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On Mappings between Algebraic Systems, II

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In the previous paper [1], we have defined the $\boldsymbol{P}$-mappings*) and the $\boldsymbol{P}$-product systems*), and shown that the algebraic Taylor's expansion theorem*) holds between the $\boldsymbol{P}$-mappings and the $\boldsymbol{P}$-product systems. And some fundamental results with respect to $\boldsymbol{P}$-mappings have been derived from this theorem.

The present paper is the continuation of the paper [1]. In the section 1 of this paper, we shall introduce the concept of the B_W-conjugate relation between families $\boldsymbol{P}$ and $\boldsymbol{Q}$ of basic mapping-formulas*), and it is a relation between $\boldsymbol{P}$-mappings and $\boldsymbol{Q}$-mappings. And, by using the algebraic Taylor's expansion theorem, we shall show that this relation is equivalent to the existence of some inner isomorphic mapping between the $\boldsymbol{P}$-product system $\boldsymbol{P}(\mathfrak{B})$ and the $\boldsymbol{Q}$-product system $\boldsymbol{Q}(\mathfrak{B})$ for every B_W-algebraic system $\mathfrak{B}$. In the section 2, we shall define the derivations between primitive algebraic systems, by using the concepts of the (A_V, B_W)-universality*) and the B_W-conjugate relation. And we shall show that one of these derivations is the usual one in the case of the commutative algebras over a field of characteristic 0. Thus the derivations can be considered as the mappings which are some natural algebraic generalization of homomorphisms.

§ 1. Some relations between families of basic mapping-formulas.

Let R be a set of relations of the form

$$b_1 = F_1(a_1, \cdots, a_m), \cdots, b_n = F_n(a_1, \cdots, a_m)$$

on a free ϕ_W-algebraic system $F(\{a_1, \cdots, a_m, b_1, \cdots, b_n\}, \phi_W)$. And let B_W be a system of composition-identities with respect to W. If there exists a set S of relations of the form

$$a_1 = F_1^*(b_1, \cdots, b_n), \cdots, a_m = F_m^*(b_1, \cdots, b_n)$$

such that

*) Cf. [1].

$$F(\{a_1, \cdots, a_m, b_1, \cdots, b_n\}, B_W, R)$$
$$= F(\{a_1, \cdots, a_m, b_1, \cdots, b_n\}, B_W, S),$$

i.e., R and S are B_W-equivalent, then the system of W-polynomials

$$F_1(x_1, \cdots, x_m), \cdots, F_n(x_1, \cdots, x_m) \tag{1.1}$$

is said to be B_W-regular, and the system of W-polynomials

$$F_1^*(y_1, \cdots, y_n), \cdots, F_m^*(y_1, \cdots, y_n)$$

is called a B_W-inverse system of (1.1). From the above definitions, it is clear that any B_W-inverse system is B_W-regular.

Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be families $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ and $\boldsymbol{Q}_{V,W}\{\eta_1, \cdots, \eta_n\}$ of basic mapping-formulas respectively. If there exists a system of W-polynomials

$$F_1(x_1, \cdots, x_m), \cdots, F_n(x_1, \cdots, x_m) \tag{1.2}$$

such that, for any system $\{\varphi_1, \cdots, \varphi_m\}$ of $\boldsymbol{P}$-mappings from any ϕ_V-algebraic system $\mathfrak{A}$ into any B_W-algebraic system $\mathfrak{B}$, the system $\{\psi_1, \cdots, \psi_n\}$ of mappings, each of which is defined by

$$\psi_\nu(a) = F_\nu(\varphi_1(a), \cdots, \varphi_m(a)),$$

is a system of $\boldsymbol{Q}$-mappings, then the system (1.2) is called a B_W-translator from $\boldsymbol{P}$ into $\boldsymbol{Q}$. In the above definition, if the system (1.2) is B_W-regular, then we say that $\boldsymbol{P}$ is B_W-conjugate to $\boldsymbol{Q}$, and denote it by $\boldsymbol{P} \overset{B_W}{\sim} \boldsymbol{Q}$.

Theorem 1.1. *Let* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *be families* $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ *and* $\boldsymbol{Q}_{V,W}\{\eta_1, \cdots, \eta_n\}$ *of basic mapping-formulas respectively. And let*

$$F_1(x_1, \cdots, x_m), \cdots, F_n(x_1, \cdots, x_m) \tag{1.3}$$

be a system of W-polynomials. Then, in order that the system (1.3) *is a* B_W*-translator from* $\boldsymbol{P}$ *into* $\boldsymbol{Q}$, *it is necessary and sufficient that*

$$\begin{cases} F_\nu\left(P_{\xi_1 v}\begin{pmatrix} \xi_1(x_1), \cdots, \xi_1(x_{N(v)}) \\ \cdots\cdots\cdots\cdots \\ \xi_m(x_1), \cdots, \xi_m(x_{N(v)}) \end{pmatrix}, \cdots, P_{\xi_m v}\begin{pmatrix} \xi_1(x_1), \cdots, \xi_1(x_{N(v)}) \\ \cdots\cdots\cdots\cdots \\ \xi_m(x_1), \cdots, \xi_m(x_{N(v)}) \end{pmatrix}\right) \\ \overset{B_W}{=} Q_{\eta_\nu v}\begin{pmatrix} F_1(\xi_1(x_1), \cdots, \xi_m(x_1)), \cdots, F_1(\xi_1(x_{N(v)}), \cdots, \xi_m(x_{N(v)})) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_n(\xi_1(x_1), \cdots, \xi_m(x_1)), \cdots, F_n(\xi_1(x_{N(v)}), \cdots, \xi_m(x_{N(v)})) \end{pmatrix} \\ \textit{for every } \nu = 1, \cdots, n \textit{ and every } v \in V. \end{cases} \tag{1.4}$$

Proof of necessity. Let $\mathfrak{A}$ be the free ϕ_V-algebraic system $F(\{x_1, \cdots, x_{N(v)}\}, \phi_V)$, and $\mathfrak{B}$ the free B_W-algebraic system $F(\{\xi_1(x_1), \cdots, \xi_1(x_{N(v)}), \cdots, \xi_m(x_1), \cdots, \xi_m(x_{N(v)})\}, B_W)$. Then it is clear by Theorem 1.3 in [1] that there exists a system $\{\varphi_1, \cdots, \varphi_m\}$ of $\boldsymbol{P}$-mappings, each of which satisfies

$$\varphi_\mu(x_N) = \xi_\mu(x_N) \qquad (N=1, \cdots, N(v)). \tag{1.5}$$

Now, let $\{\psi_1, \cdots, \psi_n\}$ be the system of mappings from $\mathfrak{A}$ into $\mathfrak{B}$, each of which is defined by

$$\psi_\nu(x) = F_\nu(\varphi_1(x), \cdots, \varphi_m(x)).$$

Then $\{\psi_1, \cdots, \psi_n\}$ is a system of $\boldsymbol{Q}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$, because the system (1.3) is a B_W-translator from $\boldsymbol{P}$ into $\boldsymbol{Q}$. Hence we have the following computation:

$$\begin{aligned}
&F_\nu\left(P_{\xi_1 v}\begin{pmatrix}\varphi_1(x_1), \cdots, \varphi_1(x_{N(v)}) \\ \cdots\cdots\cdots\cdots \\ \varphi_m(x_1), \cdots, \varphi_m(x_{N(v)})\end{pmatrix}, \cdots, P_{\xi_m v}\begin{pmatrix}\varphi_1(x_1), \cdots, \varphi_1(x_{N(v)}) \\ \cdots\cdots\cdots\cdots \\ \varphi_m(x_1), \cdots, \varphi_m(x_{N(v)})\end{pmatrix}\right) \\
&= F_\nu(\varphi_1(v(x_1, \cdots, x_{N(v)})), \cdots, \varphi_m(v(x_1, \cdots, x_{N(v)}))) \\
&= \psi_\nu(v(x_1, \cdots, x_{N(v)})) \\
&= Q_{\eta_\nu v}(\psi_1(x_1), \cdots, \psi_1(x_{N(v)}), \cdots, \psi_n(x_1), \cdots, \psi_n(x_{N(v)})) \\
&= Q_{\eta_\nu v}\begin{pmatrix}F_1(\varphi_1(x_1), \cdots, \varphi_m(x_1)), \cdots, F_1(\varphi_1(x_{N(v)}), \cdots, \varphi_m(x_{N(v)})) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_n(\varphi_1(x_1), \cdots, \varphi_m(x_1)), \cdots, F_n(\varphi_1(x_{N(v)}), \cdots, \varphi_m(x_{N(v)}))\end{pmatrix}.
\end{aligned}$$

Hence, by (1.5), the identity

$$\begin{aligned}
&F_\nu\left(P_{\xi_1 v}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_{N(v)}) \\ \cdots\cdots\cdots\cdots \\ \xi_m(x_1), \cdots, \xi_m(x_{N(v)})\end{pmatrix}, \cdots, P_{\xi_m v}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_{N(v)}) \\ \cdots\cdots\cdots\cdots \\ \xi_m(x_1), \cdots, \xi_m(x_{N(v)})\end{pmatrix}\right) \\
&= Q_{\eta_\nu v}\begin{pmatrix}F_1(\xi_1(x_1), \cdots, \xi_m(x_1)), \cdots, F_1(\xi_1(x_{N(v)}), \cdots, \xi_m(x_{N(v)})) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_n(\xi_1(x_1), \cdots, \xi_m(x_1)), \cdots, F_n(\xi_1(x_{N(v)}), \cdots, \xi_m(x_{N(v)}))\end{pmatrix}
\end{aligned}$$

is valid in $\mathfrak{B}$. This identity can be considered as the one with respect to $\overset{B_W}{=}$, because $\mathfrak{B}$ is a free B_W-algebraic system.

Proof of sufficiency. Let $\mathfrak{A}$ be any ϕ_V-algebraic system, and $\mathfrak{B}$ any B_W-algebraic system. And let $\{\varphi_1, \cdots, \varphi_m\}$ be any system of $\boldsymbol{P}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$. Moreover, let $\psi_1, \cdots, \psi_n$ be the mappings from $\mathfrak{A}$ into $\mathfrak{B}$, each of which is defined by

$$\psi_\nu(a) = F_\nu(\varphi_1(a), \cdots, \varphi_m(a)).$$

Then, by using (1.4), for any $v \in V$ and any $a_1, \cdots, a_{N(v)} \in \mathfrak{A}$, we have

$$\begin{aligned}
&\psi_\nu(v(a_1, \cdots, a_{N(v)}))\\
&= F_\nu(\varphi_1(v(a_1, \cdots, a_{N(v)})), \cdots, \varphi_m(v(a_1, \cdots, a_{N(v)})))\\
&= F_\nu\left(P_{\xi_1 v}\begin{pmatrix}\varphi_1(a_1), \cdots, \varphi_1(a_{N(v)})\\ \cdots\cdots\cdots\cdots\\ \varphi_m(a_1), \cdots, \varphi_m(a_{N(v)})\end{pmatrix}, \cdots, P_{\xi_m v}\begin{pmatrix}\varphi_1(a_1), \cdots, \varphi_1(a_{N(v)})\\ \cdots\cdots\cdots\cdots\\ \varphi_m(a_1), \cdots, \varphi_m(a_{N(v)})\end{pmatrix}\right)\\
&= Q_{\eta_\nu v}\begin{pmatrix}F_1(\varphi_1(a_1), \cdots, \varphi_m(a_1)), \cdots, F_1(\varphi_1(a_{N(v)}), \cdots, \varphi_m(a_{N(v)}))\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ F_n(\varphi_1(a_1), \cdots, \varphi_m(a_1)), \cdots, F_n(\varphi_1(a_{N(v)}), \cdots, \varphi_m(a_{N(v)}))\end{pmatrix}\\
&= Q_{\eta_\nu v}(\psi_1(a_1), \cdots, \psi_1(a_{N(v)}), \cdots, \psi_n(a_1), \cdots, \psi_n(a_{N(v)}))\,.
\end{aligned}$$

Hence $\{\psi_1, \cdots, \psi_n\}$ is a system of $\boldsymbol{Q}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$. This completes the proof.

Let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas, and let $\mathfrak{B}$ be a ϕ_W-algebraic system. Now let ψ be a mapping from $\boldsymbol{P}(\mathfrak{B})$ into $\mathfrak{B}$. If there exists a W-polynomial $F(x_1, \cdots, x_m)$ such that

$$\psi([b_1, \cdots, b_m]) = F(b_1, \cdots, b_m)$$

for every element $[b_1, \cdots, b_m]$ in $\boldsymbol{P}(\mathfrak{B})$, then ψ is called an inner mapping defined by $F(x_1, \cdots, x_m)$. Moreover, let $\boldsymbol{Q}$ be a family $\boldsymbol{Q}_{V,W}\{\eta_1, \cdots, \eta_n\}$ of basic mapping-formulas. And let $\psi_1, \cdots, \psi_n$ be mappings from $\boldsymbol{P}(\mathfrak{B})$ into $\mathfrak{B}$, and Ψ the mapping from $\boldsymbol{P}(\mathfrak{B})$ into $\boldsymbol{Q}(\mathfrak{B})$ which is defined by

$$\Psi([b_1, \cdots, b_m]) = [\psi_1([b_1, \cdots, b_m]), \cdots, \psi_n([b_1, \cdots, b_m])]$$

for all elements $[b_1, \cdots, b_m] \in \boldsymbol{P}(\mathfrak{B})$. If each ψ_ν is an inner mapping defined by a W-polynomial $F_\nu(x_1, \cdots, x_m)$, then Ψ is called an inner mapping defined by the system of W-polynomials $F_\nu(x_1, \cdots, x_m)$ ($\nu=1, \cdots, n$).

