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On Mappings between Algebraic Systems, II

By Tsuyoshi Fujiwara

In the previous paper [1], we have defined the $P$-mappings* and the $P$-product systems*, and shown that the algebraic Taylor's expansion theorem* holds between the $P$-mappings and the $P$-product systems. And some fundamental results with respect to $P$-mappings have been derived from this theorem.

The present paper is the continuation of the paper [1]. In the section 1 of this paper, we shall introduce the concept of the $B_w$-conjugate relation between families $P$ and $Q$ of basic mapping-formulas*, and it is a relation between $P$-mappings and $Q$-mappings. And, by using the algebraic Taylor's expansion theorem, we shall show that this relation is equivalent to the existence of some inner isomorphic mapping between the $P$-product system $P(B)$ and the $Q$-product system $Q(B)$ for every $B_w$-algebraic system $B$. In the section 2, we shall define the derivations between primitive algebraic systems, by using the concepts of the $(A_y, B_w)$-universality* and the $B_w$-conjugate relation. And we shall show that one of these derivations is the usual one in the case of the commutative algebras over a field of characteristic 0. Thus the derivations can be considered as the mappings which are some natural algebraic generalization of homomorphisms.

§ 1. Some relations between families of basic mapping-formulas.

Let $R$ be a set of relations of the form

$$b_1 = F_1(a_1, \cdots, a_m), \cdots, b_n = F_n(a_1, \cdots, a_m)$$

on a free $\phi_w$-algebraic system $F(\{a_1, \cdots, a_m, b_1, \cdots, b_n\}, \phi_w)$. And let $B_w$ be a system of composition-identities with respect to $W$. If there exists a set $S$ of relations of the form

$$a_1 = F^*_1(b_1, \cdots, b_n), \cdots, a_m = F^*_m(b_1, \cdots, b_n)$$

such that

*) Cf. [1].
\[ F(\{a_1, \ldots , a_m, b_1, \ldots , b_n\}, B_W, R) = F(\{a_1, \ldots , a_m, b_1, \ldots , b_n\}, B_W, S), \]

i.e., \( R \) and \( S \) are \( B_W \)-equivalent, then the system of \( W \)-polynomials

(1.1) \[ F(x_1, \ldots , x_m), \ldots , F_n(x_1, \ldots , x_m) \]

is said to be \( B_W \)-regular, and the system of \( W \)-polynomials

\[ F^*(y_1, \ldots , y_n), \ldots , F^*_n(y_1, \ldots , y_n) \]

is called a \( B_W \)-inverse system of (1.1). From the above definitions, it is clear that any \( B_W \)-inverse system is \( B_W \)-regular.

Let \( P \) and \( Q \) be families \( P_{v,w}(\xi_1, \ldots , \xi_m) \) and \( Q_{v,w}(\eta_1, \ldots , \eta_n) \) of basic mapping-formulas respectively. If there exists a system of \( W \)-polynomials

(1.2) \[ F(x_1, \ldots , x_m), \ldots , F_n(x_1, \ldots , x_m) \]

such that, for any system \( \{\varphi_1, \ldots , \varphi_m\} \) of \( P \)-mappings from any \( \varphi_v \)-algebraic system \( \mathfrak{A} \) into any \( B_W \)-algebraic system \( \mathfrak{B} \), the system \( \{\psi_1, \ldots , \psi_n\} \) of mappings, each of which is defined by

\[ \psi_v(a) = F_v(\varphi_1(a), \ldots , \varphi_m(a)), \]

is a system of \( Q \)-mappings, then the system (1.2) is called a \( B_W \)-translator from \( P \) into \( Q \). In the above definition, if the system (1.2) is \( B_W \)-regular, then we say that \( P \) is \( B_W \)-conjugate to \( Q \), and denote it by \( P \overset{B_W}{\sim} Q \).

**Theorem 1.1.** Let \( P \) and \( Q \) be families \( P_{v,w}(\xi_1, \ldots , \xi_m) \) and \( Q_{v,w}(\eta_1, \ldots , \eta_n) \) of basic mapping-formulas respectively. And let

(1.3) \[ F(x_1, \ldots , x_m), \ldots , F_n(x_1, \ldots , x_m) \]

be a system of \( W \)-polynomials. Then, in order that the system (1.3) is a \( B_W \)-translator from \( P \) into \( Q \), it is necessary and sufficient that

\[ Q_{v,w}(F_1(\xi_1(x_1), \ldots , \xi_m(x_1)), \ldots , F_n(\xi_1(x_N(v)), \ldots , \xi_m(x_N(v)))) \]

(1.4) \[ B_W \]

for every \( v=1, \ldots , n \) and every \( v \in V \).
Proof of necessity. Let \( \mathfrak{A} \) be the free \( \phi_v \)-algebraic system \( F(\{x_1, \ldots, x_{N(v)}\}, \phi_v) \), and \( \mathfrak{B} \) the free \( B_w \)-algebraic system \( F(\{\xi_1(x_1), \ldots, \xi_{(A)}(x_{N(v)}), \ldots, \xi_m(x_1), \ldots, \xi_{(A)}(x_{N(v)})\}, B_w) \). Then it is clear by Theorem 1.3 in [1] that there exists a system \( \{\varphi_1, \ldots, \varphi_m\} \) of \( P \)-mappings, each of which satisfies

\[
(1.5) \quad \varphi_\mu(x_N) = \xi_\mu(x_N) \quad (N=1, \ldots, N(v)).
\]

Now, let \( \{\psi_1, \ldots, \psi_n\} \) be the system of mappings from \( \mathfrak{A} \) into \( \mathfrak{B} \), each of which is defined by

\[
\psi_\nu(x) = F_\nu(\varphi_1(x), \ldots, \varphi_m(x)).
\]

