i^*(\zeta^1 - 1) = z\gamma$ in $K^0(CP^1)$, where $\gamma \in \tilde{K}^2(CP^1) = \tilde{K}^2(S^2)$ is represented by the unit u^K . Then we have

$$K_* = \mathbb{Z}[z, z^{-1}] \quad \text{and} \quad H_*(K) = \mathbb{Q}[t, t^{-1}],$$

where $t=h(z)$.

In $\tilde{K}^2(CP^\infty)$, choose x to be $z^{-1}(\zeta^1 - 1)$. As is well known, there is a unique ring spectrum morphism $g: MU \rightarrow K$ such that $g_*(cf_i) = z^{-1}(\zeta^1 - 1)$.

In $\tilde{H}^4(HP^\infty)$, choose x to be p_1 , the first symplectic Pontrjagin class of the universal $Sp(1)$ -bundle ξ^1 over HP^∞ . In this case we denote y_i by q_i . Then we have

$$H_*(MSp) = \mathbb{Z}[q_1, q_2, \dots, q_i, \dots], q_i \in H_{4i}(MSp).$$

In $\widetilde{MSp}^*(HP^\infty)$, choose x to be $p_1 f_i$, the first Conner-Floyd symplectic Pontrjagin class of ξ^1 , represented by the homotopy equivalence $HP^\infty \simeq MSp(1)$. In this case, we denote y_i by $q f_i$.

Put $\kappa_i = (gr)_*(q f_i) \in K_*(MSp)$, where $r: MSp \rightarrow MU$ is the morphism induced by the inclusion $Sp \rightarrow U$. Then we have

$$K_*(MSp) = K_*[\kappa_1, \kappa_2, \dots, \kappa_i, \dots], \kappa_i \in K_{4i}(MSp).$$

Let bu denote the connective BU -spectrum and $\psi: bu \rightarrow K$ the canonical morphism. Then we have

$$\psi_*: bu_* \cong K_* \quad \text{if } n \geq 0, bu_* = 0 \quad \text{if } n < 0.$$

And let $\tilde{\kappa}_i \in bu_*(MSp)$ be the unique class such that $\psi_*(\tilde{\kappa}_i) = \kappa_i \in K_*(MSp)$. Then we have

$$bu_*(MSp) = bu_*[\tilde{\kappa}_1, \tilde{\kappa}_2, \dots, \tilde{\kappa}_i, \dots].$$

Therefore $\psi_*: bu_*(MSp) \rightarrow K_*(MSp)$ is a split monomorphism, so that we have

$$(2.1) \quad W_*^{bu} = W_*^K.$$

Similarly we have

$$(2.2) \quad W_*^{bo} = W_*^{KO},$$

where bo denotes the connective BO -spectrum.

We have a Künneth isomorphism

$$H_*() \otimes H_*(MSp) \cong H_*(\wedge MSp)$$

since $H_*(MSp)$ is torsion free. By this isomorphism we identify $H_*() \otimes H_*(MSp)$ and $H_*(\wedge MSp)$.

Lemma 2.1. Consider the commutative diagram

$$\begin{array}{ccc} MSp_* \otimes Q & \xrightarrow[\cong]{h} & H_*(MSp) \otimes Q \\ \downarrow h^K & & \downarrow j \\ K_*(MSp) \otimes Q & \xrightarrow[\cong]{h} & H_*(K) \otimes H_*(MSp) \otimes Q, \end{array}$$

where $j = (u^K \wedge 1)_* : H_*(MSp) \rightarrow H_*(K \wedge MSp) = H_*(K) \otimes H_*(MSp)$. Then we have

$$j(x) = 1 \otimes x$$

for any $x \in H_*(MSp)$ and

$$\begin{aligned} h(h^K(W_*^K)) &= h(K_*(MSp)) \cap j(H_*(MSp)) \\ &= h(\mathbb{Z}[z^2, \kappa_1, \kappa_2, \dots, \kappa_i, \dots]) \cap j(H_*(MSp)) . \end{aligned}$$

Proof. It is proven by diagram chasing that

$$j(x) = 1 \otimes x$$

for any $x \in H_*(MSp)$.

We have the following commutative diagram

$$\begin{array}{ccccc} MSp_* \otimes \mathbb{Q} & \xrightarrow[h]{\simeq} & H_*(MSp) \otimes \mathbb{Q} & \xrightarrow{j} & \text{---} \\ \downarrow h^{bu} & & \downarrow \tilde{j} & & \text{---} \\ bu_*(MSp) \otimes \mathbb{Q} & \xrightarrow[h]{\simeq} & H_*(bu) \otimes H_*(MSp) \otimes \mathbb{Q} & \xrightarrow{\psi_* \otimes 1} & \text{---} \\ \downarrow \psi_* & & \downarrow & & \text{---} \\ K_*(MSp) \otimes \mathbb{Q} & \xrightarrow[h]{\simeq} & H_*(K) \otimes H_*(MSp) \otimes \mathbb{Q} & \xrightarrow{j} & \text{---} \end{array}$$

where $\tilde{j} = (u^{bu} \wedge 1)_* : H_*(MSp) \rightarrow H_*(bu \wedge MSp) = H_*(bu) \otimes H_*(MSp)$.

Now let $x \in W_*^K = W_*^{bu}$ (Cf. (2.1)). Then there is an integer $n \neq 0$ such that $nx \in MSp_*/\text{Tors}$. We have

$$\begin{aligned} nh(h^{bu}(x)) &= h(h^{bu}(nx)) \\ &= \tilde{j}(h(nx)) \in \tilde{j}(H_*(MSp)) . \end{aligned}$$

Since $\tilde{j}/\text{Tors} : H_*(MSp) \rightarrow H_*(bu)/\text{Tors} \otimes H_*(MSp)$ is a split monomorphism, $h(h^{bu}(x)) \in \tilde{j}(H_*(MSp))$. Therefore we obtain

$$h(h^K(x)) \in j(H_*(MSp)) .$$

By (2.1) and dimensional reason, we obtain

$$\begin{aligned} h^K(x) &\in \mathbb{Z}[z^2, \kappa_1, \kappa_2, \dots, \kappa_i, \dots] , \\ h(h^K(x)) &\in h(\mathbb{Z}[z^2, \kappa_1, \kappa_2, \dots, \kappa_i, \dots]) . \end{aligned}$$

Conversely let $y \in K_*(MSp)$ and $h(y) \in i(H_*(MSp))$. Then

$$h(y) \in j(h(MSp_* \otimes \mathbb{Q})) = h(h^K(MSp_* \otimes \mathbb{Q})) ,$$

so that $y \in h^K(MSp_* \otimes \mathbb{Q})$. Consequently we obtain

$$y \in h^K(W_*^K), h(y) \in h(h^K(W_*^K)).$$

Corollary 2.2. $h(W_*^K) \subset H_*(MSp)$.

It is well known that

$$(2.3) \quad g_*(b_i) = t^i/(i+1)! ,$$

where $g_*: H_*(MU) \rightarrow H_*(K)$. And we have

Lemma 2.3.

$$(gr)_*(q_i) = 2t^{2i}/[2(i+1)]! ,$$

where $(gr)_*: H_*(MSp) \rightarrow H_*(K)$.

