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On Homotopy Type Problems of Special Kinds of Polyhedra II

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§ 1. Introduction

This paper is a continuation of my previous paper [14] of the same title, where I gave detailed accounts of homotopy types of *a* A_n^2-complex and of some special A_n^3-complex. They are completely determined by their cohomology groups, some homomorphisms μ, Δ, defined among them, and Steenrod's squaring operations, so that their homotopy invariants should be also determined by them. Homotopy type problems and related subjects are dealt with in this paper.

First, the exact sequence of J.H.C. Whitehead [4] is generalized in order to compute formally $\Gamma_{n+1}(0)$, $\Gamma_{n+2}(0)$ (§ 3) under some restrictions in dimensions. In case of cohomotopy groups this is accomplished by M. Nakaoka to get a generalization of the exact sequence of Spanier (refer to [15]). Utilizing this, we can compute up to group extension homotopy groups $\pi_{n+1}(P)$, $\pi_{n+2}(P)$ of a polyhedron P with vanishing homotopy groups $\pi_i(P)=0$ for each $i<n$. This calculation suggests us to compute combinatorially $\pi_{n+1}(P)$, $\pi_{n+2}(P)$ of an A_n^2-complex and also π_{n+2} of a special kind of polyhedron (see § 6). The study of reduced complexes in my previous paper and of J.H.C. Whitehead's secondary boundary operations (see § 4) enables us to solve thoroughly how $\pi_{n+1}(P)$※, $\pi_{n+2}(P)$ of an A_n^2-complex are computed by the aids of homology groups, of Steenrod's squaring homomorphisms, and of some homomorphisms μ, Δ, (see § 5), and also to get the way of calculation of $\Gamma_{n+1}(P)$, $\Gamma_{n+2}(P)$. In § 6 we restate concisely the results of my previous paper [14] through this sequence.

Until § 6 we assume $n>3$, or $n>2$.

We proceed to attack more complicated lower dimensional cases related to the subjects discussed till § 6. Recently Hirsch [16] gave a very elegant expression of the kernel of the homomorphism $i: \pi_3(P) \to H_3(P)$, where P is a simply connected polyhedron without 2-dimensional

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※ I have been informed of the existence of Hilton's paper on $\pi_{n+1}(P)$ through Chang's paper.

torsion. In § 7 we calculate the fourth homotopy group of a polyhedron whose third homotopy group vanishes besides Hirsch's assumptions on P. This is a step towards the solution of homotopy type problems of lower dimensional cases. Finally, calculations such as $\pi_{2n-2}(S^n \vee S^n)$, which are utilized in course of our discussions, are studied in preparation for my forthcoming paper. I hope, I shall come back shortly to the homotopy type of a five dimensional simply connected polyhedron in connection with brilliant results obtained recently by N. Shimada.

I would like to express my sincere gratitude to my teacher Astuo Komatsu for his constant encouragements during this study, and I thank Mr. M. Nakaoka for his kind criticisms.

§ 2. Generalization of the exact sequence of J.H.C. Whitehead

Let K be a connected CW-complex and let e^0 be a 0-cell, which is taken to be a base point of all the homotopy groups. Let

$$C_p(q) = \pi_{p+q}(K^p, K^{p-1}) \quad \text{and} \quad A_p(q) = \pi_{p+q}(K^p),$$

where K^r denotes the r-skelton of K. Then, let us consider the following sequence designated by $(C, A)(q)$

$$\longrightarrow A_p(q) \xrightarrow{j_p(q)} C_p(q) \xrightarrow{\beta_p(q)} A_{p-1}(q) \xrightarrow{j_{p-1}(q)} C_{p-1}(q) \longrightarrow$$

, where $\beta_p(q)$ is the homotopy boundary operator and $j_p(q)$ is the relativization. Evidently $\beta_p(q)\cdot j_p(q) = 0$. If we put $j_{p-1}(q)\cdot\beta_p(q) = \partial_p(q)$, we have three groups $H_p(q)$, $\Gamma_p(q)$, $\Pi_p(q)$ as follows;

$H_p(q) = Z_p(q) | B_p(q)$, where $Z_p(q)$ is the kernel of $\partial_p(q)$, and $B_p(q)$ is the image of $\partial_{p+1}(q)$;

$\Gamma_p(q)$ is a kernel of $j_p(q)$;

$\Pi_p(q) = A_p(q) | D_p(q)$, where $D_p(q)$ is the image of $\beta_{p+1}(q)$. As J.H.C. Whitehead defined the exact sequence in [4], we have the following exact sequence $\sum_q(K)$ with three operations $\mathfrak{B}$, $\mathfrak{J}$, $\overline{\mathfrak{J}}$,

$$\cdots \longrightarrow H_{p+1}(q) \xrightarrow{\mathfrak{B}_{p+1}(q)} \Gamma_p(q) \xrightarrow{\mathfrak{J}_p(q)} \Pi_p(q) \xrightarrow{\overline{\mathfrak{J}}_p(q)} H_p(q) \xrightarrow{\mathfrak{B}_p(q)} \Gamma_{p-1}(q) \longrightarrow \cdots$$

It is obvious that $\sum_0(K)$ is the sequence of J.H.C. Whitehead used in [4]. It is also verified through an analogous way as is shown in [4] that $\sum_q(K)$ is a homotopy invariant of K. Then we have several formal properties;

Theorem 1.

(2.1) $\rho_p(q): \Gamma_p(q) \cong \Pi_{p-1}(q+1)$,

(2.2) $\Pi_p(0) \cong \pi_p(K)$, *where* $\pi_p(K)$ *denotes the p-dimensional homotopy group of* K.

$$\text{(2.3)} \qquad \Gamma_p(q) = 0 \quad \text{for each} \quad p \leq n, \qquad \Pi_p(q) = 0 \quad \text{for each} \quad p \leq n-1,$$

if $\pi_i(K)$ *vanishes foreach* $i < n$.

(2.4) *If* K *is aspherical in dimension less than* n *and if* $q \leq p-4$ *and* $q \leq Min.\ (p-3,\ n-1)-2$, *we have*

$$\rho_p(q);\ \mathfrak{H}_p(K, p^{p+q}) \simeq H_p(q),$$

where $\mathfrak{H}_p(K, p^{p+q})$ *is the* p *dimensional homology group of* K *with the* $(p+q)$ *dim. homotopy group* p^{p+q} *of* p*-dim. sphere as its coefficient group.*

Proof. (2.1), (2.2) are direct consequences of definition. (2.3) can be easily verified from the fact that K is of the same homotopy type as a complex, the $(n-1)$-skelton of which is a single point. In order to prove (2.4) we show that $C_p(q) \simeq \mathfrak{C}_p(K, p^{p+q})$, if $q \leq$ Min. $(p\text{-}2, n-1) -2$ and $q \leq p-3$, where $\mathfrak{C}_p(K, p^{p+q})$ denotes the p-dim. chain group with p^{p+q} as its coefficient group. Let t_i be an arc joining in K^{p-1} the base point e^0 to a point on the boundary of a p-cell σ_i^p and let ε^p be the union $\bigvee_i (t_i + \sigma_i^p)$. Then we have a triad $(K^p;\ \varepsilon^p, K^{p-1})$. Let us consider the sequence

$$\rightarrow \pi_i(\dot{\varepsilon}^p) \rightarrow \pi_i(K^{p-1}) \rightarrow \pi_i(K^{p-1}, \dot{\varepsilon}^p) \rightarrow \pi_{i-1}(\dot{\varepsilon}^p) \rightarrow$$

