<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>A note on the relation of $\mathbb{Z}_2$-graded complex cobordism to complex K-theory</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Yosimura, Zen-ichi</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 12(3) P.583-P.595</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1975</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/11134">https://doi.org/10.18910/11134</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/11134</td>
</tr>
</tbody>
</table>

*Osaka University Knowledge Archive : OUKA*

[https://ir.library.osaka-u.ac.jp/repo/ouka/all/](https://ir.library.osaka-u.ac.jp/repo/ouka/all/)

Osaka University
A NOTE ON THE RELATION OF $\mathbb{Z}_2$-GRADED COMPLEX COBORDISM TO COMPLEX K-THEORY

Zen-ichi Yosimura

(Received December 16, 1974)

Let $MU^*(\ )$ and $K^*(\ )$ denote the $\mathbb{Z}_2$-graded complex cobordism theory and the complex $K$-theory respectively. The Thom homomorphism $\mu_*: \pi_0(MU) \to \pi_0(K)$ on coefficient groups is identified (up to sign) with the classical Todd genus $Td: \Lambda \to \mathbb{Z}$. We denote by $I$ the ideal of $\Lambda$ to be the kernel of $Td: \Lambda \to \mathbb{Z}$. Wolff [7] proved that the decreasing filtration $\{I^q MU^*(\ )\}$ of $MU^*(\ )$ consists of cohomology theories defined on the category of based finite $CW$-complexes, and the associated quotients $I^q MU^*(\ )/I^{q+1} MU^*(\ )$ are determined by the complex $K$-theories $KG^*(\ )$ with coefficients $G_q = I^q/I^{q+1}$.

The purpose of this note is to extend the Wolff's result to the category of based $CW$-complexes. Let $F_q MU$ be the $CW$-spectrum associated with the cohomology theory $I^q MU^*(\ )$, i.e., $\{Y, F_q MU\}^* \simeq I^q MU^*(Y)$ for any based finite $CW$-complex (or finite $CW$-spectrum). We show that $\{F_q MU^*(\ )\}$ is a decreasing filtration of $MU^*(\ )$ consisting of $\Lambda$-modules so that the associated quotients are equal to $KG^*(\ )$, and in addition that $F_{q+1} MU^*(\ )$ is a direct summand of $F_q MU^*(\ )$.

Moreover we give a tower

$$MU \to \cdots \to Q_0 MU \to Q_1 MU \to \cdots \to Q_q MU = K$$

of $MU$-module spectra such that $KG_q \to Q_q MU \to Q_{q-1} MU$ is a cofiber sequence of $MU$-module spectra, which factorizes the Thom map $\mu_: MU \to K$.

Baas [3] constructed a tower of $CW$-spectra

$$MU \to \cdots \to MU\langle n\rangle \to MU\langle n-1\rangle \to \cdots \to MU\langle 0\rangle = H$$

factorizing the Thom map $\mu_: MU \to H$. In appendix we show that the tower is of $MU$-module spectra and the sequence $\Sigma^2 MU\langle n\rangle \to MU\langle n\rangle \to MU\langle n-1\rangle$ is a cofiber sequence where $m_x^n$ is the multiplication by $x^n$ a ring generator of $\Lambda$ with degree $2n$.

1. Decreasing filtration of $MU^*(\ )$

1.1. A pair $(E, \rho)$ is called a $\mathbb{Z}_2$-graded $CW$-spectrum if $E$ is a $CW$-spectrum
and \( \rho: \Sigma^2 E \to E \) is a homotopy equivalence. Such a pair \((E, \rho)\) gives rise to natural isomorphisms
\[
\rho_*: E_*(X) \to E_{*+2}(X), \quad \rho^*: E^{*+2}(X) \to E^*(X)
\]
for any CW-spectrum \(X\). So we can define \(\mathbb{Z}_2\)-graded homology and cohomology theories \(E_*(\ ), E^*(\ )\) by putting
\[
E_0(X) = E_0(X) \oplus E_2(X), \quad E^0(X) = E^0(X) \oplus E^2(X).
\]

For a CW-spectrum \(E\) we put
\[
E = \bigvee \Sigma^{2n} E, \quad \bar{E} = \prod \Sigma^{2n} E.
\]

Taking the canonical identifications \(\rho: \Sigma^2 E \to E\) and \(\rho: \Sigma^2 \bar{E} \to \bar{E}\) as structure morphisms \(E\) and \(\bar{E}\) admit structures of \(\mathbb{Z}_2\)-graded CW-spectra respectively. From definition it follows that
\[
\begin{align*}
E_0(X) &\cong \sum E_{2n}(X), \quad E_1(X) \cong \sum E_{2n+1}(X), \\
\bar{E}^0(X) &\cong \prod E^{2n}(X), \quad \bar{E}^1(X) \cong \prod E^{2n+1}(X)
\end{align*}
\]
for all CW-spectra \(X\). In particular, the canonical morphism \(H \to \overline{H}\) becomes a homotopy equivalence for the Eilenberg-MacLane spectrum \(H\).