Proof. We have

$$r_*(q_i) = 2[b_{2i} - b_1 b_{2i-1} + \dots + (-1)^{i-1} b_{i-1} b_{i+1}] + (-1)^i b_i^2 ,$$

so that the lemma follows immediately from (2.3).

Consider the commutative diagram

$$\begin{array}{ccc} MSp_*(MSp) & \xrightarrow{h} & H_*(MSp) \otimes H_*(MSp) \\ \downarrow (gr)_* & & \downarrow (gr)_* \otimes 1 \\ K_*(MSp) & \xrightarrow{h} & H_*(K) \otimes H_*(MSp) . \end{array}$$

By definition, $(gr)_*(qf_i) = \kappa_i$. Therefore we have

$$h(\kappa_i) = (gr)_* \otimes 1(h(qf_i)) ,$$

so that, by Ray [9], (5.6) and Lemma 2.3, we can calculate the Hurewicz map

$$h: K_*(MSp) \rightarrow H_*(K) \otimes H_*(MSp) .$$

Therefore, by Lemma 2.1 and the fact that $h^K: W_*^K \rightarrow K_*(MSp)$ is a monomorphism, we obtain

Proposition 2.4. W_*^K is generated by elements

$$x_i (1 \leq i \leq 7), y_4, y_6 \text{ and } y_7$$

in dimensions < 32 , where x_i ($1 \leq i \leq 6$) are defined by

$$\begin{aligned}
h^K(x_1) &= z^2 + 12\kappa_1, \\
h^K(x_2) &= z^2\kappa_1 - 4\kappa_1^2 + 10\kappa_2, \\
h^K(x_3) &= z^2(-3\kappa_1^2 + 4\kappa_2) + 12\kappa_1^3 - 36\kappa_1\kappa_2 + 28\kappa_3, \\
h^K(x_4) &= z^2(\kappa_1^3 - 2\kappa_1\kappa_2 + \kappa_3) - 4\kappa_1^4 + 14\kappa_1^2\kappa_2 - 4\kappa_2^2 - 12\kappa_1\kappa_3 + 6\kappa_4, \\
h^K(x_5) &= z^2(-7\kappa_1^4 + 18\kappa_1^2\kappa_2 - 4\kappa_2^2 - 11\kappa_1\kappa_3 + 4\kappa_4) \\
&\quad + 28\kappa_1^5 - 112\kappa_1^3\kappa_2 + 66\kappa_1\kappa_2^2 + 96\kappa_1^2\kappa_3 - 38\kappa_2\kappa_3 - 62\kappa_1\kappa_4 + 22\kappa_5, \\
h^K(x_6) &= z^4(-2\kappa_1^4 + 5\kappa_1^2\kappa_2 - \kappa_2^2 - 3\kappa_1\kappa_3 + \kappa_4) \\
&\quad + z^2(-3\kappa_1^5 - 10\kappa_1^3\kappa_2 + 24\kappa_1\kappa_2^2 + 13\kappa_1^2\kappa_3 - 18\kappa_2\kappa_3 - 14\kappa_1\kappa_4 + 8\kappa_5) \\
&\quad + 44\kappa_1^6 - 150\kappa_1^4\kappa_2 + 15\kappa_1^2\kappa_2^2 + 25\kappa_2^3 + 140\kappa_1^3\kappa_3 + 36\kappa_1\kappa_2\kappa_3 - 12\kappa_3^2 - 84\kappa_1^2\kappa_4 \\
&\quad - 45\kappa_2\kappa_4 + 18\kappa_1\kappa_5 + 13\kappa_6
\end{aligned}$$

and

$$y_4 = (-x_2^2 + x_1x_3)/4, \quad y_6 = (-x_2x_4 + x_1x_5)/2 \quad \text{and} \quad y_7 = (-x_3x_4 + x_2x_5)/2.$$

And we have

Lemma 2.5. Let $x \in W_*^K$, and

$$h^K(x) = f(z, \kappa_1, \kappa_2, \dots, \kappa_i, \dots) \in \mathbb{Z}[z, \kappa_1, \kappa_2, \dots, \kappa_i, \dots].$$

Then

$$h(x) = f(0, q_1, q_2, \dots, q_i, \dots) \in H_*(MSp).$$

For example,

$$\begin{aligned}
h(x_1) &= 12q_1, \\
h(x_2) &= -4q_1^2 + 10q_2, \\
h(x_3) &= 12q_1^3 - 36q_1q_2 + 28q_3, \\
h(x_4) &= -4q_1^4 + 14q_1^2q_2 - 4q_2^2 - 12q_1q_3 + 6q_4.
\end{aligned}$$

Proof. Notice that

$$h(\kappa_i) \equiv 1 \otimes q_i \pmod{t \otimes 1} \quad \text{in} \quad \mathbb{Q}[t] \otimes H_*(MSp)$$

where $h: K_*(MSp) \rightarrow H_*(K) \otimes H_*(MSp)$. Then the lemma follows from Lemma 2.1.

Let $c: KO \rightarrow K$ be the complexification morphism. As is well known, KO_* is generated by the classes

$$e \in KO_1, \quad x \in KO_4, \quad y \in KO_8 \quad \text{and} \quad y^{-1} \in KO_{-8}$$

subject to the relations

$$2e = e^3 = ex = 0, \quad x^2 = 4y \quad \text{and} \quad yy^{-1} = 1$$

such that

$$c_*(x) = 2x^2 \quad \text{and} \quad c_*(y) = x^4 \quad \text{in } K_*.$$

Let $\sigma_i \in KO_{*i}(MSp)$ be the unique class such that $c_*(\sigma_i) = \kappa_i \in K_{*i}(MSp)$. Then we have

$$KO_*(MSp) = KO_*[\sigma_1, \sigma_2, \dots, \sigma_i, \dots],$$

and

$$(2.4) \quad W_*^{KO} \subset W_*^K.$$

As a corollary to Proposition 2.4, we obtain

Proposition 2.6. W_{4k}^{KO} has the following generators for $k \leq 7$.

$$\begin{aligned} k = 1: & 2x_1. \\ k = 2: & x_1^2, 2x_2. \\ k = 3: & 2x_1^3, x_1x_2, 2x_3. \\ k = 4: & x_1^4, 2x_1^2x_2, x_1x_3, 2y_4, 2x_4. \\ k = 5: & 2x_1^5, x_1^3x_2, 2x_1^2x_3, 2x_1y_4, x_2x_3, x_1x_4, 2x_5. \\ k = 6: & x_1^6, 2x_1^4x_2, x_1^3x_3, 2x_1x_2x_3, 2x_2y_4, x_3^2, 2x_1^2x_4, \\ & x_1x_2x_3 + x_1^2(y_4 + x_4), x_2x_4, x_1x_5, 2x_6. \\ k = 7: & 2x_1^7, x_1^5x_2, 2x_1^4x_3, 2x_1^3y_4, x_1^2x_2x_3, 2x_1x_3^2, 2x_3y_4, x_1^3x_4, \\ & x_1x_3^2 + x_1x_2(y_4 + x_4), x_3x_4, 2x_1^2x_5, x_1y_6, x_2x_5, 2x_1x_6, \tilde{x}_7. \end{aligned}$$