Theorem 1.2. *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be families $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ and $\boldsymbol{Q}_{V,W}\{\eta_1, \cdots, \eta_n\}$ of basic mapping-formulas respectively. And let*

$$F_1(x_1, \cdots, x_m), \cdots, F_n(x_1, \cdots, x_m) \tag{1.6}$$

be a system of W-polynomials. Then, in order that the system (1.6) *is a B_W-translator from $\boldsymbol{P}$ into $\boldsymbol{Q}$, it is necessary and sufficient that, for any B_W-algebraic system $\mathfrak{B}$, the inner mapping Ψ from $\boldsymbol{P}(\mathfrak{B})$ into $\boldsymbol{Q}(\mathfrak{B})$, which is defined by the system* (1.6) *of W-polynomials, is a homomorphism.*

Proof of necessity. Let $\mathfrak{B}$ be any B_W-algebraic system. And let $\varphi_1, \cdots, \varphi_m$ be the mappings from $\boldsymbol{P}(\mathfrak{B})$ into $\mathfrak{B}$, each of which is defined by

$$\varphi_\mu([b_1, \cdots, b_m]) = b_\mu\,.$$

Then it is clear that $\{\varphi_1, \cdots, \varphi_m\}$ is a system of $\boldsymbol{P}$-mappings from $\boldsymbol{P}(\mathfrak{B})$ into $\mathfrak{B}$. Now let $\psi_1, \cdots, \psi_n$ be mappings from $\boldsymbol{P}(\mathfrak{B})$ into $\mathfrak{B}$, each of which is defined by

$$\psi_\nu([b_1, \cdots, b_m]) = F_\nu(\varphi_1([b_1, \cdots, b_m]), \cdots, \varphi_m([b_1, \cdots, b_m])), \quad \text{i.e.,}$$
$$\psi_\nu([b_1, \cdots, b_m]) = F_\nu(b_1, \cdots, b_m).$$

Then, $\{\psi_1, \cdots, \psi_n\}$ is a system of $\boldsymbol{Q}$-mappings from $\boldsymbol{P}(\mathfrak{B})$ into $\mathfrak{B}$, because the system (1. 6) is a B_W-translator from $\boldsymbol{P}$ into $\boldsymbol{Q}$. Hence, by Theorem 1. 1 in [1], the inner mapping

$$\Psi: \quad [b_1, \cdots, b_m] \rightarrow [F_1(b_1, \cdots, b_m), \cdots, F_n(b_1, \cdots, b_m)]$$

is a homomorphism from $\boldsymbol{P}(\mathfrak{B})$ into $\boldsymbol{Q}(\mathfrak{B})$.

Proof of sufficiency. Let $\mathfrak{A}$ be any ϕ_V-algebraic system, and $\mathfrak{B}$ any B_W-algebraic system. Now suppose that $\{\varphi_1, \cdots, \varphi_m\}$ is a system of $\boldsymbol{P}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$. Then, by Theorem 1. 1 in [1], the mapping

$$\Phi: \quad a \rightarrow \Phi(a) = [\varphi_1(a), \cdots, \varphi_m(a)]$$

is a homomorphism from $\mathfrak{A}$ into $\boldsymbol{P}(\mathfrak{B})$. Since the inner mapping

$$\Psi: \quad [b_1, \cdots, b_m] \rightarrow [F_1(b_1, \cdots, b_m), \cdots, F_n(b_1, \cdots, b_m)]$$

is a homomorphism from $\boldsymbol{P}(\mathfrak{B})$ into $\boldsymbol{Q}(\mathfrak{B})$, it is clear that the mapping

$$\Psi\Phi: \quad a \rightarrow \Psi\Phi(a) = [F_1(\varphi_1(a), \cdots, \varphi_m(a)), \cdots, F_n(\varphi_1(a), \cdots, \varphi_m(a))]$$

is a homomorphism from $\mathfrak{A}$ into $\boldsymbol{Q}(\mathfrak{B})$. Hence, by Theorem 1. 1 in [1], the system $\{\psi_1, \cdots, \psi_n\}$ of mappings from $\mathfrak{A}$ into $\mathfrak{B}$, each of which is defined by

$$\psi_\nu(a) = F_\nu(\varphi_1(a), \cdots, \varphi_m(a)),$$

is a system of $\boldsymbol{Q}$-mappings. Thus, the system (1. 6) of W-polynomials is a B_W-translator from $\boldsymbol{P}$ into $\boldsymbol{Q}$. This completes the proof.

Theorem 1. 3. *Let* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *be families* $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ *and* $\boldsymbol{Q}_{V,W}\{\eta_1, \cdots, \eta_n\}$ *of basic mapping-formulas respectively, and let*

$$(1.7) \qquad F_1(x_1, \cdots, x_m), \cdots, F_n(x_1, \cdots, x_m)$$

be a B_W*-regular system of W-polynomials. And let* $\mathfrak{B}$ *be any* B_W*-algebraic system. Now suppose that the inner mapping* Ψ *from* $\boldsymbol{P}(\mathfrak{B})$ *into* $\boldsymbol{Q}(\mathfrak{B})$, *which is defined by the system* (1. 7) *of W-polynomials, is a homomorphism. Then* Ψ *is an isomorphism from* $\boldsymbol{P}(\mathfrak{B})$ *onto* $\boldsymbol{Q}(\mathfrak{B})$, *moreover the inverse mapping* Ψ^{-1} *is an inner mapping defined by a* B_W*-inverse system*

$$(1.8) \qquad F_1^*(y_1, \cdots, y_n), \cdots, F_m^*(y_1, \cdots, y_n)$$

of the system (1.7).

Proof. Let $[b_1, \cdots, b_m]$ be any element in $\boldsymbol{P}(\mathfrak{B})$. Then, by the definition of the inner mapping Ψ, we have

$$\Psi([b_1, \cdots, b_m]) = [F_1(b_1, \cdots, b_m), \cdots, F_n(b_1, \cdots, b_m)].$$

On the other hand, it is clear that

$$F_\mu^*(F_1(b_1, \cdots, b_m), \cdots, F_n(b_1, \cdots, b_m)) = b_\mu \qquad (\mu = 1, \cdots, m).$$

Hence we have

$$\Psi^{-1}([c_1, \cdots, c_n]) = \Phi([c_1, \cdots, c_n])$$

for every element $[c_1, \cdots, c_n]$ in the domain of Ψ^{-1}, where Φ denotes the inner mapping from $\boldsymbol{Q}(\mathfrak{B})$ into $\boldsymbol{P}(\mathfrak{B})$ which is defined by the B_W-inverse system (1.8). Therefore the inner mapping Ψ is a one to one mapping. Hence it is the rest of our proof to show that Ψ maps $\boldsymbol{P}(\mathfrak{B})$ onto $\boldsymbol{Q}(\mathfrak{B})$. Now let $[c_1, \cdots, c_n]$ be any element in $\boldsymbol{Q}(\mathfrak{B})$. Then we have

$$F_\nu(F_1^*(c_1, \cdots, c_n), \cdots, F_m^*(c_1, \cdots, c_n)) = c_\nu \qquad (\nu = 1, \cdots, n).$$

Hence we have

$$\Psi([F_1^*(c_1, \cdots, c_n), \cdots, F_m^*(c_1, \cdots, c_n)]) = [c_1, \cdots, c_n].$$

Therefore Ψ maps $\boldsymbol{P}(\mathfrak{B})$ onto $\boldsymbol{Q}(\mathfrak{B})$. This completes our proof.

Theorem 1.4. *Let* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *be families* $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ *and* $\boldsymbol{Q}_{V,W}\{\eta_1, \cdots, \eta_n\}$ *of basic mapping-formulas respectively. Then the following three propositions are equivalent*:

(a) $\boldsymbol{P}$ *is* B_W*-conjugate to* $\boldsymbol{Q}$.