The \(BU\)-spectrum \(K\) may be regarded as a \(\mathbb{Z}_2\)-graded CW-spectrum because it possesses the Bott map \(\beta: \Sigma^2 K \to K\) which is a homotopy equivalence.

Denote by \(F_n\) the direct sum of \(n\)-copies of the integers \(\mathbb{Z}\) and by \(F\) the direct limit of \(F_n\), i.e., \(F\) is a free abelian group with countably many factors. Putting
\[
BU_{F_n} = BU \times \cdots \times BU, \text{ the product of } n\text{-copies of } BU,
\]
\[
BU_F = \bigcup_n BU_{F_n}, \text{ the union of } BU_{F_n},
\]
we obtain

**Proposition 1.** There exists a natural isomorphism
\[
[X, BU_F] \to KF^0(X)
\]
for any based connected CW-complex \(X\).

Proof. Let \(Y\) be a based connected finite CW-complex. Then we have a sequence of natural isomorphisms
\[
[Y, BU_F] \leftarrow \lim [Y, BU_{F_n}] \leftarrow \lim [Y, BU] \otimes F_n \to \lim K^n(Y) \otimes F_n \to K^n(Y) \otimes F \to KF^0(Y).
\]
Therefore the contravariant functor \(KF^0\) defined on the category of based connected CW-complexes is represented by \(BU_F\) (use [1, Addendum 1.5]).
Proposition 1 implies that $BU_F$ is homotopy equivalent to $\Omega^2 BU_F$ where $\Omega^2$ means the component of the base point in the double loop space. Hence we have

\[(1.1) \text{ in the } BU\text{-spectrum } KF \text{ with the coefficients } F \text{ every even term is the based } CW\text{-complex } BU_F.\]

1.2. Let us denote by $MU$ the unitary Thom spectrum and by $\mu_c: MU \to K$ the Thom map which is a ring morphism. The composition $\mu_c: MU \to K 
\to K$

of $\Sigma^{2\pi} \mu_c$ and $\beta^n$ is a morphism of $Z_2$-graded ring-spectra, called the $Z_2$-graded Thom map. As is well known, it is characterized by the coefficient homomorphism $\mu_c: \pi_0(MU) \to \pi_0(K)$ which coincides (up to sign) with the classical Todd genus $Td$. Putting $\Lambda = \pi_0(MU)$, $\pi_0(K) = Z$ is viewed as a $Z_2$-graded $\Lambda$-module via $\mu_c = Td$ and it is written $Z_Td$ for emphasis.

Using the kernel $I$ of $Td: \Lambda \to Z$ we define a decreasing filtration $\{I^q\}_{q \geq 0}$ consisting of ideals of $\Lambda$. Denoting by $G_q$ the associated $Z_2$-graded $\Lambda$-module $I^q/I^{q+1}$, we see easily [7, Satz 3.8] that

\[(1.2) G_0 \cong Z_{Td} \text{ and } G_q \text{ is a free abelian group with countably many factors for } q \geq 1.\]

For a $Z_2$-graded $\Lambda$-module $A$ we have a decreasing filtration $\{I^qA\}_{q \geq 0}$ consisting of submodules of $A$, whose associated $Z_2$-graded $\Lambda$-module $I^qA/I^{q+1}A$ is written $G_q(A)$. Applying the commutative diagram

\[0 \to \text{Tor}^1(\Lambda, A) \to I \otimes A \to A \to Z_{Td} \otimes A \to 0 \]

\[0 \to IA \to A \to G_q(A) \to 0\]

with exact rows, we get an isomorphism

\[(1.3) G_q \otimes A \overset{\cong}{\to} G_q(\Lambda \otimes A) \overset{\cong}{\to} G_q \otimes G_0(A)\]

by means of "4 lemma".

**Proposition 2.** Let $A$ be a $\Lambda$-module with $\text{Tor}^1(\Lambda, A) = 0$ for all $k \geq 1$. Then, for every $q \geq 0$ both $I^q \otimes A \to I^q A$ and $G_q \otimes A \to G_q(A)$ are isomorphisms and $\text{Tor}^1(I^q, A) = \text{Tor}^1(G_q, A) = 0$ for all $k \geq 1$.