REMARK.

$$W_*^{MSU} = W_*^{KO}, h^{MSU}(W_*^{MSU}) = H\text{-}Sp_*,$$

where $H\text{-}Sp_*$ is the algebra of Ray [10], (2.1), and

$$h(2x_i) = h_i \in H_*(MSp)$$

for $i \leq 4$, where h_i are the classes in [10], (3.7) (Cf. Lemma 2.5)

3. Adams spectral sequence maps

For any connective spectrum X such that X_r is finitely generated for each r , we denote by $E_{*r}^{**}(X)$ the mod 2 Adams spectral sequence for X_* (Cf. [3], 2.2). For an integer n , we denote by $F^s X_n$ the s -th filtration in the mod 2 Adams spectral sequence. Then we have

$$F^s X_n / F^{s+1} X_n = E_{\infty}^{s, s+n}(X) = E_r^{s, s+n}(X) \quad (r \text{ large})$$

Let H be a graded vector space over \mathbb{Z}_2 . We define a graded vector space H' from H by

$$H'_{2n} = H_n, H'_{2n+1} = 0$$

for any integer n . For any connected Hopf algebra H over \mathbf{Z}_2 , we denote the augmentation ideal $\sum_{i>0} H_i$ by \bar{H} .

We denote the mod 2 Steenrod Algebra by A . Let A'' be endowed with structure as a graded A -module by the following A -action.

$$A \otimes A'' \xrightarrow{\beta \otimes 1} A'' \otimes A'' \xrightarrow{\mu} A''.$$

Here $\beta: A \rightarrow A''$ is the map such that $\beta^*(x) = x^* \in A^*$ for any $x \in A''^*$ and μ is the product map in A . Using the notation of Milnor [4], we denote $(\zeta_{j+1}^m)^*$ by m_j for any integers $m, j \geq 0$. For any n ($0 \leq n \leq \infty$), let $B(n)$ be the Hopf subalgebra of A (multiplicatively) generated by the elements $1_0, 2_j$ for $j < n$. The map β induces the isomorphism

$$(3.1) \quad A//B \cong A'',$$

where $B = B(\infty)$.

Let R be a Hopf subalgebra of A , and (C, d_C, ε_C) a R -free resolution of \mathbf{Z}_2 . As is well known, A is free as a right R -module and we have the isomorphism $A/A\bar{R} \cong A \otimes_R \mathbf{Z}_2$ of A -modules. So we obtain

Lemma 3.1. There is an A -free resolution of $A/A\bar{R}$:

$$A/A\bar{R} \xleftarrow{1 \otimes \varepsilon_C} A \otimes_R C_0 \xleftarrow{1 \otimes d_C} A \otimes_R C_1 \xleftarrow{1 \otimes d_C} \cdots \xleftarrow{1 \otimes d_C} A \otimes_R C_i \xleftarrow{1 \otimes d_C} \cdots.$$

The following proposition is well known.

Proposition 3.2.

$$(1) \quad (\text{Serre [15]}) (H\mathbf{Z}_2)^*(H) \cong A/A\bar{B}(0)$$

as graded A -modules.

$$(2) \quad (\text{Cf. [1], §16}) (H\mathbf{Z}_2)^*(bo) \cong A/A\bar{B}(1)$$

as graded A -modules.

$$(3) \quad (\text{Cf. [3], THEOREM II. 4}) (H\mathbf{Z}_2)^*(MSp) \cong A'' \otimes S''$$

as graded coalgebra and A -modules (A operating on S'' trivially), where S is the graded coalgebra over \mathbf{Z}_2 such that

$$S^* \cong \mathbf{Z}_2[V_2, V_4, V_5, \dots, V_i, \dots], \quad i \neq 2^a - 1, \quad \deg V_i = i.$$

As a result of Proposition 3.2, the following proposition is obtained by (3.1) and Lemma 3.1.

Proposition 3.3.

- (1) $E_2(H) \cong \text{Ext}_{B(0)}(\mathbf{Z}_2, \mathbf{Z}_2).$
- (2) $E_2(\text{bo}) \cong \text{Ext}_{B(1)}(\mathbf{Z}_2, \mathbf{Z}_2).$
- (3) $E_2(\text{MSp}) \cong \text{Ext}_B(\mathbf{Z}_2, \mathbf{Z}_2) \otimes \mathbf{Z}_2[v_2, v_4, v_5, \dots, v_i, \dots],$
 $i \neq 2^a - 1, v_i = [V_i] \in E_2^{0,4i}(\text{MSp}).$

A $B(n)$ -free resolution of \mathbf{Z}_2 has been constructed by Liulevicius [3]. Let $Y(n)$ be the \mathbf{Z}_2 -vector space with basis

$$\left\{ I \otimes J; \begin{array}{l} I = (i_0, i_1, \dots, i_{n-1}), J = (j_0, j_1, \dots, j_n), \text{ where } I, J \text{ are} \\ \text{sequences of non-negative, finitely non-zero integers.} \end{array} \right\}.$$

Let

$$\deg I \otimes J = (\sum (i_r + j_r), \sum [i_r(2^{r+2} - 2) + j_r(2^{r+1} - 1)]).$$

We define a $B(n)$ -homomorphism $d(n): B(n) \otimes Y(n) \rightarrow B(n) \otimes Y(n)$ by

$$\begin{aligned} d(n)(I \otimes J) = & \sum_k [1_k I \otimes (J - \Delta_k) + 2_k (I - \Delta_k) \otimes J \\ & + (j_{k+1} + 1)(I - \Delta_k) \otimes (J - \Delta_0 + \Delta_{k+1}) \\ & + (j_{k+1} + 1)1_0 (I - \Delta_0 - \Delta_k) \otimes (J + \Delta_{k+1}) \\ & + \binom{j_{k+1} + 2}{2} (I - \Delta_0 - 2\Delta_k) \otimes (J + 2\Delta_{k+1})] \\ & + \sum_{k < t} (j_{k+1} + 1)(j_{t+1} + 1)(I - \Delta_0 - \Delta_k - \Delta_t) \otimes (J + \Delta_{k+1} + \Delta_{t+1}). \end{aligned}$$

Here we set $I - \Delta_r = 0$ if $i_r = 0$ and $J - \Delta_r = 0$ if $j_r = 0$. Then

$$B(n) \otimes Y(n) = (B(n) \otimes Y(n), d(n), \varepsilon(n))$$

is the $B(n)$ -free resolution of \mathbf{Z}_2 constructed by him, where $\varepsilon(n): B(n) \otimes Y(n)_0 \rightarrow \mathbf{Z}_2$ is the unique $B(n)$ -homomorphism. Put

$$\langle J \rangle = (0) \otimes J.$$

Then we have

$$d(n)\langle J \rangle = \sum 1_k \langle J - \Delta_k \rangle.$$

Using the notation of [3] for $\text{Hom}_{B(n)}(B(n) \otimes Y(n), \mathbf{Z}_2) = Y(n)^*$, let

$$\begin{aligned} k_j &= [x_j] \in \text{Ext}_{B(n)}^{1, 2^{j+2}-2}(\mathbf{Z}_2, \mathbf{Z}_2), \\ q_0 &= [y_0] \in \text{Ext}_{B(n)}^{1, 1}(\mathbf{Z}_2, \mathbf{Z}_2), \\ \tau_j &= [y_0 y_{j+1}^2 + x_0 x_j y_{j+1}] \in \text{Ext}_{B(n)}^{3, 2^{j+3}-1}(\mathbf{Z}_2, \mathbf{Z}_2), \\ \omega_0 &= [y_1^4] \in \text{Ext}_{B(n)}^{4, 12}(\mathbf{Z}_2, \mathbf{Z}_2). \end{aligned}$$