Then \( \{\psi_1, \ldots, \psi_n\} \) is a system of \( Q \)-mappings from \( \mathfrak{A} \) into \( \mathfrak{B} \), because the system (1.3) is a \( B_w \)-translator from \( P \) into \( Q \). Hence we have the following computation:

\[
\begin{align*}
F_\nu(P_{\xi_1}(\varphi_1(x_1), \ldots, \varphi_m(x_{N(v)})), \ldots, P_{\xi_m}(\varphi_1(x_1), \ldots, \varphi_m(x_{N(v)}))) \\
= F_\nu(\varphi_1(v(x_1, \ldots, x_{N(v)})), \ldots, \varphi_m(v(x_1, \ldots, x_{N(v)}))) \\
= Q_\psi(\varphi_1(x_1), \ldots, \varphi_1(x_{N(v)}), \ldots, \varphi_1(x_1), \ldots, \varphi_m(x_{N(v)})) \\
= Q_\psi(F_1(\varphi_1(x_1), \ldots, \varphi_1(x_1)), \ldots, F_m(\varphi_1(x_1), \ldots, \varphi_1(x_1)), \ldots, F_1(\varphi_1(x_{N(v)}), \ldots, \varphi_1(x_{N(v)}))) \\
\end{align*}
\]

Hence, by (1.5), the identity

\[
\begin{align*}
F_\nu(P_{\xi_1}(\xi_1(x_1), \ldots, \xi_1(x_{N(v)})), \ldots, P_{\xi_m}(\xi_1(x_1), \ldots, \xi_m(x_{N(v)}))) \\
= Q_\psi(F_1(\xi_1(x_1), \ldots, \xi_1(x_1)), \ldots, F_1(\xi_1(x_{N(v)}), \ldots, \xi_m(x_{N(v)})), \ldots, F_1(\xi_1(x_{N(v)}), \ldots, \xi_m(x_{N(v)}))) \\
\end{align*}
\]

is valid in \( \mathfrak{B} \). This identity can be considered as the one with respect to \( B_w \), because \( \mathfrak{B} \) is a free \( B_w \)-algebraic system.

Proof of sufficiency. Let \( \mathfrak{A} \) be any \( \phi_v \)-algebraic system, and \( \mathfrak{B} \) any \( B_w \)-algebraic system. And let \( \{\varphi_1, \ldots, \varphi_m\} \) be any system of \( P \)-mappings from \( \mathfrak{A} \) into \( \mathfrak{B} \). Moreover, let \( \psi_1, \ldots, \psi_n \) be the mappings from \( \mathfrak{A} \) into \( \mathfrak{B} \), each of which is defined by

\[
\psi_\nu(a) = F_\nu(\varphi_1(a), \ldots, \varphi_m(a)).
\]

Then, by using (1.4), for any \( v \in V \) and any \( a_1, \ldots, a_{N(v)} \in \mathfrak{A} \), we have
Hence \( \{ \psi_1, \ldots, \psi_n \} \) is a system of \( Q \)-mappings from \( \mathcal{A} \) into \( \mathcal{B} \). This completes the proof.

Let \( P \) be a family \( P_{v,w}\{\xi_1, \ldots, \xi_m\} \) of basic mapping-formulas, and let \( \mathcal{B} \) be a \( \phi_w \)-algebraic system. Now let \( \psi \) be a mapping from \( P(\mathcal{B}) \) into \( \mathcal{B} \). If there exists a \( \Gamma \)-polynomial \( F(x_1, \ldots, x_m) \) such that for every element \( [b_1, \ldots, b_m] \) in \( P(\mathcal{B}) \), then \( \psi \) is called an inner mapping defined by \( F(x_1, \ldots, x_m) \). Moreover, let \( Q \) be a family \( Q_{v,w}\{\eta_1, \ldots, \eta_n\} \) of basic mapping-formulas. And let \( \psi_1, \ldots, \psi_n \) be mappings from \( P(\mathcal{B}) \) into \( \mathcal{B} \), and \( \Psi \) the mapping from \( P(\mathcal{B}) \) into \( Q(\mathcal{B}) \) which is defined by

\[
\Psi([b_1, \ldots, b_m]) = [\psi_1([b_1, \ldots, b_m]), \ldots, \psi_n([b_1, \ldots, b_m])]
\]

for all elements \( [b_1, \ldots, b_m] \in P(\mathcal{B}) \). If each \( \psi_v \) is an inner mapping defined by a \( W \)-polynomial \( F_v(x_1, \ldots, x_m) \), then \( \Psi \) is called an inner mapping defined by the system of \( W \)-polynomials \( F_v(x_1, \ldots, x_m) \) \( (v=1, \ldots, n) \).

**Theorem 1.2.** Let \( P \) and \( Q \) be families \( P_{v,w}\{\xi_1, \ldots, \xi_m\} \) and \( Q_{v,w}\{\eta_1, \ldots, \eta_n\} \) of basic mapping-formulas respectively. And let

\[
F_1(x_1, \ldots, x_m), \ldots, F_n(x_1, \ldots, x_m)
\]

be a system of \( W \)-polynomials. Then, in order that the system (1.6) is a \( B_w \)-translator from \( P \) into \( Q \), it is necessary and sufficient that, for any \( B_w \)-algebraic system \( \mathcal{B} \), the inner mapping \( \Psi \) from \( P(\mathcal{B}) \) into \( Q(\mathcal{B}) \), which is defined by the system (1.6) of \( W \)-polynomials, is a homomorphism.