Proof. Choose a free $\Lambda$-module $F$ such that $A$ is isomorphic to a quotient $F/B$. By induction on $q$ we shall show that the sequences

\[0 \to I \otimes B \to I \otimes F \to I \otimes A \to 0, \quad 0 \to G_q(B) \to G_q(F) \to G_q(A) \to 0\]

are exact. The $q=0$ case is evident because of (1.3). Applying induction
hypotesis and "3 x 3 lemma" we find easily that $0 \to I^q B \to I^q F \to I^q A \to 0$ is exact. So we have a commutative diagram

$$
\begin{array}{c}
0 \to G_q \otimes G_0(B) \to G_q \otimes G_0(F) \to G_q \otimes G_0(A) \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
G_0(I^q B) \to G_0(I^q F) \to G_0(I^q A) \to 0
\end{array}
$$

with exact rows. Since all vertical arrows are epimorphisms and in particular the central one is an isomorphism, all vertical arrows become isomorphisms. Consequently we get that $0 \to G_0(B) \to G_0(F) \to G_0(A) \to 0$ is exact.

Next, we consider the commutative diagrams

$$
\begin{array}{c}
0 \to \text{Tor}_k^I(I^q, A) \to I^q \otimes B \to I^q \otimes F \to I^q \otimes A \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \to I^q B \to I^q F \to I^q A \to 0
\end{array}
$$

$$
\begin{array}{c}
0 \to \text{Tor}_k^I(G_q, A) \to G_q \otimes B \to G_q \otimes F \to G_q \otimes A \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \to G_q \otimes G_0(B) \to G_q \otimes G_0(F) \to G_q \otimes G_0(A) \to 0
\end{array}
$$

with exact rows. Remark that $\text{Tor}_k^I(Z_I, B)=0$ for all $k \geq 1$. By use of "4 lemma" and (1.3) we see that all vertical arrows are isomorphisms, and hence we obtain the required results.

For a $\Lambda$-module $A$ we put $J_q(A)=A/I^{q+1}A$ and abbreviate $J_q=J_q(\Lambda)$ when $A=\Lambda$. As an immediate corollary of Proposition 2 we have

**Corollary 3.** Let $A$ be a $\Lambda$-module with $\text{Tor}_k^I(Z_{T_\delta}, A)=0$ for all $k \geq 1$. Then $J_q \otimes A \to J_q(A)$ is an isomorphism and $\text{Tor}_k^I(J_q, A)=0$ for all $k \geq 1$.

1.3. Let $\mathcal{MU}$ denote the category of comodules over $MU_*(MU)$ which are finitely presented as $\Lambda$-modules. Notice that $\mathcal{MU}$ is an abelian category which has enough projectives, and also that $MU_*(Y)$ lies in the category $\mathcal{MU}$ whenever $Y$ is a finite CW-spectrum. Since the functor $M \to Z_{T_\delta} \otimes M$ is exact on $\mathcal{MU}$ [5, Example 3.3] it follows immediately that

$$\text{(1.4)} \quad \text{Tor}_k^I(Z_{T_\delta}, M)=0 \quad \text{for all } k \geq 1 \text{ if } M \text{ lies in } \mathcal{MU}.\]

Proposition 1 and Corollary 2 combined with (1.4) say that

$$\text{(1.5)} \quad \text{the functors } M \to I^q M \simeq I^q \otimes M, \quad M \to G_0(M) \simeq G_q \otimes M \quad \text{and} \quad M \to J_q(M) \simeq J_q \otimes M \text{ on } \mathcal{MU} \text{ are exact.}\]

**Theorem 1** (Wolff [7]). i) Both $I^q MU_*(\ )$ and $MU_*(\ )/I^{q+1}MU_*(\ )$ are
homology theories defined on the category of CW-spectra, so that \( I^q \otimes \Lambda MU(X) \to I^m MU(X) \) and \( \Lambda/I^{q+1} \otimes MU(X) \to MU(X)/I^{q+1}MU(X) \) are natural isomorphisms for all CW-spectra \( X \).

ii) \( I^g MU(\_)/I^{q+1}MU(\_) \) is a homology theory defined on the category of CW-spectra such that there exists a natural isomorphism \( I^g MU(X)/I^{q+1}MU(X) \to KG_g(X) \) for any CW-spectrum \( X \) which is induced by the \( Z_2 \)-graded Thom map \( \mu_c \).

Proof. i) and the first half of ii) are immediate from (1.5). The latter half of ii) is also valid because we have a natural isomorphism

\[
G_q(MU_\star(X)) \cong G_q \otimes MU_\star(X) \cong G_q \otimes (Z^{T_q} \otimes MU_\star(X))
\]

\[
\cong G_q \otimes K_\star(X) \to KG_\star(X).
\]

Let \( \phi: E_\star(X) \to F_\star(X) \) be a natural transformation for any CW-spectrum \( X \). According to [1, Addendum 1.5] there exists a morphism \( f: E \to F \) inducing \( \phi \), and it is unique up to weak homotopy. The proof in [1] is actually given for the category of based connected CW-complexes, but it is easily extended to that of CW-spectra. Such a morphism \( f \) is uniquely chosen (up to homotopy) under the assumption that \( F_\star(E) \) is Hausdorff.

