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1. Introduction

Let $P=P(x, D)$ be a self-adjoint pseudo-differential operator with the symbol $p(x, \xi)$ in the class $S^1_{1,0}$ of Hörmander. The positive part of $P$ is defined by

$$P^+ = \int_0^\infty \lambda dE(\lambda),$$

where $dE(\lambda)$ is the spectral measure of $P$. We shall be concerned with the following question: To what extent the correspondence; $u \rightarrow P^+u$ can be localized? We shall prove a localization principle for the operator $P^+$ which is analogous to Theorem 6.3 of Hörmander [5]. If we combine this with our previous discussions in [2], we can explicitly construct an operator $B$ such that we have estimate

$$|(A^+-B)u, v| \leq C\|u\|_{L^2} \|v\|_{L^2},$$

where $u$ and $v$ are arbitrary functions in $\mathcal{D}(\mathbb{R}^n)$ and $C$ is a positive constant independent of $u$ and $v$.

2. Localized operators

Let us repeat our notations. $p(x, \xi)$ is a function in the class $S^1_{1,0}$ which vanishes unless $x$ lies in a compact set $K$ in $\mathbb{R}^n$. We treat pseudo-differential operator $P(x, D)$ defined as

$$P(x, D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} p(x, \xi)u(y) e^{i(x-y)\cdot \xi} dy d\xi.$$  

We assume that $P=P(x, D)$ is self-adjoint in Hilbert space $L^2(\mathbb{R}^n)$.

Now we make use of the partition of unity of Hörmander [5]. Let $g_0=0, g_1, g_2, \ldots$ be the unit lattice points in $\mathbb{R}^n$. Then $\mathbb{R}^n$ is covered by open cubes of side 2 with center at these points. Let $\Theta(x)$ be a non-negative $C^2_0$ function which equals 1 on $|x_j| \leq 1$ and 0 outside $|x_j| \leq 3/2$, $j=1, 2, 3, \ldots, n$. We set
\[ (2.2) \quad \varphi_k(x) = \Theta(x-g_k)/\left(\sum_{i=0}^{\infty} \Theta(x-g_k)^i \right) \]

and

\[ \hat{\varphi}_k(x) = \varphi_k \left( \frac{1}{2} (x-g_k) + g_k \right). \]

Note that \( \hat{\varphi}(x) = 1 \) on \( \text{supp} \varphi_k \). We, by definition, have

\[ (2.4) \quad \sum_k \varphi_k(x)^2 = 1 \]

and

\[ (2.5) \quad \sum_k |D^\alpha \varphi_k(x)|^2 \leq C_{\alpha} \]

for any multi-index \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \).

\[ (2.6) \quad |x-y| \leq 3\sqrt{n} \quad \text{if } x \text{ and } y \text{ are in } \text{supp} \varphi_k. \]

We set

\[ (2.7) \quad \psi_k(\xi) = \varphi_k(\xi/|\xi|^\rho) \quad \text{and} \]

\[ (2.8) \quad \hat{\psi}_k(\xi) = \hat{\varphi}_k(\xi/|\xi|^\rho), \quad \frac{1}{2} \leq \rho \leq 1. \]

Then we have

\[ (2.9) \quad \sum_k \psi_k(\xi)^2 = 1, \]

\[ (2.10) \quad |\xi|^2(\sum_k |D^\alpha \psi_k(\xi)|^2 \leq C, \]

and

\[ (2.11) \quad |\xi-\eta| \leq C |\xi|^\rho \quad \text{if } \xi \text{ and } \eta \text{ are in } \text{supp} \psi_k. \]

Here and hereafter \( C \) stands for positive constants which are different from time to time.

\[ (2.12) \quad \sum_k |\psi_k(\xi) - \psi_k(\eta)|^2 \leq \frac{C(\xi-\eta)^2}{(1+|\xi|)(1+|\eta|)^\rho} \quad \text{for any } \xi, \eta \in \mathbb{R}^n. \]

Let \( \delta_j = |g_j|^{\rho/(1-\rho)}. \) Then \( g_j \delta_j \subseteq \text{supp} \psi_j \). We shall denote by \( \psi_j(D) \) the pseudo-differential operator corresponding to the symbol \( \psi_j(\xi) \). Then we have

\[ (2.13) \quad \sum_j |\psi_j(D)|^2 = I. \]

The Sobolev norm \( ||u||_{L^1} \) of \( u \) is equivalent to \( (\sum_j \delta_j^{2(1-\rho)}||\psi_j(D)u||^2)\).