Proposition 3.4. (Liulevicius [3])

- (1) $\text{Ext}_{B(0)}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[q_0]$.
- (2) $\text{Ext}_{B(1)}(\mathbb{Z}_2, \mathbb{Z}_2)$ has multiplicative generators q_0, k_0, τ_0 and ω_0 with bidegrees $(1,1), (1,2), (3,7)$ and $(4,12)$ respectively subject to the relations

$$q_0 k_0 = 0, k_0^3 = 0, k_0 \tau_0 = 0 \quad \text{and} \quad \tau_0^2 = q_0^2 \omega_0.$$

Corollary 3.5.

- (1) $E_\infty(H) = E_2(H)$.
- (2) $E_\infty(\text{bo}) = E_2(\text{bo})$.

Lemma 3.6. For any integer n , there is an integer $s_0 = s_0(n)$ such that

$$\text{Ext}_B^{s,s+n}(\mathbb{Z}_2, \mathbb{Z}_2) = (\mathbb{Z}_2[q_0, \{\tau_j\}])^{s,s+n} \quad \text{if } s \geq s_0.$$

Proof. Let $\tilde{B}(m)$ be the Hopf subalgebra of B (multiplicatively) generated by $B(m), 1_{m+1}$ ($0 \leq m < \infty$). By Segal [12], PROPOSITION 2.3, there is a spectral sequence ${}_m E_*^{***}$ such that

$$\begin{aligned} {}_m E_1 &= \text{Ext}_{\tilde{B}(m)}(\mathbb{Z}_2, \mathbb{Z}_2) \otimes F(\Omega^*) \quad (\Omega = B(m+1)/\tilde{B}(m)), \\ {}_m E_\infty &\stackrel{s,t}{\cong} \text{Ext}_{B(m+1)}^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2). \end{aligned}$$

Since $\Omega = E_2[k_m]$, $k_m' = [2_m]$, we have $F(\Omega^*) = \mathbb{Z}_2[k_m]$, $\deg k_m = (1, 2^{m+2} - 2)$. And $\text{Ext}_{\tilde{B}(m)}(\mathbb{Z}_2, \mathbb{Z}_2) = \text{Ext}_{B(m)}(\mathbb{Z}_2, \mathbb{Z}_2) \otimes \mathbb{Z}_2[q_{m+1}]$, $\deg q_{m+1} = (1, 2^{m+2} - 1)$. Therefore

$${}_m E_1 = \text{Ext}_{B(m)}(\mathbb{Z}_2, \mathbb{Z}_2) \otimes \mathbb{Z}_2[k_m] \otimes \mathbb{Z}_2[q_{m+1}].$$

Then we have

$$d_1(q_{m+1}) = q_0 k_m$$

and all d_s in ${}_m E$ are trivial on $\text{Ext}_{B(m)}(\mathbb{Z}_2, \mathbb{Z}_2) \otimes \mathbb{Z}_2[k_m]$ (Cf. [12]).

Now we prove by induction on m that there is an integer $s_0 = s_0(n, m)$ such that

$$\text{Ext}_{B(m)}^{s,s+n}(\mathbb{Z}_2, \mathbb{Z}_2) = (\mathbb{Z}_2[q_0, \{\tau_j; j \leq m-1\}])^{s,s+n} \quad \text{if } s \geq s_0.$$

For $m=0$, it is true by Proposition 3.4, (1). Assume that it is true for m . Since $\deg q_{m+1} = (1, 1 + (2^{m+2} - 2))$, $2^{m+2} - 2 \geq 1$ and $\deg k_m = (1, 1 + (2^{m+2} - 3))$, $2^{m+2} - 3 \geq 1$, there is an integer $s_0' = s_0'(n, m)$ such that

$$({}_m E_2)^{s,s+n} = (\mathbb{Z}_2[q_0, \{\tau_j; j \leq m-1\}, q_{m+1}^2])^{s,s+n} \quad \text{if } s \geq s_0'.$$

Clearly there is an integer $s_0'' = s_0''(n, m) \geq s_0'$ such that

$$({}_m E_2)^{s,s+n} = (\mathbb{Z}_2[q_0, \{\tau_j; j \leq m-1\}, q_0 q_{m+1}^2])^{s,s+n} \quad \text{if } s \geq s_0''.$$

$q_0q_{m+1}^2$ is a permanent cycle and τ_m is represented by $q_0q_{m+1}^2$. Put $s_0(n, m+1) = s_0''(n, m)$ then

$$\text{Ext}_{B(m+1)}^{s, s+n}(\mathbf{Z}_2, \mathbf{Z}_2) = (\mathbf{Z}_2[q_0, \{\tau_j; j \leq m\}])^{s, s+n} \quad \text{if } s \geq s_0(n, m+1).$$

From the fact that $\text{Ext}_B^{s, s+n}(\mathbf{Z}_2, \mathbf{Z}_2) \cong \text{Ext}_{B(m)}^{s, s+n}(\mathbf{Z}_2, \mathbf{Z}_2)$ if $2^{m+2} - 3 > n$, the lemma follows.

Let

$$G = {}_m G = A/\overline{AB(m)} \otimes (H\mathbf{Z}_2)^*(MSp) = (H\mathbf{Z}_2)^*({}_m M) \otimes (H\mathbf{Z}_2)^*(MSp)$$

(A operating on $(H\mathbf{Z}_2)^*(MSp)$ trivially), where $m=0$ or 1 and ${}_0 M = H$, ${}_1 M = bo$. And we define a map

$$\Phi = {}_m \Phi: G \rightarrow (H\mathbf{Z}_2)^*(M \wedge MSp) \quad (M = {}_m M)$$

by $\Phi([a] \otimes u) = \sum [a'_i] \cdot a''_i u$ for $a \in A$, $u \in (H\mathbf{Z}_2)^*(MSp)$, where $\psi(a) = \sum a'_i \otimes a''_i$. Then we have

Lemma 3.7. (Cf. [1], §16) Φ is an isomorphism of graded coalgebras and A -modules.

We identify G and $(H\mathbf{Z}_2)^*(M \wedge MSp)$ by Φ .

Corollary 3.8.

- (1) $E_2(H \wedge MSp) = \mathbf{Z}_2[q_0, v_1, v_2, \dots, v_i, \dots]$.
- (2) $E_2(bo \wedge MSp) = E_2(bo) \otimes \mathbf{Z}_2[v_1, v_2, \dots, v_i, \dots]$.

Here $v_i \in E_2^{0,4i}({}_m M \wedge MSp)$, where

$$v_i = [\zeta_j] \quad \text{if } i = 2^j - 1, v_i = [V_i] \quad \text{if } i \neq 2^a - 1$$

$$((H\mathbf{Z}_2)^*(MSp) = A''^* \otimes S''^*).$$

Corollary 3.9.

