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AN APPROXIMATE POSITIVE PART OF A SELF-ADJOINT PSEUDO-DIFFERENTIAL OPERATOR II

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1. Introduction

Let $P=P(x, D)$ be a self-adjoint pseudo-differential operator with the symbol $p(x, \xi)$ in the class $S^1_{1,0}$ of Hörmander. The positive part of P is defined by

$$P^+ = \int_0^\infty \lambda \, dE(\lambda),$$

where $dE(\lambda)$ is the spectral measure of P. We shall be concerned with the following question: To what extent the correspondence; $u \rightarrow P^+u$ can be localized? We shall prove a localization principle for the operator P^+ which is analogous to Theorem 6.3 of Hörmander [5]. If we combine this with our previous discussions in [2], we can explicitly construct an operator B such that we have estimate

$$|((A^+ - B)u, v)| \leqq C \|u\|_{1/6} \|v\|_{1/6},$$

where u and v are arbitrary functions in $\mathcal{D}(\boldsymbol{R}^n)$ and C is a positive constant independent of u and v.

2. Localized operators

Let us repeat our notations. $p(x, \xi)$ is a function in the class $S^1_{1,0}$ which vanishes unless x lies in a compact set K in $\boldsymbol{R}^n$. We treat pseudo-differential operator $P(x, D)$ defined as

$$P(x, D)u(x) = (2\pi)^{-n} \iint_{\boldsymbol{R}^{2n}} p(x, \xi) u(y) e^{i(x-y)\cdot\xi} dy \, d\xi. \tag{2.1}$$

We assume that $P=P(x, D)$ is self-adjoint in Hilbert space $L^2(\boldsymbol{R}^n)$.

Now we make use of the partition of unity of Hörmander [5]. Let $g_0=0$, g_1, g_2, $\cdots$ be the unit lattice points in $\boldsymbol{R}^n$. Then $\boldsymbol{R}^n$ is covered by open cubes of side 2 with center at these points. Let $\Theta(x)$ be a non-negative C_0^∞ function which equals 1 on $|x_j| \leqq 1$ and 0 outside $|x_j| \leqq 3/2$, $j=1, 2, 3, \cdots, n$. We set

$$\varphi_k(x) = \Theta(x-g_k)/(\sum_{k=0}^{\infty}\Theta(x-g_k)^2)^{\frac{1}{2}} \tag{2.2}$$

and

$$\mathring{\varphi}_k(x) = \varphi_k\left(\frac{1}{2}(x-g_k)+g_k\right).$$

Note that $\mathring{\varphi}(x)=1$ on supp φ_k. We, by definition, have

$$\sum_k \varphi_k(x)^2 = 1 \tag{2.4}$$

and

$$\sum_k |D^\alpha \varphi_k(x)|^2 \leqq C_\alpha \tag{2.5}$$

for any multi-index $\alpha=(\alpha_1, \alpha_2, \cdots \alpha_n)$.

$$|x-y| \leqq 3\sqrt{n} \qquad \text{if } x \text{ and } y \text{ are in supp } \varphi_k. \tag{2.6}$$

We set

$$\psi_k(\xi) = \varphi_k(\xi/|\xi|^\rho) \quad \text{and} \tag{2.7}$$

$$\mathring{\psi}_k(\xi) = \mathring{\varphi}_k(\xi/|\xi|^\rho), \qquad \frac{1}{2} \leqq \rho \leqq 1. \tag{2.8}$$

Then we have

$$\sum_k \psi_k(\xi)^2 = 1, \tag{2.9}$$

$$|\xi|^{2|\alpha|\rho} \sum_k |D^\alpha \psi_k(\xi)|^2 \leqq C, \tag{2.10}$$

and

$$|\xi-\eta| \leqq C|\xi|^\rho \qquad \text{if } \xi \text{ and } \eta \text{ are in supp } \psi_k. \tag{2.11}$$

Here and hereafter C stands for positive constants which are different from time to time.

$$\sum_k |\psi_k(\xi)-\psi_k(\eta)|^2 \leqq \frac{C(\xi-\eta)^2}{(1+|\xi|)^\rho(1+|\eta|)^\rho} \qquad \text{for any } \xi, \eta \in \boldsymbol{R}^n. \tag{2.12}$$

Let $\delta_j=|g_j|^{\rho/(1-\rho)}$. Then $g_j\delta_j \in \text{supp}\,\psi_j$. We shall denote by $\psi_j(D)$ the pseudo-differential operator corresponding to the symbol $\psi_j(\xi)$. Then we have

$$\sum_j \psi_j(D)^2 = I. \tag{2.13}$$

The Sobolef norm $||u||_t$ of u is equivalent to $(\sum_j \delta_j^{2t/\rho}||\psi_j(D)u||^2)^{\frac{1}{2}}$.