Proof of necessity. Let \( \mathcal{B} \) be any \( B_w \)-algebraic system. And let \( \varphi_1, \ldots, \varphi_m \) be the mappings from \( P(\mathcal{B}) \) into \( \mathcal{B} \), each of which is defined by

\[
\varphi_v([b_1, \ldots, b_m]) = b_v.
\]
Then it is clear that \( \{ \varphi_1, \ldots, \varphi_m \} \) is a system of \( P \)-mappings from \( P(\mathfrak{B}) \) into \( \mathfrak{B} \). Now let \( \psi_1, \ldots, \psi_n \) be mappings from \( P(\mathfrak{B}) \) into \( \mathfrak{B} \), each of which is defined by

\[
\psi_\nu([b_1, \ldots, b_m]) = F_\nu(\varphi_\nu([b_1, \ldots, b_m])), \quad \text{i.e.,}
\]

\[
\psi_\nu([b_1, \ldots, b_m]) = F_\nu(b_1, \ldots, b_m).
\]

Then, \( \{ \psi_1, \ldots, \psi_n \} \) is a system of \( Q \)-mappings from \( P(\mathfrak{B}) \) into \( \mathfrak{B} \), because the system (1.6) is a \( B_w \)-translator from \( P \) into \( Q \). Hence, by Theorem 1.1 in [1], the inner mapping

\[
\Psi: [b_1, \ldots, b_m] \rightarrow [F_1(b_1, \ldots, b_m), \ldots, F_n(b_1, \ldots, b_m)]
\]

is a homomorphism from \( P(\mathfrak{B}) \) into \( Q(\mathfrak{B}) \).

Proof of sufficiency. Let \( \mathfrak{A} \) be any \( \phi_v \)-algebraic system, and \( \mathfrak{B} \) any \( B_w \)-algebraic system. Now suppose that \( \{ \varphi_1, \ldots, \varphi_m \} \) is a system of \( P \)-mappings from \( \mathfrak{A} \) into \( \mathfrak{B} \). Then, by Theorem 1.1 in [1], the mapping

\[
\Phi: a \rightarrow \Phi(a) = [\varphi_\nu(a), \ldots, \varphi_m(a)]
\]

is a homomorphism from \( \mathfrak{A} \) into \( P(\mathfrak{B}) \). Since the inner mapping

\[
\Psi: [b_1, \ldots, b_m] \rightarrow [F_1(b_1, \ldots, b_m), \ldots, F_n(b_1, \ldots, b_m)]
\]

is a homomorphism from \( P(\mathfrak{B}) \) into \( Q(\mathfrak{B}) \), it is clear that the mapping

\[
\Psi\Phi: a \rightarrow \Psi\Phi(a) = [F_1(\varphi_1(a), \ldots, \varphi_m(a)), \ldots, F_n(\varphi_1(a), \ldots, \varphi_m(a))]
\]

is a homomorphism from \( \mathfrak{A} \) into \( Q(\mathfrak{B}) \). Hence, by Theorem 1.1 in [1], the system \( \{ \psi_1, \ldots, \psi_n \} \) of mappings from \( \mathfrak{A} \) into \( \mathfrak{B} \), each of which is defined by

\[
\psi_\nu(a) = F_\nu(\varphi_\nu(a), \ldots, \varphi_m(a))
\]

is a system of \( Q \)-mappings. Thus, the system (1.6) of \( W \)-polynomials is a \( B_w \)-translator from \( P \) into \( Q \). This completes the proof.

**Theorem 1.3.** Let \( P \) and \( Q \) be families \( P_{v,w}\{\xi_1, \ldots, \xi_m\} \) and \( Q_{v,w}\{\eta_1, \ldots, \eta_n\} \) of basic mapping-formulas respectively, and let

\[
(1.7) \quad F_i(x_1, \ldots, x_m), \ldots, F_n(x_1, \ldots, x_m)
\]

be a \( B_w \)-regular system of \( W \)-polynomials. And let \( \mathfrak{B} \) be any \( B_w \)-algebraic system. Now suppose that the inner mapping \( \Psi \) from \( P(\mathfrak{B}) \) into \( Q(\mathfrak{B}) \), which is defined by the system (1.7) of \( W \)-polynomials, is a homomorphism. Then \( \Psi \) is an isomorphism from \( P(\mathfrak{B}) \) onto \( Q(\mathfrak{B}) \), moreover the inverse mapping \( \Psi^{-1} \) is an inner mapping defined by a \( B_w \)-inverse system.
(1.8) \[ F_1^*(y_1, \ldots, y_n), \ldots, F_m^*(y_1, \ldots, y_n) \]

of the system (1.7).

Proof. Let \([b_1, \ldots, b_m]\) be any element in \(P(B)\). Then, by the definition of the inner mapping \(\Psi\), we have

\[ \Psi([b_1, \ldots, b_m]) = [F_1(b_1, \ldots, b_m), \ldots, F_n(b_1, \ldots, b_m)]. \]

On the other hand, it is clear that

\[ F_\mu^*(F_1(b_1, \ldots, b_m), \ldots, F_n(b_1, \ldots, b_m)) = b_\mu \quad (\mu = 1, \ldots, m). \]

Hence we have

\[ \Psi^{-1}([c_1, \ldots, c_n]) = \Phi([c_1, \ldots, c_n]) \]

for every element \([c_1, \ldots, c_n]\) in the domain of \(\Psi^{-1}\), where \(\Phi\) denotes the inner mapping from \(Q(B)\) into \(P(B)\) which is defined by the \(B_w\)-inverse system (1.8). Therefore the inner mapping \(\Psi\) is a one to one mapping. Hence it is the rest of our proof to show that \(\Psi\) maps \(P(B)\) onto \(Q(B)\). Now let \([c_1, \ldots, c_n]\) be any element in \(Q(B)\). Then we have

\[ F_\nu(F_1^*(c_1, \ldots, c_n), \ldots, F_m^*(c_1, \ldots, c_n)) = c_\nu \quad (\nu = 1, \ldots, n). \]

Hence we have

\[ \Psi([F_1^*(c_1, \ldots, c_n), \ldots, F_m^*(c_1, \ldots, c_n)]) = [c_1, \ldots, c_n]. \]

Therefore \(\Psi\) maps \(P(B)\) onto \(Q(B)\). This completes our proof.

**Theorem 1.4.** Let \(P\) and \(Q\) be families \(P_{v,w}\{\xi_1, \ldots, \xi_m\}\) and \(Q_{v,w}\{\eta_1, \ldots, \eta_n\}\) of basic mapping-formulas respectively. Then the following three propositions are equivalent:

(a) \(P\) is \(B_w\)-conjugate to \(Q\).