If $i \leq p-2$, we have $\pi_i(K^{p-1}) \simeq \pi_i(K^{p-1}, \dot{\varepsilon}^p)$ and $\pi_i(K^{p-1}) \simeq \pi_i(K)$, so that $\pi_i(K^{p-1}, \dot{\varepsilon}^p) = 0$ for $i \leq$ Min. $(p-2,\ n-1)$. In virtue of a main theorem of triad it is seen that $(K^p;\ \varepsilon^p, K^{p-1})$ is $(p+q+1)$-connected if $p+q+1 \leq$ Min. $(p-2, n-1)+p-1$. Therefore, we have

$$\pi_{p+q}(\varepsilon^p, \dot{\varepsilon}^p) \simeq \pi_{p+q}(K^p, K^{p-1}),$$

if $q \leq$ Min. $(p-2, n-1)-2$. Furthermore, if $q \leq p-3$, we have $\pi_{p+q}(\varepsilon^p, \dot{\varepsilon}^p) \simeq \pi_{p+q-1}(\bigvee_i S_i^{p-1}) \simeq \sum_i \pi_{p+q-1}(S_i^{p-1}) \simeq \sum_i \pi_{p+q}(S_i^p)$, where the last iromorphism is established by suspension. Thus we have i_p; $\pi_{p+q}(K^p, K^{p-1}) \simeq \mathfrak{C}_p(K, p^{p+q})$ if $q \leq$ Min. $(p-2, n-1)-2$ and $q \leq p-3$. Now let us consider the following diagram,

$$\begin{array}{ccccccc}
\pi_{p+q+1}(K^{p+1}, K^p) & \xrightarrow{\beta_{p+1}(q)} & \pi_{p+q}(K^p) \xrightarrow{j_p(q)} & \pi_{p+q}(K^p, K^{p-1}) & \xrightarrow{\beta_p(q)} & \pi_{p+q-1}(K^{p-1}) & \xrightarrow{j_{p-1}(q)} \\
\downarrow i_{p+1} & & & \downarrow i_p & & & \\
\mathfrak{C}_{p+1}(K, (p+1)^{p+q+1}) & & \xrightarrow{\tilde{\partial}_{p+1}(q)} & \mathfrak{C}_p(K, p^{p+q}) & & \xrightarrow{\tilde{\partial}_p(q)} &
\end{array}$$

$$\begin{array}{l}
\longrightarrow \pi_{p+q-1}(K^{p-1}, K^{p-2}) \\
\qquad\qquad \downarrow i_{p-1} \\
\longrightarrow \mathfrak{C}_{p-1}(K, (p-1)^{p+q-1})
\end{array}$$

If $q \leq$ Min. $(p-3,\ n-1)-2$ and $q \leq p-4$, i_{p-1}, i_p, and i_{p+1} are all isomorphisms onto. As is easily seen, we have

$$\partial_{p+1}(q) = j_p(q)\cdot\beta_{p+1}(q) = i_p^{-1}\cdot\tilde{\partial}_{p+1}(q)\cdot i_{p+1}$$
$$\partial_p(q) = j_{p-1}(q)\cdot\beta_p(q) = i_{p-1}^{-1}\cdot\tilde{\partial}_p(q)\cdot i_p\,.$$

Thus, $\partial_p(q)$, $\partial_{p+1}(q)$ may be regarded as ordinary homological boundary operators. Notice that coefficient groups are identified by isomorphisms by suspension, when homological boundary operators $\tilde{\partial}_{p+1}(q)$, $\tilde{\partial}_p(q)$ are considered. This proves

$$\sigma_p(q):\ \ \mathfrak{H}_p(K, p^{p+q}) \cong H_p(q)\,.$$

§ 3. Formal calculations of $\Gamma_{n+1}(0)$, $\Gamma_{n+2}(0)$.

In this section we assume that K is a connected complex aspherical in dimensions less than n. Then we have

(3.1) $$\Gamma_{n-1}(0) = 0\,,\ \ \Gamma_n(0) = 0 \ \ \text{from (2.3)}\,.$$

It is seen from (2.4) that if $n \geqq 5$, we have

(3.2) $$\begin{aligned} H_n(0) &\cong \mathfrak{H}_n(K, I)\,,\\ H_{n+1}(0) &\cong \mathfrak{H}_{n+1}(K, I)\,,\\ H_{n+2}(0) &\cong \mathfrak{H}_{n+2}(K, I)\,,\\ H_{n+3}(0) &\cong \mathfrak{H}_{n+3}(K, I)\,,\end{aligned}$$

where I denotes the group of integers. Let us consider the sequence $\sum_1(K)$;

$$\longrightarrow \Gamma_n(1) \longrightarrow \Pi_n(1) \xrightarrow{\overline{\mathfrak{J}}_n(1)} H_n(1) \longrightarrow \Gamma_{n-1}(1) \longrightarrow$$

, then we have $\overline{\mathfrak{J}}_n(1): \Pi_n(1) \cong H_n(1)$ from $\Gamma_n(1) = \Gamma_{n-1}(1) = 0$. Since $\rho_{n+1}(0): \Gamma_{n+1}(0) \cong \Pi_n(1)$ from (2.1) and since $\sigma_n(1): \mathfrak{H}_n(K, I_2) \cong H_n(1)$, for $n \geqq 6$, from (2.4), we have

(3.3) $$\sigma_n^{-1}(1)\ \overline{\mathfrak{J}}_n(1)\ \rho_{n+1}(0)\,;\ \Gamma_{n+1}(0) \cong \mathfrak{H}_n(K, I_2)\,,$$

if $n \geqq 6$, where I_2 is the group of integers reduced mod. 2.

Next we calculate $\Gamma_{n+2}(0)$ by the sequence $\sum_q(K)$. In the sequence $\sum_1(K)$

$$H_{n+1}(1) \xrightarrow{\mathfrak{B}_{n+2}(1)} \Gamma_{n+1}(1) \xrightarrow{\mathfrak{J}_{n+1}(1)} \Pi_{n+1}(1) \xrightarrow{\overline{\mathfrak{J}}_{n+1}(1)} H_{n+1}(1) \longrightarrow \Gamma_n(1) \longrightarrow\,,$$

we have
$$\begin{aligned} &\sigma_{n+2}(1): \mathfrak{H}_{n+2}(K, I_2) \cong H_{n+2}(1)\,,\ \ \text{for } n \geqq 6\,,\\ &\sigma_{n+1}(1): \mathfrak{H}_{n+1}(K, I_2) \cong H_{n+1}(1)\,,\ \ \text{for } n \geqq 6\,,\\ &\Gamma_n(1) = 0 \ \ \text{from (2.3)},\\ &\rho_{n+2}(0): \Gamma_{n+2}(0) \cong \Pi_{n+1}(1) \ \ \text{from (2.1)}.\end{aligned}$$

Let us denote by A $\Gamma_{n+1}(1) \mid \mathfrak{B}_{n+2}(1)\cdot\sigma_{n+2}(1)\ \mathfrak{H}_{n+2}(K, I_2)$, then we have from the exactness of $\sum_1(K)$

(3.4) $$\overline{\mathfrak{J}}_{n+1}(1)\,;\ \rho_{n+2}(0)\ \Gamma_{n+2}(0) \mid \mathfrak{J}_{n+1}(1) A \cong \sigma_{n+1}(1)\ \mathfrak{H}_{n+1}(K, I_2)\,.$$

In order to calculate $\Gamma_{n+1}(1)$ involved in A we consider $\sum_2(K)$

$$\longrightarrow \Gamma_n(2) \longrightarrow \Pi_n(2) \xrightarrow{\overline{\mathfrak{J}}_n(2)} H_n(2) \longrightarrow \Gamma_{n-1}(2) \longrightarrow ,$$

where we have $\overline{\mathfrak{J}}_n(2): \Pi_n(2) \cong H_n(2)$ from $\Gamma_n(2) = \Gamma_{n-1}(2) = 0$. Since $\Gamma_{n+1}(1) \cong \Pi_n(2)$ from (2.1) and since $H_n(2) \cong \mathfrak{H}_n(K, I_2)$ for $n \geq 7$, we have

$$\text{(3.5)} \qquad \sigma_n^{-1}(2)\cdot\overline{\mathfrak{J}}_n(2)\cdot\rho_{n+1}(1)\,;\ \Gamma_{n+1}(1) \cong \mathfrak{H}_n(K, I_2)\,.$$

Theorem 2. *In a connected complex K aspherical in dimensions less than n we have*

(3.6) $\Gamma_n(0) = 0$,

(3.7) $\sigma_n^{-1}(1)\ \overline{\mathfrak{J}}_n(1)\ \rho_{n+1}(0): \Gamma_{n+1}(0) \cong \mathfrak{H}_n(K, I_2)$ for $n \geq 6$,

(3.8) $\overline{\mathfrak{J}}_{n+1}(1): \rho_{n+2}(0)\cdot\Gamma_{n+2}(0)|\mathfrak{J}_{n+1}(1)(\rho_{n+1}^{-1}(2)\ \overline{\mathfrak{J}}_n^{-1}(2)\ \sigma_n(2)\ \mathfrak{H}_n(K, I_2)|$
$\mathfrak{B}_{n+2}(1)\ \sigma_{n+2}(1)\ \mathfrak{H}_{n+2}(K, I_2)) \cong \sigma_{n+1}(1)\ \mathfrak{H}_{n+1}(K, I_2)$ *for* $n \geq 7$.