We put $\varphi_{jk}(x)=\varphi_j(\delta_k^\sigma x)$ and $\phi_{jk}(x, \xi)=\varphi_{jk}(x)\psi_k(\xi)$, $\mathring{\phi}_{jk}(x, \xi)=\mathring{\varphi}_{jk}(x)\mathring{\psi}_k(\xi)$ where $\sigma=(1-\rho)/\rho$. It is obvious from definition that

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial \xi}\right)^\beta \phi_{jk}(x, \xi)\right| \leqq C\delta_k^{|\alpha|\sigma}\delta_k^{-|\beta|} \leqq C|\xi|^{|\alpha|(1-\rho)-|\beta|\rho}, \tag{2.14}$$

This means that the set $\{\phi_{jk}\}_{jk}$ is bounded in the class $S^0_{\rho,1-\rho}$. We shall frequently use the inequality

$$C\|u\|_s^2 \leqq \sum_{jk} \delta_k^{2s/\rho} \|\phi_{jk}(x, D)u\|_s^2 \leqq C^{-1}\|u\|_s^2 . \tag{2.15}$$

Choosing a point (x^{jk}, ξ^k) in supp ϕ_{jk}, we set

$$Q_{jk}(x, D) = \sum_{|\alpha|+|\beta|<N} \frac{x^\alpha D^\beta}{\alpha!\beta!} p^{(\beta)}_{(\alpha)}(x^{jk}, \xi^k),\ N \geqq \rho/(1-\rho)\,, \tag{2.16}$$

and $P_{jk}(x, D) = \frac{1}{2}(Q_{jk}(x, D) + Q_{jk}(x, D)^*)$, where $Q(x, D)^*$ is the formal adjoint of $Q_{jk}(x, D)$. We call these $P_{jk}(x, D)$ localized operators.

3. Statement of results

Theorem 1. *For any given $\gamma > \frac{1}{2}(1-\rho)$, there exists a constant $C_\gamma > 0$ such that inequality*

$$|(P^+u, u) - \sum_{jk} (P^+\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)| \leqq C_\gamma \|u\|_\gamma \|u\| \tag{3.1}$$

holds for any $u \in C_0^\infty(\boldsymbol{R}^n)$.

Theorem 2. *Assume that the localized operators $P_{jk}(x, D)$ are self-adjoint. Let P^+_{jk} denote the non-negative part of P_{jk}. Then, for any $\gamma > \frac{1}{2}(1-\rho)$, there exists a constant $C_\gamma > 0$ such that we have estimate*

$$|(P^+u, u) - \sum_{jk} (P^+_{jk}\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)| \leqq C_\gamma(\|u\|_\gamma \|u\| + \|u\|^2_{\frac{1}{2}(1-\rho)}) \tag{3.2}$$

for any u in $C_0^\infty(\boldsymbol{R}^n)$.

REMARK 3.1. When $\rho = 2/3$ and $N = 2$, the assumption that $P_{jk}(x, D)$ is self-adjoint is satisfied and P^+_{jk} is easily constructed. See [2] for the details. We can construct operator B for which the estimate $|((P^+ - B)u, v)| \leqq C\|u\|_{1/6}\|v\|_{1/6}$ holds for any u and v in C_0^∞.

4. Proofs

We begin our proof by the following lemma.