We put \( \varphi_{jk}(x) = \varphi_j(\delta_j x) \) and \( \phi_{jk}(x, \xi) = \varphi_{jk}(x) \psi_k(\xi), \hat{\phi}_{jk}(x, \xi) = \hat{\varphi}_{jk}(x)\hat{\psi}_k(\xi) \) where \( \sigma = (1-\rho)/\rho. \) It is obvious from definition that

\[ (2.14) \quad \left| \left( \frac{\partial}{\partial x} \right)^\alpha \left( \frac{\partial}{\partial \xi} \right)^\beta \phi_{jk}(x, \xi) \right| \leq C \delta_j^{\sigma |\beta|} \delta_j^{-|\beta|} \leq C |\xi|^{\sigma(1-\rho)-|\beta|\rho}, \]
This means that the set \( \{\phi_{jk}\}_{jk} \) is bounded in the class \( S_{p,1}^0 \). We shall frequently use the inequality

\[
C||u||^2 \leq \sum_{jk} \delta^{\gamma/p} ||\phi_{jk}(x, D)u||^2 \leq C^{-1}||u||^2.
\]

Choosing a point \((x^j, \xi^k)\) in \( \text{supp} \phi_{jk} \), we set

\[
Q_{jk}(x, D) = \sum_{|\alpha| + |\beta| < N} \frac{x^\alpha D^\beta}{\alpha! \beta!} p_{(\alpha)}(x^j, \xi^k), \quad N \geq \rho/(1 - \rho),
\]

and

\[
P_{jk}(x, D) = \frac{1}{2} (Q_{jk}(x, D) + Q_{jk}(x, D)^*),
\]

where \( Q(x, D)^* \) is the formal adjoint of \( Q_{jk}(x, D) \). We call these \( P_{jk}(x, D) \) localized operators.

3. Statement of results

**Theorem 1.** For any given \( \gamma > \frac{1}{2} (1 - \rho) \), there exists a constant \( C_\gamma > 0 \) such that inequality

\[
|\langle P^+ u, u \rangle - \sum_{jk} \langle P^+ \phi_{jk}(x, D)u, \phi_{jk}(x, D)u \rangle| \leq C_\gamma ||u||_\gamma ||u||
\]

holds for any \( u \in C^0_\gamma (\mathbb{R}^n) \).

**Theorem 2.** Assume that the localized operators \( P_{jk}(x, D) \) are self-adjoint. Let \( P^+ \) denote the non-negative part of \( P_{jk} \). Then, for any \( \gamma > \frac{1}{2} (1 - \rho) \), there exists a constant \( C_\gamma > 0 \) such that we have estimate

\[
|\langle P^+ u, u \rangle - \sum_{jk} \langle P^+ \phi_{jk}(x, D)u, \phi_{jk}(x, D)u \rangle| \leq C_\gamma (||u||_\gamma ||u|| + ||u||^3_{1-\rho})
\]

for any \( u \in C^0_\gamma (\mathbb{R}^n) \).

**Remark 3.1.** When \( \rho = 2/3 \) and \( N = 2 \), the assumption that \( P_{jk}(x, D) \) is self-adjoint is satisfied and \( P^+ \) is easily constructed. See [2] for the details. We can construct operator \( B \) for which the estimate \( |\langle (P^+ - B)u, v \rangle| \leq C ||u||_\gamma ||v||_{1/6} \) holds for any \( u \) and \( v \) in \( C^0_\gamma \).