- (1) $E_\infty(H \wedge MSp) = E_2(H \wedge MSp)$.

Therefore we have

$$F^s H_n(MSp) = \{x \in H_n(MSp); 2^s \mid x\}.$$

- (2) $E_\infty(bo \wedge MSp) = E_2(bo \wedge MSp)$.

Lemma 3.10. For any $u \in (H\mathbf{Z}_2)^*(MSp)$, we have

$$(u^M \wedge 1)^*(1 \otimes u) = u,$$

where $(u^M \wedge 1)^*: G \rightarrow (H\mathbf{Z}_2)^*(MSp)$.

Proof. For any $v \in (HZ_2)_*(MSp)$, we can prove by diagram chaising that

$$(u^M \wedge 1)_*(v) = 1 \cdot v \in (HZ_2)_*(M \wedge MSp).$$

Therefore we have

$$(u^M \wedge 1)^*(1 \cdot u) = u$$

for any $u \in (HZ_2)_*(MSp)$, where $(u^M \wedge 1)^*: (HZ_2)_*(M \wedge MSp) \rightarrow (HZ_2)_*(MSp)$. Since $\Phi^{-1}(1 \cdot u) = 1 \otimes u$, the lemma follows.

For any ring spectrum X and any spectrum Y , $u^X \wedge 1: Y \rightarrow X \wedge Y$ induces the spectral sequence map

$$h^X: E_*^{**}(Y) \rightarrow E_*^{**}(X \wedge Y).$$

For $X = H$, we denote h^X simply by h .

Lemma 3.11.

$$(1\text{-a}) \quad h(v_i) = v_i \quad \text{if } i \neq 2^a - 1.$$

$$(1\text{-b}) \quad h(Ext_B(Z_2, Z_2)) \text{ is contained in the ring}$$

$$_0R = Z_2[q_0, v_1, v_3, \dots, v_{2^a-1}, \dots].$$

$$(1\text{-c}) \quad h(\tau_j) = q_0^3(v_{2^j+1-1} + \text{decomposables in } Z_2[v_1, v_3, \dots, v_{2^a-1}, \dots]) \in _0R.$$

$$(2\text{-a}) \quad h^{bo}(v_i) = v_i \quad \text{if } i \neq 2^a - 1.$$

$$(2\text{-b}) \quad h^{bo}(Ext_B(Z_2, Z_2)) \text{ is contained in the ring}$$

$$_1R = Ext_{B(1)}(Z_2, Z_2) \otimes Z_2[v_1, v_3, \dots, v_{2^a-1}, \dots].$$

$$(2\text{-c}) \quad h^{bo}(\tau_j) = \tau_0(v_{2^j-1}^2 + \text{other terms in } Z_2[v_1, v_3, \dots, v_{2^j-1}]) + q_0^3(v_{2^j+1-1} + \text{decomposables in } Z_2[v_1, v_3, \dots, v_{2^a-1}, \dots]) \in _1R,$$

where $v_0 = 1$.

$$(2\text{-c}') \quad \text{Let } u \in (Z_2[q_0, \{\tau_a\}])^{s,t} \subset Ext_B^{s,t}(Z_2, Z_2). \text{ Then we have}$$

$$h^{bo}(u) \in Z_2[q_0, \tau_0, \{v_{2^a-1}\}]$$

and

$$h^{bo}(u) \in Z_2[q_0, \{v_{2^a-1}\}] \quad \text{if } u \notin Z_2[q_0].$$

$$(2\text{-d}) \quad h^{bo}(k_j) = k_0(v_{2^j-1} + \text{decomposables in } Z_2[v_1, v_3, \dots, v_{2^a-1}, \dots]) \in _1R.$$

Proof. We porve only (2). We can prove (1) in the same way. Applying Lemma 3.1 to the resolution $B(n) \otimes Y(n)$, we obtain an A -free resolurion of $A/A\overline{B(n)}$:

$$A/A\overline{B(n)} \xleftarrow{\epsilon} A \xleftarrow{d} A \otimes Y(n)_1 \xleftarrow{d} \dots \xleftarrow{d} A \otimes Y(n)_s \xleftarrow{d} \dots.$$

Then $(A \otimes Y(1) \otimes A'' \otimes S'', d \otimes 1 \otimes 1, \varepsilon \otimes 1 \otimes 1)$ is an A -free resolution of

$$_1G = A/\overline{AB(1)} \otimes A'' \otimes S''$$

and $(A \otimes Y(\infty) \otimes S'', d \otimes 1, \beta \otimes 1)$ an A -free resolution of

$$(HZ_2)^*(MSp) = A'' \otimes S''.$$

We can define an A -homomorphism $f_s: A \otimes Y(1)_s \otimes A'' \rightarrow A \otimes Y(\infty)_s$ for each $s \geq 0$ such that

$$\{f_s \otimes 1; A \otimes Y(1)_s \otimes A'' \otimes S'' \rightarrow A \otimes Y(\infty)_s \otimes S''\}$$

is a homomorphism of A -free resolutions, that is,

$$(u^{bo} \wedge 1)^*(\varepsilon \otimes 1 \otimes 1) = (\beta \otimes 1)(f_0 \otimes 1)$$

and $(f_s \otimes 1)(d \otimes 1 \otimes 1) = (d \otimes 1)(f_{s+1} \otimes 1)$ for any $s \geq 0$,

where $(u^{bo} \wedge 1)^*: A/\overline{AB(1)} \otimes A'' \otimes S'' \rightarrow A'' \otimes S''$ (Cf. Lemma 3.10). Partial construction of $\{f_s\}$ is given as the following ((i)~(iii), (i')).

(i) For $(\zeta_1^{n_1} \zeta_2^{n_2} \cdots \zeta_j^{n_j} \cdots)^* \in A'' = Y(1)_0 \otimes A''$,

$$f_0[(\zeta_1^{n_1} \zeta_2^{n_2} \cdots \zeta_j^{n_j} \cdots)^*] = (\zeta_1^{4n_1} \zeta_2^{4n_2} \cdots \zeta_j^{4n_j} \cdots)^* \in A = A \otimes Y(\infty)_0.$$

(i) $f_1(\langle \Delta_0 \rangle \otimes 2_{j-1}) = 8_{j-1} \langle \Delta_0 \rangle + 6_{j-1} \langle \Delta_j \rangle$ for $j \geq 2$,

$$f_1(\langle \Delta_0 \rangle \otimes 2_0) = 8_0 \langle \Delta_0 \rangle + 6_0 \langle \Delta_1 \rangle + 2_0 \langle \Delta_2 \rangle,$$

$$f_1(\langle \Delta_1 \rangle \otimes 2_{j-1}) = 8_{j-1} \langle \Delta_1 \rangle + 4_{j-1} \langle \Delta_{j+1} \rangle,$$

$$f_1(\langle \Delta_0 \rangle \otimes 1_j) = 4_j \langle \Delta_0 \rangle + 2_j \langle \Delta_{j+1} \rangle.$$