For the sake of brevity we shall often use the way of expression

$$\text{(3.9)} \qquad \Gamma_{n+2}(0)|\mathfrak{H}_n(I_2)|\mathfrak{B}_{n+2}(1)\ \mathfrak{H}_{n+2}(I_2) \cong \mathfrak{H}_{n+1}(I_2)$$

for (3.8), abbreviating all the isomorphisms in (3.8). As we stated in the introduction, Theorem 2 is established in the sense that it helps us in suggesting the complete solution of computations of homotopy groups and of homotopy type problems. It should be noted that we shall give full accounts of $\Gamma_{n+1}(0)$, $\Gamma_{n+2}(0)$ without restriction as to dimension in the sequel, utilizing reduced complexes together with the study of $\mathfrak{B}$-operation.

§ 4. $\mathfrak{B}_{n+2}(0)$, $\mathfrak{B}_{n+2}(1)$, and $\mathfrak{B}_{n+3}(0)$

i) $\mathfrak{B}_{n+2}(0)$ Let K be a A_n^2-complex, and let $Sq_{n-2}: \mathfrak{H}^n(K, I_2) \to \mathfrak{H}^{n+2}(K, I_2)$ be Steenrod's Squaring homomorphism. As is shown in [4] by J.H.C. Whitehead, we have

$$\lambda: \ \Gamma_{n+1}(0) \cong \mathfrak{H}_n(K, I_2) \quad \text{for } n > 2,$$
$$\mu: \ H_{n+2}(0) \cong \mathfrak{H}_{n+2}(K, I) \quad \text{for } n > 2.$$

Then $\nu = \lambda\cdot\mathfrak{B}_{n+2}(0)\cdot\mu^{-1}: \mathfrak{H}_{n+2}(I) \to \mathfrak{H}_{n+2}(I_2)$ for $n > 2$ can be determined by Steenrod's operation as follows. If $x \in \mathfrak{H}^n(I_2)$ and $y \in \mathfrak{H}_{n+2}(I)$, we have

$$\text{(4.1)} \qquad KI[\nu y, x] = KI[y, Sq_{n-2}\, x]\,,$$

where KI denotes Kronecker index, and, as to group multiplication, two groups I, I_2 are paired to I_2. From (4.1) νy may be regarded as an element in Hom $[\mathfrak{H}^n(K, I_2), I_2]$ such that

$$\nu y: \ {}^{r}x \to KI[y, Sq_{n-2}x]\,.$$

Therefore ν is determined in the sense that νy represents an element in $\mathfrak{H}_n(K, I_2)$. (refer to [11] or [12])

ii) $\mathfrak{B}_{n+2}(1)$ Let K be the same as before. From (3.5) we have $\lambda = \sigma_n^{-1}(2)\cdot\mathfrak{Z}_n(2)\,\rho_{n+1}(1): \Gamma_{n+1}(1) \cong \mathfrak{H}_n(K, I_2)$, and put $\mu = \sigma_{n+2}(1)$: $\mathfrak{H}_{n+2}(I_2) \to H_{n+2}(1)$. Then $\nu = \lambda\,\mathfrak{B}_{n+2}(1)\,\mu: \mathfrak{H}_{n+2}(I_2) \to \mathfrak{H}_n(I_2)$ can be also determined analogously by Steenrod's operation. Two cases i), ii) can be easily verified by the aid of reduced complexes

iii) $\mathfrak{B}_{n+3}(0)$ Since no account of $\mathfrak{B}_{n+3}(0)$ is in print and since it is applied to the homotopy type problem discussed in the sequel, we give here detailed account of it in case where K is such a complex as was dealt with in [14]. K is of the same homotopy type as the following complex L.

$$L^{n+2} = (S_1^n \cup e_1^{n+1}) + \cdots + (S_k^n \cup e_k^{n+1}) + (S_{k+1}^n \cup e_{k+1}^{n+1} \cup e_{k+1}^{n+2}) + \cdots + (S_{k+l}^n \cup e_{k+l}^{n+1} \cup e_{k+l}^{n+2})$$
$$+ (S_{k+l+1}^n \cup e_{k+l+1}^{n+2}) + \cdots + (S_\kappa^n \cup e_\kappa^{n+2}) + S_1^{n+2} + \cdots + S_t^{n+2},$$

where $e_i^{n+1}(i=1,\dots,k)$ is attached to S_i^n by a map $f_i: \partial e_i^{n+1} \to S_i^n$ of odd degree σ_i, $e_i^{n+1}(i=k+1,\dots k+l)$ is attached to S_i^n by a map g_i: $\partial e_i^{n+1} \to S_i^n$ of degree 2^{p_i}, and $e_i^{n+2}(i=k+1,\dots,\kappa)$ is attached to S_i^n by an essential may $\eta_i: \partial e_i^{n+2} \to S_i^n$. L is constructed by attaching to L^{n+2} a number of $(n+3)$ cells $e_i^{n+3}(i=1,\dots,\alpha)$ by $\beta e_i^{n+3} = \sum_{j=1}^{t} \lambda_{ij} S_j^{n+2} + \sum_{j=k+1}^{k+l} \mu_{ij}\omega_j + \sum_{j=k+1}^{k+l} \nu_{ij} v_j + \sum_{j=k+l+1}^{\kappa} \nu_{ij}\omega_j$, where S_j^{n+3} denotes the generator of $\pi_{n+2}(S_j^{n+2})$, $\omega_j(j=k+1,\dots,\kappa)$ is the free generator of $\pi_{n+2}(S_j^n \cup e_j^{n+2})$, and $v_j(j=k+1, \dots, k+l)$ is the generator of $\pi_{n+2}(S_j^n \cup e_j^{n+1} \cup e_j^{n+2})$ of order two. By definition $\Gamma_{n+2}(0)$ is the image of the injection $i: \pi_{n+2}(L^{n+1}) \to \pi_{n+2}(L^{n+2})$. Since $\pi_{n+2}(S_j^n \cup e_j^{n+1}) = 0$ $(j=1,\dots,k)$, $\Gamma_{n+2}(0)$ is generated by $v_j(j=k+1, \dots, k+l)$. A base of $\mathfrak{H}_{n+1}(L, I_2)$ is $\{j_2 e_j^{n+1}, (j=k+1,\dots,k+l)\}$, where j_2 is the natural homomorphism of a group of cycles mod. 2 into the corresponding homology group with integral group reduced mod. 2 as its coefficient group. A mapping $\lambda: j_2 e_j^{n+1} \to v_j$ induces an isomorphism $\lambda: \mathfrak{H}_{n+1}(L, I_2) \to \Gamma_{n+2}(0)$. If a base of $\mathfrak{H}_{n+3}(L, I)$ is $\{j_0 e_i^{n+3}, (i=1,\dots,m)\}$, we have

$$\beta e_i^{n+3} = \sum_{j=k+1}^{k+l} \nu_{ij} V_j.$$

If a base of $\mathfrak{H}^{n+1}(L, I_2)$ is $\{j_2\varphi_j^{n+1}, (j=k+1,\dots,k+l)\}$, and if a base of $\mathfrak{H}^{n+3}(K, I_2)$ is $\{j_2\varphi_j^{n+3}, (j=1,\dots,\alpha)\}$, we can choose them such that

$$KI\left[j_2 e_i^{n+1}, j_2 \varphi_j^{n+1}\right] = \delta_{ij}, \quad KI\left[j_0 e_i^{n+3}, j_2 \varphi_i^{n+3}\right] = \delta_{ij}.$$

