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§ 1. Introduction

This paper is a continuation of my previous paper [14] of the same title, where I gave detailed accounts of homotopy types of a $A^2$-complex and of some special $A^2$-complex. They are completely determined by their cohomology groups, some homomorphisms $\mu$, $\Delta$, defined among them, and Steenrod’s squaring operations, so that their homotopy invariants should be also determined by them. Homotopy type problems and related subjects are dealt with in this paper.

First, the exact sequence of J.H.C. Whitehead [4] is generalized in order to compute formally $\Gamma_{n+1}(0)$, $\Gamma_{n+2}(0)$ (§ 3) under some restrictions in dimensions. In case of cohomotopy groups this is accomplished by M. Nakaoka to get a generalization of the exact sequence of Spanier (refer to [15]). Utilizing this, we can compute up to group extension homotopy groups $\pi_{n+1}(P)$, $\pi_{n+2}(P)$ of a polyhedron $P$ with vanishing homotopy groups $\pi_i(P) = 0$ for each $i < n$. This calculation suggests us to compute combinatorially $\pi_{n+1}(P)$, $\pi_{n+2}(P)$ of an $A^2$-complex and also $\pi_{n+2}$ of a special kind of polyhedron (see § 6). The study of reduced complexes in my previous paper and of J.H.C. Whitehead’s secondary boundary operations (see § 4) enables us to solve thoroughly how $\pi_{n+1}(P)$, $\pi_{n+2}(P)$ of an $A^2$-complex are computed by the aids of homology groups, of Steenrod’s squaring homomorphisms, and of some homomorphisms $\mu$, $\Delta$, (see § 5), and also to get the way of calculation of $\Gamma_{n+1}(P)$, $\Gamma_{n+2}(P)$. In § 6 we restate concisely the results of my previous paper [14] through this sequence.

Until § 6 we assume $n > 3$, or $n > 2$.

We proceed to attack more complicated lower dimensional cases related to the subjects discussed till § 6. Recently Hirsch [16] gave a very elegant expression of the kernel of the homomorphism $i: \pi_3(P) \rightarrow H_3(P)$, where $P$ is a simply connected polyhedron without 2-dimensional

* Yukawa Fellow.

I have been informed of the existence of Hilton’s paper on $\pi_{n+1}(P)$ through Chang’s paper.
torsion. In § 7 we calculate the fourth homotopy group of a polyhedron whose third homotopy group vanishes besides Hirsch's assumptions on $P$. This is a step towards the solution of homotopy type problems of lower dimensional cases. Finally, calculations such as $\pi_4$ of a polyhedron whose third homotopy group vanishes besides Hirsch's assumptions on $P$. This is a step towards the solution of homotopy type problems of lower dimensional cases. Finally, calculations such as $\pi_4$ of a polyhedron $P$, which are utilized in course of our discussions, are studied in preparation for my forthcoming paper. I hope, I shall come back shortly to the homotopy type of a five dimensional simply connected polyhedron in connection with brilliant results obtained recently by N. Shimada.

I would like to express my sincere gratitude to my teacher Astuo Komatsu for his constant encouragements during this study, and I thank Mr. M. Nakaoka for his kind criticisms.

§ 2. Generalization of the exact sequence of J.H.C. Whitehead

Let $K$ be a connected CW-complex and let $e^0$ be a 0-cell, which is taken to be a base point of all the homotopy groups. Let

$$C_p(q) = \pi_{p+q}(K^r, K^{r-1}) \quad \text{and} \quad A_p(q) = \pi_{p+q}(K^r),$$

where $K^r$ denotes the $r$-skelton of $K$. Then, let us consider the following sequence designated by $(C, A)(q)$

$\xrightarrow{j_p(q)} A_p(q) \xrightarrow{\beta_p(q)} C_p(q) \xrightarrow{\partial_p(q)} A_{p-1}(q) \xrightarrow{j_{p-1}(q)} C_{p-1}(q) \xrightarrow{\beta_{p-1}(q)} A_{p-2}(q) \xrightarrow{j_{p-2}(q)} C_{p-2}(q) \rightarrow \ldots$

where $\beta_p(q)$ is the homotopy boundary operator and $j_p(q)$ is the relativization. Evidently $\beta_p(q) \cdot j_p(q) = 0$. If we put $j_{p-1}(q) \cdot \beta_p(q) = \partial_p(q)$, we have three groups $H_p(q)$, $\Gamma_p(q)$, $\Pi_p(q)$ as follows;

$$H_p(q) = Z_p(q) / B_p(q),$$

where $Z_p(q)$ is the kernel of $\partial_p(q)$, and $B_p(q)$ is the image of $\beta_{p+1}(q)$.

$$\Gamma_p(q) \text{ is a kernel of } j_p(q);$$

$$\Pi_p(q) = A_p(q) / D_p(q),$$

where $D_p(q)$ is the image of $\beta_{p+1}(q)$. As J.H.C. Whitehead defined the exact sequence in [4], we have the following exact sequence $\sum_p(K)$ with three operations $\mathcal{B}$, $\mathcal{S}$, $\mathcal{N}$,

$$\ldots \rightarrow H_{p+1}(q) \xrightarrow{\mathcal{B}_{p+1}(q)} \Gamma_p(q) \xrightarrow{\mathcal{S}_p(q)} \Pi_p(q) \xrightarrow{\mathcal{N}_p(q)} H_p(q) \xrightarrow{\mathcal{B}_p(q)} \Gamma_{p-1}(q) \rightarrow \ldots$$

It is obvious that $\sum_0(K)$ is the sequence of J.H.C. Whitehead used in [4]. It is also verified through an analogous way as is shown in [4] that $\sum_p(K)$ is a homotopy invariant of $K$. Then we have several formal properties:

**Theorem 1.**

1. $\rho_p(q) : \Gamma_p(q) \cong \Pi_{p-1}(q + 1),$
2. $\Pi_p(0) \cong \pi_p(K)$, where $\pi_p(K)$ denotes the $p$-dimensional homotopy group of $K$. 
(2.3) \( \Gamma_p(q) = 0 \) for each \( p \leq n \),
\( \Pi_p(q) = 0 \) for each \( p \leq n-1 \),
if \( \pi_i(K) \) vanishes for each \( i < n \).

(2.4) If \( K \) is aspherical in dimension less than \( n \) and if \( q \leq p-4 \) and \( q \leq \text{Min.} (p-3, n-1)-2 \), we have
\[
\rho_p(q); \delta_p(K, p^{n+q}) \cong H_p(q),
\]
where \( \delta_p(K, p^{n+q}) \) is the \( p \) dimensional homology group of \( K \) with the \((p+q)\) dim. homotopy group \( p^{n+q} \) of \( p \)-dim. sphere as its coefficient group.