Lemma 4.1. *Let A be a self-adjoint operator in a Hilbert space X. Let e^{isA} be the corresponding one-parameter group of unitary operators. Then the non-negative part A^+ of A is given by the formula*

$$A^+ x = -(2\pi)^{-1} \int_{-\infty}^{\infty} \frac{e^{isA}}{(s - i0)^2} x\, ds \tag{4.1}$$

for any x in $D(A^2)$. *Here* $(s-i0)^{-2}$ *is the distribution* $\lim_{\varepsilon\downarrow 0}(s-i\varepsilon)^{-2}$. (cf. Gelfand-Silov [3])

Proof. Let $\lambda^+=\max(\lambda, 0)$. Then we have

$$\int_{-\infty}^{\infty}(s-i0)^{-2}e^{is\lambda}ds=-2\pi\lambda^+. \tag{4.2}$$

If φ is in $\mathscr{B}(\boldsymbol{R}^n)$, then

$$\langle(s-i0)^{-2}, \varphi(s)\rangle=\int_0^{\infty}(\varphi(s)+\varphi(-s)-2\varphi(0))/s^2ds+i\pi\varphi'(0). \tag{4.3}$$

This and (4.2) mean that

$$-2\pi\lambda^+=\int_0^{\infty}(e^{is\lambda}+e^{-is\lambda}-2)/s^2ds-\pi\lambda. \tag{4.4}$$

Now we need spectral representation $A=\int_{-\infty}^{\infty}\lambda dE(\lambda)$ of A. Integrating (4.4) with respect to λ by measure $d_\lambda E(\lambda)x$, we have

$$-2\pi A^+x=\int_0^{\infty}(e^{isA}+e^{-isA}-2)/s^2ds\,x-\pi Ax=\int_{-\infty}^{\infty}e^{isA}x/(s-i0)^2ds.$$

Proof of Theorem 1. We have to deal with the difference

$$\begin{aligned}&(P^+u, u)-\sum_{jk}(P^+\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)\\&=\sum_{jk}([P^+, \phi^*_{jk}(x, D)]\phi_{jk}(x, D)u, u).\end{aligned} \tag{4.5}$$

Putting

$$\begin{aligned}&\phi_{jk}(s; x, D)=e^{i\frac{1}{2}sP}\phi_{jk}(x, D)e^{-i\frac{1}{2}sP} \quad\text{and}\\&\phi^*_{jk}(s; x, D)=e^{i\frac{1}{2}sP}\phi_{jk}(x, D)^*e^{-i\frac{1}{2}sP},\end{aligned} \tag{4.6}$$

we have

$$\begin{aligned}&[e^{isP}, \phi_{jk}(x, D)^*]\phi_{jk}(x, D)\\&=e^{i\frac{1}{2}sP}(\phi^*_{jk}(s; x, D)-\phi^*_{jk}(-s; x, D))\phi_{jk}(s; x, D)e^{i\frac{1}{2}sP}.\end{aligned} \tag{4.7}$$

Therefore by lemma 4.1,

$$\begin{aligned}&[P^+, \phi_{jk}(x, D)^*]\phi_{jk}(x, D)\\&=-(2\pi)^{-1}\int_{-\infty}^{\infty}(s-i0)^{-2}e^{i\frac{1}{2}sP}(\phi^*_{jk}(s; x, D)-\phi^*_{jk}(-s; x, D))\phi_{jk}(s; x, D)e^{i\frac{1}{2}sP}ds.\end{aligned} \tag{4.8}$$

The operator $\phi_{jk}(s; x, D)$ is a pseudo-differential operator whose symbol is given in the following manner; Let $(y(t; x, \xi), \eta(t; x, \xi))$ be the solution of the Hamilton-Jacobi equations

$$\frac{d\eta}{dt}=\frac{\partial p(y, \eta)}{\partial y}, \quad \frac{dy}{dt}=-\frac{\partial p(y, \eta)}{\partial \eta} \tag{4.9}$$

with initial conditions $y(0; x, \xi)=x$, and $\eta(0; x, \xi)=\xi$. The symbol of $\phi_{jk}(s; x, D)$ is

$$\phi_{jk}(s; x, \xi) = \phi_{jk}(y(s; x, \xi), \eta(s, x, \xi)) . \tag{4.10}$$

(cf. Egoroff [1], Hörmander [6] and Nirenberg-Trèves [7].). As a consequence, the sequence $\phi_{jk}(s; x, \xi)$ is bounded in $S^{0}_{\rho,1-\rho}$ and the number of overlaps of supp $\phi_{jk}(s; x, \xi)$ is bounded. Set

$$\Phi_{jk}(s; x, D) = (\phi^{*}_{jk}(s; x, D)-\phi^{*}_{jk}(-s; x, D))\phi_{jk}(s; x, D) . \tag{4.11}$$

Then we have

Lemma 4.2.