δ_{ij} is 0 or the generator of I_2 according as $i \neq j$ or $i = j$. If $x \in \mathfrak{H}^{n+1}(I_2)$, $y \in \mathfrak{H}_{n+3}(I)$, we have

$$x = \sum_{j=k+1}^{k+l} p_j j_2 \varphi_j^{n+1}, \qquad p_j \in I_2,$$
$$y = \sum_{j=1}^{m} q_j j_0 \varphi_j^{n+3}, \qquad q_j \in I.$$

Then it is seen that

$$Sq_{n-1} x = \sum_j p_j^2 Sq_{n-1} j_2 \varphi_j^{n+1} = \sum_j p_j^2 \sum_{i=1}^{\alpha} \nu_{ij} j_2 \varphi_i^{n+3},$$

so that we have

$$KI [y, Sq_{n-1} x] = \sum_{j=k+1}^{k+l} \sum_{i=1}^{m} q_i p_j^2 \nu_{ij} \in I_2.$$

By definition of $\mathfrak{B}_{n+3}(0)$ we have

$$\lambda^{-1}\mathfrak{B}_{n+3}(0) y = \lambda^{-1}(\sum_{j=1}^{m} q_j \beta e_j^{n+3}) = \lambda^{-1}(\sum_{i=1}^{m} q_j \sum_{j=k+1}^{k+l} \nu_{ji} v_i) = \sum_{j=k+1}^{k+l} \sum_{i=1}^{m} q_i \nu_{ij} j_2 e_j^{n+1},$$

so that $KI [\lambda^{-1} \mathfrak{B}_{n+3}(0) y, x] = \sum_{j=k+1}^{k+l} \sum_{i=1}^{m} q_i p_j \nu_{ij} \in I_2$.

This proves $KI [y, Sq_{n-1} x] = KI [\lambda^{-1} \mathfrak{B}_{n+3}(0) y, x]$.

Since $\lambda^{-1} \mathfrak{B}_{n+3}(0) y : {}^r x \rightarrow KI [y, Sq_{n-1} x] \in I_2$ may be regarded as an element of $\mathfrak{H}_{n+1}(L, I_2)$, $\mathfrak{B}_{n+3}(0)$ can be determined effectively by squaring homomorphism Sq_{n-1}. The sequence of Whitehead $\sum_0(L)$ is a homotopy invariant so that all the discussions are available for K as well.

§ 5. Computation of π_{n+1}, π_{n+2}

In this section we assume that K is a A_n^2-complex. Let us consider the sequence

$$\longrightarrow H_{n+2}(0) \xrightarrow{\mathfrak{B}_{n+2}(0)} \Gamma_{n+1}(0) \xrightarrow{\mathfrak{J}_{n+1}(0)} \Pi_{n+1}(0) \xrightarrow{\overline{\mathfrak{J}}_{n+1}(0)} H_{n+1}(0) \longrightarrow 0$$

It is seen that $\overline{\mathfrak{J}}_{n+1}(0)$ is onto and that the kernel of $\overline{\mathfrak{J}}_{n+1}(0)$ is isomorphic to $\Gamma_{n+1}(0) | \mathfrak{B}_{n+2}(0) \mu^{-1} \mathfrak{H}_{n+2}(K, I)$ by $\overline{\mathfrak{J}}_{n+1}(0)$. By definition we have $\lambda : \Gamma_{n+1}(0) \approx \mathfrak{H}_n(K, I_2)$ for $n > 2$. Thus the kernel of $\mathfrak{J}_{n+1}(0)$ is isomorphic to $\mathfrak{H}_n(K, I_2) | \nu \mathfrak{H}_{n+2}(K, I)$ by $\mathfrak{J}_{n+1}(0) \lambda$, so that $\pi_{n+1}(K)$, isomorphic to $\Pi_{n+1}(0)$, is a group extension of $\mathfrak{H}_n(K, I_2) | \nu \mathfrak{H}_{n+2}(K, I)$ by $\mathfrak{H}_{n+1}(K, I)$. Thus π_{n+1} is determined combinatorially up to group extension.

Now we proceed to show how $\pi_{n+1}(K)$ is calculated completely. This is treated by Chang in his exciting paper [5], but the method of his is different from mine. To do this we apply a reduced complex obtained by Chang. Without loss of generality we assume that K is a reduced complex. For convenience of calculation in the sequel it seems desirable for us to put down here nine types of elementary polyhedra;

i) $Q_1^n = S^n$, $Q_1^{n+1} = S^{n+1}$, $Q_1^{n+1} = S^{n+2}$,

ii) $Q_2 = S^n \cup e^{n+1}$, where e^{n+1} is attached to S^n by a map f: $\partial e^{n+1} \to S^n$ odd degree σ, a power of a prime,

iii) $Q_3 = S^n \cup e^{n+2}$, where e^{n+2} is attached to S^n by an essential map f: $\partial e^{n+2} \to S^n$,

iv) $Q_4 = (S^n \vee S^{n+1}) \cup e^{n+2}$, where e^{n+2} is attached to $S^n \vee S^{n+1}$ by a map f: $\partial e^{n+2} \to S^n \vee S^{n+1}$ of the form $a+b$, where a denotes an essential map of ∂e^{n+2} onto S^n and b maps ∂e^{n+2} onto S^{n+1} with degree 2^{q_i},

v) $Q_5 = S^n \cup e^{n+1} \cup e^{n+2}$ where e^{n+1} is attached to S^n by a map f: $\partial e^{n+1} \to S^n$ of degree 2^p and e^{n+2} is attached to S^n by an essential map of ∂e^{n+2} onto S^n,

vi) $Q_6 = (S^n \vee S^{n+1}) \cup e^{n+1} \cup e^{n+2}$, where e^{n+1} is attached to $(S^n \vee S^{n+1}) \cup e^{n+2}$ by a map f: $\partial e^{n+1} \to S^n$ of degree 2^q,

vii) $Q_7 = S^n \cup e^{n+1}$, where e^{n+1} is attached to S^n by a map f: $\partial e^{n+1} \to S^n$ of degree 2^p,

viii) $Q_8 = S^{n+1} \cup e^{n+2}$, where e^{n+2} is attached to S^{n+1} by a map f: $\partial e^{n+2} \to S^{n+1}$ of odd degree σ, a power of prime,

ix) $Q_9 = S^{n+1} \cup e^{n+2}$, where e^{n+2} is attached to S^{n+1} by a map f: $\partial e^{n+2} \to S^{n+1}$ of degree 2^r.

A A_n^2-complex is a complex which consists of a collection of nine types of elementary polyhedra. A base of $\mathfrak{H}_n(K, I_2)$ is represented by n cells belonging to Q_1^n, Q_3, Q_4, Q_5, Q_6, Q_7, which are denoted by $e_{1,i}^n$, $e_{3,i}^n$, $e_{4,i}^n$, $e_{5,i}^n$, $e_{6,i}^n$, $e_{7,i}^n$, where i represents the number of n cells. A base of $\mathfrak{H}_{n+2}(K, I_2)$ is represented by $(n+2)$ cells belonging to Q_1^{n+2}, Q_3, Q_4, Q_5, Q_6, Q_8, Q_9, which are denoted by $e_{1,i}^{n+2}$, $e_{3,i}^{n+2}$, $e_{4,i}^{n+2}$, $e_{5,i}^{n+2}$, $e_{6,i}^{n+2}$, $e_{9,i}^{n+2}$. As we consider $\nu: \mathfrak{H}_{n+2}(K, I_2) \to \mathfrak{H}_n(K, I_2)$ in §4 ii), it is seen that

$$\nu j_2 e_{1,i}^{n+2} = 0, \quad \nu j_2 e_{3,i}^{n+2} = j_2 e_{3,i}^n, \quad \nu j_2 e_{4,i}^{n+2} = j_2 e_{4,i}^n$$
$$\nu j_2 e_{5,i}^{n+2} = j_2 e_{5,i}^n, \quad \nu j_2 e_{6,i}^{n+2} = j_2 e_{6,i}^n, \quad \nu j_2 e_{9,i}^{n+2} = 0.$$