(b) There exists a \(B_w\)-regular system of \(W\)-polynomials

\[ F_1(x_1, \ldots, x_m), \ldots, F_n(x_1, \ldots, x_m) \]

such that, for any \(B_w\)-algebraic system \(B\), the inner mapping from \(P(B)\) into \(Q(B)\), which is defined by the system (1.9), is an isomorphism from \(P(B)\) onto \(Q(B)\).

(c) There exists a \(B_w\)-regular system of \(W\)-polynomials

\[ F_1(x_1, \ldots, x_m), \ldots, F_n(x_1, \ldots, x_m) \]

such that
for every $v = 1, \ldots, n$ and every $v \in V$.

Proof. (a) $\iff$ (c) is clear from Theorem 1.1. (a) $\iff$ (b) is obvious from Theorems 1.2 and 1.3.

**Theorem 1.5.** The $B_w$-conjugate relation $B_w$ is an equivalence relation.

Proof of reflexive law is easy.

Proof of symmetric law. Let $P$ and $Q$ be families $P_{v,w}\{\xi_1, \ldots, \xi_m\}$ and $Q_{v,w}\{\eta_1, \ldots, \eta_n\}$ of basic mapping-formulas respectively. Now suppose that $P \overset{B_w}{\sim} Q$. Then, by Theorem 1.4, there exists a $B_w$-regular system of $W$-polynomials

$$(1.10) \quad F_i(x_1, \ldots, x_m), \ldots, F_n(x_1, \ldots, x_m)$$

such that, for any $B_w$-algebraic system $S$, the inner mapping $\Psi$ from $P(S)$ into $Q(S)$, which is defined by the system $(1.10)$, is an isomorphism from $P(S)$ onto $Q(S)$. Moreover, by Theorem 1.3, $\Psi^{-1}$ is an inner mapping defined by a $B_w$-inverse system of $(1.10)$. Hence $Q \overset{B_w}{\sim} P$ follows from Theorem 1.4, because the $B_w$-inverse system is $B_w$-regular.

Proof of transitive law. Let $P$, $Q$, and $R$ be families $P_{v,w}\{\xi_1, \ldots, \xi_m\}$, $Q_{v,w}\{\eta_1, \ldots, \eta_n\}$, and $R_{v,w}\{\zeta_1, \ldots, \zeta_l\}$ of basic mapping-formulas respectively. Now suppose that $P \overset{B_w}{\sim} Q$ and $Q \overset{B_w}{\sim} R$. Then, by Theorem 1.4, there exist two systems

$$(1.11) \quad F_i(x_1, \ldots, x_m), \ldots, F_n(x_1, \ldots, x_m)$$
$$(1.12) \quad G_i(y_1, \ldots, y_n), \ldots, G_l(y_1, \ldots, y_n)$$

of $W$-polynomials such that, for any $B_w$-algebraic system $S$, the inner mappings $\Psi: P(S) \rightarrow Q(S)$ and $\Theta: Q(S) \rightarrow R(S)$, which are defined by the systems $(1.11)$ and $(1.12)$ respectively, are onto isomorphisms. Hence it is clear that the mapping $\Theta\Psi$ is an isomorphism from $P(S)$ onto $R(S)$ and it is an inner mapping defined by the system of $W$-polynomials

$$(1.13) \quad \{G_i(F_i(x_1, \ldots, x_m), \ldots, F_n(x_1, \ldots, x_m)), \ldots, G_l(F_i(x_1, \ldots, x_m), \ldots, F_n(x_1, \ldots, x_m))\}.$$
Now let
\[ F_1^\phi(y_1, \ldots, y_n), \ldots, F_m^\phi(y_1, \ldots, y_n) \quad \text{and} \quad G_1^\phi(z_1, \ldots, z_l), \ldots, G_m^\phi(z_1, \ldots, z_l) \]
be \( B_w \)-inverse systems of the systems (1.11) and (1.12) respectively.
Then it is easily obtained that the system of \( W \)-polynomials
\[
F_1^\phi(G_1^\phi(z_1, \ldots, z_l), \ldots, G_m^\phi(z_1, \ldots, z_l)) \\
\ldots \\
F_m^\phi(G_1^\phi(z_1, \ldots, z_l), \ldots, G_m^\phi(z_1, \ldots, z_l))
\]
is a \( B_w \)-inverse system of (1.13). Hence the system (1.13) is \( B_w \)-regular. Therefore \( P \sim B_w R \) follows from Theorem 1.4. This completes the proof.

Finally we shall introduce the concept of \( B_w \)-similarity as a special case of the concept of \( B_w \)-conjugate. Now let \( P \) and \( Q \) be families \( P_{V,W}\{\xi_1, \ldots, \xi_m\} \) and \( Q_{V,W}\{\eta_1, \ldots, \eta_m\} \) of basic mapping-formulas respectively. If, for any \( \phi_V \)-algebraic system \( \mathfrak{A} \) and any \( B_w \)-algebraic system \( \mathfrak{B} \), any system of \( P \)-mappings from \( \mathfrak{A} \) into \( \mathfrak{B} \) is a system of \( Q \)-mappings, and conversely, then we say that \( P \) and \( Q \) are \( B_w \)-similar. As an easy consequence of the above definition we obtain

**Theorem 1.6.** Let \( P \) and \( Q \) be families \( P_{V,W}\{\xi_1, \ldots, \xi_m\} \) and \( Q_{V,W}\{\eta_1, \ldots, \eta_m\} \) of basic mapping-formulas respectively. Then, in order that \( P \) and \( Q \) are \( B_w \)-similar, it is necessary and sufficient that

\[
P_{\xi_{\mu,v}}(y_{1,1}, \ldots, y_{1,N(v)}), \ldots, y_{m,1}, \ldots, y_{m,N(v)}) \\
B_w \\
Q_{\eta_{\mu,v}}(y_{1,1}, \ldots, y_{1,N(v)}) \ldots, y_{m,1}, \ldots, y_{m,N(v)})
\]
for every \( \mu = 1, \ldots, m \) and every \( v \in V \).