4. Proofs

We begin our proof by the following lemma.

**Lemma 4.1.** Let \( A \) be a self-adjoint operator in a Hilbert space \( X \). Let \( e^{isA} \) be the corresponding one-parameter group of unitary operators. Then the non-negative part \( A^+ \) of \( A \) is given by the formula

\[
A^+ x = -(2\pi)^{-1} \int_{-\infty}^\infty \frac{e^{isA}}{(s - i0)^2} x \, ds
\]
for any \( x \) in \( D(A^*) \). Here \( (s-i0)^{-2} \) is the distribution \( \lim_{\varepsilon \to 0^+} (s-i\varepsilon)^{-2} \). (cf. Gelfand-Silov [3])

Proof. Let \( \lambda^+=\max(\lambda, 0) \). Then we have

\[
\int_{-\infty}^{\infty} (s-i0)^{-2} e^{is\lambda} ds = -2\pi \lambda^+ .
\]

If \( \varphi \) is in \( \mathcal{B}(\mathbb{R}^n) \), then

\[
\langle (s-i0)^{-2}, \varphi(s) \rangle = \int_{0}^{\infty} (\varphi(\varepsilon) + \varphi(-\varepsilon) - 2\varphi(0))/\varepsilon^2 ds + i\pi \varphi'(0) .
\]

This and (4.2) mean that

\[
-2\pi \lambda = \int_{0}^{\infty} (e^{is\lambda} + e^{-is\lambda} - 2)/\varepsilon^2 ds - \pi \lambda .
\]

Now we need spectral representation \( A=\int_{-\infty}^{\infty} \lambda dE(\lambda) \) of \( A \). Integrating (4.4) with respect to \( \lambda \) by measure \( d\lambda E(\lambda)x \), we have

\[
-2\pi A^+ x = \int_{0}^{\infty} (e^{isA} + e^{-isA} - 2)/\varepsilon^2 ds x - \pi Ax = \int_{-\infty}^{\infty} e^{isA} x/(s-i0)^2 ds .
\]

Proof of Theorem 1. We have to deal with the difference

\[
(P^+ u, u) - \sum_{j,k} (P^+ \phi_{jk}(x, D)u, \phi_{jk}(x, D)u)
\]

\[
= \sum_{j,k} \left( [P^+, \phi_{jk}(x, D)]\phi_{jk}(x, D)u, u \right) .
\]

Putting

\[
\phi_{jk}(s; x, D) = e^{isP}\phi_{jk}(x, D)e^{-isP} \quad \text{and} \quad \phi_{jk}^*(s; x, D) = e^{isP}\phi_{jk}^*(x, D)e^{-isP} ,
\]

we have

\[
[e^{isP}, \phi_{jk}(x, D)^*]\phi_{jk}(x, D)
\]

\[
= e^{isP}(\phi_{jk}^*(s; x, D) - \phi_{jk}^*(-s; x, D))\phi_{jk}(s; x, D)e^{isP} .
\]

Therefore by lemma 4.1,

\[
[P^+, \phi_{jk}(x, D)^*]\phi_{jk}(x, D)
\]

\[
= -(2\pi)^{-1} \int_{-\infty}^{\infty} (s-i0)^{-2} e^{isP}(\phi_{jk}^*(s; x, D) - \phi_{jk}^*(-s; x, D))\phi_{jk}(s; x, D)e^{isP}ds .
\]

The operator \( \phi_{jk}(s; x, D) \) is a pseudo-differential operator whose symbol is given in the following manner; Let \( (y(t; x, \xi), \eta(t; x, \xi)) \) be the solution of the Hamilton-Jacobi equations

\[
\frac{d\eta}{dt} = \frac{\partial p(y, \eta)}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial p(y, \eta)}{\partial \eta}.
\]
with initial conditions \( y(0; x, \xi) = x \), and \( \eta(0; x, \xi) = \xi \). The symbol of \( \phi_{jk}(s; x, D) \) is

\[
\phi_{jk}(s; x, \xi) = \phi_{jk}(y(s; x, \xi), \eta(s, x, \xi)).
\]