Thus $\mathfrak{H}_n(K, I_2) | \nu \mathfrak{H}_{n-2}(K, I_2)$ is freely generated by $j_2 e_{1,i}^n$, $j_2 e_{7,i}^n$.
It is easily verified that $\pi_{n+1}(K) \cong \sum_{i,j} \pi_{n+1}(Q_{i,j})$ for $n > 2$, where i denotes the type and j represents the number of Q_i. Therefore $A = \sum_i \pi_{n+1}(Q_{1,i}^n) + \sum_i \pi_{n+1}(Q_{7,i})$ is a direct factor of $\pi_{n+1}(K)$. Since $\pi_{n+1}(Q_{1,i}^n) = I_2$ and $\pi_{n+1}(Q_{7,i}) = I_2$, A is isomorphic to $\mathfrak{H}_n(I_2) | \nu \mathfrak{H}_{n+2}(I_2)$. Now we put down here the homotopy groups of $Q_{j,i}^{n+1}$ $(j = 1, 2, 3, 4, 5, 6, 7, 8, 9)$ as follows:

$$\pi_{n+1}(Q_{1,i}^{n+1}) = I, \quad \pi_{n+1}(Q_{1,i}^{n+2}) = 0, \quad \pi_{n+1}(Q_{2,i}) = 0,$$
$$(5.1) \qquad \pi_{n+1}(Q_{3,i}^{n+1}) = 0, \quad \pi_{n+1}(Q_{4,i}) = I_{2^{q_i+1}}, \quad \pi_{n+1}(Q_{5,i}) = 0,$$
$$\pi_{n+1}(Q_{6,i}^{n+1}) = I_{2^{q_i+1}}, \quad \pi_{n+1}(Q_{8,i}) = I_{\sigma_i}, \quad \pi_{n+1}(Q_{9,i}) = I_{2^{r_i}}.$$

From this table it is seen that the rank of $\pi_{n+1}(K)$ is equal to that of $\mathfrak{H}_{n+1}(K, I)$ so that we have

$$\pi_{n+1}(K) \cong \mathfrak{H}_n(I_2) | \nu \mathfrak{H}_{n+2}(I_2) + B + \mathfrak{X}$$

where B denotes the free group of $\mathfrak{H}_{n+1}(K, I)$. Let us determine $\mathfrak{X}$. Let T be the torsion group of $\mathfrak{H}_{n+1}(I)$, and let $(h_1, h_2, \dots, h_\lambda)$ be the invariant system of $(n+1)$ dimensional torsion coefficients, where h_i is a power of a prime. From (5.1) we have $\mathfrak{X} \cong C + \mathfrak{Y}$, where C denotes the subgroup of T consisting of all the cyclic groups of odd degree h_i. Choosing even torsion coefficients $\{h_{i_1}, \dots, h_{i_\alpha}\}$ out of the system $(h_1, \dots, h_\lambda)$, we consider the following operation with respect to h_{i_ν} $(\nu = 1, \dots, \alpha)$,

$$\mathfrak{H}_{n+1}(K, I) \xleftarrow{\Delta_{h_{i_\nu}}} \mathfrak{H}_{n+2}(K, I_{h_{i_\nu}}) \xrightarrow{\mu_{h_{i_\nu}, 2}} \mathfrak{H}_{n+2}(K, I_2) \xrightarrow{\mathfrak{B}_{n+2}(1)} \mathfrak{H}_n(K, I_2).$$

Let us define two homomorphisms

$$\Delta_p : \quad \mathfrak{H}_{\alpha+1}(K, I_p) \to \mathfrak{H}_\alpha(K, I),$$
$$\mu_{p,q} : \quad \mathfrak{H}_\alpha(K, I_p) \to \mathfrak{H}_\alpha(K, I_q).$$

The first operation Δ_p is $\frac{1}{p}\partial$. Let $x \in \mathfrak{H}_\alpha(K, I_p)$ and let x' be a representative of x, then $\frac{q}{(p, q)} x'$ is a cycle mod. q. $\mu_{p,q}\, x$ is represented by $\frac{q}{(p, q)} x'$. If $\nu_{h_{i_\nu}} = \mathfrak{B}_{n+2}(1) \cdot \mu_{h_{i_\nu}, 2}$, the kernel of $\nu_{h_{i_\nu}}$ does not contain $j_2 e_{4,i}^{n+2}$, $j_2 e_{6,i}^{n+2}$. Putting $D = \bigcup_{h_{i_\nu}} \Delta_{h_{i_\nu}} \cdot \nu_{h_{i_\nu}}^{-1}(0)$, D is a subgroup of T, which is generated by $j_0 e_{9,i}^{n+1}$. Together with $\pi_{n+1}(Q_{9,i}) = I_{2^{r_i}}$ we have

$$\mathfrak{Y} \cong D + \mathfrak{Z}.$$

Let the invariant system of $T | C + D$ be represented by $\{2^{\eta_1}, 2^{\eta_2}, \dots, 2^{\eta_\kappa}\}$ and let E be an abelian group, the invariant system of which is $\{2^{\eta_1+1}, 2^{\eta_2+1}, \dots, 2^{\eta_\kappa+1}\}$. Since $\pi_{n+1}(Q_{4,i}) = I_{2^{\eta_i+1}}$, $\pi_{n+1}(Q_{6,i}) = I_{2^{\eta_i+1}}$, we have $\mathfrak{Z} \cong E$. In virtue of $\pi_{n+1}(Q_1^{n+2}) = \pi_{n+1}(Q_{2,i}) = \pi_{n+1}(Q_{3,i}) = \pi_{n+1}(Q_{5,i}) = 0$ it is concluded that

$$\pi_{n+1}(K) \cong \mathfrak{H}_n(K, I_2) | \nu \mathfrak{H}_{n+2}(K, I_2) + B + C + D + E.$$

Theorem 3. *The $(n+1)$-dimensional homotopy group of a A_n^2-complex can be calculated combinatorially by homology groups; $\mathfrak{H}_n$, $\mathfrak{H}_{n+1}$, $\mathfrak{H}_{n+2}$, by homomorphisms; μ, Δ, and by $\mathfrak{B}_{n+2}(1)$: $\mathfrak{H}_{n+2}(I_2) \to \mathfrak{H}_n(I_2)$.*

Now we give a more detailed account of Theorem 2, 3.9 and then compute $\pi_{n+2}(K)$ combinatorially. First $\Gamma_{n+2}(Q_i)$ $(i = 1, 2, \dots, 9)$ are calculated as follows:

$$
(5.2)\qquad
\begin{aligned}
&\Gamma_{n+2}(Q_1^n)=I_2, \quad \Gamma_{n+2}(Q_1^{n+1})=I_2, \quad \Gamma_{n+2}(Q_1^{n+2})=0\\
&\Gamma_{n+2}(Q_2)=0, \quad \Gamma_{n+2}(Q_3)=0, \quad \Gamma_{n+2}(Q_4)=I_2,\\
&\Gamma_{n+2}(Q_5)=I_2, \quad \Gamma_{n+2}(Q_6)=I_2+I_2, \quad \Gamma_{n+2}(Q_7)=I_4,^{*}\\
&\Gamma_{n+2}(Q_8)=0, \quad \Gamma_{n+2}(Q_9)=I_2.
\end{aligned}
$$

i) Let h_i be an n-dimensional even torsion coefficient, a power of 2, and consider $\varphi_{h_i}=\mu_{0,2}\Delta_{h_i}: \mathfrak{H}_{n+1}(I_{h_i})\xrightarrow{\Delta_{h_i}}\mathfrak{H}_n(I)\xrightarrow{\mu_{0,2}}\mathfrak{H}_n(I_2)$, then $\bigcup_{h_i}\varphi_{h_i}\mathfrak{H}_{n+1}(I_{h_i})$ is generated by $j_2e^n_{5,i}$, $j_2e^n_{6,i}$, $j_2e^n_{7,i}$.