**§ 2. Families of (\( A_V, B_w \))-homomorphism type and families of (\( A_V, B_w \))-derivation type.**

Let \( P \) be a family \( P_{V,W}\{\xi_1, \ldots, \xi_m\} \) of basic mapping-formulas. If the basic mapping-formulas of \( P \) are of the form

\[
\tilde{\xi}_{\mu}(v(x_1, \ldots, x_{N(v)})) = P_{\xi_{\mu,v}}(\xi_{\mu}(x_1), \ldots, \xi_{\mu}(x_{N(v)})) \quad (\mu = 1, \ldots, m; \ v \in V),
\]
then \( P \) is called a family of \( (\phi_V, \phi_w) \)-homomorphism type. Moreover let \( A_V \) and \( B_w \) be systems of composition-identities with respect to \( V \) and \( W \) respectively. If \( P \) is \( (A_V, B_w) \)-universal and \( B_w \)-similar to some family of \( (\phi_V, \phi_w) \)-homomorphism type, then \( P \) is called a family of \( (A_V, B_w) \)-homomorphism type.
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Next let \( P \) be a family \( P_{v,w}\{\xi_1, \cdots, \xi_m, \delta\} \) of basic mapping-formulas. If the basic mapping-formulas of \( P \) are of the form

\[
\xi_\mu(v(x_1, \cdots, x_{N(v)})) = P_{\xi_\mu}(\xi_1(x_1), \cdots, \xi_m(x_{N(v)})) \quad (\mu = 1, \cdots, m; v \in V)
\]

and

\[
\delta(v(x_1, \cdots, x_{N(v)})) = P_{\delta}(\xi_1(x_1), \cdots, \xi_m(x_{N(v)}))
\]

then \( P \) is called a family of \((\phi_v, \phi_w)\)-derivation type. Moreover let \( A_v \) and \( B_w \) be systems of composition-identities. If \( P \) is \((A_v, B_w)\)-universal and \( B_w\)-similar to some family of \((\phi_v, \phi_w)\)-derivation type, then \( P \) is called a family of \((A_v, B_w)\)-derivation type.

Let \( P \) be a family of \((A_v, B_w)\)-derivation type. If there exists a family \( Q \) of \((A_v, B_w)\)-homomorphism type such that \( P \) and \( Q \) are \( B_w\)-conjugate, then \( P \) is called a family of improper \((A_v, B_w)\)-derivation type. Otherwise, \( P \) is called a family of proper \((A_v, B_w)\)-derivation type.

Let \( K \) be a commutative field of characteristic 0, and \( V \) the set-sum of \{+ , \cdot\} and \( K \). And let \( R_v \) be the system of composition-identities with respect to \( V \), which define the commutative algebras over \( K \). In the following, we shall determine the form of the family \( P_{v,v}\{\phi_1, \cdots, \phi_m\}^{*} \) of \( R_v\{-\} \)-homomorphism type, and that of the family \( P_{v,v}\{\phi, \delta\} \) of \( R_v\{-\} \)-derivation type.

**Theorem 2.1.** Let \( P \) be a family \( P_{v,v}\{\phi_1, \cdots, \phi_m\} \) whose basic mapping-formulas concerning the compositions different from \( \cdot \) are of homomorphism type. Then, in order that \( P \) is a family of \( R_v\{-\} \)-homomorphism type, it is necessary and sufficient that the basic mapping-

*) For convenience, we use below the letters \( \phi, \psi \) in places of the letters \( \xi, \eta \).
formulas of $P$ concerning $\cdot$ are of the form

$$\varphi_\mu(xy) = P_{\varphi_\mu}(\varphi_\mu(x), \varphi_\mu(y)) \equiv h_\mu \varphi_\mu(x) \varphi_\mu(y), \quad h_\mu \in K \quad (\mu = 1, \ldots, m).$$

Proof. The sufficiency can be easily obtained by Theorem 3.2 in [1]. In the following, we shall prove the necessity. Since the composition-identity $(x+y)z = xz + yz$ is contained in $R_V$, and $P$ is $R_V$-universal, it follows from Theorem 3.2 in [1] that

$$F_{\varphi_\mu(x+y)z}(\varphi_\mu(x), \varphi_\mu(y), \varphi_\mu(z)).$$

Hence, by Theorem 2.1 in [1], we have

$$P_{\varphi_\mu}(\varphi_\mu(x) + \varphi_\mu(y), \varphi_\mu(z)) \equiv P_{\varphi_\mu}(\varphi_\mu(x), \varphi_\mu(y), \varphi_\mu(z)).$$

Similarly we have

$$P_{\varphi_\mu}(\varphi_\mu(x), \varphi_\mu(y) + \varphi_\mu(z)) \equiv P_{\varphi_\mu}(\varphi_\mu(x), \varphi_\mu(y), \varphi_\mu(z)),$$

because the composition-identity $x(y+z) = xy + xz$ is contained in $R_V$. Therefore we have

$$P_{\varphi_\mu}(\varphi_\mu(x), \varphi_\mu(y)) \equiv h_\mu \varphi_\mu(x) \varphi_\mu(y), \quad h_\mu \in K.$$

This completes the proof.

**Theorem 2.2.** Let $P$ be a family $P_{V,V}\{\varphi, \delta\}$ whose basic mapping-formulas concerning the compositions different from $\cdot$ are of homomorphism type. Then, in order that $P$ is a family of $R_V\{-\}$-derivation type, it is necessary and sufficient that the basic mapping-formulas of $P$ concerning $\cdot$ are of the form

$$\varphi(xy) = P_{\varphi}(\varphi(x), \varphi(y)) \equiv h\varphi(x)\varphi(y) \quad \text{and}$$

$$\delta(xy) = P_{\delta}(\varphi(x), \varphi(y), \delta(x), \delta(y)) \equiv a\varphi(x)\varphi(y) + b\varphi(x)\delta(y) + b\delta(x)\varphi(y) + d\delta(x)\delta(y),$$

where $a, b, d, h \in K$ and $bh + ad = b^2$.