Proof. (2.1), (2.2) are direct consequences of definition. (2.3) can be easily verified from the fact that \( K \) is of the same homotopy type as a complex, the \((n-1)\)-skelton of which is a single point. In order to prove (2.4) we show that \( C_p(q) \cong \mathbb{C}_p(K, p^{n+q}) \), if \( q \leq \text{Min.} (p-2, n-1)-2 \) and \( q \leq p-3 \), where \( \mathbb{C}_p(K, p^{n+q}) \) denotes the \( p \)-dim. chain group with \( p^{n+q} \) as its coefficient group. Let \( t_i \) be an arc joining in \( K^{p-1} \) the base point \( e^0 \) to a point on the boundary of a \( p \)-cell \( \sigma_i \) and let \( \hat{e}^p \) be the union \( \bigsqcup (t_i + \sigma_i^p) \). Then we have a triad \( (K^p; \hat{e}^p, K^{p-1}) \). Let us consider the sequence
\[
\pi_i(\hat{e}^p) \to \pi_i(K^{p-1}) \to \pi_i(\hat{e}^p) \to \pi_{i-1}(\hat{e}^p) \to
\]
If \( i \leq p-2 \), we have \( \pi_i(K^{p-1}) \cong \pi_i(\hat{e}^p) \) and \( \pi_i(K^{p-1}) \cong \pi_i(K) \), so that \( \pi_i(K^{p-1}, \hat{e}^p) = 0 \) for \( i \leq \text{Min.} (p-2, n-1) \). In virtue of a main theorem of triad it is seen that \((K^p; \hat{e}^p, K^{p-1}) \) is \((p+q+1)\)-connected if \( p+q+1 \leq \text{Min.} (p-2, n-1)+p-1 \). Therefore, we have
\[
\pi_{p+q}(\hat{e}^p, \hat{e}^p) \cong \pi_{p+q}(K^p, K^{p-1}) \]
if \( q \leq \text{Min.} (p-2, n-1)-2 \). Furthermore, if \( q \leq p-3 \), we have \( \pi_{p+q}(\hat{e}^p, \hat{e}^p) \cong \pi_{p+q-1}(\bigvee S_i^{p-1}) \cong \sum \pi_{p+q-1}(S_i^{p-1}) \cong \sum \pi_{p+q} (S_i^p) \), where the last isomorphism is established by suspension. Thus we have \( i_p \); \( \pi_{p+q}(K^p, K^{p-1}) \cong \mathbb{C}_p(K, p^{n+q}) \) if \( q \leq \text{Min.} (p-2, n-1)-2 \) and \( q \leq p-3 \). Now let us consider the following diagram,
\[
\begin{array}{ccccccccc}
\pi_{p+q+1}(K^{p+1}, K^p) & \xrightarrow{\delta_{p+1}(q)} & \pi_{p+q}(K^p) & \xrightarrow{i_p(q)} & \pi_{p+q}(K^p, K^{p-1}) & \xrightarrow{\delta_p(q)} & \pi_{p+q-1}(K^{p-1}) & \xrightarrow{i_{p-1}(q)} & \\
\mathbb{C}_{p+1}(K, (p+1)^{n+q+1}) & \xrightarrow{\delta_{p+1}(q)} & \mathbb{C}_p(K, p^{n+q}) & \xrightarrow{\delta_p(q)} & & & & & \\
\end{array}
\]
\[
\to \pi_{p+q-1}(K^{p-1}, K^{p-2}) \to \mathbb{C}_p(K, (p-1)^{n+q-1})
\]
If \( q \leq \text{Min.} (p-3, n-1)-2 \) and \( q \leq p-4 \), \( i_{p-1}, i_p, \) and \( i_{p+1} \) are all isomorphisms onto. As is easily seen, we have
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\[ \partial_{p+1}(q) = j_p(q) \cdot \beta_{p+1}(q) = \tilde{i}_{p+1} \cdot \tilde{\partial}_{p+1}(q) \cdot i_{p+1} \]

\[ \partial_p(q) = j_{p-1}(q) \cdot \beta_p(q) = \tilde{i}_{p-1} \cdot \tilde{\partial}_p(q) \cdot i_p. \]

Thus, \( \partial_p(q) \), \( \partial_{p+1}(q) \) may be regarded as ordinary homological boundary operators. Notice that coefficient groups are identified by isomorphisms by suspension, when homological boundary operators \( \tilde{\partial}_{p+1}(q), \tilde{\partial}_p(q) \) are considered. This proves

\[ \sigma_p(q) : \tilde{\Phi}_p(K, p^{p+1}) \simeq H_p(q). \]

§ 3. Formal calculations of \( \Gamma_{n+1}(0), \Gamma_{n+2}(0) \).

In this section we assume that \( K \) is a connected complex aspherical in dimensions less than \( n \). Then we have

\[ \Gamma_{n-1}(0) = 0, \quad \Gamma_n(0) = 0 \quad \text{from} \quad (2.3). \]

It is seen from (2.4) that if \( n \geq 5 \), we have

\[
\begin{align*}
H_n(0) & \simeq \tilde{\Phi}_n(K, I), \\
H_{n+1}(0) & \simeq \tilde{\Phi}_{n+1}(K, I), \\
H_{n+2}(0) & \simeq \tilde{\Phi}_{n+2}(K, I), \\
H_{n+3}(0) & \simeq \tilde{\Phi}_{n+3}(K, I),
\end{align*}
\]

where \( I \) denotes the group of integers. Let us consider the sequence \( \Sigma_1(K) \):

\[
\cdots \rightarrow \Gamma_n(1) \rightarrow \Pi_n(1) \rightarrow H_n(1) \rightarrow \Gamma_{n-1}(1) \rightarrow \cdots
\]

then we have \( \tilde{\Phi}_n(1) : \Pi_n(1) \simeq H_n(1) \) from \( \Gamma_n(1) = \Gamma_{n-1}(1) = 0 \). Since \( \rho_{n+1}(0) : \Gamma_{n+1}(0) \simeq \Pi_n(1) \) from (2.1) and since \( \sigma_n(1) : \tilde{\Phi}_n(K, I_2) \simeq H_n(1) \), for \( n \geq 6 \), from (2.4), we have

\[
\sigma_n^{-1}(1) \tilde{\Phi}_n(1) \rho_{n+1}(0) ; \quad \Gamma_{n+1}(0) \simeq \tilde{\Phi}_n(K, I_2),
\]

if \( n \geq 6 \), where \( I_2 \) is the group of integers reduced mod. 2.

Next we calculate \( \Gamma_{n+2}(0) \) by the sequence \( \Sigma_2(K) \). In the sequence \( \Sigma_1(K) \)

\[
\begin{array}{c}
H_{n+1}(1) \xrightarrow{\rho_{n+2}(1)} \Gamma_{n+1}(1) \xrightarrow{\sigma_{n+1}(1)} \Pi_{n+1}(1) \xrightarrow{\tilde{\Phi}_{n+1}(1)} H_{n+1}(1) \xrightarrow{\rho_{n+2}(1)} \Gamma_n(1) \\
\end{array}
\]

we have

\[
\begin{align*}
\sigma_{n+2}(1) : & \tilde{\Phi}_{n+2}(K, I_2) \simeq H_{n+2}(1), \quad \text{for} \quad n \geq 6, \\
\sigma_{n+1}(1) : & \tilde{\Phi}_{n+1}(K, I_2) \simeq H_{n+1}(1), \quad \text{for} \quad n \geq 6, \\
\Gamma_n(1) = 0 & \quad \text{from} \quad (2.3), \\
\rho_{n+2}(0) : & \Gamma_{n+2}(0) \simeq \Pi_{n+1}(1) \quad \text{from} \quad (2.1).
\end{align*}
\]

Let us denote by \( A \) \( \Gamma_{n+1}(1) \mid \tilde{\Phi}_{n+2}(1) \cdot \sigma_{n+2}(1) \tilde{\Phi}_{n+2}(K, I_2) \), then we have from the exactness of \( \Sigma_1(K) \)

\[
\tilde{\Phi}_{n+1}(1) ; \rho_{n+2}(0) \Gamma_{n+2}(0) \tilde{\Phi}_{n+1}(1) A \simeq \sigma_{n+1}(1) \tilde{\Phi}_{n+1}(K, I_2).
\]
In order to calculate $\Gamma_{n+1}(1)$ involved in $A$ we consider $\Sigma_k G\delta \rightarrow \Gamma$. (2)

where we have $\Sigma_n (2): \Pi_n (2) \approx H_n (2)$ from $\Gamma_n (2) = \Gamma_{n-1} (2) = 0$. Since $\Gamma_{n+1} (1) = \Pi_n (2)$ from (2.1) and since $H_n (2) = \delta_n (K, I_2)$ for $n \geq 7$, we have

$$\sigma_n (2) \cdot \Sigma_n (2) \cdot \rho_{n+1} (1); \Gamma_{n+1} (1) = \delta_n (K, I_2).$$