(b) *There exists a* B_W*-regular system of W-polynomials*

$$(1.9) \qquad F_1(x_1, \cdots, x_m), \cdots, F_n(x_1, \cdots, x_m)$$

such that, for any B_W*-algebraic system* $\mathfrak{B}$, *the inner mapping from* $\boldsymbol{P}(\mathfrak{B})$ *into* $\boldsymbol{Q}(\mathfrak{B})$, *which is defined by the system* (1.9), *is an isomorphism from* $\boldsymbol{P}(\mathfrak{B})$ *onto* $\boldsymbol{Q}(\mathfrak{B})$.

(c) *There exists a* B_W*-regular system of W-polynomials*

$$F_1(x_1, \cdots, x_m), \cdots, F_n(x_1 \cdots, x_m)$$

such that

$$F_\nu\begin{pmatrix}P_{\xi_1 v}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_{N(v)})\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \xi_m(x_1), \cdots, \xi_m(x_{N(v)})\end{pmatrix}, \cdots, P_{\xi_m v}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_{N(v)})\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \xi_m(x_1), \cdots, \xi_m(x_{N(v)})\end{pmatrix}\end{pmatrix}$$

$$\overset{B_W}{=} Q_{\eta_\nu v}\begin{pmatrix}F_1(\xi_1(x_1), \cdots, \xi_m(x_1)), \cdots, F_1(\xi_1(x_{N(v)}), \cdots, \xi_m(x_{N(v)}))\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ F_n(\xi_1(x_1), \cdots, \xi_m(x_1)), \cdots, F_n(\xi_1(x_{N(v)}), \cdots, \xi_m(x_{N(v)}))\end{pmatrix}$$

for every $\nu=1, \cdots, n$ *and every* $v\in V$.

Proof. (a)$\Leftrightarrow$(c) is clear from Theorem 1.1. (a)$\Leftrightarrow$(b) is obvious from Theorems 1.2 and 1.3.

Theorem 1.5. *The B_W-conjugate relation $\overset{B_W}{\sim}$ is an equivalence relation.*

Proof of reflexive law is easy.

Proof of symmetric law. Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be families $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ and $\boldsymbol{Q}_{V,W}\{\eta_1, \cdots, \eta_n\}$ of basic mapping-formulas respectively. Now suppose that $\boldsymbol{P}\overset{B_W}{\sim}\boldsymbol{Q}$. Then, by Theorem 1.4, there exists a B_W-regular system of W-polynomials

$$F_1(x_1, \cdots, x_m), \cdots, F_n(x_1, \cdots, x_m) \tag{1.10}$$

such that, for any B_W-algebraic system $\mathfrak{B}$, the inner mapping Ψ from $\boldsymbol{P}(\mathfrak{B})$ into $\boldsymbol{Q}(\mathfrak{B})$, which is defined by the system (1.10), is an isomorphism from $\boldsymbol{P}(\mathfrak{B})$ onto $\boldsymbol{Q}(\mathfrak{B})$. Moreover, by Theorem 1.3, Ψ^{-1} is an inner mapping defined by a B_W-inverse system of (1.10). Hence $\boldsymbol{Q}\overset{B_W}{\sim}\boldsymbol{P}$ follows from Theorem 1.4, because the B_W-inverse system is B_W-regular.

Proof of transitive law. Let $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ be families $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$, $\boldsymbol{Q}_{V,W}\{\eta_1, \cdots, \eta_n\}$ and $\boldsymbol{R}_{V,W}\{\zeta_1, \cdots, \zeta_l\}$ of basic mapping-formulas respectively. Now suppose that $\boldsymbol{P}\overset{B_W}{\sim}\boldsymbol{Q}$ and $\boldsymbol{Q}\overset{B_W}{\sim}\boldsymbol{R}$. Then, by Theorem 1.4, there exist two systems

$$F_1(x_1, \cdots, x_m), \cdots, F_n(x_1, \cdots, x_m) \text{ and} \tag{1.11}$$

$$G_1(y_1, \cdots, y_n), \cdots, G_l(y_1, \cdots, y_n) \tag{1.12}$$

of W-polynomials such that, for any B_W-algebraic system $\mathfrak{B}$, the inner mappings Ψ: $\boldsymbol{P}(\mathfrak{B})\to\boldsymbol{Q}(\mathfrak{B})$ and Θ: $\boldsymbol{Q}(\mathfrak{B})\to\boldsymbol{R}(\mathfrak{B})$, which are defined by the systems (1.11) and (1.12) respectively, are onto isomorphisms. Hence it is clear that the mapping $\Theta\Psi$ is an isomorphism from $\boldsymbol{P}(\mathfrak{B})$ onto $\boldsymbol{R}(\mathfrak{B})$ and it is an inner mapping defined by the system of W-polynomials

$$\begin{cases}G_1(F_1(x_1, \cdots, x_m), \cdots, F_n(x_1, \cdots, x_m)),\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\ G_l(F_1(x_1, \cdots, x_m), \cdots, F_n(x_1, \cdots, x_m)).\end{cases} \tag{1.13}$$

Now let

$$F_1^*(y_1, \cdots, y_n), \cdots, F_m^*(y_1, \cdots, y_n) \quad \text{and}$$
$$G_1^*(z_1, \cdots, z_l), \cdots, G_n^*(z_1, \cdots, z_l)$$

be B_W-inverse systems of the systems (1.11) and (1.12) respectively. Then it is easily obtained that the system of W-polynomials

$$\begin{gathered} F_1^*(G_1^*(z_1, \cdots, z_l), \cdots, G_n^*(z_1, \cdots, z_l)), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ F_m^*(G_1^*(z_1, \cdots, z_l), \cdots, G_n^*(z_1, \cdots, z_l)) \end{gathered}$$

is a B_W-inverse system of (1.13). Hence the system (1.13) is B_W-regular. Therefore $\boldsymbol{P} \overset{B_W}{\sim} \boldsymbol{R}$ follows from Theorem 1.4. This completes the proof.

Finally we shall introduce the concept of B_W-similarity as a special case of the concept of B_W-conjugate. Now let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be families $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ and $\boldsymbol{Q}_{V,W}\{\eta_1, \cdots, \eta_m\}$ of basic mapping-formulas respectively. If, for any ϕ_V-algebraic system $\mathfrak{A}$ and any B_W-algebraic system $\mathfrak{B}$, any system of $\boldsymbol{P}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$ is a system of $\boldsymbol{Q}$-mappings, and conversely, then we say that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are B_W-similar. As an easy consequence of the above definition we obtain

Theorem 1.6. *Let* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *be families* $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ *and* $\boldsymbol{Q}_{V,W}\{\eta_1, \cdots, \eta_m\}$ *of basic mapping-formulas respectively. Then, in order that* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *are* B_W*-similar, it is necessary and sufficient that*

$$P_{\xi_\mu v}\begin{pmatrix} y_{11}, \cdots, y_{1N(v)} \\ \cdots\cdots\cdots\cdots \\ y_{m1}, \cdots, y_{mN(v)} \end{pmatrix} \overset{B_W}{=} Q_{\eta_\mu v}\begin{pmatrix} y_{11}, \cdots, y_{1N(v)} \\ \cdots\cdots\cdots\cdots \\ y_{m1}, \cdots, y_{mN(v)} \end{pmatrix}$$

for every $\mu = 1, \cdots, m$ *and every* $v \in V$.