(4.12) 1° $\Phi_{jk}(0; x, D) = 0$,

(4.13) 2° $$\frac{d}{ds}\Phi_{jk}(s; x, D) = \frac{1}{2}i\{[P, \Phi^{*}_{jk}(x, D)]_{jk}+[P, \phi^{*}_{jk}(x, D)]_{-(s)}\}\phi_{jk}(s; x, D) + \frac{1}{2}i(\phi^{*}_{jk}(s; x, D)-\phi_{jk}(-s; x, D))[P, \phi_{jk}]_{(s)} .$$

(4.15) 3° $$|s|^{-\alpha}\left\{\frac{d}{ds}\Phi_{jk}(s; x, D)-2i[P, \phi^{*}_{jk}(x, D)]\phi_{jk}(x, D)\right\},\quad j, k=0, 1, 2,\cdots,$$

is a bounded sequence in the space $L^{(1+\alpha)(1-\rho)}_{\rho,1-\rho}$, *if* $0\leq\alpha<1$. *Here we have used the notation* $[P, \phi^{*}_{jk}(x, D)]_{(s)}=e^{i\frac{1}{2}sP}[P, \phi^{*}_{jk}(x, D)]e^{-i\frac{1}{2}sP}$.

Proof.

1° is obvious.

2° $$\frac{d}{ds}\phi^{*}_{jk}(s; x, D) = \frac{1}{2}ie^{i\frac{1}{2}sP}[P, \phi^{*}_{jk}]e^{-i\frac{1}{2}sP} = \frac{1}{2}i[P, \phi^{*}_{jk}(x, D)]_{(s)} .$$

3° $$\begin{aligned}\frac{d^2}{ds^2}\Phi_{jk}(s; x, D) = \\ &= (i/2)^2\{[P, [P, \phi^{*}_{jk}]]_{(s)}-[P, [P, \phi^{*}_{jk}(x, D)]]_{(-s)}\}\phi_{jk}(s; x, D) \\ &\quad +2(i/2)^2\{[P, \phi_{jk}(x, D)^{*}]_{(s)}+[P, \phi^{*}_{jk}(x, D)]_{(-s)}\}[P, \phi_{jk}]_{(+s)} \\ &\quad +(i/2)^2(\phi^{*}_{jk}(s; x, D)-\phi_{jk}(-s; x, D))[P, [P, \phi_{jk}]]_{(s)} .\end{aligned}$$

This implies that the set $\left\{\frac{d^2}{ds^2}\Phi_{jk}(s; x, D)\right\}_{jk}$ is bounded in $S^{2(1-\rho)}_{\rho,1-\rho}$. Applying convexity argument, we can prove that the set $\left\{\frac{d}{ds}\Phi_{jk}(s; x, D)-\frac{d}{ds}\Phi_{jk}(0; x, D)\right\}|s|^{-\alpha}$ is bounded in $S^{(1+\alpha)(1-\rho)}_{\rho,1-\rho}(\boldsymbol{R}^n)$. This proves 3°.

Now we come back to the proof of Theorem 1. We divide integral (4.8) into two parts;

$$(4.16)\qquad A_{jk}=\int_t^\infty s^{-2}(e^{i\frac{1}{2}sP}\Phi_{jk}(s;\,x,\,D)e^{i\frac{1}{2}sP}+e^{-i\frac{1}{2}sP}\Phi_{jk}(-s;\,x,\,D)e^{-i\frac{1}{2}sP})ds$$

and

$$(4.17)\qquad B_{jk}=-2\pi[P,\,\phi^*_{jk}(x,\,D)]\phi_{jk}(x,\,D)+ \\ +\int_0^t s^{-2}(e^{i\frac{1}{2}sP}\Phi_{ij}(s;\,x,\,D)e^{i\frac{1}{2}sP}+e^{-i\frac{1}{2}sP}\Phi_{jk}(-s;\,x,\,D)e^{-i\frac{1}{2}sP})ds\,.$$

We have to prove estimate

$$(4.18)\qquad |\sum_{jk}(A_{jk}u,\,u)+\sum_{jk}(B_{jk}u,\,u)|\leqq C_\gamma\|u\|_\gamma\|u\|\,.$$