**Theorem 2.** In a connected complex $K$ aspherical in dimensions less than $n$ we have

$$\Gamma_n (0) = 0,$$

$$\sigma_n^{-1} (1) \cdot \Sigma_n (1) \cdot \rho_{n+1} (0); \Gamma_{n+1} (0) = \delta_n (K, I_2) \tag{3.6} \text{ for } n \geq 6,$$

$$\Sigma_{n+1} (1); \rho_{n+2} (0) \cdot \Gamma_{n+2} (0) | \Sigma_{n+1} (1) \cdot \rho_{n+2} (2) \cdot \Sigma_{n+1} (2) \cdot \rho_{n+1} (2); \delta_n (K, I_2) | \delta_{n+2} (1) \cdot \sigma_{n+2} (1) \cdot \delta_{n+2} (K, I_2) \approx \delta_{n+1} (1) \cdot \delta_{n+1} (K, I_2) \tag{3.7} \text{ for } n \geq 7.$$

For the sake of brevity we shall often use the way of expression

$$\Gamma_{n+2} (0) | \delta_n (I_2) | \delta_{n+2} (I_2) \approx \delta_{n+1} (I_2) \tag{3.8}$$

for (3.8), abbreviating all the isomorphisms in (3.8). As we stated in the introduction, Theorem 2 is established in the sense that it helps us in suggesting the complete solution of computations of homotopy groups and of homotopy type problems. It should be noted that we shall give full accounts of $\Gamma_{n+1} (0), \Gamma_{n+2} (0)$ without restriction as to dimension in the sequel, utilizing reduced complexes together with the study of $\delta$-operation.

§ 4. $\delta_{n+2} (0), \delta_{n+2} (1)$, and $\delta_{n+3} (0)$

i) $\delta_{n+2} (0)$ Let $K$ be a $A_2$-complex, and let $S_{q_{n-2}}: \delta^n (K, I_2) \rightarrow \delta^{n+2} (K, I_2)$ be Steenrod's Squaring homomorphism. As is shown in [4] by J.H.C. Whitehead, we have

$$\lambda: \Gamma_{n+1} (0) \approx \delta_n (K, I_2) \tag{4.1} \text{ for } n > 2,$$

$$\mu: H_{n+2} (0) \approx \delta_{n+2} (K, I_2) \tag{4.2} \text{ for } n > 2.$$

Then $\nu = \lambda \cdot \delta_{n+2} (0) \cdot \mu^{-1}: \delta_{n+2} (I) \rightarrow \delta_{n+2} (I_2)$ for $n > 2$ can be determined by Steenrod's operation as follows. If $x \in \delta^n (I_2)$ and $y \in \delta_{n+2} (I)$, we have

$$KI [\nu y, x] = KI [y, S_{q_{n-2}} x],$$

where $KI$ denotes Kronecker index, and, as to group multiplication, two groups $I, I_2$ are paired to $I_2$. From (4.1) $\nu y$ may be regarded as an element in $\text{Hom} [\delta^n (K, I_2), I_2]$ such that

$$\nu y: \ x \rightarrow KI [y, S_{q_{n-2}} x].$$
Therefore \( \nu \) is determined in the sense that \( \nu y \) represents an element in \( \mathbb{S}_n(K, I_2) \). (Refer to [11] or [12])

ii) \( \mathbb{S}_{n+2}(I) \) Let \( K \) be the same as before. From (3.5) we have

\[ \lambda = \sigma_n^{-1}(2) \cdot \rho_{n+1}(1) : \Gamma_{n+1}(1) \approx \mathbb{S}_n(K, I_2), \]

and put \( \mu = \sigma_{n+2}(1) : \mathbb{S}_{n+2}(I_2) \rightarrow \mathbb{S}_n(I_2) \) can be also determined analogously by Steenrod’s operation. Two cases i), ii) can be easily verified by the aid of reduced complexes.

iii) \( \mathbb{S}_{n+3}(0) \) Since no account of \( \mathbb{S}_{n+3}(0) \) is in print and since it is applied to the homotopy type problem discussed in the sequel, we give here detailed account of it in case where \( K \) is such a complex as was dealt with in [14]. \( K \) is of the same homotopy type as the following complex \( L \).

\[
L^{n+2} = (S^n \cup e_1^{n+1}) + \cdots + (S^n \cup e_k^{n+1}) + (S^n \cup e_k^{n+1} \cup e_{k+1}^{n+1}) + \cdots + (S^n \cup e_{k+1}^{n+1} \cup e_{k+2}^{n+1}) + \cdots + (S^n \cup e_{k+2}^{n+1} \cup e_{k+3}^{n+1}) + \cdots + (S^n \cup e_{k+3}^{n+1} \cup e_{k+4}^{n+1}) + \cdots + (S^n \cup e_{k+4}^{n+1} \cup e_{k+5}^{n+1}) + \cdots ,
\]

where \( e_i^{n+1} (i = 1, \ldots, k) \) is attached to \( S_i^n \) by a map \( f_i : \partial e_i^{n+1} \rightarrow S_i^n \) of odd degree \( \sigma_i \), \( e_i^{n+2} (i = k+1, \ldots, k+l) \) is attached to \( S_i^n \) by a map \( g_i : \partial e_i^{n+2} \rightarrow S_i^n \) of degree \( 2^p_i \), and \( e_i^{n+3} (i = k+1, \ldots, \kappa) \) is attached to \( S_i^n \) by an essential map \( \eta_i : \partial e_i^{n+3} \rightarrow S_i^n \). \( L \) is constructed by attaching to \( L^{n+2} \) a number of \((n+3)\) cells \( e_i^{n+3} (i = 1, \ldots, \alpha) \) by \( \beta e_i^{n+3} = \sum_{j=1}^\alpha \chi_{ij} S_j^{n+2} + \sum_{j=1}^{n+1} \mu_{ij} \omega_j + \sum_{j=1}^{n+1} \nu_{ij} \nu_j + \sum_{j=1}^{n+1} \nu_{ij} \nu_j \omega_j \), where \( S_j^{n+3} \) denotes the generator of \( \pi_{n+2}(S_j^{n+3}) \), \( \omega_j (j = k+1, \ldots, \kappa) \) is the free generator of \( \pi_{n+2}(S_j^n \cup e_j^{n+2}) \), and \( v_j (j = k+1, \ldots, k+l) \) is the generator of \( \pi_{n+2}(S_j^n \cup e_j^{n+1} \cup e_j^{n+2}) \) of order two. By definition \( \Gamma_{n+2}(0) \) is the image of the injection \( i : \pi_{n+2}(L^{n+2}) \rightarrow \pi_{n+2}(L^{n+2}) \). Since \( \pi_{n+2}(S_j^n \cup e_j^{n+2}) = 0 (j = 1, \ldots, k) \), \( \Gamma_{n+2}(0) \) is generated by \( v_j (j = k+1, \ldots, k+l) \). A base of \( \mathbb{S}_{n+1}(L, I_2) \) is \( \{ f_2 e_i^{n+1}, (i = k+1, \ldots, k+l) \} \), where \( f_2 \) is the natural homomorphism of a group of cycles mod. 2 into the corresponding homology group with integral group reduced mod. 2 as its coefficient group. A mapping \( \lambda : f_2 e_i^{n+1} \rightarrow v_j \) induces an isomorphism \( \lambda : \mathbb{S}_{n+1}(L, I_2) \rightarrow \Gamma_{n+2}(0) \). If a base of \( \mathbb{S}_{n+3}(L, I) \) is \( \{ j_2 e_i^{n+3}, (i = 1, \ldots, m) \} \), we have

\[
\beta e_i^{n+3} = \sum_{j=k+1}^{k+l} \nu_{ij} V_j .
\]

If a base of \( \mathbb{S}_{n+1}(L, I_2) \) is \( \{ f_2 e_i^{n+1}, (j = k+1, \ldots, k+l) \} \), and if a base of \( \mathbb{S}_{n+3}(K, I_2) \) is \( \{ j_2 e_i^{n+3}, (j = 1, \ldots, \alpha) \} \), we can choose them such that

\[
KI \left[ j_2 e_i^{n+1}, j_2 e_j^{n+1} \right] = \delta_{ij} , \quad KI \left[ j_0 e_i^{n+3}, j_2 e_j^{n+3} \right] = \delta_{ij} .
\]