§2. Families of (A_V, B_W)-homomorphism type and families of (A_V, B_W)-derivation type.

Let $\boldsymbol{P}$ be a family $P_{V,W}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas. If the basic mapping-formulas of $\boldsymbol{P}$ are of the form

$$\xi_\mu(v(x_1, \cdots, x_{N(v)})) = P_{\xi_\mu v}(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)})) \qquad (\mu = 1, \cdots, m\,;\ v \in V),$$

then $\boldsymbol{P}$ is called a family of (ϕ_V, ϕ_W)-homomorphism type. Moreover let A_V and B_W be systems of composition-identities with respect to V and W respectively. If $\boldsymbol{P}$ is (A_V, B_W)-universal and B_W-similar to some family of (ϕ_V, ϕ_W)-homomorphism type, then $\boldsymbol{P}$ is called a family of (A_V, B_W)-homomorphism type.

Next let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m, \delta\}$ of basic mapping-formulas. If the basic mapping-formulas of $\boldsymbol{P}$ are of the form

$$\xi_\mu(v(x_1, \cdots, x_{N(v)})) = P_{\xi_\mu v}(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)})) \qquad (\mu=1, \cdots, m\,;\; v\in V)$$

and

$$\delta(v(x_1, \cdots, x_{N(v)})) = P_{\delta v}\begin{pmatrix} \xi_1(x_1), \cdots, \xi_1(x_{N(v)}) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \xi_m(x_1), \cdots, \xi_m(x_{N(v)}) \\ \delta(x_1), \cdots, \delta(x_{N(v)}) \end{pmatrix} \qquad (v\in V),$$

then $\boldsymbol{P}$ is called a family of (ϕ_V, ϕ_W)-derivation type. Moreover let A_V and B_W be systems of composition-identities. If $\boldsymbol{P}$ is (A_V, B_W)-universal and B_W-similar to some family of (ϕ_V, ϕ_W)-derivation type, then $\boldsymbol{P}$ is called a family of (A_V, B_W)-derivation type.

Let $\boldsymbol{P}$ be a family of (A_V, B_W)-derivation type. If there exists a family $\boldsymbol{Q}$ of (A_V, B_W)-homomorphism type such that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are B_W-conjugate, then $\boldsymbol{P}$ is called a family of improper (A_V, B_W)-derivation type. Otherwise, $\boldsymbol{P}$ is called a family of proper (A_V, B_W)-derivation type.

If $V=W$ and $A_V=B_W$ in the above definitions, then we simply say "A_V-homomorphism" or "A_V-derivation" in place of "(A_V, B_W)-homomorphism" or "(A_V, B_W)-derivation". Let $\boldsymbol{P}$ be a family of A_V-homomorphism (or A_V-derivation) type, and let U be a subset of V. If the family, which consists of all the basic mapping-formulas of $\boldsymbol{P}$ concerning all the compositions $v\in V-U$, is of homomorphism type, then $\boldsymbol{P}$ is called a family of A_V-U-homomorphism (or A_V-U-derivation) type.

Let $\boldsymbol{P}$ be a family of A_V-U-derivation type. If $\boldsymbol{P}$ is A_V-conjugate to some family of A_V-U-homomorphism type, then $\boldsymbol{P}$ is called a family of U-improper A_V-U-derivation type. Otherwise, $\boldsymbol{P}$ is called a family of U-proper A_V-U-derivation type.

Let K be a commutative field of characteristic 0, and V the set-sum of $\{+, \cdot\}$ and K. And let R_V be the system of composition-identities with respect to V, which define the commutative algebras over K. In the following, we shall determine the form of the family $\boldsymbol{P}_{V,V}\{\varphi_1, \cdots, \varphi_m\}$*) of R_V-$\{\cdot\}$-homomorphism type, and that of the family $\boldsymbol{P}_{V,V}\{\varphi, \delta\}$*) of R_V-$\{\cdot\}$-derivation type.

Theorem 2.1. *Let* $\boldsymbol{P}$ *be a family* $\boldsymbol{P}_{V,V}\{\varphi_1, \cdots, \varphi_m\}$ *whose basic mapping-formulas concerning the compositions different from* $\cdot$ *are of homomorphism type. Then, in order that* $\boldsymbol{P}$ *is a family of* R_V-$\{\cdot\}$*-homomorphism type, it is necessary and sufficient that the basic mapping-*

*) For convienence, we use below the letters φ, ψ in places of the letters ξ, η.

formulas of $\boldsymbol{P}$ *concerning* $\cdot$ *are of the form*

$$\varphi_\mu(xy) = P_{\varphi_\mu\cdot}(\varphi_\mu(x),\ \varphi_\mu(y)) \overset{R_V}{=} h_\mu\varphi_\mu(x)\varphi_\mu(y)\,, \quad h_\mu \in K \qquad (\mu = 1, \cdots, m)\,.$$

Proof. The sufficiency can be easily obtained by Theorem 3. 2 in [1]. In the following, we shall prove the necessity. Since the composition-identity $(x+y)z=xz+yz$ is contained in R_V, and $\boldsymbol{P}$ is R_V-universal, it follows from Theorem 3. 2 in [1] that

$$\begin{aligned} &F_{\varphi_\mu((x+y)z)}(\varphi_\mu(x),\ \varphi_\mu(y),\ \varphi_\mu(z)) \\ &\overset{R_V}{=} F_{\varphi_\mu(xz+yz)}(\varphi_\mu(x),\ \varphi_\mu(y),\ \varphi_\mu(z))\,. \end{aligned}$$

Hence, by Theorem 2. 1 in [1], we have

$$\begin{aligned} &P_{\varphi_\mu\cdot}(\varphi_\mu(x)+\varphi_\mu(y),\ \varphi_\mu(z)) \\ &\overset{R_V}{=} P_{\varphi_\mu\cdot}(\varphi_\mu(x),\ \varphi_\mu(z))+P_{\varphi_\mu\cdot}(\varphi_\mu(y),\ \varphi_\mu(z))\,. \end{aligned}$$

Similarly we have

$$\begin{aligned} &P_{\varphi_\mu\cdot}(\varphi_\mu(x),\ \varphi_\mu(y)+\varphi_\mu(z)) \\ &\overset{R_V}{=} P_{\varphi_\mu\cdot}(\varphi_\mu(x),\ \varphi_\mu(y))+P_{\varphi_\mu\cdot}(\varphi_\mu(x),\ \varphi_\mu(z))\,, \end{aligned}$$

because the composition-identity $x(y+z)=xy+xz$ is contained in R_V. Therefore we have

$$P_{\varphi_\mu\cdot}(\varphi_\mu(x),\ \varphi_\mu(y)) \overset{R_V}{=} h_\mu\varphi_\mu(x)\varphi_\mu(y)\,, \qquad h_\mu \in K\,.$$

This completes the proof.