Proof. The sufficiency can be easily obtained by Theorem 3.2 in [1]. In the following, we shall prove the necessity. Now suppose that $P$ is a family of $R_V\{-\}$-derivation type. Then (2.1) can be similarly obtained as in the proof of Theorem 2.1. Next, since the composition-
identity \((x + y)z = xz + yz\) is contained in \(R_v\), and \(P\) is \(R_v\)-universal, it follows from Theorem 3.2 in [1] that

\[
F_{\delta; (x+y)z}((\varphi(x), \varphi(y), \varphi(z), \delta(x), \delta(y), \delta(z)) \equiv F_{\delta; yz}(\varphi(x), \varphi(y), \varphi(z), \delta(x), \delta(y), \delta(z)).
\]

Hence, by Theorem 2.1 in [1], we have

\[
P_\delta.(\varphi(x) + \varphi(y), \varphi(z), \delta(x) + \delta(y), \delta(z)) \equiv P_\delta.(\varphi(x), \varphi(z), \delta(x), \delta(z)) + P_\delta.(\varphi(y), \varphi(z), \delta(y), \delta(z)).
\]

Similarly we have

\[
P_\delta.(\varphi(x), \varphi(y) + \varphi(z), \delta(x), \delta(y) + \delta(z)) \equiv P_\delta.(\varphi(x), \varphi(y), \delta(x), \delta(y)) + P_\delta.(\varphi(x), \varphi(z), \delta(x), \delta(z)),
\]

because the composition-identity \(x(y + z) = xy + xz\) is contained in \(R_v\). Therefore we can easily obtain

\[
P_\delta.(\varphi(x), \varphi(y), \delta(x), \delta(y)) \equiv a\varphi(x)\varphi(y) + b\varphi(x)\delta(y) + c\delta(x)\varphi(y) + d\delta(x)\delta(y),
\]

where \(a, b, c, d \in K\). Moreover we have \(b = c\), because the composition-identity \(xy = yx\) is contained in \(R_v\). Hence we have

\[
(2.3) \quad P_\delta.(\varphi(x), \varphi(y), \delta(x), \delta(y)) \equiv a\varphi(x)\varphi(y) + b\varphi(x)\delta(y) + b\delta(x)\varphi(y) + d\delta(x)\delta(y).
\]

Since the composition-identity \((xy)z = x(yz)\) is contained in \(R_v\), it follows from Theorem 3.2 in [1] that

\[
F_{\delta; (xy)z}((\varphi(x), \varphi(y), \varphi(z), \delta(x), \delta(y), \delta(z)) \equiv F_{\delta; x(yz)}((\varphi(x), \varphi(y), \varphi(z), \delta(x), \delta(y), \delta(z)).
\]

Hence, by using (2.3) and Theorem 2.1 in [1], we have

\[
bh + ad = b^2.
\]

This completes the proof.

**Theorem 2.3.** Let \(P\) be a family \(P_{V\cdot V}\{\varphi, \delta\}\) whose basic mapping-formulas concerning the compositions different from \(\cdot\) are of homomorphism type. Then, in order that \(P\) is a family of \(\{\cdot\}\)-proper \(R_v\)\{-\}\-derivation type, it is necessary and sufficient that the basic mapping-formulas of \(P\) concerning \(\cdot\) are of the form
\[
\begin{aligned}
\phi(xy) &= P_{xy}(\phi(x), \phi(y)) R_v = h \phi(x) \phi(y) \quad \text{and} \\
\delta(xy) &= P_{xy}(\phi(x), \phi(y), \delta(x), \delta(y)) \\
&\equiv a \phi(x) \phi(y) + h \phi(x) \delta(y) + h \delta(x) \phi(y),
\end{aligned}
\]
where \(a, h \in K\), and at least one of them is not \(0\).

Proof of sufficiency. Suppose that \(P\) is of the form (2.4). Then it is clear from Theorem 2.2 that \(P\) is a family of \(R_v\{-\}\)-derivation type. Hence it is sufficient to prove that \(P\) is not \(R_v\)-conjugate to any family \(Q = Q_{v,v}\{\psi_1, \ldots, \psi_m\}\) of \(R_v\{-\}\)-homomorphism type, i.e., there exists no \(R_v\)-regular \(R_v\)-translator from \(Q\) into \(P\). Now, by Theorem 2.1, we may assume that the basic mapping-formulas of \(Q\) concerning \(\psi\) are of the form

\[
(2.5) \quad \psi_\mu(xy) = Q_{\psi_\mu}(\psi_\mu(x), \psi_\mu(y)) R_v = h_\mu \psi_\mu(x) \psi_\mu(y) \quad (\mu = 1, \ldots, m).
\]

And let

\[
(2.6) \quad F_1(x_1, \ldots, x_m), F_2(x_1, \ldots, x_m)
\]
be an \(R_v\)-translator from \(Q\) into \(P\). Then, by Theorem 1.1, we have

\[
R_v F_\nu(\psi_\nu(x), \ldots, \psi_\nu(m)) = F_\nu(\psi_\nu(x), \ldots, \psi_\nu(m)) + F_\nu(\psi_\nu(y), \ldots, \psi_\nu(m)) \quad (\nu = 1, 2).
\]

Hence we have

\[
R_v F(x_1, \ldots, x_m) = \alpha_v x_1 + \beta_v x_m, \quad \alpha_v, \ldots, \beta_v \in K \quad (\nu = 1, 2).
\]

Therefore the \(R_v\)-translator (2.6) is not \(R_v\)-regular in the case of \(m = 2\). Hence, in the following, we may assume that \(m = 2\), i.e.,

\[
F_1(x_1, \ldots, x_m) = F_1(x_1, x_2) R_v = \alpha_1 x_1 + \beta_1 x_2, \quad F_2(x_1, \ldots, x_m) = F_2(x_1, x_2) R_v = \alpha_2 x_1 + \beta_2 x_2 \quad \text{and}
\]

\[
Q = Q_{v,v}\{\psi_1, \ldots, \psi_m\} = Q_{v,v}\{\psi_1, \psi_2\}.
\]