\( \delta_{ij} \) is 0 or the generator of \( I_2 \) according as \( i \neq j \) or \( i = j \). If \( x \in \mathbb{S}_{n+1}(I_2), y \in \mathbb{S}_{n+3}(I) \), we have
\[ x = \sum_{j=1}^{k+1} p_j j_2 \varphi_j^{q+1}, \quad p_j \in I_2, \]

\[ y = \sum_{j=1}^{m} q_j j_0 \varphi_j^{q+3}, \quad q_j \in I. \]

Then it is seen that

\[ S_{q_{n-1}} x = \sum_j p_j^2 S_{q_{n-1}} j_2 \varphi_j^{q+1} = \sum_j p_j^2 \sum_{i=1}^{a} \nu_{ij} j_2 \varphi_i^{q+3}, \]

so that we have

\[ KI[y, S_{q_{n-1}} x] = \sum_{j=1}^{k+1} \sum_{i=1}^{m} q_j p_j^2 \nu_{ij} \in I_2. \]

By definition of \( \mathcal{B}_{n+3}(0) \) we have

\[ \lambda^{-1} \mathcal{B}_{n+3}(0) y = \lambda^{-1} \left( \sum_{j=1}^{m} q_j \beta e_j^{q+3} \right) = \lambda^{-1} \left( \sum_{j=1}^{m} q_j \sum_{i=1}^{k+1} \nu_{ij} v_j \right) = \sum_{j=1}^{k+1} \sum_{i=1}^{m} q_i \nu_{ij} \beta e_j^{q+1}, \]

so that

\[ KI[\lambda^{-1} \mathcal{B}_{n+3}(0) y, x] = \sum_{j=1}^{k+1} \sum_{i=1}^{m} q_i p_j \nu_{ij} \in I_2. \]

This proves \( KI[y, S_{q_{n-1}} x] = KI[\lambda^{-1} \mathcal{B}_{n+3}(0) y, x] \).

Since \( \lambda^{-1} \mathcal{B}_{n+3}(0) y : x \rightarrow KI[y, S_{q_{n-1}} x] \in I_2 \) may be regarded as an element of \( \mathcal{S}_{n+1}(L, I_2) \), \( \mathcal{B}_{n+3}(0) \) can be determined effectively by squaring homomorphism \( S_{q_{n-1}} \). The sequence of Whitehead \( \Sigma_0(L) \) is a homotopy invariant so that all the discussions are available for \( K \) as well.

### §5. Computation of \( \pi_{n+1}, \pi_{n+2} \)

In this section we assume that \( K \) is a \( A_n \)-complex. Let us consider the sequence

\[
\cdots \to H_{n+2}(0) \mathcal{B}_{n+2}(0) \to I_{n+1}(0) \mathcal{S}_{n+1}(0) \to \Pi_{n+1}(0) \mathcal{S}_{n+1}(0) \to H_{n+1}(0) \to 0
\]

It is seen that \( \mathcal{S}_{n+1}(0) \) is onto and that the kernel of \( \mathcal{S}_{n+1}(0) \) is isomorphic to \( \Gamma_{n+1}(0) \mathcal{S}_{n+2}(0) \rho^{-1} \mathcal{S}_{n+2}(K, I) \) by \( \mathcal{S}_{n+1}(0) \). By definition we have \( \lambda : \Gamma_{n+1}(0) \cong \mathcal{S}_{n}(K, I_2) \) for \( n > 2 \). Thus the kernel of \( \mathcal{S}_{n+1}(0) \) is isomorphic to \( \mathcal{S}_{n}(K, I_2) \varphi \mathcal{S}_{n+2}(K, I) \) by \( \mathcal{S}_{n+1}(0) \lambda \), so that \( \pi_{n+1}(K) \), isomorphic to \( \Pi_{n+1}(0) \), is a group extension of \( \mathcal{S}_{n}(K, I_2) \) by \( \mathcal{S}_{n+2}(K, I) \) by \( \mathcal{S}_{n+1}(K, I) \). Thus \( \pi_{n+1} \) is determined combinatorially up to group extension.

Now we proceed to show how \( \pi_{n+1}(K) \) is calculated completely. This is treated by Chang in his exciting paper [5], but the method of his is different from mine. To do this we apply a reduced complex obtained by Chang. Without loss of generality we assume that \( K \) is a reduced complex. For convenience of calculation in the sequel it seems desirable for us to put down here nine types of elementary polyhedra;
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i) $Q^i = S^n$, $Q^{i+1} = S^{n+1}$, $Q^{n+1} = S^{n+2}$,

ii) $Q_2 = S^n \cup e^{n+1}$, where $e^{n+1}$ is attached to $S^n$ by a map $f$; $\sigma e^{n+1} \rightarrow S^n$ odd degree $\sigma$, a power of a prime,

iii) $Q_3 = S^n \cup e^{n+2}$, where $e^{n+2}$ is attached to $S^n$ by an essential map $f : e^{n+2} \rightarrow S^n$,

iv) $Q_4 = (S^n \cup S^{n+1}) \cup e^{n+2}$, where $e^{n+2}$ is attached to $(S^n \cup S^{n+1})$ by a map $f : e^{n+2} \rightarrow S^n \setminus S^{n+1}$ of the form $a + b$, where $a$ denotes an essential map of $e^{n+2}$ onto $S^n$ and $b$ maps $e^{n+2}$ onto $S^{n+1}$ with degree $2^n$,

v) $Q_5 = S^n \cup e^{n+1} \cup e^{n+2}$, where $e^{n+1}$ is attached to $S^n$ by a map $f : e^{n+1} \rightarrow S^n$ of degree $2^n$ and $e^{n+2}$ is attached to $S^n$ by an essential map of $e^{n+2}$ onto $S^n$,

vi) $Q_6 = (S^n \cup S^{n+1}) \cup e^{n+2}$, where $e^{n+1}$ is attached to $(S^n \cup S^{n+1})$ by a map $f : e^{n+2} \rightarrow S^n$ of degree $2^n$,

vii) $Q_7 = S^n \cup e^{n+1}$, where $e^{n+1}$ is attached to $S^n$ by a map $f : e^{n+1} \rightarrow S^n$ of odd degree $\sigma$, a power of prime,

viii) $Q_8 = S^n \cup e^{n+2}$, where $e^{n+2}$ is attached to $S^n$ by a map $f : e^{n+2} \rightarrow S^n$ of degree $2^n$,

ix) $Q_9 = S^{n+1} \cup e^{n+2}$, where $e^{n+2}$ is attached to $S^{n+1}$ by a map $f : e^{n+2} \rightarrow S^{n+1}$ of degree $2^n$.

A $A^n_2$-complex is a complex which consists of a collection of nine types of elementary polyhedra. A base of $\S_n(K, I_2)$ is represented by $n$ cells belonging to $Q^i_1, Q^i_3, Q^i_4, Q^i_5, Q^i_6, Q^i_7$, which are denoted by $e^{i,1}_i, e^{i,3}_i, e^{i,4}_i, e^{i,5}_i, e^{i,6}_i, e^{i,7}_i$, where $i$ represents the number of $n$ cells. A base of $\S_{n+2}(K, I_2)$ is represented by $(n+2)$ cells belonging to $Q^{n+2}_1, Q^3_1, Q^3_5, Q^3_6, Q^3_8, Q^3_9$, which are denoted by $e^{3,1}_i, e^{3,3}_i, e^{3,5}_i, e^{3,6}_i, e^{3,8}_i, e^{3,9}_i, e^{3,10}_i$. As we consider $\nu : \S_{n+2}(K, I_2) \rightarrow \S_{n}(K, I_2)$ in §4 ii), it is seen that

$$
\nu_j^2 e^{i,1}_i = 0, \quad \nu_j^2 e^{i,3}_i = j^2 e^{i,3}_i, \quad \nu_j^2 e^{i,4}_i = j^2 e^{i,4}_i, \quad \nu_j^2 e^{i,5}_i = j^2 e^{i,5}_i, \quad \nu_j^2 e^{i,6}_i = j^2 e^{i,6}_i, \quad \nu_j^2 e^{i,7}_i = j^2 e^{i,7}_i, \quad \nu_j^2 e^{i,8}_i = j^2 e^{i,8}_i, \quad \nu_j^2 e^{i,9}_i = j^2 e^{i,9}_i, \quad \nu_j^2 e^{i,10}_i = 0.
$$

Thus $\S_n(K, I_2)|\nu\S_{n+2}(K, I_2)$ is freely generated by $j^2 e^{i,1}_i, j^2 e^{i,3}_i$.