Since $\{\Phi_{jk}(s;\,x,\,\xi)\}_{jk}$ is bounded in $S^0_{\rho,1-\rho}$ and the number of overlaps of supp Φ_{jk} is bounded, the series $\sum_{jk}\Phi_{jk}(s;\,x,\,D)$ converges to an operator $T(s;\,x,\,D)$ in $L^0_{\rho,1-\rho}$ of Hörmander [5]. Thus we have

$$(4.19)\qquad |\sum_{jk}(A_{jk}u,\,u)|=\left|\int_t^\infty s^{-2}\{(T(s;\,x,\,D)e^{i\frac{1}{2}sP}u,\,e^{-i\frac{1}{2}sP}u) \\ +(T(-s;\,x,\,D)e^{-i\frac{1}{2}sP}u,\,e^{i\frac{1}{2}sP}u)\}ds\right| \\ \leqq Ct^{-1}\|u\|^2\,.$$

We get estimate of $\sum_{jk}(B_{jk}u,\,u)$ by virtue of lemma 4.2. The set $\left\{|s|^{-(1+\alpha)}\left(\Phi_{jk}(s;\,x,\,D)-s\frac{d}{ds}\Phi_{jk}(0;\,x,\,D)\right)\right\}_{jk}$ is bounded in $S^{(1+\alpha)(1-\rho)}_{\rho,1-\rho}$. If we set $\Lambda=(1-\Delta)^{\frac{1}{2}}$ and

$$S_{jk}(s;\,x,\,D)=\Lambda^{-\frac{1}{2}(1+\alpha)(1-\rho)}s^{-(1+\alpha)}\left(\Phi_{jk}(s;\,x,\,D)-s\frac{d}{ds}\Phi_{jk}(0;\,x,\,D)\right)\Lambda^{-\frac{1}{2}(1+\alpha)(1-\rho)}\,,$$

the sequence of their symbols $S_{jk}(s;\,x,\,D)$ is bounded in $S^0_{\rho,1-\rho}$ and the number of overlaps of supports of them is also bounded. The series $\sum_{kj}S_{jk}(s;\,x,\,D)$ thus converges to an operator $S(s;\,x,\,D)$ in the space $L^0_{\rho,1-\rho}$. Hence we have

$$(4.20)\qquad \sum_{jk}(B_{jk}u,\,u)= \\ =\int_0^t s^{\alpha-1}(S(s;\,x,\,D)e^{i\frac{1}{2}sP}\Lambda^{\frac{1}{2}(1+\alpha)(1-\rho)}(s)u,\,e^{-i\frac{1}{2}sP}\Lambda^{\frac{1}{2}(1+\alpha)(1-\rho)}(-s)u)ds \\ +\int_0^t s^{\alpha-1}(S(-s;\,x,\,D)e^{-i\frac{1}{2}sP}\Lambda^{\frac{1}{2}(1+\alpha)(1-\rho)}(-s)u,\,e^{-i\frac{1}{2}sP}\Lambda^{\frac{1}{2}(1+\alpha)(1-\rho)}(-s)u)ds\,,$$

where $\Lambda(s)=e^{i\frac{1}{2}sP}\Lambda e^{-i\frac{1}{2}sP}$.

Since $\Lambda(s)$ and $\Lambda(-s)$ are elliptic operators of order 1, we have

$$(4.21)\qquad |\sum_{jk}(B_{jk}u,\,u)|\leqq C\int_0^t s^{\alpha-1}ds\|u\|^2_{\frac{1}{2}(1+\alpha)(1-\rho)} \\ =Ct^\alpha\|u\|^2_{\frac{1}{2}(1+\alpha)(1-\rho)}$$

Setting $\gamma=\frac{1}{2}(1+\alpha)(1-\rho)$ and adding (4.19) and (4.21), we obtain

$$|\sum_{jk}(A_{jk}u, u)+\sum_{jk}(B_{jk}u, u)|\leqq C(t^{\alpha}\|u\|_{\gamma}^{2}+t^{-1}\|u\|^{2}).$$

Since t was arbitrary positive number we take the minimum of the right side with respect to t. This completes proof of Theorem I.

Proof of Theorem II.