Theorem 2. 2. *Let* $\boldsymbol{P}$ *be a family* $\boldsymbol{P}_{V,V}\{\varphi,\ \delta\}$ *whose basic mapping-formulas concerning the compositions different from* $\cdot$ *are of homomorphism type. Then, in order that* $\boldsymbol{P}$ *is a family of* R_V-$\{\cdot\}$*-derivation type, it is necessary and sufficient that the basic mapping-formulas of* $\boldsymbol{P}$ *concerning* $\cdot$ *are of the form*

$$\varphi(xy) = P_{\varphi\cdot}(\varphi(x),\ \varphi(y)) \overset{R_V}{=} h\varphi(x)\varphi(y) \quad \textit{and} \tag{2. 1}$$

$$\begin{aligned} \delta(xy) &= P_{\delta\cdot}(\varphi(x),\ \varphi(y),\ \delta(x),\ \delta(y)) \\ &\overset{R_V}{=} a\varphi(x)\varphi(y)+b\varphi(x)\delta(y)+b\delta(x)\varphi(y)+d\delta(x)\delta(y)\,, \end{aligned} \tag{2. 2}$$

where $a,\ b,\ d,\ h \in K$ *and* $bh+ad=b^2$.

Proof. The sufficiency can be easily obtained by Theorem 3. 2 in [1]. In the following, we shall prove the necessity. Now suppose that $\boldsymbol{P}$ is a family of R_V-$\{\cdot\}$-derivation type. Then (2. 1) can be similarly obtained as in the proof of Theorem 2. 1. Next, since the composition-identity $(x+y)z=xz+yz$ is contained in R_V, and $\boldsymbol{P}$ is R_V-universal, it follows from Theorem 3.2 in [1] that

$$F_{\delta((x+y)z)}(\varphi(x), \varphi(y), \varphi(z), \delta(x), \delta(y), \delta(z))$$
$$\overset{R_V}{=} F_{\delta(xz+yz)}(\varphi(x), \varphi(y), \varphi(z), \delta(x), \delta(y), \delta(z)) .$$

Hence, by Theorem 2.1 in [1], we have

$$P_{\delta\cdot}(\varphi(x)+\varphi(y), \varphi(z), \delta(x)+\delta(y), \delta(z))$$
$$\overset{R_V}{=} P_{\delta\cdot}(\varphi(x), \varphi(z), \delta(x), \delta(z))+P_{\delta\cdot}(\varphi(y), \varphi(z), \delta(y), \delta(z)) .$$

Similarly we have

$$P_{\delta\cdot}(\varphi(x), \varphi(y)+\varphi(z), \delta(x), \delta(y)+\delta(z))$$
$$\overset{R_V}{=} P_{\delta\cdot}(\varphi(x), \varphi(y), \delta(x), \delta(y))+P_{\delta\cdot}(\varphi(x), \varphi(z), \delta(x), \delta(z)) ,$$

because the composition-identity $x(y+z)=xy+xz$ is contained in R_V. Therefore we can easily obtain

$$P_{\delta\cdot}(\varphi(x), \varphi(y), \delta(x), \delta(y))$$
$$\overset{R_V}{=} a\varphi(x)\varphi(y)+b\varphi(x)\delta(y)+c\delta(x)\varphi(y)+d\delta(x)\delta(y) ,$$

where a, b, c, $d \in K$. Moreover we have $b=c$, because the composition-identity $xy=yx$ is contained in R_V. Hence we have

$$\text{(2.3)} \qquad P_{\delta\cdot}(\varphi(x), \varphi(y), \delta(x), \delta(y))$$
$$\overset{R_V}{=} a\varphi(x)\varphi(y)+b\varphi(x)\delta(y)+b\delta(x)\varphi(y)+d\delta(x)\delta(y) .$$

Since the composition-identity $(xy)z=x(yz)$ is contained in R_V, it follows from Theorem 3.2 in [1] that

$$F_{\delta((xy)z)}(\varphi(x), \varphi(y), \varphi(z), \delta(x), \delta(y), \delta(z))$$
$$\overset{R_V}{=} F_{\delta(x(yz))}(\varphi(x), \varphi(y), \varphi(z), \delta(x), \delta(y), \delta(z)) .$$

Hence, by using (2.3) and Theorem 2.1 in [1], we have

$$bh+ad = b^2 .$$

This completes the proof.

Theorem 2.3. *Let* $\boldsymbol{P}$ *be a family* $\boldsymbol{P}_{V,V}\{\varphi, \delta\}$ *whose basic mapping-formulas concerning the compositions different from* $\cdot$ *are of homomorphism type. Then, in order that* $\boldsymbol{P}$ *is a family of* $\{\cdot\}$*-proper* R_V-$\{\cdot\}$*-derivation type, it is necessary and sufficient that the basic mapping-formulas of* $\boldsymbol{P}$ *concerning* $\cdot$ *are of the form*

$$(2.4)\quad \begin{cases} \varphi(xy) = P_{\varphi\cdot}(\varphi(x), \varphi(y)) \overset{R_V}{=} h\varphi(x)\varphi(y) \quad \textit{and} \\ \delta(xy) \;= P_{\delta\cdot}(\varphi(x), \varphi(y), \delta(x), \delta(y)) \\ \qquad \overset{R_V}{=} a\varphi(x)\varphi(y) + h\varphi(x)\delta(y) + h\delta(x)\varphi(y)\,, \\ \textit{where } a, h \in K, \textit{ and at least one of them is not } 0\,. \end{cases}$$

Proof of sufficiency. Suppose that $\boldsymbol{P}$ is of the form (2.4). Then it is clear from Theorem 2.2 that $\boldsymbol{P}$ is a family of R_V-$\{\cdot\}$-derivation type. Hence it is sufficient to prove that $\boldsymbol{P}$ is not R_V-conjugate to any family $\boldsymbol{Q}=\boldsymbol{Q}_{V,V}\{\psi_1, \cdots, \psi_m\}$ of R_V-$\{\cdot\}$-homomorphism type, i.e., there exists no R_V-regular R_V-translator from $\boldsymbol{Q}$ into $\boldsymbol{P}$. Now, by Theorem 2.1, we may assume that the basic mapping-formulas of $\boldsymbol{Q}$ concerning $\cdot$ are of the form

$$(2.5)\qquad \psi_\mu(xy) = Q_{\psi_\mu\cdot}(\psi_\mu(x), \psi_\mu(y)) \overset{R_V}{=} h_\mu\psi_\mu(x)\psi_\mu(y) \qquad (\mu = 1, \cdots, m)\,.$$

And let

$$(2.6)\qquad F_1(x_1, \cdots, x_m),\; F_2(x_1, \cdots, x_m)$$

be an R_V-translator from $\boldsymbol{Q}$ into $\boldsymbol{P}$. Then, by Theorem 1.1, we have

$$\begin{aligned} &F_\nu(\psi_1(x)+\psi_1(y), \cdots, \psi_m(x)+\psi_m(y)) \\ &\overset{R_V}{=} F_\nu(\psi_1(x), \cdots, \psi_m(x)) + F_\nu(\psi_1(y), \cdots, \psi_m(y)) \qquad (\nu = 1, 2)\,. \end{aligned}$$

Hence we have

$$F_\nu(x_1, \cdots, x_m) \overset{R_V}{=} \alpha_\nu x_1 + \cdots + \beta_\nu x_m, \quad \alpha_\nu, \cdots, \beta_\nu \in K \qquad (\nu = 1, 2)\,.$$