It is easily verified that $\pi_{n+1}(K) \cong \sum I^i_{i+1}(Q^i)$ for $n \geq 2$, where $i$ denotes the type and $j$ represents the number of $Q_i$. Therefore $A = \sum I^i_{i+1}(Q^i) + I^i_{i+1}(Q^i)$ is a direct factor of $\pi_{n+1}(K)$. Since $\pi_{n+1}(Q^i_{n+1}) = I_1$ and $\pi_{n+1}(Q^i_{n+2}) = I_2$, $A$ is isomorphic to $\S_n(I_2)|\nu\S_{n+2}(I_2)$.

Now we put down here the homotopy groups of $Q^{n+1}(j = 1, 2, 3, 4, 5, 6, 7, 8, 9)$ as follows:

$$
\begin{align*}
\pi_{n+1}(Q^1_{n+1}) &= I_1, \quad \pi_{n+1}(Q^{2}_{n+1}) = 0, \quad \pi_{n+1}(Q^{3}_{n+1}) = 0, \\
\pi_{n+1}(Q^{2}_{n+2}) &= I_1, \quad \pi_{n+1}(Q^{3}_{n+2}) = I_1, \quad \pi_{n+1}(Q^{4}_{n+2}) = 0, \\
\pi_{n+1}(Q^{3}_{n+3}) &= I_1, \quad \pi_{n+1}(Q^{4}_{n+3}) = I_1, \quad \pi_{n+1}(Q^{5}_{n+3}) = I_1, \\
\pi_{n+1}(Q^{5}_{n+4}) &= I_1, \quad \pi_{n+1}(Q^{6}_{n+4}) = I_1, \quad \pi_{n+1}(Q^{7}_{n+4}) = I_1, \\
\end{align*}
$$

(5.1)
From this table it is seen that the rank of \( \pi_{n+1}(K) \) is equal to that of \( \mathfrak{S}_{n+1}(K, I) \) so that we have

\[
\pi_{n+1}(K) \simeq \mathfrak{S}_n(I_2) | \nu \mathfrak{S}_{n+2}(I_2) + B + \xi
\]

where \( B \) denotes the free group of \( \mathfrak{S}_{n+1}(K, I) \). Let us determine \( \xi \). Let \( T \) be the torsion group of \( \mathfrak{S}_{n+1}(I) \), and let \( \{ h_1, h_2, \ldots, h_a \} \) be the invariant system of \( (n+1) \) dimensional torsion coefficients, where \( h_i \) is a power of a prime. From (5.1) we have \( \xi \simeq C + \mathfrak{Y} \), where \( C \) denotes the subgroup of \( T \) consisting of all the cyclic groups of odd degree \( h_i \). Choosing even torsion coefficients \( \{ h_1, \ldots, h_a \} \) out of the system \( \{ h_1, \ldots, h_a \} \), we consider the following operation with respect to \( h_i \) \((\nu = 1, \ldots, \alpha)\),

\[
\mathfrak{S}_{n+1}(K, I) \xleftarrow{\Delta_{h_i\nu}} \mathfrak{S}_{n+2}(K, I_{h_i\nu}) \xrightarrow{\mu_{h_i\nu}, 2} \mathfrak{S}_{n+2}(K, I_2) \xrightarrow{\mathfrak{Y}_{h_i, 2}(1)} \mathfrak{S}_n(K, I_2).
\]

Let us define two homomorphisms

\[
\Delta_p : \mathfrak{S}_a(K, I_p) \to \mathfrak{S}_a(K, I), \\
\mu_{p, q} : \mathfrak{S}_a(K, I_p) \to \mathfrak{S}_a(K, I_q).
\]

The first operation \( \Delta_p \) is \( \frac{1}{p} \cdot \). Let \( x \in \mathfrak{S}_a(K, I_p) \) and let \( x' \) be a representative of \( x \), then \( \frac{q}{(p, q)} \cdot x' \) is a cycle mod. \( q \). \( \mu_{p, q} \cdot x \) is represented by \( \frac{q}{(p, q)} \cdot x' \). If \( \nu_{\Delta_{h_i\nu}} = \mathfrak{Y}_{n+2}(1) \cdot \mu_{h_i\nu, 2}, \)

the kernel of \( \nu_{\Delta_{h_i\nu}} \) does not contain \( j_2 e_i^{n+2}, j_2 e_i^{n+2} \). Putting \( D = \bigcup_{h_i\nu} \Delta_{h_i\nu} \cdot \nu_{h_i\nu}^{-1}(0) \), \( D \) is a subgroup of \( T \), which is generated by \( j_0 e_i^{n+1} \). Together with \( \pi_{n+1}(Q_{\nu, i}) = I_{2v_i} \) we have \( \mathfrak{Y} \simeq D + \mathfrak{Y} \).

Let the invariant system of \( T | C + D \) be represented by \( \{ 2^i, 2^i, \ldots, 2^i \} \) and let \( E \) be an abelian group, the invariant system of which is \( \{ 2^{i+1}, 2^{i+1}, \ldots, 2^{i+1} \} \). Since \( \pi_{n+1}(Q_{\nu, i}) = I_{2v_i+1}, \pi_{n+1}(Q_{\nu, i}) = I_{2v_i+1} \), we have \( \mathfrak{Y} \simeq E \). In virtue of \( \pi_{n+1}(Q_{\nu, i}) = \pi_{n+1}(Q_{\nu, i}) = \pi_{n+1}(Q_{\nu, i}) = \pi_{n+1}(Q_{\nu, i}) = 0 \) it is concluded that

\[
\pi_{n+1}(K) \simeq \mathfrak{S}_n(K, I_2) | \nu \mathfrak{S}_{n+2}(K, I_2) + B + C + D + E.
\]

**Theorem 3.** The \((n+1)\)-dimensional homotopy group of a \( A^2 \)-complex can be calculated combinatorially by homology groups; \( \mathfrak{S}_n, \mathfrak{S}_{n+1}, \mathfrak{S}_{n+2}, \) by homomorphisms; \( \mu, \Delta, \) and by \( \mathfrak{Y}_{n+2}(1) : \mathfrak{S}_{n+2}(I_2) \to \mathfrak{S}_n(I_2) \).

Now we give a more detailed account of Theorem 2, 3.9 and then compute \( \pi_{n+2}(K) \) combinatorially. First \( \Gamma_{n+2}(Q_i) (i = 1, 2, \ldots, 9) \) are calculated as follows:
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(5.2)

\[ \Gamma_{n+2}(Q_i^3) = I_2, \quad \Gamma_{n+2}(Q_i^{r+1}) = I_2, \quad \Gamma_{n+2}(Q_i^{r+2}) = 0 \]

\[ \Gamma_{n+2}(Q_2) = 0, \quad \Gamma_{n+2}(Q_3) = 0, \quad \Gamma_{n+2}(Q_4) = I_2, \]

\[ \Gamma_{n+2}(Q_5) = I_2, \quad \Gamma_{n+2}(Q_6) = I_2 + I_2, \quad \Gamma_{n+2}(Q_7) = I_4, \quad \gamma_{n+2}(Q_8) = I_2 \]

i) Let \( h_i \) be an \( n \)-dimensional even torsion coefficient, a power of 2, and consider \( \varphi_{h_i} = \mu_{\Delta h_i} : \mathcal{D}_{n+1}(I_{h_i}) \xrightarrow{\Delta h_i} \mathcal{D}_n(I) \xrightarrow{\mu_{h_i}} \mathcal{D}_n(I_2) \), then \( \bigvee_{h_i} \varphi_{h_i} \mathcal{D}_{n+1}(I_{h_i}) \) is generated by \( j_2 e_{6}^{n+1} \), \( j_2 e_{6}^{n+2} \).