Let \( E_\star(\_) \) be a \( Z_2 \)-graded homology theory defined on the category of CW-spectra, i.e., a homology theory equipped with a natural isomorphism \( E_\star(X) \to E_{\star+1}(X) \) for any CW-spectrum \( X \). Then it gives \( E \) a structure of \( Z_2 \)-graded CW-spectrum. In particular, the induced structure is unique if \( E_\star(E) \) is Hausdorff.

Recall that the \( Z_2 \)-graded CW-spectrum \( MU \) is equipped with the canonical identification \( \rho: \Sigma MU \to MU \) as structure morphism. Since \( MU^\text{H}(MU) \) is Hausdorff (use Proposition 6 below), the \( Z_2 \)-graded CW-spectrum \( (MU, \rho) \) is characterized only by the \( Z_2 \)-graded homology theory \( MU_\star(\_) \).

Denote by \( F_q MU \) and \( Q_q MU \) the representing spectra of the new homology theories \( I^q MU_\star(\_) \) and \( MU_\star(\_)/I^{q+1}MU_\star(\_) \) respectively, i.e.,

\[
I^q MU_\star(X) \simeq \{ \Sigma^*, F_q MU \wedge X \}, \quad MU_\star(X)/I^{q+1}MU_\star(X) \simeq \{ \Sigma^*, Q_q MU \wedge X \}
\]

for any CW-spectrum \( X \). Of course, they are both \( Z_2 \)-graded CW-spectra. Then there exist morphisms

\[
i_q: F_{q+1}MU \to F_q MU, \quad j_q: Q_q MU \to Q_{q-1}MU, \quad t_q: F_{q+1}MU \to MU, \quad \pi_q: MU \to Q_q MU
\]

which induce the canonical morphisms in homology groups, and moreover we have morphisms.
\[ \mu_q: F_qMU \to KG_q, \quad \nu_q: KG_q \to Q_qMU \]
such that \( \mu_q: F_qMU_*(X) \to KG_q(X) \) and \( \nu_q: KG_q(X) \to Q_qMU_*(X) \) in homology groups are natural homomorphisms induced by the \( \mathbb{Z}_2 \)-graded Thom map \( \mu_c \).

**Lemma 4.** Let \( E \to F \to G \) be a sequence which satisfies the property that \( 0 \to E_*(X) \to F_*(X) \to G_*(X) \to 0 \) is a short exact sequence for every CW-spectrum \( X \). Then it is a cofiber sequence.

**Proof.** Let \( C_f \) be the mapping cone of \( f \), i.e., \( E \to F \to C_f \) a cofiber sequence. Then for any CW-spectrum \( X \) we have a commutative diagram

\[
\begin{array}{ccc}
0 & \to & E_*(X) \\
\downarrow & & \downarrow \\
0 & \to & F_*(X) \\
\downarrow \phi = h_* & & \downarrow \\
0 & \to & G_*(X) & \to & 0
\end{array}
\]

with exact rows. Clearly \( h: C_f \to G \) which induces \( \phi \) is a homotopy equivalence.

By virtue of Lemma 4 we verify that

\[ (1.6) \quad F_{q+1}MU \to MU \to Q_qMU, \quad F_{q+1}MU \to F_qMU \to KG_q \text{ and } KG_q \to Q_qMU \to Q_{q-1}MU \text{ are all cofiber sequences.} \]

2. \( \mathbb{Z}_2 \)-graded \( MU \)-module spectra

2.1. The inclusion \( Z \subset Q \) induces a natural transformation \( ch: E^*(X) \to EQ^*(X) \) for any CW-spectrum \( X \), called the Chern-Dold character.

**Proposition 5.** If \( ch: E^*(X) \to EQ^*(X) \) is a monomorphism, then \( E^*(X) \) is Hausdorff.

**Proof.** Since \( EQ^*(X) \) is always Hausdorff [8, Proposition 4], the result is immediate.

Let \( W \) be a connective CW-spectrum with \( H_*(W) \) free and assume that \( \pi_*(E) \) is torsion free. Then \( H^*(W; \pi_*(E)) \to H^*(W; \pi_*(E) \otimes Q) \) is a monomorphism, and hence the Atiyah-Hirzebruch spectral sequences for \( E^*(W) \) and \( EQ^*(W) \) collapse. Therefore we get that

\[ (2.1) \quad ch: E^*(W) \to EQ^*(W) \text{ is a monomorphism.} \quad (Cf., [8, Lemma 11]). \]

Applying Proposition 5 we obtain

**Proposition 6.** Let \( W \) be a connective CW-spectrum with \( H_*(W) \) free. If \( \pi_*(E) \) is torsion free, then \( E^*(W) \) is Hausdorff.