(ii) $f_2(\langle \Delta_0 + \Delta_1 \rangle \otimes 2_{j-1}) = 8_{j-1} \langle \Delta_0 + \Delta_1 \rangle + 6_{j-1} \langle \Delta_1 + \Delta_j \rangle + 4_{j-1} \langle \Delta_0 + \Delta_{j+1} \rangle + 2_{j-1} \langle \Delta_j + \Delta_{j+1} \rangle$ for $j \geq 2$,

$$f_2(\langle \Delta_0 + \Delta_1 \rangle \otimes 2_0) = 8_0 \langle \Delta_0 + \Delta_1 \rangle + 4_0 \langle \Delta_0 + \Delta_2 \rangle,$$

$$f_2(\langle 2\Delta_1 \rangle \otimes 2_{j-1}) = 8_{j-1} \langle 2\Delta_1 \rangle + 4_{j-1} \langle \Delta_1 + \Delta_{j+1} \rangle + \langle 2\Delta_{j+1} \rangle,$$

$$f_2(\langle 2\Delta_0 \rangle \otimes 1_j) = 4_j \langle 2\Delta_0 \rangle + 2_j \langle \Delta_0 + \Delta_{j+1} \rangle + \langle 2\Delta_{j+1} \rangle.$$

(iii) $f_3(\langle \Delta_0 + 2\Delta_1 \rangle \otimes 2_{j-1}) = 8_{j-1} \langle \Delta_0 + 2\Delta_1 \rangle + 6_{j-1} \langle 2\Delta_1 + \Delta_j \rangle + 4_{j-1} \langle \Delta_0 + \Delta_1 + \Delta_{j+1} \rangle + 2_{j-1} \langle \Delta_1 + \Delta_j + \Delta_{j+1} \rangle + \langle \Delta_0 + 2\Delta_{j+1} \rangle$,

$$f_3(\langle 3\Delta_0 \rangle \otimes 1_j) = 4_j \langle 3\Delta_0 \rangle + 2_j \langle 2\Delta_0 + \Delta_{j+1} \rangle + \langle \Delta_0 + 2\Delta_{j+1} \rangle.$$

(i') $f_1([\Delta_0 \otimes (0)] \otimes 1_{j-1}) = 4_{j-1} \Delta_0 \otimes (0) + 1_0 1_j \Delta_{j-1} \otimes (0) + \Delta_j \otimes (0)$ for $j \geq 2$,

$$f_1([\Delta_0 \otimes (0)] \otimes 1_0) = 4_0 \Delta_0 \otimes (0) + \Delta_1 \otimes (0).$$

We have

$$Hom_A(f_s \otimes 1, 1) = f_s^* \otimes 1: Y(\infty)_s^* \otimes S'' \rightarrow Y(1)_s^* \otimes A'' \otimes S'',$$

where $f_s^*: Y(\infty)_s^* \rightarrow Y(1)_s^* \otimes A''$ and $1: S''^* \rightarrow S''^*$. So we obtain (2-a) and (2-b).

By (iii), we obtain

$$f_s^*(y_0 y_{j+1}^2 + x_0 x_j y_{j+1}) = y_0 y_1^2 \otimes \zeta_j^2 + \text{other terms in } Y(1)_s^* \otimes A'' \quad \text{for } j \geq 1$$

and

$$f_s^*(y_0 y_{j+1}^2 + x_0 x_j y_{j+1}) = y_0^3 \otimes \zeta_{j+1} + \text{other terms in } Y(1)_s^* \otimes A''.$$

Obviously we have $f_s^*(y_0 y_1^2 + x_0^2 y_1) = y_0 y_1^2 \otimes 1 + \text{other terms}$, so that

$$f_s^*(y_0 y_{j+1}^2 + x_0 x_j y_{j+1}) = y_0 y_1^2 \otimes \zeta_j^2 + y_0^3 \otimes \zeta_{j+1} + \text{other terms in } Y(1)_s^* \otimes A'' \quad \text{for } j \geq 1,$$

where $\zeta_0 = 1$. No $y_0 y_1^2 + \text{other terms}$ in $(Y(1)_s^*)^y$ is coboundary and $Ext_{B_{(1)}^3}^3(Z_2, Z_2) = \{0, \tau_0\}$. $(Y(1)_s^*)^3 = \{0, y_0^3\}$. Therefore we have

$$h^{bo}(\tau_j) = \tau_0 v_{2^j-1}^2 + q_0^3 v_{2^{j+1}-1} + \text{other terms in } R.$$

From the dimensional reason, (2-c) follows.

(2-d) can be proven by (i').

Now we prove (2-c'). We define a ring homomorphism

$$\gamma: Z_2[q_0, \tau_0, \{v_{2^a-1}\}] \rightarrow Z_2[\tau_0, \{v_{2^a-1}\}]$$

by $\gamma(q_0) = 0$, $\gamma(\tau_0) = \tau_0$, $\gamma(v_{2^a-1}) = v_{2^a-1}$. And we define a decreasing filtration $\{F^s\}$ in $Z_2[\tau_0, \{v_{2^a-1}\}]$ by

$$F^0 = Z_2[\tau_0, \{v_{2^a-1}\}],$$

$$F^{s+1} = (\text{the ideal of } F^0 \text{ generated by } \{v_{2^a-1}; a \geq 1\}) F^s$$

Then $F^s F^t \subset F^{s+t}$ and $\gamma h^{bo}(\tau_j) \equiv \tau_0 v_{2^j-1}^2 \pmod{\text{higher filtration}}$.

Let

$$u = q_0^{s'} u', s' \geq 0, u' \in Z_2[q_0, \{\tau_a\}],$$

$$u' \text{ is not divisible by } q_0 \text{ in } Z_2[q_0, \{\tau_a\}].$$

If $u \in Z_2[q_0]$ then u' has the form

$$u' = \sum_{\substack{0 \leq s_0 \leq m, \\ 0 \leq s_1, \dots, s_j, \dots, \\ (s_0, s_1, \dots, s_j, \dots) \neq (0)}} b^{(s_0, s_1, \dots, s_j, \dots)} \tau_0^{s_0} \tau_1^{s_1} \dots \tau_{s_j}^{s_j} \dots + q_0 u'',$$

$$b^{(s_0, s_1, \dots, s_j, \dots)} \in Z_2, u'' \in Z_2[q_0, \{\tau_a\}]$$

($m \geq 0$ and there is (s_1, \dots, s_j, \dots) such that $b^{(m, s_1, \dots, s_j, \dots)} \neq 0$). We have

$$s_0 + s_1 + \cdots + s_j + \cdots = (s - s')/3 \quad \text{if} \quad b^{(s_0, s_1, \dots, s_j, \dots)} \neq 0.$$

Therefore we obtain

$$\gamma h^{bo}(u') \equiv \sum_{\substack{0 \leq s_1, \dots, s_j, \dots, \\ (m, s_1, \dots, s_j, \dots) \neq (0)}} b^{(m, s_1, \dots, s_j, \dots)} \tau_0^m (v_0 v_1^2)^{s_1} \cdots (\tau_0 v_{2^j-1}^2)^{s_j} \cdots \pmod{\text{mod higher filtration}},$$

so that (2-c') is proven.