ii) In virtue of the operation ν, $\mathfrak{H}_n(I_2)|(\bigcup_{h_i}\varphi_{h_i}\mathfrak{H}_{n+1}(I_{h_i}))^{\cup}\nu\cdot\mathfrak{H}_{n+2}(I_2)$ is generated by $j_2e^n_{1,i}$.

iii) Let k_i be an $(n+1)$-dimensional even torsion coefficient, a power of 2, and consider the operation $\psi_{k_i}=\mu_{0,2}\Delta_{k_i}: \mathfrak{H}_{n+2}(I_{k_i})\xrightarrow{\Delta_{k_i}}\mathfrak{H}_{n+1}(I)\xrightarrow{\mu_{0,2}}\mathfrak{H}_{n+1}(I_2)$, then $\bigcup_{k_i}\psi_{k_i}\mathfrak{H}_{n+2}(I_{k_i})$ is generated by $j_2e^{n+1}_{4,i}$, $j_2\tilde{e}^{n+1}_{6,i}$, $j_2e^{n+1}_{9,i}$ where $\tilde{e}^{n+1}_{6,i}$ denotes an $(n+1)$ cell bounded by an $(n+2)$ cell.

iv) Let us denote by B an abelian group, which is the direct sum of ρ integral groups mod. 2, where ρ is the $(n+1)$-th Betti number of K.

v) $\bigcup_{h_i}\varphi_{h_i}\mathfrak{H}_{n+1}(I_{h_i})\frown(\nu\mathfrak{H}_{n+2}(I_2))$ is generated by $j_2e^n_{5,i}$, $j_2e^n_{6,i}$.

iv) $A'=\bigcup_{h_i}\varphi_{h_i}\mathfrak{H}_{n+1}(I_{h_i})|(\bigcup_{h_i}\varphi_{h_i}\mathfrak{H}_{n+1}(I_{h_i}))\frown(\nu\mathfrak{H}_{n+2}(I_2))$ is generated by $j_2e^n_{7,i}$. Let the invariant system of A' be $\{\underbrace{2,\dots,2}_{6}\}$ and let us denote by $\{\underbrace{4,\dots,4}_{6}\}$ that of A.

Then from (5.2) and from $\Gamma_{n+2}(K)\cong\sum_{i,j}\Gamma_{n+2}(Q_{i,j})$ it is concluded that we have

$$
\begin{aligned}
\Gamma_{n+2}(K)\cong{}&(\bigcup_{h_i}\varphi_{h_i}\mathfrak{H}_{n+1}(I_{h_i}))\frown(\nu\mathfrak{H}_{n+2}(I_2))+\mathfrak{H}_n(I_2)|(\bigcup_{h_i}\varphi_{h_i}\mathfrak{H}_{n+1}(I_{h_i}))^{\cup}(\nu\mathfrak{H}_{n+2}(I_2))\\
&+\bigcup_{k_i}\varphi_{k_i}\mathfrak{H}_{n+2}(I_{k_i})+A+B.
\end{aligned}
$$

Theorem 4. *$\Gamma_{n+2}(K)$ of a A_n^2-complex K can be calculated combinatorially by homology groups; $\mathfrak{H}_n$, $\mathfrak{H}_{n+1}$, $\mathfrak{H}_{n+2}$, by homorphisms; μ, Δ, and by $\mathfrak{B}_{n+2}(1): \mathfrak{H}_{n+2}(I_2)\to\mathfrak{H}_n(I_2)$.*

Now that $\Gamma_{n+2}(K)$ has been computed, it is easy to compute $\pi_{n+2}(K)$. We give a table of $\pi_{n+2}(Q_i)$;

$$
(5.3)\qquad
\begin{aligned}
&\pi_{n+2}(Q_1^n)=I_2, \quad \pi_{n+2}(Q_1^{n+1})=I_2, \quad \pi_{n+2}(Q_1^{n+2})=I,\\
&\pi_{n+2}(Q_2)=0, \quad \pi_{n+2}(Q_3)=I, \quad \pi_{n+2}(Q_4)=I_2,
\end{aligned}
$$

---

* It is well known that $\pi_{n+2}(S^n\cup e^{n+1})$ is a group extension of I_2 by I_2, where e^{n+1} is attached to S^n by a map f: $\partial e^{n+1}\to S^n$ of even degree σ. According to [20], $\pi_{n+2}(S^n\cup e^{n+1})=I_4$ if $\sigma=2$. Here we assume, $\pi_{n+2}(S^n\cup e^{n+1})=I_4$ for $n>3$ in case $\sigma=2$.

$$\pi_{n+2}(Q_5) = I_2 + I, \quad \pi_{n+2}(Q_6) = I_2 + I_2, \quad \pi_{n+2}(Q_7) = I_4,$$
$$\pi_{n+2}(Q_8) = 0, \quad \pi_{n+2}(Q_9) = I_2.$$

It is clear that $\mathfrak{H}_{n+2}(I)$ is generated by $j_0 e_{1,i}^{n+2}$, $j_0 e_{3,i}^{n+2}$, $j_0 e_{5,i}^{n+2}$. From (5.2), (5.3) and from $\pi_{n+2}(K) \cong \sum_{i,j} \pi_{n+2}(Q_{i,j})$, for $n > 3$, we have

$$\pi_{n+2}(K) \cong \Gamma_{n+2}(K) + \mathfrak{H}_{n+2}(I).$$

Theorem 5. *The $(n+2)$-dimensional homotopy group of a A_n^2-complex can be calculated combinatorially from $\mathfrak{H}_n$, $\mathfrak{H}_{n+1}$, $\mathfrak{H}_{n+2}$, μ, Δ, $\mathfrak{B}_{n+2}$.*

§ 6. $\bar{A}_n^3$-complex

In [14] *I* solved the homotopy type problem of a $\bar{A}_n^3$-complex. Making use of the sequence of Whitehead, we restate the problem. Let us consider the sequence

$$(6.1)\quad \begin{array}{l} \to H_{n+2} \xrightarrow{\mathfrak{B}_{n+3}(0)} \Gamma_{n+2}(0) \to \Pi_{n+2}(0) \to H_{n+2}(0) \xrightarrow{\mathfrak{B}_{n+2}(0)} \Gamma_{n+1}(0) \to \\ 0 \to H_{n+1}(0) \to 0 \to \Pi_n(0) \to H_n(0) \to 0. \end{array}$$

It was proved in § 4 ii) iii) that the homomorphisms $\mathfrak{B}_{n+3}(0)$, $\mathfrak{B}_{n+2}(0)$ are determined by Steenrod's Squaring homomorphisms Sq_{n-1}, Sq_{n-2} respectively and that $\Gamma_{n+2}(0)$ is isomorphic to $\mathfrak{H}_{n+1}(I_2)$. Following Whitehead [4], we can establish analogously geometrical realizability, so that all the results in [14] are obtained by the aid of the sequence (6.1).

By the sequence we have

Theorem 6. *$\pi_{n+2}(K)$ of $\bar{A}_n^3$-complex is a group extension of $\mathfrak{H}_{n+1}(I_2)| \mathfrak{B}_{n+3}(0)\cdot\mathfrak{H}_{n+3}(I)$ by the kernel of $\mathfrak{B}_{n+2}(0)$; $\mathfrak{H}_{n+2}(0) \to \Gamma_{n+1}(0)$.*

§ 7. Lower dimensional case

In this section we assume that K is a simply connected complex without 2-dimensional torsion. Besides this, we assume $\pi_3(K) = 0$. We shall show how $\pi_4(K)$ can be calculated in terms of homology. As was proved by Whitehead [2],

$$K^3 \sim L^3 = S_1^2 + S_2^2 + \cdots + S_\rho^2 + S_1^3 + \cdots + S_\sigma^3 + S_{\sigma+1}^3 + \cdots + S_{\sigma+t}^3,$$

where 2-spheres and 3-spheres are attached at a point. Since K is free from 2nd torsion, the 3-skelton of K is of the same homotopy type as L^3. Then we have

$$K^4 \sim \{L^3; R_1, \ldots, R_\lambda\}^* = L^4,$$

* This notation is often used in [14]; if an n cell e^n is attached to a space P by a map $f: \partial e^n \to P$, the attached space is denoted by $\{P; \alpha\}$, where α is an element of $\pi_{n-1}(P)$ represented by f.

where $R_i = \alpha_i + b_{ij}S_j^3$, $\alpha_i \in \pi_3(S_1^2 \vee \cdots \vee S_\rho^2)$, and S_j^3 $(j = 1, \ldots, \sigma)$ is the generator of $\pi_3(S_j^3)$. Notice that $b_{ij}S_j^3$ is not summed with respect to j, and that b_{ij} is greater than unity.* If e_{ii} denotes the generator of $\pi_3(S_i^2)$ and if $e_{ij}(i \neq j)$ is represented by the Whitehead product $[a_i, a_j]$, where the generator is represented by a_i, we have

$$\alpha_i = \sum_{i \leqq j} p_{ij} e_{ij}$$

where p_{ij} are integers.