By means of Proposition 5 we get the following lemmas.
Lemma 7. Assume that $\pi_0(E)$ is torsion free and $\pi_i(E)=0$. Then $E^0(KG_q \wedge MU \wedge \cdots \wedge MU)$ and $E^0(Q_qMU \wedge MU \wedge \cdots \wedge MU)$ are Hausdorff and $E^0(F_qMU \wedge MU \wedge \cdots \wedge MU)=0$.

Proof. Since $E^0(KG_q \wedge MU \wedge \cdots \wedge MU)=E^0(Q_qMU \wedge MU \wedge \cdots \wedge MU)=0$ we have a commutative square

$$
\begin{array}{ccc}
E^0(KG_q \wedge MU \wedge \cdots \wedge MU) & \rightarrow & E^0(KG_q \wedge MU \wedge \cdots \wedge MU) \\
\downarrow & & \downarrow \\
E^0(Q_qMU \wedge MU \wedge \cdots \wedge MU) & \rightarrow & E^0(Q_qMU \wedge MU \wedge \cdots \wedge MU)
\end{array}
$$

such that the horizontal arrows are isomorphisms. The left arrow is a monomorphism because so is the right one by use of (2.1). Since Theorem 1 implies that $0 \rightarrow EQ^0(Q_qMU \wedge MU \wedge \cdots \wedge MU) \rightarrow EQ^0(Q_qMU \wedge MU \wedge \cdots \wedge MU)$ is exact, an induction on $q$ involving “4 lemma” shows that $E^0(Q_qMU \wedge MU \wedge \cdots \wedge MU)$ is a monomorphism. Then we find that $E^0(Q_qMU \wedge MU \wedge \cdots \wedge MU)\rightarrow E^0(MU \wedge \cdots \wedge MU)$ is a monomorphism because so is $EQ^0(Q_qMU \wedge MU \wedge \cdots \wedge MU)$ is a monomorphism. Therefore $E^0(MU \wedge \cdots \wedge MU)\rightarrow F_qMU \wedge \cdots \wedge MU)$ is an epimorphism, and hence $E^0(F_qMU \wedge MU \wedge \cdots \wedge MU)=0$.

Lemma 8. $KG_q(F_qMU \wedge MU \wedge \cdots \wedge MU)$ and $Q_pMU(F_qMU \wedge MU \wedge \cdots \wedge MU)$ are Hausdorff and $KG_q(F_qMU \wedge MU \wedge \cdots \wedge MU)=0$.

Proof. Putting $X=F_qMU \wedge MU \wedge \cdots \wedge MU$, we note by Theorem 1 i) that $K_q(X)$ is free and $K_q(X)=0$. Applying the universal coefficient sequence

$$
0 \rightarrow \text{Ext}(K_{q-1}(X), G_p) \rightarrow KG_p(X) \rightarrow \text{Hom}(K_q(X), G_p) \rightarrow 0
$$

for $K$ (see [9, (3.1)]) we get immediately that $ch: KG_p(X) \rightarrow KG_p \otimes Q^0(X)$ is a monomorphism and $KG_p(X)=0$. By induction on $p$ we obtain that $ch: Q_pMU^0(X) \rightarrow Q_pMUQ^0(X)$ is a monomorphism and $Q_pMU^0(X)=0$ because $0 \rightarrow KG_p \otimes Q^0(X) \rightarrow Q_pMUQ^0(X) \rightarrow Q_{p-1}MUQ^0(X) \rightarrow 0$ is exact.

Assume that $\pi_0(E)$ is free and of finite type and put again $X=F_qMU \wedge MU \wedge \cdots \wedge MU$. Using the universal coefficient sequence for $E$ [9, (1.8)] we have a commutative diagram

$$
\begin{array}{ccc}
0 \rightarrow \text{Ext}(\hat{E}_{q-1}(X), Z) & \rightarrow & E^0(X) \\
\downarrow & & \downarrow \\
0 \rightarrow \text{Ext}(\hat{E}_{q-1}(X), Q) & \rightarrow & EQ^0(X)
\end{array}
$$

with exact rows where $\hat{E}$ is the dual of $E$ constructed in [9]. Note that $\pi_0(\hat{E})$ is free and hence so is $\hat{E}_0(X)$. Then the central arrow becomes a monomorphism. Considering the commutative square
in which the upper arrow is an isomorphism, we find that the left one is a monomorphism. Thus we get that

\[(2.2) \quad \tilde{E}^t(F_q^*MU \wedge MU \wedge \cdots \wedge MU) \text{ is Hausdorff.}\]