By Lemma 3.11, (2-a) and (2-d), we obtain

Proposition 3.12. *Let k be an integer, and $x \in MSp_{4k+1}$ represented by an element $\neq 0$ of $E_\infty^{1,1+(4k+1)}(MSp)$. Then $h^{KO}(x) \neq 0$ in $KO_{4k+1}(MSp)$.*

Proof. $Ext_B^1(\mathbb{Z}_2, \mathbb{Z}_2)$ is a \mathbb{Z}_2 -vector space generated by $\{q_0, k_0, k_1, \dots, k_j, \dots\}$.

By Lemma 3.6 and Lemma 3.11, (1), we obtain

Lemma 3.13. *Let s, t be integers, and $u \in E_\infty^{s,t}(MSp)$ such that $q_n^s u \neq 0$ for any integer $n \geq 0$. Then $h(u) \neq 0$ in $E_\infty^{s,t}(H \wedge MSp)$.*

REMARK. Lemma 3.13 follows also from [12], PROPOSITION 3.2.

4. Proof of Theorem 1.1

For any integer k , we denote by g_k the composition of the following sequence of homomorphism

$$MSp_{4k} \otimes \mathbb{Q} \xrightarrow{h} H_{4k}(MSp) \otimes \mathbb{Q} \xrightarrow{p_k} \mathbb{Q},$$

where $p_k(x)$ is the coefficient of q_k in x for any $x \in H_{4k}(MSp) \otimes \mathbb{Q}$. We have the commutative diagram

$$\begin{array}{ccccc} MSp_{4k}/\text{Tors} & \xrightarrow{q} & \mathbb{Q}_{4k}(MSp_*/\text{Tors}) & \xrightarrow{u_1} & \mathbb{Z} \\ \downarrow i & & \downarrow i_* & & \nearrow u_2 \\ W_{4k}^{KO} & \xrightarrow{q} & \mathbb{Q}_{4k}(W_*^{KO}) & & \end{array}$$

Here q denotes the quotient map, and u_1, u_2 are the maps such that

$$g_k|MSp_{4k}/\text{Tors} = u_1 \circ q, g_k|W_{4k}^{KO} = u_2 \circ q$$

(Cf. Corollary 2.2 and (2.4)). Since u_2 is a monomorphism, Theorem 1.1 is equivalent to

$$(4.1) \quad g_k(MSp_{4k}/\text{Tors}) \supset g_k(W_{4k}^{KO}) \quad \text{for any integer } k.$$

By [9], (6.4), we have

$$(4.2) \quad MSp_*/\text{Tors} \otimes \mathbf{Z}[\frac{1}{2}] = W_*^{KO} \otimes \mathbf{Z}[\frac{1}{2}].$$

Therefore (4.1) is equivalent to

$$(4.3) \quad 2^s | g_k(MSp_{4k}/\text{Tors}) \Rightarrow 2^s | g_k(W_{4k}^{KO}) \quad \text{for any integer } k.$$

Let E be a ring spectrum. Then, obviously, we have

$$(4.4) \quad f_*(W_*^E) \subset W_{*-n}^E.$$

for any morphism $f: MSp \rightarrow MSp$ of degree $-n$, where $f_*: MSp_* \otimes \mathbf{Q} \rightarrow MSp_{*-n} \otimes \mathbf{Q}$.

Making use of Proposition 2.6, Lemma 2.5 and (4.4), we can prove the following proposition in the same way as that of Segal [13].

Proposition 4.1.

(1) For any integer k , $g_k(W_{4k}^{KO})$ is divisible by 2. If k is a power of 2 then it is divisible by 4.

(2) Let k be an odd integer. Then $h(W_{4k}^{KO})$ is divisible by 4 in $H_{4k}(MSp)$. In particular, $g_k(W_{4k}^{KO})$ is divisible by 4.

And further, making use of some results in §3, we obtain

Proposition 4.2. If $k=2^j-1$, j an integer >0 , then $g_k(W_{4k}^{KO})$ is divisible by 8.

Proof. Let $x \in W_{4k}^{KO}$ and $x \neq 0$. By Lemma 3.6, there is an integer $n \geq 0$ such that $2^n x \in MSp_{4k}/\text{Tors} \subset MSp_{4k}$ is represented by an element of $\mathbf{Z}_2[q_0, \{\tau_m\}, \{v_i; i \neq 2^a-1\}] \cap E_\infty(MSp)$. Let $2^n x$ be represented by $u \in E_\infty^s(MSp)$, $s \geq 0$.

(i) In case $u \in \mathbf{Z}_2[q_0, \{v_i; i \neq 2^a-1\}]$: There is a decomposable element $y \in H_{4k}(MSp)$ such that

$$h(2^n x) \equiv y \pmod{F^{s+1} H_{4k}(MSp)}.$$

Therefore, by Corollary 3.9, (1), $g_k(2^n x)$ is divisible by 2^{s+1} , so that $g_k(x)$ is divisible by 2^{s+1-n} . By Lemma 3.13, $h(2^n x)$ is not divisible by 2^{s+1} , so that $h(x)$ is not divisible by 2^{s+1-n} . By Proposition 4.1, (2), we have $s+1-n \geq 3$. Consequently $g_k(x)$ is divisible by 8.

(ii) In case $u \notin \mathbf{Z}_2[q_0, \{v_i; i \neq 2^a-1\}]$: By Lemma 3.11, (2-a) and (2-c'), we have

$$h^{bo}(u) \in \mathbf{Z}_2[q_0, \{v_i\}] \subset E_\infty(bo \wedge MSp).$$

By (2.2), $h^{bo}(x) \in bo_{4k}(MSp)$. Let $h^{bo}(x)$ be represented by $w \in E_\infty^{*, *+4k}(bo \wedge MSp)$. By Proposition 3.4, (2), $h^{bo}(2^n x) = 2^n h^{bo}(x)$ is represented by $q_0^n w$, so that

$$h^{bo}(u) = q^s w, w \in E_*^{s-n}(bo \wedge MSp).$$

Then $w \in \mathbb{Z}_2[q_0, \{v_i\}]$, so that $s-n \geq 3$. $h(2^n x)$ is divisible by 2^s , so that $h(x)$ is divisible by 2^{s-n} . Consequently $h(x)$ is divisible by 8.

Let $n_j(n_1, n_2, \dots, n_r) \in MSp_{2N-4j}$ be the Stong-Ray classes in [11], where $N = \sum_{i=1}^r (2n_i - 1)$.

Proposition 4.3.

(1) (Segal [13]) For an even integer $k > 0$, we define integers s_k and t_k as follows. If k is not a power of 2 then we define $s_k = 2^u + 1$, 2^u the largest power of 2 less than k , and $t_k = k - s_k + 2$. If $k = 2^j$ then we define $s_k = t_k = 2^{j-1} + 1$. Then we have

$$g_k(n_1(s_k, t_k)) \equiv \begin{cases} 2 \pmod{4} & \text{if } k \equiv 0 \pmod{2}, k \neq 2^j \\ 4 \pmod{8} & \text{if } k = 2^j \end{cases}$$

(2) Using the notation of (1), we have

$$g_k(n_2(s_{k+1}, t_{k+1})) \equiv \begin{cases} 4 \pmod{8} & \text{if } k \equiv 1 \pmod{2}, k \neq 2^j - 1 \\ 8 \pmod{16} & \text{if } k = 2^j - 1 \end{cases}$$

(Segal [13] has proven the fact that $g_k(MSp_{4k}/\text{Tors})$ is not divisible by 8 if $k \equiv 1 \pmod{2}$, $k \neq 2^j - 1$.)