(cf. Egoroff [1], Hörmander [6] and Nirenberg-Trèves [7]). As a consequence, the sequence \( \phi_{jk}(s; x, \xi) \) is bounded in \( S^0_{p,1-p} \) and the number of overlaps of \( \text{supp} \phi_{jk}(s; x, \xi) \) is bounded. Set

\[
\Phi_{jk}(s; x, D) = (\phi_{jk}(s; x, D) - \phi_{jk}(-s; x, D))\phi_{jk}(s; x, D).
\]

Then we have

**Lemma 4.2.**

\[
\begin{aligned}
(4.12) & \quad 1^o \quad \Phi_{jk}(0; x, D) = 0, \\
(4.13) & \quad 2^o \quad \frac{d}{ds} \Phi_{jk}(s; x, D) = \frac{i}{2} \{[P, \Phi_{jk}^*(x, D)]_{jk} + [P, \phi_{jk}^*(x, D)]_{-\omega}\} \phi_{jk}(s; x, D) \\
& \quad \quad \quad \quad \quad + \frac{1}{2} i(\phi_{jk}(s; x, D) - \phi_{jk}(-s; x, D))\{P, \phi_{jk}\}_{(2,\omega)}. \\
(4.15) & \quad 3^o \quad |s|^{-a}\left\{\frac{d}{ds} \Phi_{jk}(s; x, D) - 2i [P, \phi_{jk}^* (x, D)] \phi_{jk}(s; x, D)\right\}, j, k = 0, 1, 2, \ldots
\end{aligned}
\]

is a bounded sequence in the space \( L^{(1+a)(1-p)}_{\omega,1-p} \), if \( 0 \leq \alpha < 1 \). Here we have used the notation \([P, \phi_{jk}^*(x, D)]_{(2,\omega)} = e^{i \omega P} [P, \phi_{jk}^*(x, D)] e^{-i \omega P} \).

**Proof.**

1° is obvious.

\[
\begin{aligned}
(4.12) & \quad 1^o \quad \Phi_{jk}(0; x, D) = 0, \\
(4.13) & \quad 2^o \quad \frac{d}{ds} \phi_{jk}^*(s; x, D) = \frac{1}{2} i e^{i \omega P} [P, \phi_{jk}^*(x, D)] e^{-i \omega P} = \frac{1}{2} i [P, \phi_{jk}^*(x, D)]_{(2,\omega)}. \\
(4.15) & \quad 3^o \quad \frac{d^2}{ds^2} \Phi_{jk}(s; x, D) = \\
& \quad \quad = (i/2)\{[P, [P, \phi_{jk}^*(x, D)]_{(2,\omega)} - [P, [P, \phi_{jk}^*(x, D)]_{(2,\omega)}]_{\omega}\} \phi_{jk}(s; x, D) \\
& \quad \quad \quad + 2(i/2)\{[P, \phi_{jk}(x, D)]_{(2,\omega)} + [P, \phi_{jk}(x, D)]_{(2,\omega)}\} [P, \phi_{jk}]_{(2,\omega)} \\
& \quad \quad \quad + (i/2)\{[\phi_{jk}(s; x, D)]_{(2,\omega)} - \phi_{jk}(-s; x, D)\} [P, [P, \phi_{jk}]_{(2,\omega)}].
\end{aligned}
\]

This implies that the set \( \left\{\frac{d^2}{ds^2} \Phi_{jk}(s; x, D)\right\}_{jk} \) is bounded in \( S^{2(1-p)}_{p,1-p} \). Applying convexity argument, we can prove that the set \( \left\{\frac{d}{ds} \Phi_{jk}(s; x, D) - \frac{d}{ds} \Phi_{jk}(0; x, D)\right\} \) is bounded in \( S^{1+a(1-p)}_{p,1-p} (R^n) \). This proves 3°.