Lemma 7.1. e_{ij}, S_j^3 $(j = 1, \ldots, \sigma + t)$ are linear combinations of $R_1, R_2, \ldots, R_\lambda$

Take 4 simplex ε_i^4 in the interior of each 4 cell e_i^4, and join a point on the boundary $\dot{\varepsilon}_i^4$ to L^0 by an arc t_i. Let us denote $\bigvee_i (\varepsilon_i^4 + t_i)$ by ε^4 and its boundary by $\dot{\varepsilon}^4$. If we put $L = L^4 - \varepsilon^4$, L^3 is a deformation retract of L. Let us consider the following diagram

$$\begin{array}{ccccc}
\uparrow & & & & \\
\pi_4(L^4; \varepsilon^3, L) & & & & \\
\uparrow j_4 & & & & \\
\pi_4(L^4, L) & \xrightarrow{\partial_3} & \pi_3(L) & \xrightarrow{k_3} & \pi_3(L^4) \rightarrow \\
\uparrow i_1 & & \uparrow p & & \\
\pi_4(\varepsilon^4, \dot{\varepsilon}^4) & \xrightarrow{\partial} & \pi_3(\dot{\varepsilon}^4) & & \\
\uparrow \beta_1 & & & & \\
\pi_5(L^4; \varepsilon^4, L) & & & &
\end{array}$$

Since the triad $(L^4; \varepsilon^4, L)$ is 4-connected and β_4 is trivial, i_4 is an isomorphism onto. From $\pi_3(L^4) \cong \pi_3(K) = 0$, ∂_3 is onto. Since ∂ is an isomorphism onto, and $\partial_3 i_4 = p\partial$, p is onto, so that e_{ij}, S_j^3 are linear combinations of $\beta e_i^4 = R_i$.

Let us denote by $M^4\{L^3; R_1, \ldots, R_\lambda; e_{11}, \ldots, e_{ij}, \ldots, e_{\rho\rho}; S_1^3, \ldots, S_{\sigma+t}^3\}$. From Lemma (7.1) and from elementary operations we have

$$M^4 \sim \left\{L^3; R_1, \ldots, R_\lambda; 0, \ldots, 0\right\} = L^4 + \underbrace{S_1^4 + \cdots + S_{\rho\rho}^4}_{\frac{\rho(\rho-1)}{2}} + \underbrace{S_1^4 + \cdots + S_{\sigma+\tau}^4}_{\sigma+\tau}.$$

From $R_i = \alpha_i + b_{ij}S_j^3$ and from elementary operations, we have

$$\begin{aligned} M^4 &\sim \left\{L^3; 0, \ldots, 0; e_{11}, \ldots, e_{\rho\rho}; S_1^3, \ldots, S_{\sigma+\tau}^3\right\} \\ &= \left\{L^3; e_{11}, \ldots, e_{\rho\rho}, S_1^3, \ldots, S_{\sigma+\tau}^3\right\} + S_1^4 + \cdots + S_\lambda^4. \end{aligned}$$

Since e_j^4 $(j = 1, \ldots, \sigma + t)$ are attached to S_j^3 with degree unity, we have

$$M^4 \sim \left\{L^2; e_{11}, \ldots, e_{\rho\rho}\right\} + S_1^4 + \cdots + S_\lambda^4,$$

* Refer to [14].

where $L^2 = S_1^2 + \cdots + S_\sigma^2$. Let us denote by N^4, $\{L^2; e_{11}, \ldots, e_{\rho\rho}\}$.

From these considerations we have

$$(7.2) \qquad L^4 + \underbrace{S_1^4 + \cdots + S_{\rho\rho}^4}_{\frac{\rho(\rho-1)}{2}} + \underbrace{S_1^4 + \cdots + S_{\sigma+\tau}^4}_{\sigma+\tau} \sim N^4 + S + \cdots {}_1^4 + S_\lambda^4 .$$

It is clear that

$$K^5 + S_1^4 + \cdots + S_\omega^4 \sim \left\{L^4 ; X_1, \ldots, X_\kappa\right\} + S_1^4 + \cdots + S_{\rho\rho}^4 + S_1^4 + \cdots + S_{\sigma+\tau}^4$$

, where $\beta e_i^5 = X_i \in \pi_4(L^4)$ and $\omega = \frac{\rho(\rho-1)}{2} + (\sigma+\tau)$.

If f is a homotopy equivalence of (7.2), we have

$$Y_i = f(X_i) \in \pi_4(N^4 + S_1^4 + \cdots + S_\lambda^4) = \pi_4(S_1^4 + \cdots + S_\lambda^4)$$

(refer to § 8). From Lemma 2 [14] we have

$$K^5 + S_1^4 + \cdots + S_\omega^4 \sim N^4 + \left\{S_1^4 + \cdots + S_\lambda^4; Y_1, \ldots, Y_\kappa\right\}.$$

Through elementary operations and change of a base $\{S_1^4, \ldots, S_\lambda\}$, it is concluded that

$$(7.3) \quad K^5 + S_1^4 + \cdots + S_\omega^4 \sim N^4 + P_{\sigma_1}^5 + \cdots + P_{\sigma_\nu}^5 + S_{\nu+1}^5 + \cdots + S_\lambda^4 + S_1^5 + \cdots + S_\mu^5 ,$$

where $P_{\sigma_i} = \{S_i^4 ; \sigma_i S_i^4\}$.

If we consider 4-th homology groups of both sides of (7.3), we have

$$H_4(K^5) + \underbrace{I + \cdots + I}_{\omega} \cong \underbrace{I + \cdots + I}_{\frac{\rho(\rho-1)}{2}} + I_{\sigma_1} + \cdots I_{\sigma_\nu} + \underbrace{I + \cdots + I}_{\lambda-\nu} .$$

Since the ranks of both sides are equal, we have

$$(7.4) \qquad \beta_4 + \sigma + \tau = \lambda - \nu ,$$

where β_4 denotes 4-th Betti number. If r is the rank of $\pi_4(K)$, we have

$$r + \omega = \lambda - \nu .$$

From (7.4) we have

$$(7.5) \qquad r = \beta_4 - \frac{\rho(\rho-1)}{2} .$$

It is also seen that the torsion group of $\pi_4(K)$ is isomorphic to that of $H_4(K)$. Thus we have

Theorem 7. The four dimensional homotopy group of a complex K such that $\pi_i(K) = 0$ for $i = 1, 3$ and K is free from 2nd torsion, is given explicity in terms of homology groups;

$$\pi_4(K) \cong \underbrace{I + \cdots + I}_{\beta_4 - \frac{\rho(\rho-1)}{2}} + I_{\sigma_1} + \cdots + I_{\sigma_\nu} ,$$

where β_4, ρ are 4-dimensional Betti number, 2-dimensional Betti number respectively, and $(\sigma_1, \dots, \sigma_\nu)$ is the 4-dimensional torsion coefficients.

In such a complex it is also seen that we have $\beta_4 \geqq \frac{\rho(\rho-1)}{2}$.