Now (4.3) follows from Propositions 4.1, 4.2 and 4.3, so that Theorem 1.1 is proven.

As a corollary to Proposition 4.3, we obtain

Proposition 4.4. $\{n_j(n_1, n_2, \dots, n_r) \in MSp_* generates $Q(MSp_*/\text{Tors}) \cong Q(W_*^{KO})$.$

Proof. From Stong [17], Theorem 1, it follows that $\{n_1(n_1, n_2, \dots, n_r)\}$ generates $Q(MSp_*/\text{Tors}) \otimes \mathbb{Z}_p$ for any odd prime p .

5. Proof of Theorem 1.2 and some remarks

For integers $k, s \geq 0$, we put

$$F_1^s = h(MSp_{4k}) \cap F^s H_{4k}(MSp)$$

and

$$F_2^s = h(W_{4k}^{KO}) \cap F^s H_{4k}(MSp).$$

The following lemma follows immediately from the definition.

Lemma 5.1. For $m=1$ or 2 , the inclusion $F_m^s \rightarrow H_{4k}(MSp)$ induces the monomorphism

$$F_m^s/F_m^{s+1} \rightarrow F^s H_{4k}(MSp)/F^{s+1} H_{4k}(MSp) = E_{\infty}^{s,s+4k}(H \wedge MSp).$$

Lemma 5.2. $MSp_{4k}/\text{Tors} = W_{4k}^{KO}$ if and only if

$$F_1^s/F_1^{s+1} = F_2^s/F_2^{s+1} \subset E_{\infty}^{s,s+4k}(H \wedge MSp) \quad \text{for any } s \geq 0.$$

Proof. By (4.2), we have

$$h(MSp_{4k}) \otimes \mathbb{Z}[\frac{1}{2}] = h(W_{4k}^{KO}) \otimes \mathbb{Z}[\frac{1}{2}].$$

Therefore there is an integer $s_0 = s_0(k)$ such that $F_1^s = F_2^s$ for any $s \geq s_0$. Then it is easy to see that $h(MSp_{4k}) = h(W_{4k}^{KO})$ if and only if

$$F_1^s/F_1^{s+1} = F_2^s/F_2^{s+1} \quad \text{for any } s \geq 0.$$

Since $h: MSp_* \otimes \mathbb{Q} \rightarrow H_*(MSp) \otimes \mathbb{Q}$ is an isomorphism, the lemma follows.

By Theorem 1.1, Proposition 2.6, Lemmas 3.13, 5.2 and Segal [12], TABLE II, we obtain

Lemma 5.3. $MSp_{4k}/\text{Tors} = W_{4k}^{KO}$ for $k \leq 7$.

By Lemma 5.3, Proposition 2.6, Lemma 3.11, (2) and [12], TABLE II, we can prove

Lemma 5.4.

$$\text{order of } MSp_n = \text{order of } h^{KO}(MSp_n)$$

for $n \leq 30$, $n \not\equiv 0 \pmod{4}$.

Since MSp_{4k} is torsion free for $k \leq 7$ by [12], Theorem 1.2 follows from Lemmas 5.3 and 5.4.

Making use of the Ray classes $\phi_i \in MSp_{8i-3}$ in [8], we can immediately calculate the ring structure of MSp_* in dimensions ≤ 30 except the values of $\alpha\tilde{x}_7$ and $\alpha^2\tilde{x}_7$, where α is the generator of $MSp_1 \cong \mathbb{Z}_2$ (Cf. Ray [10], (5.25)). For example, we have

Proposition 5.5. For $k \leq 5$,

$$x_1^2 MSp_{4k+1} \subset \alpha MSp_{4k+8} \quad \text{and} \quad x_1^2 MSp_{4k+2} \subset \alpha^2 MSp_{4k+8}.$$

We can calculate the Hurewicz map (1.1) for $n = 17$:

Proposition 5.6. There is an indecomposable element $\tau \in MSp_{17}$ such that

$$h^{KO}(\tau) = e(\sigma_2^2 + y\sigma_2).$$

Proof. Using the notation of [12], x_1^2 is represented by ω_0 and $2y_4$ by $q_0v_2^2$. Therefore $2x_1^2y_4$ is represented by $q_0\omega_0v_2^2$. Since

$$2(x_1x_2x_3 + x_1^2(x_2^2 + y_4 + x_4)) \equiv 2x_1^2y_4 \pmod{F^6MSp_{24}},$$

$x_1x_2x_3 + x_1^2(x_2^2 + y_4 + x_4)$ is represented by $\omega_0v_2^2$.

Let $\tau' \in MSp_{17}$ be a class represented by $k_0v_2^2$. Then $x_1^2\tau'$ is represented by $k_0\omega_0v_2^2$, so that

$$x_1^2\tau' \equiv \alpha(x_1x_2x_3 + x_1^2(x_2^2 + y_4 + x_4)) \pmod{F^6MSp_{25}}.$$

Therefore

$$yh^{KO}(\tau') = h^{KO}(x_1^2\tau') \equiv ye(\sigma_2^2 + y\sigma_2) \pmod{h^{KO}(F^6MSp_{25})}.$$

Since $h^{KO}(F^6MSp_{25}) = yh^{KO}(F^2MSp_{17})$, there is an element $\lambda \in F^2MSp_{17}$, such that

$$\begin{aligned} yh^{KO}(\tau') &= ye(\sigma_2^2 + y\sigma_2) + yh^{KO}(\lambda), \\ h^{KO}(\tau') &= e(\sigma_2^2 + y\sigma_2) + h^{KO}(\lambda). \end{aligned}$$

We may take $\tau = \tau' + \lambda$.

Let ${}_U E_*^{**}(MSp)$ denote the Adams-Novikov spectral sequence for MSp_* (Cf. [5]). Proposition 2.6 shows us the structure of

$$MSp_*/\text{Tors} = {}_U E_*^{0*}(MSp) \subset {}_U E_2^{0*}(MSp)$$

in low dimensions:

Proposition 5.7.

- (1) (Porter [6]) ${}_U E_2^{0*}(MSp) \cong \{x \in MSp_* \otimes \mathbb{Q}; r_*(x) \in MU_*\}$.
- (2) $\{x \in MSp_* \otimes \mathbb{Q}; r_*(x) \in MU_*\} = W_*^K$.

Proof of (2). Consider the commutative diagram

$$\begin{array}{ccc} MSp_* \otimes \mathbb{Q} & \xrightarrow{h^K} & K_*(MSp) \otimes \mathbb{Q} \\ \downarrow r_* & & \downarrow r_* \\ MU_* \otimes \mathbb{Q} & \xrightarrow{h^K} & K_*(MU) \otimes \mathbb{Q}. \end{array}$$

Then $r_*: K_*(MSp) \rightarrow K_*(MU)$ is a split monomorphism. And, by Hattori [2] or Stong [16], $h^K: MU_* \rightarrow K_*(MU)$ is a split monomorphism.

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