Now we come back to the proof of Theorem 1. We divide integral (4.8) into two parts;
(4.16) \[ A_{jk} = \int_0^t s^{-2} (e^{t\xi P} \Phi_{jk}(s; x, D) e^{t\xi P} + e^{-t\xi P} \Phi_{jk}(-s; x, D) e^{-t\xi P}) ds \]

and

(4.17) \[ B_{jk} = -2\pi [P, \Phi_{jk}(x, D)] \Phi_{jk}(x, D) + \int_0^t s^{-2} (e^{t\xi P} \Phi_{jk}(s; x, D) e^{t\xi P} + e^{-t\xi P} \Phi_{jk}(-s; x, D) e^{-t\xi P}) ds. \]

We have to prove estimate

(4.18) \[ | \sum_{jk} (A_{jk} u, u) + \sum_{jk} (B_{jk} u, u) | \leq C \| u \| \| u \|. \]

Since \( \{ \Phi_{jk}(s; x, D) \}_{jk} \) is bounded in \( S^0_{\rho,1-p} \) and the number of overlaps of \( \text{supp} \Phi_{jk} \) is bounded, the series \( \sum_{jk} \Phi_{jk}(s; x, D) \) converges to an operator \( T(s; x, D) \) in \( L^p_{\rho,1-p} \) of Hörmander [5]. Thus we have

(4.19) \[ | \sum_{jk} (A_{jk} u, u) | = \int_0^t s^{-2} \{ (T(s; x, D) e^{t\xi P} u, e^{-t\xi P} u) + (T(-s; x, D) e^{-t\xi P} u, e^{t\xi P} u) \} ds \leq C t^{-1} \| u \|^2. \]

We get estimate of \( \sum_{jk} (B_{jk} u, u) \) by virtue of lemma 4.2. The set

\[ \{ |s|^{-1+\alpha} \left( \Phi_{jk}(s; x, D) - s \frac{d}{ds} \Phi_{jk}(0; x, D) \right) \}_{jk} \]

is bounded in \( S^0_{\rho,1-p} \). If we set \( \Lambda = (1-\Delta)^{1/2} \) and

\[ S_{jk}(s; x, D) = \Lambda^{-1/(1+\alpha)(1-p)} S^{-1/(1+\alpha)} \left( \Phi_{jk}(s; x, D) - s \frac{d}{ds} \Phi_{jk}(0; x, D) \right) \Lambda^1/(1+\alpha)(1-p), \]

the sequence of their symbols \( S_{jk}(s; x, D) \) is bounded in \( S^0_{\rho,1-p} \) and the number of overlaps of supports of them is also bounded. The series \( \sum_{jk} S_{jk}(s; x, D) \) thus converges to an operator \( S(s; x, D) \) in the space \( L^p_{\rho,1-p} \). Hence we have

(4.20) \[ \sum_{jk} (B_{jk} u, u) = \int_0^t s^{-1} (S(s; x, D) e^{t\xi P} \Lambda^{1/(1+\alpha)(1-p)}(s) u, e^{-t\xi P} \Lambda^{1/(1+\alpha)(1-p)}(-s) u) ds \]

\[ + \int_0^t s^{-1} (S(-s; x, D) e^{-t\xi P} \Lambda^{1/(1+\alpha)(1-p)}(-s) u, e^{t\xi P} \Lambda^{1/(1+\alpha)(1-p)}(s) u) ds, \]

where \( \Lambda(s) = e^{t\xi P} \Lambda e^{-t\xi P} \).

Since \( \Lambda(s) \) and \( \Lambda(-s) \) are elliptic operators of order 1, we have

(4.21) \[ | \sum_{jk} (B_{jk} u, u) | \leq C \int_0^t s^{-1} ds \| u \|_{L^2(1+\alpha)(1-p)}^p \leq C t^p \| u \|_{L^2(1+\alpha)(1-p)}^2. \]
Setting $\gamma=\frac{1}{2}(1+\alpha)(1-\rho)$ and adding (4.19) and (4.21), we obtain

$$|\sum_{j_k} (A_{j_k} u, u) + \sum_{j_k} (B_{j_k} u, u)| \leq C(t^\rho \|u\|_r^2 + t^{-1}\|u\|^2) .$$

Since $t$ was arbitrary positive number we take the minimum of the right side with respect to $t$. This completes proof of Theorem I.