This time we have to deal with

$$\begin{aligned}(4.22)\qquad &|(P^{+}u, u)-\sum_{jk}(P_{jk}^{+}\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)|\\ &\leqq\sum_{jk}|((P^{+}-P_{jk}^{+})\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)|.\end{aligned}$$

Using Lemma 4.1 again, we have

$$\begin{aligned}(4.23)\qquad &((P^{+}-P_{jk}^{+})\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)\\ &=\int_{-\infty}^{\infty}(s-i0)^{-2}((e^{isP}-e^{isP_{jk}})\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)ds.\end{aligned}$$

We put

$$L(s)=((e^{isP}-e^{isP_{jk}})\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)\quad\text{and}$$

divide the integral in (4.23) into two parts;

$$(4.24)\qquad M_{jk}=\int_{0}^{|\xi_k|^{\rho-1}}s^{-2}(L(s)+L(-s))ds\quad\text{and}$$

$$(4.25)\qquad N_{jk}=\pi iL'(0)+\int_{|\xi_k|^{\rho-1}}^{\infty}s^{-2}(L(s)+L(-s))ds.$$

The latter is easily majorized. In fact, unitarity of operators e^{isP} and $e^{isP_{jk}}$ imply that

$$\begin{aligned}(4.26)\qquad \int_{|\xi_k|^{\rho-1}}^{\infty}s^{-2}|L(s)+L(-s)|ds&\leqq 2\int_{|\xi_k|^{\rho-1}}^{\infty}s^{-2}\|\phi_{jk}(x, D)u\|^{2}ds\\ &\leqq C|\xi_k|^{1-\rho}\|\phi_{jk}(x, D)u\|^{2},\end{aligned}$$

while

$$\begin{aligned}(4.27)\qquad |L'(0)|&=|((P-P_{jk})\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)|\\ &\leqq C|\xi_k|^{1-\rho}\|\phi_{jk}(x, D)u\|^{2}.\end{aligned}$$

And we have

$$(4.28)\qquad N_{jk}\leqq C|\xi_k|^{1-\rho}\|\phi_{jk}(x, D)u\|^{2}.$$

$L(s)$ can be written in the form

$$(4.29)\qquad L(s)=\int_0^s \frac{d}{dt}((e^{itP}e^{-i(s-t)P_{jk}})\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)dt$$
$$=\int_0^s (e^{itP}(P-P_{jk})e^{i(s-t)P_{jk}}\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)dt\,.$$

The integrand can be divided into two parts

$$(4.30)\qquad J(t)=e^{itP}\overset{\circ}{\phi}{}^*_{jk}(2t;\, x, D)(P-P_{jk})e^{i(s-t)P_{jk}}$$

and

$$(4.31)\qquad K(t)=e^{itP}(I-\overset{\circ}{\phi}{}^*_{jk}(2t;\, x, D))(P-P_{jk})e^{i(s-t)P_{jk}}\,.$$

Here $\overset{\circ}{\phi}{}^*_{jk}(2t;\, x, D)=e^{-itP}\overset{\circ}{\phi}_{jk}(x, D)^*e^{itP}$. The symbol $\overset{\circ}{\phi}_{jk}(2t;\, x, \xi)^*$ of it is obtained from $\overset{\circ}{\phi}_{jk}(x, \xi)^*$ in exactly the same manner as $\phi_{jk}(t;\, x, \xi)^*$ is obtained from $\phi^*_{jk}(x, \xi)$. A consequence of this is that there exists constant $C>0$ such that $|x-x^{jk}|\leqq C|\xi_k|^{\rho-1}$ and $|\xi-\xi^k|\leqq C|\xi_k|^{\rho}$ hold if (x, ξ) is in supp $\phi^*_{jk}(2t;\, x, \xi)$ and $|t|\leqq|\xi_k|^{\rho-1}$. This fact together with definition of P_{jk} imply that $\{\phi^*_{jk}(2t;\, x, \xi)(P-P_{jk})\}_{jk}$ is bounded in $S^{1-\rho}_{\rho,1-\rho}$ and at most bounded number of them have non-empty intersection.