Let \(\overline{MU}\) denote the mapping cone of the canonical morphism \(MU \to \overline{MU}\). Since \(\prod Z/\Sigma Z \to \prod Q/\Sigma Q\) is a monomorphism we remark that

\[(2.3) \quad \pi_0(\overline{MU}) = \prod \pi_{2m}(MU)/\Sigma^m \pi_{2m}(MU) \text{ is torsion free and } \pi_1(\overline{MU}) = 0,\]

(see [4, Exercise IV 20]). Then \(\overline{MU}\) has a unique structure of \(Z_2\)-graded \(CW\)-spectrum so that the cofiber sequence \(MU \to \overline{MU} \to \overline{MU}\) is of \(Z_2\)-graded \(CW\)-spectra.

**Lemma 9.** \(F_p^*MU(F_q^*MU \wedge MU \wedge \cdots \wedge MU)\) is Hausdorff and \(F_p^*MU(F_q^*MU \wedge MU \wedge \cdots \wedge MU) = 0\)

Proof. We put \(X = F_q^*MU \wedge MU \wedge \cdots \wedge MU\). From Lemma 7 it follows that \(F_p^*MU(X) = 0\). In the sequence

\[
F_p^*MU(X) \to MU^\circ(X) \to \overline{MU}^\circ(X)
\]

the former arrow is a monomorphism because of Lemma 8 and the latter one is so by means of (2.3) and Lemma 7. Thus the above composition is a monomorphism. On the other hand, (2.2) says that \(\overline{MU}^\circ(X)\) is Hausdorff. So we get the remaining result.

2.2. Since \(F_q^*MU(F_q^*MU)\) and \(Q_q^*MU(Q_q^*MU)\) are both Hausdorff, we verify that

\[(2.4) \quad \text{the } Z_2\text{-graded homology theories } I^*MU_*(\ ) \text{ and } MU_*(\ )/I^*+1MU_*(\ ) \text{ give } F_q^*MU \text{ and } Q_q^*MU \text{ unique structures of } Z_2\text{-graded } CW\text{-spectra respectively.}\]

Moreover, by virtue of Lemmas 7, 8 and 9 we see that

\[
(2.5) \quad i_q: F_{q+1}MU \to F_qMU, \quad \iota_q: F_{q+1}MU \to MU, \quad \mu_q: F_qMU \to KG_q
\]

\[
j_q: Q_qMU \to Q_{q-1}MU, \quad \pi_q: MU \to Q_qMU, \quad \nu_q: KG_q \to Q_qMU
\]

are uniquely determined (up to homotopy), which induce the canonical morphisms in homology groups. In particular, the composition \(\iota_{q-1} \cdot i_q\) is homotopic to \(\iota_q\) and \(j_q \cdot \pi_q\) is so to \(\pi_{q-1}\).
Consider the diagram consisting of cofiber sequences. With an application of Verdier's lemma (see [2, Lemma 6.8]) we get a cofiber sequence $K\gamma_\uparrow \frak{Q} \frak{M} \frak{U} \rightarrow \frak{Q} \frak{M} \frak{U} \rightarrow \frak{Q} \frak{M} \frak{U}$ which makes the diagram homotopy commutative. Clearly this yields the canonical exact sequence

$$0 \rightarrow K\gamma_\uparrow \frak{Q} \frak{M} \frak{U} \rightarrow \frak{Q} \frak{M} \frak{U} \rightarrow \frak{Q} \frak{M} \frak{U} \rightarrow 0.$$ By uniqueness of $\nu_\uparrow, j_\uparrow$, the above cofiber sequence coincides with $K\gamma_\uparrow \frak{Q} \frak{M} \frak{U} \rightarrow \frak{Q} \frak{M} \frak{U} \rightarrow \Sigma K\gamma_\uparrow$.