Proof of Theorem II.

This time we have to deal with

(4.22) $$(P^+ u, u) - \sum_{j_k} (P^+_j \phi_{j_k}(x, D) u, \phi_{j_k}(x, D) u)$$

$$\leq \sum_{j_k} \|((P^+-P^+_j)\phi_{j_k}(x, D) u, \phi_{j_k}(x, D) u)\| .$$

Using Lemma 4.1 again, we have

(4.23) $$(P^+ - P^+_j)\phi_{j_k}(x, D) u, \phi_{j_k}(x, D) u)$$

$$= \int_{-\infty}^{\infty} (s-i\theta)^{-2}(e^{isP}-e^{isP_j})\phi_{j_k}(x, D) u, \phi_{j_k}(x, D) u) ds .$$

We put

$$L(s) = ((e^{isP}-e^{isP_j})\phi_{j_k}(x, D) u, \phi_{j_k}(x, D) u)$$

and divide the integral in (4.23) into two parts;

(4.24) $$M_{j_k} = \int_{0}^{1}|s|^{\rho-1} |L(s)+L(-s)| ds$$

and

(4.25) $$N_{j_k} = \pi i L'(0) + \int_{|s|^{\rho-1}}^{\infty} \int_{-\infty}^{\infty} |L(s)+L(-s)| ds .$$

The latter is easily majorized. In fact, unitarity of operators $e^{isP}$ and $e^{isP_j}$ imply that

(4.26) $$\int_{|s|^{\rho-1}}^{\infty} \int_{-\infty}^{\infty} |L(s)+L(-s)| ds \leq 2 \int_{|s|^{\rho-1}}^{\infty} \int_{-\infty}^{\infty} |\phi_{j_k}(x, D) u|^2 ds$$

$$\leq C \|\phi_{j_k}(x, D) u\|^2 ,$$

while

(4.27) $$|L'(0)| = \|((P-P_j)\phi_{j_k}(x, D) u, \phi_{j_k}(x, D) u)\|$$

$$\leq C \|\phi_{j_k}(x, D) u\|^2 .$$

And we have

(4.28) $$N_{j_k} \leq C \|\phi_{j_k}(x, D) u\|^2 .$$

$L(s)$ can be written in the form
The integrand can be divided into two parts

\[ J(t) = e^{itP} \phi^*_j(2t; x, D)(P - P_{jk})e^{is(t)P_{jk}} \]

and

\[ K(t) = e^{itP}(I - \phi^*_j(2t; x, D))(P - P_{jk})e^{is(t)P_{jk}}. \]

Here \( \phi^*_j(2t; x, D) = e^{-itP}\phi_j(x, D)e^{itP} \). The symbol \( \phi^*_j(2t; x, \xi)^* \) of it is obtained from \( \phi^*_j(x, \xi)^* \) in exactly the same manner as \( \phi^*_j(t; x, \xi)^* \) is obtained from \( \phi^*_j(x, \xi) \). A consequence of this is that there exists constant \( C > 0 \) such that \( |x - x^*| \leq C \xi k |^p - 1 \) and \( |\xi - \xi^*| \leq C \xi k |^p \) hold if \( (x, \xi) \) is in \( \text{supp } \phi^*_j(2t; x, \xi) \) and \( |t| \leq |\xi k| |^p - 1 \). This fact together with definition of \( P_{jk} \) imply that \( \{\phi^*_j(2t; x, \xi)(P - P_{jk})\}_{jk} \) is bounded in \( S_{\alpha, \gamma}^{1, -p} \) and at most bounded number of them have non-empty intersection.