Lemma 4.3. *We have the following estimates;*

$$(4.32)\quad (1)\quad |(J(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)|\leqq C|\xi_k|^{1-\rho}\|\phi_{jk}(x, D)u\|^2\,,$$

$$(4.33)\quad (2)\quad \big||t|^{-\alpha}((J(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)-(J(0)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u))\big|$$
$$\leqq C|\xi_k|^{(1+\alpha)(1-\beta)}\|\phi_{jk}(x, D)u\|^2\,.$$

Proof.

(1) Since $\{\overset{\circ}{\phi}{}^*_{jk}(2t;\, x, D)(P-P_{jk})\}_{jk}$ is a bounded set in $L^{1-\rho}_{\rho,1-\rho}$, we have

$$|(J(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)|$$
$$=|(e^{itP}\Lambda^{\rho-1}\overset{\circ}{\phi}{}^*_{jk}(2t;\, x, D)(P-P_{jk})e^{i(s-t)P_{jk}}\phi_{jk}(x, D)u, \Lambda^{1-\rho}(-2t)\phi_{jk}(x, D)u)|$$
$$\leqq C\|\phi_{jk}(x, D)u\|\,\|\Lambda^{1-\rho}(-2t)\phi_{jk}(x, D)u\|$$
$$\leqq C\|\phi_{jk}(x, D)u\|^2|\xi_k|^{1-\rho}\,.$$

(2) Differentiating (4.30), we have

$$\frac{d}{dt}J(t)=e^{itP}\overset{\circ}{\phi}{}^*_{jk}(2t;\, x, D)(P(P-P_{jk})-(P-P_{jk})P_{jk})e^{i(s-t)P_{jk}}$$
$$=e^{itP}\phi^*_{jk}(2t;\, x, D)\{(P-P_{jk})^2+[P, P-P_{jk}]\}e^{i(s-t)P_{jk}}\,.$$

We know, just as above, that

$$\overset{\circ}{\phi}{}^*_{jk}(2t;\, x, D)\{(P-P_{jk})^2+[P, P-P_{jk}]\}\Lambda^{-(1-\rho)}$$

is bounded. This fact implies that

$$\left|\left(\frac{d}{dt}J(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u\right)\right| \leqq C\,|\xi_k|^{2(1-\rho)}\|\phi_{jk}(x, D)u\|^2 .$$

Convexity argument again proves

$$\begin{aligned}&|\,|t|^{-\alpha}\{(J(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)-(J(0)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)\}\,| \\ &\leqq C\,|\xi_k|^{(1+\alpha)(1-\rho)}\|\phi_{jk}(x, D)u\|^2 .\end{aligned}$$

Lemma 4.4.

$$|(K(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)| \leqq C|\xi_k|^{-4n}\|\phi_{jk}(x, D)u\|\,\|u\| \tag{4.34}$$

and

$$\left|\left(\frac{d}{dt}K(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u\right)\right| \leqq C\,|\xi_k|^{-4n}\|\phi_{jk}(x, D)u\|\,\|u\| . \tag{4.35}$$

Proof. By definition (4.31) we have

$$\phi^*_{jk}(x, D)K(t) = e^{itP}\phi^*_{jk}(2t; x, D)(1-\mathring{\phi}^*_{jk}(2t; x, D))(P-P_{jk})e^{i(s-t)P_{jk}} .$$

Lemma 4.4 is a consequence of this and the fact that $\phi^*_{jk}(2t; x, D)(1-\mathring{\phi}^*_{jk}(2t; x, D))$ belongs to $L^{-\infty}$.

Now we are able to manage (4.23). $L(s)$ turns out to be

$$\begin{aligned} L(s) = &\int_0^s ((J(t)-J(0))\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)\,dt \\ &+s(J(0)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\ &+\int_0^s (s-t)\left(\frac{d}{dt}K(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u\right)dt \\ &+s(K(0)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) . \end{aligned} \tag{4.36}$$

The first term is estimated as a consequence of Lemma 4.3.

$$\begin{aligned} &\left|\int_0^s ((J(t)-J(0))\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)\,dt\right| \\ &=\left|\int_0^s t^{\alpha}t^{-\alpha}(J(t)-J(0))(\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)\,dt\right| \\ &\leqq Cs^{\alpha+1}|\xi_k|^{(1+\alpha)(1-\rho)}\|\phi_{jk}(x, D)u\|^2, \qquad \alpha>0 . \end{aligned} \tag{4.37}$$