The multiplication $\phi: \frak{M} \frak{U} \wedge \frak{M} \frak{U} \rightarrow \frak{M} \frak{U}$ gives rise to natural $\mathbb{Z}_2$-graded homomorphisms

$$m_\uparrow: F_\frak{M} \frak{U}(X) \otimes \frak{M} \frak{U}(Y) \rightarrow F_\frak{M} \frak{U}(X \wedge Y)$$

$$\bar{m}_\uparrow: Q_\frak{M} \frak{U}(X) \otimes \frak{M} \frak{U}(Y) \rightarrow Q_\frak{M} \frak{U}(X \wedge Y)$$

for all CW-spectra $X$ and $Y$. By use of Lemmas 7 and 9 there exist unique pairings

$$\phi_\uparrow: F_\frak{M} \frak{U} \wedge \frak{M} \frak{U} \rightarrow F_\frak{M} \frak{U}, \quad \bar{\phi}_\uparrow: Q_\frak{M} \frak{U} \wedge \frak{M} \frak{U} \rightarrow Q_\frak{M} \frak{U}$$

which induce the above $m_\uparrow$ and $\bar{m}_\uparrow$ respectively. Then it follows that

(2.6) both $F_\frak{M} \frak{U}$ and $Q_\frak{M} \frak{U}$ are (associative) $\mathbb{Z}_2$-graded $\frak{M} \frak{U}$-module spectra.

**Proposition 10.** Let $M$ be a $\mathbb{Z}_2$-graded ring spectrum, $E$, $F$ and $G$ $\mathbb{Z}_2$-graded $M$-module spectra and $E \rightarrow F \rightarrow G$ a cofiber sequence. Assume that $E^0(E)$, $E^0(E \wedge M)$, $G^0(F)$ and $G^0(F \wedge M)$ are Hausdorff, or that $F^0(E)$, $F^0(E \wedge M)$, $G^0(G)$ and $G^0(G \wedge M)$ are Hausdorff. If for any CW-spectrum $X 0 \rightarrow E_\frak{Q}(X) \rightarrow F_\frak{Q}(X) \rightarrow G_\frak{Q}(X) \rightarrow 0$ is a short exact sequence of $\mathbb{Z}_2$-graded $M$-modules, then the cofiber sequence $E \rightarrow F \rightarrow G$ is of $\mathbb{Z}_2$-graded $M$-module spectra.

**Proof.** Assuming that $F^0(E)$, $F^0(E \wedge M)$, $G^0(G)$ and $G^0(G \wedge M)$ are Hausdorff, we consider the diagrams

$$\Sigma^2 E \rightarrow \Sigma^2 F \rightarrow \Sigma^2 G \rightarrow \Sigma^2 E \quad E \wedge M \rightarrow F \wedge M \rightarrow G \wedge M \rightarrow \Sigma E \wedge M$$

$$E \rightarrow F \rightarrow G \rightarrow \Sigma E \quad E \rightarrow F \rightarrow G \rightarrow \Sigma E$$
with cofiber sequences. Under the first two assumptions two left squares become homotopy commutative because they induce the $\mathbb{Z}_2$-graded homomorphism $E_\ell(\ ) \to F_\ell(\ )$ of $M_\ell(\ )$-modules. Therefore there exist morphisms $\Sigma^i G \to G$ and $G \wedge M \to G$ which make the above diagrams into morphisms of cofiber sequences. As is easily checked, they give $G_\ell(\ )$ a structure of $\mathbb{Z}_2$-graded $M_\ell(\ )$-module, which coincides with the original one. So, using the remaining assumptions again we see that the above morphisms are homotopic to the given ones respectively. Consequently the cofiber sequence $E \to F \to G$ becomes the required one.

Another case is similarly proved.

The ring spectrum $K$ may be regarded as a $\mathbb{Z}_2$-graded $MU$-module spectrum via the $\mathbb{Z}_2$-graded Thom map $\mu_c : MU \to K$.

Applying Proposition 10 to three cofiber sequences of (1.6) we get

**Theorem 2.** The sequences $F_{q+1}MU \to MU \to Q_qMU$, $F_{q+1}MU \to F_qMU \to KG_q$ and $KG_q \to Q_qMU \to Q_{q-1}MU$ are cofiber sequences of $\mathbb{Z}_2$-graded $MU$-module spectra.

Proof. The assumptions needed in Proposition 10 are satisfied by Lemmas 7, 8 and 9.

As a result we have a tower

$$(2.7) \quad MU \to \cdots \to Q_qMU \to Q_{q-1}MU \to \cdots \to Q_0MU = K$$

of $\mathbb{Z}_2$-graded $MU$-module spectra such that $KG_q \to Q_qMU \to Q_{q-1}MU$ is a cofiber sequence, which factorizes the $\mathbb{Z}_2$-graded Thom map $\mu_c : MU \to K$.

2.3. Here we extend the Wolff’s result to the case of based CW-complexes.

**Proposition 11.** There exists an (unstable) natural homomorphism

$$\Phi_q : KG^*_q(X) \to F_qMU^*(X)$$

for any based CW-complex $X$, which satisfies the equality that $\mu_q \circ \Phi_q = \text{id}$.