ii) In virtue of the operation \( \nu, \mathcal{D}_{n}(I_2) \sqcup \bigvee_{h_i} \varphi_{h_i} \mathcal{D}_{n+1}(I_{h_i}) \sqcup \nu \mathcal{D}_{n+2}(I_2) \) is generated by \( j_2 e_{6}^{n+1} \).

iii) Let \( k_i \) be an \((n+1)\)-dimensional even torsion coefficient, a power of 2, and consider the operation \( \psi_{k_i} = \mu_{\Delta k_i} : \mathcal{D}_{n+2}(I_{k_i}) \xrightarrow{\Delta k_i} \mathcal{D}_{n+1}(I) \xrightarrow{\mu_{k_i}} \mathcal{D}_{n+1}(I_2) \), then \( \bigvee_{k_i} \psi_{k_i} \mathcal{D}_{n+2}(I_{k_i}) \) is generated by \( j_2 e_{6}^{n+1} \), \( j_2 e_{6}^{n+2} \), \( j_2 e_{6}^{n+3} \)

\( e_{6}^{n+1} \) denotes an \((n+1)\) cell bounded by an \((n+2)\) cell.

iv) Let us denote by \( B \) an abelian group, which is the direct sum of \( \rho \) integral groups mod. 2, where \( \rho \) is the \((n+1)\)-th Betti number of \( K \).

v) \( \bigvee_{h_i} \varphi_{h_i} \mathcal{D}_{n+1}(I_{h_i}) \sqcup \nu \mathcal{D}_{n+2}(I_2) \) is generated by \( j_2 e_{6}^{n+1} \), \( j_2 e_{6}^{n+2} \).

vi) \( A' = \bigvee_{h_i} \varphi_{h_i} \mathcal{D}_{n+1}(I_{h_i}) \sqcup \nu \mathcal{D}_{n+2}(I_2) \) is generated by \( j_2 e_{6}^{n+1} \).

Let the invariant system of \( A' \) be \( \{2, \ldots, 2\} \) and let us denote by \( \{4, \ldots, 4\} \) that of \( A \).

Then from (5.2) and from \( \Gamma_{n+2}(K) \approx \bigoplus_{i,j} \Gamma_{n+2}(Q_{ij}) \) it is concluded that we have

\[ \Gamma_{n+2}(K) = \bigvee_{h_i} \mathcal{D}_{n+1}(I_{h_i}) \sqcup \bigvee_{k_i} \mathcal{D}_{n+2}(I_{k_i}) + A' \]

\[ + \bigvee_{k_i} \psi_{k_i} \mathcal{D}_{n+2}(I_{k_i}) + A + B \]

Theorem 4. \( \Gamma_{n+2}(K) \) of a \( A^2 \)-complex \( K \) can be calculated combinatorially by homology groups; \( \mathcal{D}_n, \mathcal{D}_{n+1}, \mathcal{D}_{n+2} \), by homomorphisms; \( \mu, \Delta, \gamma \), and by \( \mathcal{B}_{n+2}(1) : \mathcal{D}_{n+2}(I_2) \to \mathcal{D}_n(I_2) \).

Now that \( \Gamma_{n+2}(K) \) has been computed, it is easy to compute \( \pi_{n+2}(K) \). We give a table of \( \pi_{n+2}(Q) \);

(5.3)

\[ \pi_{n+2}(Q_1^n) = I_2, \quad \pi_{n+2}(Q_1^{r+1}) = I_2, \quad \pi_{n+2}(Q_1^{r+2}) = I_2, \]

\[ \pi_{n+2}(Q_2) = 0, \quad \pi_{n+2}(Q_3) = I, \quad \pi_{n+2}(Q_4) = I_2, \]

* It is well known that \( \pi_{n+2}(S^n \cup e^{n+1}) \) is a group extension of \( I_2 \) by \( I_2 \), where \( e^{n+1} \) is attached to \( S^n \) by a map \( f : \partial e^{n+1} \to S^n \) of even degree \( \alpha \). According to [20], \( \pi_{n+2}(S^n \cup e^{n+1}) = I_4 \) if \( \alpha = 2 \). Here we assume, \( \pi_{n+2}(S^n \cup e^{n+1}) = I_4 \) for \( n \geq 3 \) in case \( \alpha = 2 \).
\[ \pi_{n+2}(Q_t) = I_2 + I_2, \quad \pi_{n+2}(Q_0) = I_2 + I_2, \quad \pi_{n+2}(Q_t) = I_4, \]
\[ \pi_{n+2}(Q_t) = 0, \quad \pi_{n+2}(Q_0) = I_2. \]

It is clear that \( \tilde{S}_{n+2}(I) \) is generated by \( j_0 e_{i_0}^{n+2}, j_0 e_{c}^{n+2}, j_0 e_{2c}^{n+2} \). From (5.2), (5.3) and from \( \pi_{n+2}(K) \approx \sum_{i, j} \pi_{n+2}(Q_{ij}) \), for \( n > 3 \), we have
\[ \pi_{n+2}(K) \approx \Gamma_{n+2}(K) + \tilde{S}_{n+2}(I). \]

**Theorem 5.** The \((n+2)\)-dimensional homotopy group of a \( \mathbb{A}^3 \)-complex can be calculated combinatorially from \( \tilde{S}_n, \tilde{S}_{n+1}, \tilde{S}_{n+2}, \mu, \Delta, \mathbb{B}_{n+2}. \)

§ 6. \( \mathbb{A}^3 \)-complex

In [14] I solved the homotopy type problem of a \( \mathbb{A}^3 \)-complex. Making use of the sequence of Whitehead, we restate the problem. Let us consider the sequence
\[ H_{n+2} \mathbb{B}_{n+2}(0) \rightarrow \Pi_{n+2}(0) \rightarrow H_{n+2}(0) \mathbb{B}_{n+2}(0) \rightarrow \Pi_{n+2}(0) \rightarrow H_{n+1}(0) \rightarrow 0. \]

It was proved in § 4 ii) iii) that the homomorphisms \( \mathbb{B}_{n+3}(0), \mathbb{B}_{n+2}(0) \) are determined by Steenrod’s Squaring homomorphisms \( Sq_{n-1}, Sq_n \) respectively and that \( \Gamma_{n+2}(0) \) is isomorphic to \( \tilde{S}_{n+1}(I_2) \). Following Whitehead [4], we can establish analogously geometrical realizability, so that all the results in [14] are obtained by the aid of the sequence (6.1).

By the sequence we have

**Theorem 6.** \( \pi_{n+2}(K) \) of \( \mathbb{A}^3 \)-complex is a group extension of \( \tilde{S}_{n+1}(I_2) \mathbb{B}_{n+3}(0) \), \( \tilde{S}_{n+3}(I) \) by the kernel of \( \mathbb{B}_{n+2}(0) \mathbb{B}_{n+2}(0) \rightarrow \Gamma_{n+1}(0) \).