**Lemma 4.3.** We have the following estimates;

\[ (4.32) \quad (1) \quad |(J(t)\phi^*_j(x, D)u, \phi^*_j(x, D)u)| \leq C \xi k |^1 \ |\phi^*_j(x, D)u| |^p, \]

\[ (4.33) \quad (2) \quad ||t|^{-\alpha}(J(t)\phi^*_j(x, D)u, \phi^*_j(x, D)u) - (J(0)\phi^*_j(x, D)u, \phi^*_j(x, D)u)|| \leq C \xi k (1 + \alpha) |\phi^*_j(x, D)u| |^p. \]

**Proof.**

(1) Since \( \{\phi^*_j(2t; x, D)(P - P_{jk})\}_{jk} \) is a bounded set in \( L_{\alpha, \gamma}^{1, -p} \), we have

\[
|\phi^*_j(x, D)u, \phi^*_j(x, D)u)| = |e^{itP} \Lambda^{p, 1}\phi^*_j(2t; x, D)(P - P_{jk})e^{is(t)P_{jk}}\phi^*_j(x, D)u, \Lambda^{1, -p}(2t)\phi^*_j(x, D)u)| \leq C|\phi^*_j(x, D)u||^p \Lambda^{1, -p}(-2t)\phi^*_j(x, D)u| \leq C|\phi^*_j(x, D)u||^p \xi k |^1 \ .
\]

(2) Differentiating (4.30), we have

\[
\frac{d}{dt} J(t) = e^{itP} \phi^*_j(2t; x, D)(P - P_{jk})e^{is(t)P_{jk}}
\]

\[
= e^{itP} \phi^*_j(2t; x, D)((P - P_{jk})^2 + [P, P - P_{jk}])e^{is(t)P_{jk}}. \]

We know, just as above, that

\[ \phi^*_j(2t; x, D)((P - P_{jk})^2 + [P, P - P_{jk}]) \Lambda^{1, -p} \]

is bounded. This fact implies that
Convexity argument again proves
\[ |t|^{-\alpha} \left\{ (J(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) - (J(0)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \right\} \leq C |\xi_k|^{\alpha + \alpha(1 - \rho)} |\phi_{jk}(x, D)u|^2. \]

**Lemma 4.4.**

\[ |(K(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)| \leq C |\xi_k|^{-\alpha} |\phi_{jk}(x, D)u||u| \]

and

\[ \left| \left( \frac{d}{dt} K(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u \right) \right| \leq C |\xi_k|^{-\alpha} |\phi_{jk}(x, D)u||u|. \]

**Proof.** By definition (4.31) we have
\[ \phi_{jk}(x, D)K(t) = e^{tP}\phi_{jk}(2t; x, D)(1 - \phi_{jk}(2t; x, D))(P - P_{jk})e^{(s-t)P_{jk}}. \]

Lemma 4.4 is a consequence of this and the fact that \( \phi_{jk}(2t; x, D)(1 - \phi_{jk}(2t; x, D)) \) belongs to \( L^{-\infty} \).

Now we are able to manage (4.23). \( L(s) \) turns out to be

\[ L(s) = \int_0^s ((J(t) - J(0))\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) dt \]

\[ + s(J(0)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \]

\[ + \int_0^s (s-t) \left( \frac{d}{dt} K(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u \right) dt \]

\[ + s(K(0)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u). \]

The first term is estimated as a consequence of Lemma 4.3.

\[ \left| \int_0^s ((J(t) - J(0))\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) dt \right| \]

\[ \leq \int_0^s t^\alpha t^{-\alpha} ((J(t) - J(0))\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) dt \]

\[ \leq C s^{\alpha+1} |\xi_k|^{\alpha + \alpha(1 - \rho)} |\phi_{jk}(x, D)u|^2, \quad \alpha > 0. \]

Estimate of the third term follows from Lemma 4.4;

\[ \left| \int_0^s (s-t) \left( \frac{d}{dt} K(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u \right) dt \right| \]

\[ \leq C |\xi_k|^{-\alpha} s^\alpha |\phi_{jk}(x, D)u||u|. \]