Estimate of the third term follows from Lemma 4.4;

$$\begin{aligned} &\left|\int_0^s (s-t)\left(\frac{d}{dt}K(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u\right)dt\right| \\ &\leqq C\,|\xi_k|^{-4n}s^2\|\phi_{jk}(x, D)u\|\,\|u\| . \end{aligned} \tag{4.38}$$

Thus we have proved that $L(s)=sW(s)+R(s)$, where

(4.39) $$W(s)=((P-P_{jk})e^{isP_{jk}}\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)$$

and

(4.40) $$|R(s)|\leqq C(s^{\alpha+1}|\xi_k|^{(1+\alpha)(1-\rho)}||\phi_{jk}(x, D)u||^2+s^2|\xi_k|^{-4n}||\phi_{jk}(x, D)u||\,||u||)\,.$$

Now we majorize M_{jk}. First we have

$$\left|\int_0^{|\xi_k|^{\rho-1}} s^{-2}(R(s)+R(-s))ds\right|$$

$$\leqq C(|\xi_k|^{\alpha(\rho-1)}|\xi_k|^{(1+\alpha)(1-\rho)}||\phi_{jk}(x, D)u||^2+|\xi_k|^{-4n+1-\rho}||\phi_{jk}(x, D)u||\,||u||)\,.$$

The remainder is

$$\int_0^{|\xi_k|^{\rho-1}} s^{-1}(\sin(sP_{jk})\phi_{jk}(x, D)u, (P-P_{jk})^*\phi_{jk}(x, D)u)ds\,.$$

Therefore we have proved estimate

(4.41) $$|M_{jk}|\leqq C(|\xi_k|^{1-\rho}||\phi_{jk}(x, D)u||^2+|\xi_k|^{-4n+1-\rho}||\phi_{jk}(x, D)u||\,||u||)$$

if we admit the following lemma that will be proved later.

Lemma 4.5. *Let A be a self-adjoint operator in a Hilbert space X, then*

$$\left\|\int_0^K s^{-1}\sin(sA)\,ds\right\|\leqq\pi\,.$$

It follows from (4.23), (4.24) and (4.26) that we must prove estimate

$$|\sum_{jk} M_{jk}+\sum_{jk} N_{jk}|\leqq C(||u||_\gamma||u||+||u||^2_{(1-\rho)/2})$$

This is proved in the following manner: Taking summation of (4.41) with respect to j and k, we have

$$\sum_{jk}|M_{jk}|\leqq C\sum_{jk}|\xi_k|^{1-\rho}||\phi_{jk}(x, D)u||^2\leqq C||u||^2_{\frac{1}{2}(1-\rho)}\,.$$

On the other hand

$$\sum_{jk}|N_{jk}|\leqq C(\sum_{jk}|\xi_k|^{1-\rho}||\phi_{jk}(x, D)u||^2+\xi_k^{-4n+1-\rho}||\phi_{jk}(x, D)u||\,||u||)$$

$$\leqq C(\sum_{jk}||\phi_{jk}(x,D)u||^2_{\frac{1}{2}(1-\rho)}+||u||^2)$$

$$\leqq C||u||^2_{\frac{1}{2}(1-\rho)}\,,$$

This is because the number of those j's for which supp $\phi_{jk}\cap K\times R^n$, k being fixed, is of order $|\xi_k|^{(1-\rho)n}\times$(the volume of the set K). Theorem II is now proved up to Lemma 4.5.

Proof of Lemma 4.5. Let $A=\int_{-\infty}^{\infty}\lambda dE(\lambda)$ be the spectral representation of A. Then we have

$$\int_0^K s^{-1}(\sin(sA)x, y)ds = \int_0^K ds \int_{-\infty}^{\infty} s^{-1}\sin(\lambda s)d(E(\lambda)x, y)$$
$$= \int_{-\infty}^{\infty} d(E(\lambda)x, y) \int_0^K s^{-1}\sin(\lambda s)ds$$
$$= \int_{-\infty}^{\infty} d(E(\lambda)x, y) \int_0^{K\lambda} s^{-1}\sin s\,ds\,.$$

Therefore,

$$\left\|\int_0^K s^{-1}\sin sA\,ds\right\| \leqq \operatorname{Sup}_T \left|\int_0^T s^{-1}\sin s\,ds\right| \leqq \pi\,.$$

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