Therefore the R_V-translator (2.6) is not R_V-regular in the case of $m \neq 2$. Hence, in the following, we may assume that $m=2$, i.e.,

$$\begin{aligned} F_1(x_1, \cdots, x_m) &= F_1(x_1, x_2) \overset{R_V}{=} \alpha_1 x_1 + \beta_1 x_2\,, \\ F_2(x_1, \cdots, x_m) &= F_2(x_1, x_2) \overset{R_V}{=} \alpha_2 x_1 + \beta_2 x_2 \quad \text{and} \\ \boldsymbol{Q} = \boldsymbol{Q}_{V,V}\{\psi_1, \cdots, \psi_m\} &= \boldsymbol{Q}_{V,V}\{\psi_1, \psi_2\}\,. \end{aligned}$$

Therefore, by using (2.4), (2.5) and Theorem 1.1, we have

$$\begin{aligned} &\alpha_1 h_1\psi_1(x)\psi_1(y) + \beta_1 h_2\psi_2(x)\psi_2(y) \\ &\overset{R_V}{=} h(\alpha_1\psi_1(x)+\beta_1\psi_2(x))(\alpha_1\psi_1(y)+\beta_1\psi_2(y)) \end{aligned}$$

and

$$\begin{aligned} &\alpha_2 h_1\psi_1(x)\psi_1(y) + \beta_2 h_2\psi_2(x)\psi_2(y) \\ &\overset{R_V}{=} a(\alpha_1\psi_1(x)+\beta_1\psi_2(x))(\alpha_1\psi_1(y)+\beta_1\psi_2(y)) \\ &\quad + h(\alpha_1\psi_1(x)+\beta_1\psi_2(x))(\alpha_2\psi_1(y)+\beta_2\psi_2(y)) \\ &\quad + h(\alpha_2\psi_1(x)+\beta_2\psi_2(x))(\alpha_1\psi_1(y)+\beta_1\psi_2(y))\,. \end{aligned}$$

Hence we have

$$(2.7)\qquad h\alpha_1^2-\alpha_1h_1 = 0\,,$$

$$(2.8)\qquad h\alpha_1\beta_1 = 0\,,$$

$$(2.9)\qquad h\beta_1^2-\beta_1h_2 = 0\,,$$

and

$$(2.10)\qquad a\alpha_1^2+2h\alpha_1\alpha_2-\alpha_2h_1 = 0\,,$$

$$(2.11)\qquad a\alpha_1\beta_1+h\alpha_1\beta_2+h\alpha_2\beta_1 = 0\,,$$

$$(2.12)\qquad a\beta_1^2+2h\beta_1\beta_2-\beta_2h_2 = 0\,.$$

By using (2.7)-(2.12), we shall prove that the R_V-translator (2.6) in not R_V-regular in any case.

(a) The case of $h=0$. By the assumption of this theorem, we have $a\neq 0$. Hence, by using (2.7), (2.9), (2.10) and (2.12), we have $\alpha_1=\beta_1=0$, and hence the R_V-translator (2.6) is not R_V-regular.

(b) The case of $h\neq 0$ and $h_1=h_2=0$. By using (2.7) and (2.9), we have $\alpha_1=\beta_1=0$. Hence the R_V-translator (2.6) is not R_V-regular.

(c) The case of $h\neq 0$, $h_1\neq 0$ and $h_2=0$. By (2.9), we have $\beta_1=0$. Hence by (2.11) we have $\alpha_1\beta_2=0$, i.e., $\alpha_1=0$ or $\beta_2=0$. Therefore the R_V-translator (2.6) is not R_V-regular.

(d) The case of $h\neq 0$, $h_1=0$ and $h_2\neq 0$. It is similarly obtained as in the case (c) that the R_V-translator (2.6) is not R_V-regular.

(e) The case of $h\neq 0$, $h_1\neq 0$ and $h_2\neq 0$. By (2.8), we have $\alpha_1=0$ or $\beta_1=0$. If $\alpha_1=0$, then by (2.11), we have $\alpha_2\beta_1=0$, i.e., $\alpha_2=0$ or $\beta_1=0$. Hence, in the case of $\alpha_1=0$, the R_V-translator (2.6) is not R_V-regular. If $\beta_1=0$, then by (2.11), we have $\alpha_1\beta_2=0$, i.e., $\alpha_1=0$ or $\beta_2=0$. Hence, in the case of $\beta_1=0$, the R_V-translator (2.6) is not R_V-regular. This completes the proof of sufficiency.

Proof of necessity. In Theorem 2.2, we have shown that, if $\boldsymbol{P}$ is a family of R_V-$\{\cdot\}$-derivation type, then the basic mapping-formulas of $\boldsymbol{P}$ concerning $\cdot$ are of the form

$$\begin{aligned}\varphi(xy) &= P_{\varphi\cdot}(\varphi(x),\,\varphi(y)) \overset{R_V}{=} h\varphi(x)\varphi(y) \quad \text{and}\\ \delta(xy) &= P_{\delta\cdot}(\varphi(x),\,\varphi(y),\,\delta(x),\,\delta(y))\\ &\overset{R_V}{=} a\varphi(x)\varphi(y)+b\varphi(x)\delta(y)+b\delta(x)\varphi(y)+d\delta(x)\delta(y)\,,\end{aligned}$$

where $bh+ad=b^2$. Hence it is sufficient to show that, if the basic mapping-formulas of $\boldsymbol{P}$ concerning $\cdot$ are not of the form (2.4), then $\boldsymbol{P}$ is not a family of $\{\cdot\}$-proper R_V-$\{\cdot\}$-derivation type in any case.

(a) The case of $d\neq 0$. Let $\boldsymbol{Q}$ be a family $\boldsymbol{Q}_{V,V}\{\psi_1, \psi_2\}$ of homomorphism type. Then it is clear from Theorem 1.4 that the system of V-polynomials

$$F_1(x_1, x_2) = x_1, \quad F_2(x_1, x_2) = bx_1 + dx_2$$

is an R_V-regular R_V-translator from $\boldsymbol{P}$ into $\boldsymbol{Q}$. Hence, in this case, $\boldsymbol{P}$ is not a family of $\{\cdot\}$-proper R_V-$\{\cdot\}$-derivation type.

(b) The case of $d=0$. From $bh+ad=b^2$, we have that $b=0$ or $b=h$. Hence we have that

$$P_{\delta\cdot}(\varphi(x), \varphi(y), \delta(x), \delta(y)) \overset{R_V}{=} a\varphi(x)\varphi(y) \quad \text{or}$$
$$P_{\delta\cdot}(\varphi(x), \varphi(y), \delta(x), \delta(y)) \overset{R_V}{=} a\varphi(x)\varphi(y) + h\varphi(x)\delta(y) + h\delta(x)\varphi(y).$$

Now it is sufficient to show that $\boldsymbol{P}$ is not a family of $\{\cdot\}$-proper R_V-$\{\cdot\}$-derivation type in the case of $a=0$ and $h=0$. Since, in this case, we have

$$P_{\delta\cdot}(\varphi(x), \varphi(y), \delta(x), \delta(y)) \overset{R_V}{=} 0,$$

it is clear that $\boldsymbol{P}$ is not a family of $\{\cdot\}$-proper R_V-$\{\cdot\}$-derivation type. This completes the proof.