Therefore, by using (2.4), (2.5) and Theorem 1.1, we have

\[
R_v \alpha_1 h_1 \psi_1(x) \psi_1(y) + \beta_1 h_2 \psi_2(x) \psi_2(y)
\]
and

\[
R_v \alpha_2 h_1 \psi_1(x) \psi_1(y) + \beta_2 h_2 \psi_2(x) \psi_2(y)
\]

\[
a(\alpha_1 \psi_1(x) + \beta_1 \psi_2(x))(\alpha_1 \psi_1(y) + \beta_1 \psi_2(y))
\]

\[
+ h(\alpha_1 \psi_1(x) + \beta_1 \psi_2(x))(\alpha_1 \psi_1(y) + \beta_1 \psi_2(y))
\]

\[
+ h(\alpha_2 \psi_1(x) + \beta_2 \psi_2(x))(\alpha_1 \psi_1(y) + \beta_1 \psi_2(y)).
\]
Hence we have

\[(2.7)\quad h\alpha_1^2 - \alpha_1 h_1 = 0 ,\]
\[(2.8)\quad h\alpha_1 \beta_1 = 0 ,\]
\[(2.9)\quad h\beta_1^2 - \beta_1 h_2 = 0 ,\]

and

\[(2.10)\quad a\alpha_1^2 + 2h\alpha_1 \alpha_2 - \alpha_1 h_1 = 0 ,\]
\[(2.11)\quad a\alpha_1 \beta_1 + h\alpha_1 \beta_2 + h\alpha_2 \beta_1 = 0 ,\]
\[(2.12)\quad a\beta_1^2 + 2h\beta_1 \beta_2 - \beta_1 h_2 = 0 .\]

By using \((2.7)-(2.12)\), we shall prove that the \(R_v\)-translator \((2.6)\) in not \(R_v\)-regular in any case.

(a) The case of \(h=0\). By the assumption of this theorem, we have \(a \not= 0\). Hence, by using \((2.7)\), \((2.9)\), \((2.10)\) and \((2.12)\), we have \(\alpha_1 = \beta_1 = 0\), and hence the \(R_v\)-translator \((2.6)\) is not \(R_v\)-regular.

(b) The case of \(h \not= 0\) and \(h_1 = h_2 = 0\). By using \((2.7)\) and \((2.9)\), we have \(\alpha_1 = \beta_1 = 0\). Hence the \(R_v\)-translator \((2.6)\) is not \(R_v\)-regular.

(c) The case of \(h = 0\), \(h_1 = 0\) and \(h_2 = 0\). By \((2.9)\), we have \(\beta_1 = 0\). Hence by \((2.11)\) we have \(\alpha_1 \beta_2 = 0\), i.e., \(\alpha_1 = 0\) or \(\beta_2 = 0\). Therefore the \(R_v\)-translator \((2.6)\) is not \(R_v\)-regular.

(d) The case of \(h = 0\), \(h_1 = 0\) and \(h_2 \not= 0\). It is similarly obtained as in the case (c) that the \(R_v\)-translator \((2.6)\) is not \(R_v\)-regular.

(e) The case of \(h = 0\), \(h_1 \not= 0\) and \(h_2 = 0\). By \((2.8)\), we have \(\alpha_1 = 0\) or \(\beta_1 = 0\). If \(\alpha_1 = 0\), then by \((2.11)\), we have \(\alpha_1 \beta_1 = 0\), i.e., \(\alpha_1 = 0\) or \(\beta_1 = 0\). Hence, in the case of \(\alpha_1 = 0\), the \(R_v\)-translator \((2.6)\) is not \(R_v\)-regular.

Proof of necessity. In Theorem 2.2, we have shown that, if \(P\) is a family of \(R_v\)-\{\} -derivation type, then the basic mapping-formulas of \(P\) concerning \(\cdot\) are of the form

\[\varphi(xy) = P_v((\varphi(x), \varphi(y))^R_v = h\varphi(x)\varphi(y)\text{ and}\]
\[\delta(xy) = P_s((\varphi(x), \varphi(y), \delta(x), \delta(y))^R_v = a\varphi(x)\varphi(y) + b\varphi(x)\delta(y) + b\delta(x)\varphi(y) + d\delta(x)\delta(y) ,\]

where \(bh + ad = b^2\). Hence it is sufficient to show that, if the basic mapping-formulas of \(P\) concerning \(\cdot\) are not of the form \((2.4)\), then \(P\) is not a family of \{\} -proper \(R_v\)-\{\} -derivation type in any case.
(a) The case of $d = 0$. Let $Q$ be a family $Q_{\nu, \nu}\{\psi_1, \psi_3\}$ of homomorphism type. Then it is clear from Theorem 1.4 that the system of $V$-polynomials

$$F_1(x_1, x_2) = x_1, \quad F_2(x_1, x_2) = bx_1 + dx_2$$

is an $R_v$-regular $R_v$-translator from $P$ into $Q$. Hence, in this case, $P$ is not a family of $\{\cdot\}$-proper $R_v\{\cdot\}$-derivation type.

(b) The case of $d = 0$. From $bh + ad = b^2$, we have that $b = 0$ or $b = h$. Hence we have that

$$P_{\delta}((\varphi(x), \varphi(y), \delta(x), \delta(y)) = a\varphi(x)\varphi(y)$$

or

$$P_{\delta}((\varphi(x), \varphi(y), \delta(x), \delta(y)) = a\varphi(x)\varphi(y) + h\varphi(x)\delta(y) + h\delta(x)\varphi(y).$$

Now it is sufficient to show that $P$ is not a family of $\{\cdot\}$-proper $R_v\{\cdot\}$-derivation type in the case of $a = 0$ and $h = 0$. Since, in this case, we have

$$P_{\delta}((\varphi(x), \varphi(y), \delta(x), \delta(y)) = 0,$$

it is clear that $P$ is not a family of $\{\cdot\}$-proper $R_v\{\cdot\}$-derivation type. This completes the proof.