§ 8. Note on homotopy groups

I owe a great deal to recent results due to Blakers and Massey [7], which enable me to calculate homotopy groups used till now. In preparation for my forthcoming paper it seems convenient to calculate $\pi_{3n-2}(S^n \vee S^n)$ for each $n \geqq 2$, which was also solved by Blakers and Massey [7]. By doing this we can prove $\pi_4(N^4) = 0$, which was essentially used in § 7. First we define a generalized Whitehead Product.

$$E^n = \left\{x\,;\ 1 \geqq x_i \geqq 0\,,\ i = 1, \dots, n\right\},$$

$$J^{n-1} = \left\{x\,;\ (x_n - 1)\prod_{i=1}^{n-1} x_i(x_i - 1) = 0\right\},$$

$$\begin{aligned} K^{p+q-1} &= \partial(E^p \times E^q) + (-1)^{p+q+1} E^p \times E^{q-1} \times 0 + (-1)^{p+1} E^{p-1} \times 0 \times E^q \\ &= (\partial E^p + (-1)^{p+1} E^{p-1} \times 0) \times E^q + (-1)^p E^p \times (\partial E^q + (-1)^{q+1} E^{q-1} \times 0) \\ &= J^{p-1} \times E^q + (-1)^p E^p \times J^{q-1}. \end{aligned}$$

Then K^{p+q-1} is a $(p+q-1)$ cell. Let X be a space such that $X = A \cup B$ and $A \frown B$ is non-void. If $\pi_p(B, A \frown B) \ni \alpha$, and $\pi_q(A, A \frown B) \ni \beta$, α and β are represented by maps f and g respectively such that

$$f : (E^p, \partial E^p, J^{p-1}) \to (B, A \frown B, *),$$

$$g : (E^q, \partial E^q, J^{q-1}) \to (A, A \frown B, *).$$

Let us define a map $f \vee g : K^{p+q-1} \to X = A \cup B$ such that

$$\begin{aligned} f \vee g(x, y) &= g(y), \quad (x, y) \in J^{p-1} \times E^q, \\ &= f(x), \quad (x, y) \in E^p \times J^{q-1}. \end{aligned}$$

If $\varphi : E^{p+q-1} \to K^{p+q-1}$ is an orientation preserving map of degree unity, a composite map $(f \vee g) \circ \varphi$ represents an element of $\pi_{p+q-1}(X\,; A, B)$. In course of verification that $(f \vee g) \circ \varphi$ represents an element of $\pi_{p+q-1}(X\,; A, B)$ it is easily seen that

$$(-1)\,[\partial\alpha, \beta] = \beta_+\,[\alpha, \beta],$$

$$(-1)^{p-1}[\alpha, \partial\beta] = \beta_-\,[\alpha, \beta].$$

By definition we have $[\alpha, \beta] = (-1)^{pq}[\beta, \alpha]$. If $\alpha \in \pi_p(S^n)$, $\beta \in \pi_q(S^n)$, we have $E[\alpha, \beta] = 0$ by definition of the generalized Whitehead product, where E denotes Freudenthal's suspension. These properties are used in calculating $\pi_{3n-2}(S_1^n \vee S_2^n)$.

Next, by a result of G.W. Whitehead [9] we have $\pi_{3n-2}(S_1^n \vee S_2^n)$

$\simeq i_1\pi_{3n-2}(S^n)+i_2\pi_{3n-2}(S^n)+\partial\pi_{3n-1}(S_1^n\times S_2^n, S_1^n\vee S_2^n)$, where ∂ is an isomorphism into and i_1, i_2 are injections. Let σ^{2n} be a 2n simplex in $S_1^n\times S_2^n = X^*$ and let X be X -Int. σ^{2n}, then $S_1^n\vee S_2^n$ is a deformation retract of X. This retraction is denoted by ψ. Consider a sequence of a triad $(X^*; \sigma^{2n}, X)$;

$\rightarrow\pi_{3n}(X^*; \sigma^{2n}, X)\xrightarrow{\beta^+_{3n-1}}\pi_{3n-1}(\sigma^{2n}, \dot\sigma^{2n})\xrightarrow{i}\pi_{3n-1}(X, X)\xrightarrow{j}\pi_{3n-1}(X^*; \sigma^{2n}, X)\rightarrow$.

If $n\geqq 2$, β_i^+ is trivial for each $i\leqq 3n-1$, so that i is an isomorphism into and j is a homomorphism onto.

$$\begin{aligned}\psi\partial i\pi_{3n-1}(\sigma^{2n}, \dot\sigma^{2n}) &= \psi i\partial\pi_{3n-1}(\sigma^{2n}, \dot\sigma^{2n})\\ &= \psi i\pi_{2n-2}(\dot\sigma^{2n}).\end{aligned}$$

From this any element of $\psi\partial i\pi_{3n-1}(\sigma^{2n}, \dot\sigma^{2n})$ is represented by a map f: $S^{3n-2}\rightarrow S^{2n-1}\xrightarrow{[i_1, i_2]} S_1^n\vee S_2^n$, where $[i_1, i_2]$ denotes the Whitehead product of i_1 and i_2. If $\alpha\in\pi_{3n-1}(X^*: \sigma^{2n}, X)$, α is represented by a map f: $(E^{3n-1}; E_+^{3n-2}, E_-^{3n-2})\rightarrow(X^*: \sigma^{2n}, X)$. Let p be an interior point of σ^{2n}, and let C^{n-1} be the inverse image of p by f, then we have $\partial C^{n-1} = D^{n-2} = C^{n-1}\frown E_+^{n-2}$. Select a point O in E_+^{3n-2} such that $OD^{n-2} = L^{n-1} \subset E_+^{3n-2}$ and $OC^{n-1} = K^n$, then

$$\partial f(K^n) = f(\partial K^n) = -f(L^{n-1})\subset\sigma^{2n}.$$

Thus $f(K^n)$ represents an element of $\mathfrak{H}_n(X^*, \sigma^{2n})$. If S_1^n, S_2^n are two generators of $\mathfrak{H}_n(X^*, \sigma^{2n})$, we have

$$f(K^n)\backsim a_1S_1^n+a_2S_2^n,$$

where (a_1, a_2) is a pair of integers. Then it is verified that if $f\sim g$, we have $(a_1, a_2)=(b_1, b_2)$, where $g(K^n)\backsim b_1S_1^n+b_2S_2^n$. Moreover it is also seen that if $a_1=a_2=0$, f is inessential. Thus it is concluded that that (a_1, a_2) is an invariant of homotopy classes. If l is the generator of $\pi_{2n}(\sigma^{2n}, \dot\sigma^{2n})$ and if $\pi_n(X, \dot\sigma^{2n})\cong\pi_n(X) = \{l_1\}+\{l_2\}$, we have

$$\begin{aligned}&[l, l_1]\in\pi_{3n-1}(X^*; *, X),\\ &[l, l_2]\in\pi_{3n-1}(X^*; *, X).\end{aligned}$$

Then it is verified that homotopy invariants of these product are $(1, 0)$, $(0, 1)$ respectively. Furthermore we have

$$\psi\partial[l, l_1] = \psi[\partial l, l] = [[l_1, l_2], l_1],$$

so that two free generators of $\partial\pi_{3n-1}(S_1^n\times S_2^n, S_1^n\vee S_2^n)$ are represented by two triple Whitehead products. This is a result announced by Blakers and Massey [7].

Now we prove $\pi_4(N^4)=0$, making use of this. Let us consider the injection map i:

$$\pi_4(S_1^2 \vee S_2^2 \vee \cdots \vee S_\rho^2) \xrightarrow{i} \pi_4(N^4).$$

Then it is seen that i is onto and that the kernel of i is generated by $e_{ij}\cdot\eta$ and triple Whitehead products $[a_i[a_j, a_k]]$, where $e_{ij}\cdot\eta$ is represented by a map $f: S^4 \xrightarrow{\eta} S^3 \xrightarrow{e_{ij}} S_i^2 \vee S_j^2$. This proves that the generators of the kernel of i is the same as those of $\pi_4(S_1^2 \vee \cdots \vee S_\rho^2)$, so that we have $\pi_4(N^4) = 0$.

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