§ 7. Lower dimensional case

In this section we assume that \( K \) is a simply connected complex without 2-dimensional torsion. Besides this, we assume \( \pi_3(K) = 0. \) We shall show how \( \pi_4(K) \) can be calculated in terms of homology. As was proved by Whitehead [2],
\[ K^3 \simeq L^3 = S_1^3 + S_2^3 + \cdots + S_3^3 + S_4^3 + \cdots + S_5^3 + S_6^3 + \cdots + S_8^3, \]
where 2-spheres and 3-spheres are attached at a point. Since \( K \) is free from 2nd torsion, the 3-skelton of \( K \) is of the same homotopy type as \( L^3 \). Then we have
\[ K^4 \simeq \{ L^3 ; R_1, \ldots, R_4 \}^* = L^4, \]

* This notation is often used in [14]: if an n cell \( e^n \) is attached to a space \( P \) by a map \( f: \partial e^n \rightarrow P \), the attached space is denoted by \( \{ P; x \} \), where \( x \) is an element of \( \pi_{n-1}(P) \) represented by \( f \).
where \( R_i = \alpha_i + b_{ij} S_j^3 \), \( \alpha_i \in \pi_3 (S_j^3 \vee \ldots \vee S_p^3) \), and \( S_j^3 (j = 1, \ldots, \sigma) \) is the generator of \( \pi_3(S_j^3) \). Notice that \( b_{ij} S_j^3 \) is not summed with respect to \( j \), and that \( b_{ij} \) is greater than unity.\(^*\) If \( e_{ij} \) denotes the generator of \( \pi_3(S_i^3) \) and if \( e_{ij} (i = j) \) is represented by the Whitehead product \([a_i, a_j] \), where the generator is represented by \( a_i \), we have

\[ \alpha_i = \sum_{i \neq j} p_{ij} e_{ij} \]

where \( p_{ij} \) are integers.

**Lemma 7.1.** \( e_{ij}, S_j^3 (j = 1, \ldots, \sigma + t) \) are linear combinations of \( R_1, R_2, \ldots, R_\lambda \).

Take 4 simplex \( \varepsilon_i^4 \) in the interior of each 4 cell \( e_i^4 \), and join a point on the boundary \( \hat{\varepsilon}_i^4 \) to \( \Theta \) by an arc \( t_i \). Let us denote \( \bigcup (\varepsilon_i^4 + t_i) \) by \( \varepsilon^4 \) and its boundary by \( \hat{\varepsilon}^4 \). If we put \( L = L^4 - \varepsilon^4 \), \( L^3 \) is a deformation retract of \( L \). Let us consider the following diagram

\[
\begin{array}{c}
\pi_4 (L^4 ; \varepsilon^3, L) \\
\uparrow j_4 \\
\pi_4 (L^4, L) \xrightarrow{\partial_3} \pi_3 (L) \xrightarrow{k_3} \pi_3 (L^4) \xrightarrow{\rho} \\
\uparrow i_i \\
\pi_4 (\varepsilon^4, \hat{\varepsilon}^4) \xrightarrow{\partial} \pi_3 (\hat{\varepsilon}^4) \\
\uparrow \hat{i}_i \\
\pi_3 (L^4 ; \varepsilon^3, L)
\end{array}
\]

Since the triad \((L^4 ; \varepsilon^3, L)\) is 4-connected and \( \beta_4 \) is trivial, \( i_4 \) is an isomorphism onto. From \( \pi_3 (L^4) \cong \pi_3 (K) = 0 \), \( \partial_3 \) is onto. Since \( \varepsilon \) is an isomorphism onto, and \( \partial_3 i_4 = p \partial_3 \), \( p \) is onto, so that \( e_{ij}, S_j^3 \) are linear combinations of \( \beta e_i^4 = R_i \).

Let us denote by \( M^4 \{L^3 ; R_1, \ldots, R_\lambda ; e_{11}, \ldots, e_{ij}, \ldots, e_{pp} ; S_1^3, \ldots, S_\sigma^3 \} \). From Lemma (7.1) and from elementary operations we have

\[
M^4 \sim \left\{ L^3 ; R_1, \ldots, R_\lambda ; 0, \ldots, 0 \right\} = L^4 + S_1^4 + \cdots + S_p^4 + S_1^4 + \cdots + S_\sigma^3.
\]

From \( R_i = \alpha_i + b_{ij} S_j^3 \) and from elementary operations, we have

\[
M^4 \sim \left\{ L^3 ; 0, \ldots, 0 ; e_{11}, \ldots, e_{pp} ; S_1^3, \ldots, S_\sigma^3 \right\}
\]

\[
= \left\{ L^3 ; e_{11}, \ldots, e_{pp}, S_1^3, \ldots, S_\sigma^3 \right\} + S_1^4 + \cdots + S_\lambda^4.
\]

Since \( e_j^4 (j = 1, \ldots, \sigma + t) \) are attached to \( S_j^3 \) with degree unity, we have

\[
M^4 \sim \left\{ L^2 ; e_{11}, \ldots, e_{pp} \right\} + S_1^4 + \cdots + S_\lambda^4.
\]

\(^*\) Refer to [14].
where \( L^2 = S_1^1 + \cdots + S_1^\nu \). Let us denote by \( N^4, \{ L^2; e_{11}, \ldots, e_{\nu \nu} \} \).

From these considerations we have

\[
(7.2) \quad L^4 + S_1^1 + \cdots + S_\rho^1 + S_1^1 + \cdots + S_\nu^{\sigma + \tau} \sim N^4 + S + \cdots + S_1^1.
\]

It is clear that

\[
K^5 + S_1^1 + \cdots + S_\mu^\omega \sim \{ L^4; X_1, \ldots, X_\nu \} + S_1^1 + \cdots + S_\nu^{\sigma + \tau} + S^\omega \beta \lambda
\]

where \( \beta e_i^5 = X_i \in \pi_4(L^4) \) and \( \omega = \frac{\rho(\rho-1) + (\sigma + \tau)}{2} \).

If \( f \) is a homotopy equivalence of (7.2), we have

\[
Y_i = f(X_i) \in \pi_4(N^4 + S_1^1 + \cdots + S_\nu^1) = \pi_4(S_1^1 + \cdots + S_\nu^1)
\]

(refer to §8). From Lemma 2 [14] we have

\[
K^5 + S_1^1 + \cdots + S_\mu^\omega \sim N^4 + \{ S_1^1 + \cdots + S_\nu^1; Y_1, \ldots, Y_\nu \}.
\]

Through elementary operations and change of a base \{\( S_1^1, \ldots, S_\nu^1 \)}, it is concluded that

\[
(7.3) \quad K^5 + S_1^1 + \cdots + S_\mu^\omega \sim N^4 + P_1^1 + \cdots + P_\sigma^1 + S_1^1 + \cdots + S_\nu^1 + S_1^1 + \cdots + S_\mu^1,
\]

where \( P_\sigma = \{ S_1^1; \sigma_1 S_1^1 \} \).

If we consider 4-th homology groups of both sides of (7.3), we have

\[
H_4(K^5) + I + \cdots + I \sim I + \cdots + I + I_\sigma + \cdots + I_\nu + I + \cdots + I.
\]

Since the ranks of both sides are equal, we have

\[
(7.4) \quad \beta_4 + \sigma + \tau = \lambda - \nu,
\]

where \( \beta_4 \) denotes 4-th Betti number. If \( r \) is the rank of \( \pi_4(K) \), we have

\[
r + \omega = \lambda - \nu.
\]

From (7.4) we have

\[
(7.5) \quad r = \beta_4 - \frac{\rho(\rho-1)}{2}.
\]

It is also seen that the torsion group of \( \pi_4(K) \) is isomorphic to that of \( H_4(K) \). Thus we have

**Theorem 5.** The four dimensional homotopy group of a complex \( K \) such that \( \pi_i(K) = 0 \) for \( i = 1, 3 \) and \( K \) is free from 2nd torsion, is given explicitly in terms of homology groups;

\[
\pi_4(K) \cong \frac{I + \cdots + I + I_\sigma + \cdots + I_\nu}{\beta_4 - \frac{\rho(\rho-1)}{2}}.
\]
where $\beta_4, \rho$ are 4-dimensional Betti number, 2-dimensional Betti number respectively, and $(\sigma_1, \ldots, \sigma_n)$ is the 4-dimensional torsion coefficients.

In such a complex it is also seen that we have $\beta_4 \geq \frac{\rho(\rho-1)}{2}$.