Let $P$ be a family $P_{\nu, \nu}\{\varphi, \delta\}$ of $\{\cdot\}$-proper $R_v\{\cdot\}$-derivation type. Then, by Theorem 2.3, the basic mapping-formulas of $P$ concerning are of the form

$$\varphi(xy) = P_{\varphi}(\varphi(x), \varphi(y)) = a\varphi(x)\varphi(y)$$

and

$$\delta(xy) = P_{\delta}(\varphi(x), \varphi(y), \delta(x), \delta(y)) = a\varphi(x)\varphi(y) + h\varphi(x)\delta(y) + h\delta(x)\varphi(y),$$

where $a = 0$ or $h = 0$ or both. Now, if $h = 0$, then $P$ is called a family of trivial $\{\cdot\}$-proper $R_v\{\cdot\}$-derivation type. And if $h = 0$, then $P$ is called a family of non-trivial $\{\cdot\}$-proper $R_v\{\cdot\}$-derivation type.

**Theorem 2.4.** (I) Any family $Q_{\nu, \nu}\{\psi, \theta\}$ of trivial $\{\cdot\}$-proper $R_v\{\cdot\}$-derivation type is $R_v$-conjugate to the family $P_{\nu, \nu}\{\varphi, \delta\}$ of trivial $\{\cdot\}$-proper $R_v\{\cdot\}$-derivation type whose basic mapping-formulas concerning are of the form

$$\varphi(xy) = 0 \quad \text{and} \quad \delta(xy) = \varphi(x)\varphi(y).$$

(II) Any family $Q_{\nu, \nu}\{\psi^*, \theta^*\}$ of non-trivial $\{\cdot\}$-proper $R_v\{\cdot\}$-derivation type is $R_v$-conjugate to the family $P_{\nu, \nu}\{\varphi^*, \delta^*\}$ of non-trivial $\{\cdot\}$-proper $R_v\{\cdot\}$-derivation type whose basic mapping-formulas concerning are of the form
\[ \varphi^*(xy) = \varphi^*(x)\varphi^*(y) \quad \text{and} \quad \delta^*(xy) = \varphi^*(x)\delta^*(y) + \delta^*(x)\varphi^*(y). \]

(III) Any family of trivial \(\{\cdot\}\)-proper \(R_v\)-\{\cdot\}-derivation type is not \(R_v\)-conjugate to any family of non-trivial \(\{\cdot\}\)-proper \(R_v\)-\{\cdot\}-derivation type.

Proof of (I). By the above definition, the basic mapping-formulas of \(Q_{v,v}\{\psi, \theta\}\) concerning \(\cdot\) are of the form

\[
\psi(xy) = Q_v.(\psi(x), \psi(y))^R_v = 0 \quad \text{and} \quad \theta(xy) = Q_v.(\psi(x), \psi(y), \theta(x), \theta(y))^R_v = a\psi(x)\psi(y).
\]

Then, by Theorem 1.4, the system of \(V\)-polynomials

\[
F_i(x_1, x_2) = x_1, \quad F_2(x_1, x_2) = ax_2
\]

is an \(R_v\)-regular \(R_v\)-translator from \(P_{v,v}\{\varphi, \delta\}\) into \(Q_{v,v}\{\psi, \theta\}\). Hence \(P_{v,v}\{\varphi, \delta\}\) is \(R_v\)-conjugate to \(Q_{v,v}\{\psi, \theta\}\).

Proof of (II). By the above definition, the basic mapping-formulas of \(Q^*_v\{\varphi^*, \theta^*\}\) concerning \(\cdot\) are of the form

\[
\psi^*(xy) = Q^*_v.(\psi^*(x), \psi^*(y))^R_v = h\psi^*(x)\psi^*(y) \quad \text{and} \quad \theta^*(xy) = Q^*_v.(\psi^*(x), \psi^*(y), \theta^*(x), \theta^*(y))^R_v = a\psi^*(x)\psi^*(y) + h\psi^*(x)\theta^*(y) + h\theta^*(x)\psi^*(y),
\]

where \(h = 0\). Then, by Theorem 1.4, the system of \(V\)-polynomials

\[
F_i(x_1, x_2) = \frac{1}{h} x_1, \quad F_2(x_1, x_2) = x_2 - \frac{a}{h^2} x_1
\]

is an \(R_v\)-regular \(R_v\)-translator from \(P^*_v\{\varphi^*, \delta^*\}\) into \(Q^*_v\{\psi^*, \theta^*\}\). Hence \(P^*_v\{\varphi^*, \delta^*\}\) is \(R_v\)-conjugate to \(Q^*_v\{\psi^*, \theta^*\}\).

Proof of (III). It is sufficient to show that \(P_{v,v}\{\varphi, \delta\}\) is not \(R_v\)-conjugate to \(P^*_v\{\varphi^*, \delta^*\}\). Now let a system of \(V\)-polynomials

(2.13)

\[
F_i(x_1, x_2), \quad F_2(x_1, x_2)
\]

be an \(R_v\)-translator from \(P^*_v\{\varphi^*, \delta^*\}\) into \(P_{v,v}\{\varphi, \delta\}\). Then it is similarly obtained as in the first part of the proof of sufficiency of Theorem 2.3 that the \(V\)-polynomials (2.13) are of the form

\[
F_1(x_1, x_2)^{R_v} = \alpha_1 x_1 + \beta_1 x_2 \quad \text{and} \quad F_2(x_1, x_2)^{R_v} = \alpha_2 x_1 + \beta_2 x_2.
\]

Hence, by Theorem 1.1, we have
and hence we have

\[ F_i(\phi^*(x)\phi^*(y), \phi^*(x)\delta^*(y) + \delta^*(x)\phi^*(y))^R = 0, \]

Therefore \( \alpha_i = \beta_i = 0 \), and therefore the system (2.13) is not \( R_V \)-regular. Hence \( P_{V,V}\{\rho, \delta\} \) is not \( R_V \)-conjugate to \( P^{\pm}_{V,V}\{\rho^*, \delta^*\} \). This completes the proof.

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Reference