Thus we have proved that \( L(s) = sW(s) + R(s) \), where
\[ W(s) = ((P - P_{jk}) e^{isP_{jk}} \phi_{jk}(x, D) u, \phi_{jk}(x, D) u) \]

and

\[ |R(s)| \leq C(s^{\alpha+1} |\xi_k| (1 + \rho)^{-\frac{1}{2}} ||\phi_{jk}(x, D) u||^2 + s^2 |\xi_k|^{-4\alpha} ||\phi_{jk}(x, D) u|| ||u||. \]

Now we majorize \( M_{jk} \). First we have

\[ \int_0^{\xi_k^{\alpha-1}} s^{-1} (R(s) + R(-s)) ds \]

\[ \leq C(|\xi_k|^\alpha (1 + \rho)^{-\frac{1}{2}} ||\phi_{jk}(x, D) u||^2 + |\xi_k|^{-4\alpha} ||\phi_{jk}(x, D) u|| ||u||). \]

The remainder is

\[ \int_0^{\xi_k^{\alpha-1}} s^{-1} (\sin (s P_{jk}) \phi_{jk}(x, D) u, (P - P_{jk}) \phi_{jk}(x, D) u) ds. \]

Therefore we have proved estimate

\[ |M_{jk}| \leq C(|\xi_k|^{1-\rho} ||\phi_{jk}(x, D) u||^2 + |\xi_k|^{-4\alpha + 1-\rho} ||\phi_{jk}(x, D) u|| ||u||) \]

if we admit the following lemma that will be proved later.

**Lemma 4.5.** Let \( A \) be a self-adjoint operator in a Hilbert space \( X \), then

\[ \left\| \int_0^K s^{-1} \sin (s A) ds \right\| \leq \pi. \]

It follows from (4.23), (4.24) and (4.26) that we must prove estimate

\[ |\sum_{jk} M_{jk} + \sum_{jk} N_{jk}| \leq C(||u||_||u|| + ||u||^{2+\frac{1}{1+\rho}}). \]

This is proved in the following manner: Taking summation of (4.41) with respect to \( j \) and \( k \), we have

\[ \sum_{jk} |M_{jk}| \leq C \sum_{jk} |\xi_k|^{1-\rho} ||\phi_{jk}(x, D) u||^2 \leq C ||u||^2_1. \]

On the other hand

\[ \sum_{jk} |N_{jk}| \leq C \left( \sum_{jk} |\xi_k|^{1-\rho} ||\phi_{jk}(x, D) u||^2 + |\xi_k|^{-4\alpha + 1-\rho} ||\phi_{jk}(x, D) u|| ||u|| \right) \]

\[ \leq C \left( \sum_{jk} ||\phi_{jk}(x, D) u||^2 + ||u||^2_1 \right) \]

This is because the number of those \( j \)'s for which \( \text{supp} \phi_{jk} \cap K \times R^n, k \) being fixed, is of order \( |\xi_k|^{1-\rho n} \times (\text{the volume of the set } K) \). Theorem II is now proved up to Lemma 4.5.

**Proof of Lemma 4.5.** Let \( A = \int_{-\infty}^{\infty} \lambda dE(\lambda) \) be the spectral representation of \( A \). Then we have
\[
\int_0^K s^{-1}(\sin(sA)x, y) ds = \int_0^K ds \int_{-\infty}^{\infty} s^{-1} \sin(\lambda s) d(E(\lambda)x, y)
\]
\[
= \int_{-\infty}^{\infty} d(E(\lambda)x, y) \int_0^K s^{-1} \sin(\lambda s) ds
\]
\[
= \int_{-\infty}^{\infty} d(E(\lambda)x, y) \int_0^K s^{-1} \sin s ds .
\]
Therefore,
\[
\left\| \int_0^K s^{-1} \sin sA ds \right\| \leq \sup_T \left| \int_0^T s^{-1} \sin s ds \right| \leq \pi .
\]

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References