Let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,V}\{\varphi, \delta\}$ of $\{\cdot\}$-proper R_V-$\{\cdot\}$-derivation type. Then, by Theorem 2.3, the basic mapping-formulas of $\boldsymbol{P}$ concerning are of the form

$$\begin{aligned}\varphi(xy) &= P_{\varphi\cdot}(\varphi(x), \varphi(y)) \overset{R_V}{=} h\varphi(x)\varphi(y) \quad \text{and}\\ \delta(xy) &= P_{\delta\cdot}(\varphi(x), \varphi(y), \delta(x), \delta(y))\\ &\overset{R_V}{=} a\varphi(x)\varphi(y) + h\varphi(x)\delta(y) + h\delta(x)\varphi(y),\end{aligned}$$

where $a\neq 0$ or $h\neq 0$ or both. Now, if $h=0$, then $\boldsymbol{P}$ is called a family of trivial $\{\cdot\}$-proper R_V-$\{\cdot\}$-derivation type. And if $h\neq 0$, then $\boldsymbol{P}$ is called a family of non-trivial $\{\cdot\}$-proper R_V-$\{\cdot\}$-derivation type.

Theorem 2.4. (I) *Any family $\boldsymbol{Q}_{V,V}\{\psi, \theta\}$ of trivial $\{\cdot\}$-proper R_V-$\{\cdot\}$-derivation type is R_V-conjugate to the family $\boldsymbol{P}_{V,V}\{\varphi, \delta\}$ of trivial $\{\cdot\}$-proper R_V-$\{\cdot\}$-derivation type whose basic mapping-formulas concerning are of the form*

$$\varphi(xy) = 0 \quad \textit{and} \quad \delta(xy) = \varphi(x)\varphi(y).$$

(II) *Any family $\boldsymbol{Q}^*_{V,V}\{\psi^*, \theta^*\}$ of non-trivial $\{\cdot\}$-proper R_V-$\{\cdot\}$-derivation type is R_V-conjugate to the family $\boldsymbol{P}^*_{V,V}\{\varphi^*, \delta^*\}$ of non-trivial $\{\cdot\}$-proper R_V-$\{\cdot\}$-derivation type whose basic mapping-formulas concerning $\cdot$ are of the form*

$$\varphi^*(xy) = \varphi^*(x)\varphi^*(y) \quad \textit{and}$$
$$\delta^*(xy) = \varphi^*(x)\delta^*(y)+\delta^*(x)\varphi^*(y)\,.$$

(III) *Any family of trivial* $\{\cdot\}$*-proper* R_V-$\{\cdot\}$*-derivation type is not* R_V*-conjugate to any family of non-trivial* $\{\cdot\}$*-proper* R_V-$\{\cdot\}$*-derivation type.*

Proof of (I). By the above definition, the basic mapping-formulas of $\boldsymbol{Q}_{V,V}\{\psi, \theta\}$ concerning $\cdot$ are of the form

$$\psi(xy) = Q_{\psi\cdot}(\psi(x), \psi(y)) \overset{R_V}{=} 0 \quad \text{and}$$
$$\theta(xy) = Q_{\theta\cdot}(\psi(x), \psi(y), \theta(x), \theta(y)) \overset{R_V}{=} a\psi(x)\psi(y)\,.$$

Then, by Theorem 1.4, the system of V-polynomials

$$F_1(x_1, x_2) = x_1\,, \quad F_2(x_1, x_2) = ax_2$$

is an R_V-regular R_V-translator from $\boldsymbol{P}_{V,V}\{\varphi, \delta\}$ into $\boldsymbol{Q}_{V,V}\{\psi, \theta\}$. Hence $\boldsymbol{P}_{V,V}\{\varphi, \delta\}$ is R_V-conjugate to $\boldsymbol{Q}_{V,V}\{\psi, \theta\}$.

Proof of (II). By the above definition, the basic mapping-formulas of $\boldsymbol{Q}^*_{V,V}\{\psi^*, \theta^*\}$ concerning $\cdot$ are of the form

$$\psi^*(xy) = Q^*_{\psi^*\cdot}(\psi^*(x), \psi^*(y)) \overset{R_V}{=} h\psi^*(x)\psi^*(y) \quad \text{and}$$
$$\theta^*(xy) = Q^*_{\theta^*\cdot}(\psi^*(x), \psi^*(y), \theta^*(x), \theta^*(y))$$
$$\overset{R_V}{=} a\psi^*(x)\psi^*(y)+h\psi^*(x)\theta^*(y)+h\theta^*(x)\psi^*(y)\,,$$

where $h\neq 0$. Then, by Theorem 1.4, the system of V-polynomials

$$F_1(x_1, x_2) = \frac{1}{h}x_1\,, \quad F_2(x_1, x_2) = x_2-\frac{a}{h^2}x_1$$

is an R_V-regular R_V-translator from $\boldsymbol{P}^*_{V,V}\{\varphi^*, \delta^*\}$ into $\boldsymbol{Q}^*_{V,V}\{\psi^*, \theta^*\}$. Hence $\boldsymbol{P}^*_{V,V}\{\varphi^*, \delta^*\}$ is R_V-conjugate to $\boldsymbol{Q}^*_{V,V}\{\psi^*, \theta^*\}$.

Proof of (III). It is sufficient to show that $\boldsymbol{P}_{V,V}\{\varphi, \delta\}$ is not R_V-conjugate to $\boldsymbol{P}^*_{V,V}\{\varphi^*, \delta^*\}$. Now let a system of V-polynomials

(2.13) $$F_1(x_1, x_2)\,, \quad F_2(x_1, x_2)$$

be an R_V-translator from $\boldsymbol{P}^*_{V,V}\{\varphi^*, \delta^*\}$ into $\boldsymbol{P}_{V,V}\{\varphi, \delta\}$. Then it is similarly obtained as in the first part of the proof of sufficiency of Theorem 2.3 that the V-polynomials (2.13) are of the form

$$F_1(x_1, x_2) \overset{R_V}{=} \alpha_1x_1+\beta_1x_2 \quad \text{and} \quad F_2(x_1, x_2) \overset{R_V}{=} \alpha_2x_1+\beta_2x_2\,.$$

Hence, by Theorem 1.1, we have

$$F_1(\varphi^*(x)\varphi^*(y),\ \varphi^*(x)\delta^*(y)+\delta^*(x)\varphi^*(y)) \overset{R_V}{=} 0\,,$$

and hence we have

$$\alpha_1\varphi^*(x)\varphi^*(y)+\beta_1(\varphi^*(x)\delta^*(y)+\delta^*(x)\varphi^*(y)) \overset{R_V}{=} 0\,.$$

Therefore $\alpha_1=\beta_1=0$, and therefore the system (2.13) is not R_V-regular. Hence $\boldsymbol{P}_{V,V}\{\varphi, \delta\}$ is not R_V-conjugate to $\boldsymbol{P}^*_{V,V}\{\varphi^*, \delta^*\}$. This completes the proof.

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Reference

[1] T. Fujiwara: On mappings between algebraic systems, Osaka Math. J. **11** (1959), 153–172.