§ 8. Note on homotopy groups

I owe a great deal to recent results due to Blakers and Massey [7], which enable me to calculate homotopy groups used till now. In preparation for my forthcoming paper it seems convenient to calculate $\pi_{3n-2}(S^n \vee S^n)$ for each $n \geq 2$, which was also solved by Blakers and Massey [7]. By doing this we can prove $\pi_4(N^4) = 0$, which was essentially used in § 7. First we define a generalized Whitehead Product.

$$E^n = \left\{ x; 1 \geq x_i \geq 0, i = 1, \ldots, n \right\},$$

$$J^{n-1} = \left\{ x; (x_n-1) \prod_{i=1}^{n-1} x_i (x_i-1) = 0 \right\},$$

$$K^{p+q-1} = \partial(E^p \times E^q) + (-1)^{p+q+1} E^p \times E^{q-1} \times 0 + (-1)^{p+1} E^{p-1} \times 0 \times E^q \times (-1)^p E^{p-1} \times (\partial E^{q} + (-1)E^{q-1}) \times 0 \times E^q \times J^{q-1}.$$

Then $K^{p+q-1}$ is a $(p+q-1)$ cell. Let $X$ be a space such that $X = A \cup B$ and $A \cap B$ is non-void. If $\pi_p(B, A \cap B) \ni \alpha$, and $\pi_q(A, A \cap B) \ni \beta$, $\alpha$ and $\beta$ are represented by maps $f$ and $g$ respectively such that

$$f : (E^p, \partial E^p, J^{p-1}) \rightarrow (B, A \cap B, *) ,$$

$$g : (E^q, \partial E^q, J^{q-1}) \rightarrow (A, A \cap B, *).$$

Let us define a map $f \vee g : K^{p+q-1} \rightarrow X = A \cup B$ such that

$$f \vee g (x, y) = g (y), \ (x, y) \in J^{p-1} \times E^q ,$$

$$= f (x), \ (x, y) \in E^p \times J^{q-1}.$$

If $\varphi : E^{p+q-1} \rightarrow K^{p+q-1}$ is an orientation preserving map of degree unity, a composite map $(f \vee g) \circ \varphi$ represents an element of $\pi_{p+q-1}(X; A, B)$. In course of verification that $(f \vee g) \circ \varphi$ represents an element of $\pi_{p+q-1}(X; A, B)$ it is easily seen that

$$(-1) \ [\partial \alpha, \beta] = \beta, [\alpha, \beta],$$

$$(-1)^{p-1} [\alpha, \partial \beta] = \beta, [\alpha, \beta].$$

By definition we have $[\alpha, \beta] = (-1)^p [\beta, \alpha]$. If $\alpha \in \pi_p(S^n), \beta \in \pi_q(S^n)$, we have $E[\alpha, \beta] = 0$ by definition of the generalized Whitehead product, where $E$ denotes Freudenthal's suspension. These properties are used in calculating $\pi_{3n-2}(S^n \vee S^n)$.

Next, by a result of G.W. Whitehead [9] we have $\pi_{3n-2}(S^n \vee S^n)$
Let $\sigma_{2n}$ be a 2n simplex in $S_1^n \times S_2^n = X^*$ and let $X$ be $X - \text{Int.} \sigma_{2n}$, then $S_{1}^{n} \vee S_{2}^{n}$ is a deformation retract of $X$. This retraction is denoted by $\psi$. Consider a sequence of a triad $(X^{*}; \sigma_{2n}, X)$; 
\[ \pi_{3n}(X^{*}; \sigma_{2n}, X) \xrightarrow{\beta_{\sigma_{2n}}} \pi_{3n-1}(\sigma_{2n}, \omega_{2n}) \xrightarrow{i} \pi_{3n-1}(X^{*}, X) \xrightarrow{j} \pi_{3n-1}(X^{*}; \sigma_{2n}, X) \rightarrow. \]
If $n \geq 2$, $\beta_{i}$ is trivial for each $i \leq 3n-1$, so that $i$ is an isomorphism into and $j$ is a homomorphism onto.

\[ \psi \partial i \pi_{3n-1}(\sigma_{2n}, \omega_{2n}) = \psi i \partial \pi_{3n-1}(\sigma_{2n}, \omega_{2n}) = \psi i \pi_{2n-2}(\omega_{2n}). \]

From this any element of $\psi \partial i \pi_{3n-1}(\sigma_{2n}, \omega_{2n})$ is represented by a map $f: S_{3n-2} \rightarrow S_{2n-2} \xrightarrow{[i_{1}, i_{2}]} S_{1}^{n} \vee S_{2}^{n}$, where $[i_{1}, i_{2}]$ denotes the Whitehead product of $i_{1}$ and $i_{2}$. If $\alpha \in \pi_{3n-1}(X^{*}; \sigma_{2n}, X)$, $\alpha$ is represented by a map $f: (E_{3n-1}, E_{3n-1}^{*}, E_{3n-2}) \rightarrow (X^{*}; \sigma_{2n}, X)$. Let $p$ be an interior point of $\sigma_{2n}$, and let $C_{n-1}^{n}$ be the inverse image of $p$ by $f$, then we have $\partial C_{n-1}^{n} = D_{n-2}^{n-2} = C_{n-1}^{n} \cap E_{3n-2}^{*}$. Select a point $O$ in $E_{3n-1}^{*}$ such that $OD^{n-2} = L^{n-1}$ $\subset E_{3n-2}^{*}$ and $OC^{n-1} = K^{n}$, then

\[ \partial f(K^{n}) = f(\partial K^{n}) = -f(L^{n-1}) \subset \sigma_{2n}^{n}. \]

Thus $f(K^{n})$ represents an element of $\Sigma_{n}(X^{*}, \sigma_{2n})$. If $S_{1}^{n}, S_{2}^{n}$ are two generators of $\Sigma_{n}(X^{*}, \sigma_{2n})$, we have

\[ f(K^{n}) \cong a_{1}S_{1}^{n} + a_{2}S_{2}^{n}, \]

where $(a_{1}, a_{2})$ is a pair of integers. Then it is verified that if $f \sim g$, we have $(a_{1}, a_{2}) = (b_{1}, b_{2})$, where $g(K^{n}) \cong b_{1}S_{1}^{n} + b_{2}S_{2}^{n}$. Moreover it is also seen that if $a_{1} = a_{2} = 0$, $f$ is inessential. Thus it is concluded that that $(a_{1}, a_{2})$ is an invariant of homotopy classes. If $l$ is the generator of $\pi_{2n}(\sigma_{2n}, \omega_{2n})$ and if $\pi_{n}(X, \omega_{2n}) \cong \pi_{n}(X) = [l_{1}] + [l_{2}]$, we have

\[ [l, l_{1}] \in \pi_{3n-1}(X^{*}; *, X), \]
\[ [l, l_{2}] \in \pi_{3n-1}(X^{*}; *, X). \]

Then it is verified that homotopy invariants of these product are $(1, 0)$, $(0, 1)$ respectively. Furthermore we have

\[ \psi \partial [l, l_{1}] = \psi [\partial l, l] = [[l_{1}, l_{2}], l_{1}], \]

so that two free generators of $\psi \partial \pi_{3n-1}(S_{1}^{n} \times S_{2}^{n}, S_{1}^{n} \vee S_{2}^{n})$ are represented by two triple Whitehead products. This is a result announced by Blakers and Massey [7].

Now we prove $\pi_{4}(N^{4}) = 0$, making use of this. Let us consider the injection map $i$: 

\[ \approx i_{1} \pi_{3n-2}(S^{n}) + i_{2} \pi_{3n-2}(S^{n}) + \partial \pi_{3n-1}(S_{1}^{n} \times S_{2}^{n}, S_{1}^{n} \vee S_{2}^{n}), \]
Then it is seen that \( i \) is onto and that the kernel of \( i \) is generated by \( e_i, \eta \) and triple Whitehead products \([a_i[a_j, a_k]]\), where \( e_i, \eta \) is represented by a map \( f: S^4 \longrightarrow S^3 \longrightarrow S^2 \vee S^1 \). This proves that the generators of the kernel of \( i \) is the same as those of \( \pi_4(S^1 \vee \cdots \vee S^p) \), so that we have \( \pi_4(N^4) = 0 \).

Mathematical Institute, Nagoya University

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Bibliography