Proof. We may assume that $X$ is connected. Let $i : BU_{G_e} \to KG_q$ be the inclusion. Then we can choose a morphism $c_q : BU_{G_q} \to F_qMU$ such that $i$ is homotopic to the composition $\mu_q \circ c_q$, because $F_{q+1}MU^*(BU_{G_q}) = 0$. In the commutative diagram

$$\begin{align*}
[X, BU_{G_q}] &\xrightarrow{J_q} \{X, BU_{G_q}\} \xrightarrow{i_*} \{X, KG_q\} = KG^*_q(X) \\
&\xrightarrow{\mu_q \circ c_q} \{X, F_qMU\} = F_qMU^*(X)
\end{align*}$$
the composition $j_* \cdot J_*$ is an isomorphism because of Proposition 2 (see [6, Theorem 14.5]). So we put that $\Phi_q = e_q \cdot J_*(i_* \cdot J_*)^{-1}$.

**Remark.** If $\Phi_q$ is stable, then we have a natural split exact sequence

$$0 \to F_{q+1}MU^*(X) \to F_qMU^*(X) \to KG^*_q(X) \to 0$$

for every $CW$-spectrum $X$. Therefore $F_qMU$ becomes homotopy equivalent to the wedge $F_{q+1}MU \vee KG_q$. However $H_\oplus(KG_q)$ is a free abelian group and $H_\oplus(KG_q)$ is a $Q$-module. This is a contradiction.

We now obtain our main result.

**Theorem 3.** For any based $CW$-complex $X$ the natural sequences

$$0 \to F_{q+1}MU^*(X) \to F_qMU^*(X) \to KG^*_q(X) \to 0$$

$$0 \to KG^*_q(X) \to Q_qMU^*(X) \to Q_{q-1}MU^*(X) \to 0$$

of $Z_2$-graded $\Lambda$-modules are split exact.

Proof. The first case is immediate from Proposition 11. On the other hand, a diagram chase shows that the second sequence is exact for any based $CW$-complex $X$ and hence it is split.

**Appendix**

Recall that $MU$ is a ring spectrum with coefficients $\Lambda = \mathbb{Z}[x_1, \ldots, x_n, \ldots]$ where $\deg x_n = 2n$. By killing certain bordism classes Baas [3] constructed homology theories $MU_{\langle n \rangle}$ with coefficient $\pi_* MU_{\langle n \rangle} = \Lambda((x_{n+1}, \ldots)$, whose representing spectrum we denote by $MU_{\langle n \rangle}$. $MU_{\langle n \rangle}$ is an (associative) $MU_*$-module spectrum, thus there exists a natural homomorphism

$$m_*: MU_*(X) \otimes MU_{\langle n \rangle} \to MU_{\langle n \rangle}(X \wedge Y)$$

for any $CW$-spectra $X$ and $Y$. This gives us a pairing

$$\phi_\langle n \rangle: MU \wedge MU_{\langle n \rangle} \to MU_{\langle n \rangle}$$

by which the above $m_*$ is induced.

An easy computation shows that $MU_{\langle n \rangle} \otimes Z^{2^{k-1}}(MU \wedge \cdots \wedge MU \wedge MU_{\langle n \rangle}) = 0$ because $\pi_{2^{k-1}}(MU_{\langle n \rangle}) = 0$ for all $l$. Then [8, Theorem 1] says that

(A.1) $MU_{\langle n \rangle} \otimes (MU \wedge \cdots \wedge MU \wedge MU_{\langle m \rangle})$ is Hausdorff (see also [9, Corollary 13]).

Hence $\phi_\langle n \rangle$ is uniquely determined (up to homotopy) and moreover

(A.2) $MU_{\langle n \rangle}$ is an (associative) $MU$-module spectrum.
An important relationship between \( MU \langle n \rangle \ast ( ) \) and \( MU \langle n-1 \rangle \ast ( ) \) is given in the form of a natural exact sequence

\[
\rightarrow MU \langle n \rangle \ast_{-2n}(X) \xrightarrow{\cdot x_n} MU \langle n \rangle \ast(X) \xrightarrow{t_n} MU \langle n-1 \rangle \ast(X) \rightarrow
\]

of \( MU \ast( ) \)-modules where \( \cdot x_n \) denotes the multiplication by \( x_n \). Because of (A.1) there exists a unique morphism \( \tau_n: MU \langle n \rangle \rightarrow MU \langle n-1 \rangle \) of \( MU \)-module spectra whose induced homomorphism is the above \( t_n \). On the other hand, the composition

\[
m_{x_n}: \Sigma^m MU \langle n \rangle \xrightarrow{x_n \wedge 1} MU \wedge MU \langle n \rangle \xrightarrow{\phi \langle n \rangle} MU \langle n \rangle
\]
is characterized by the above multiplication \( \cdot x_n \).

**Lemma A.** Let \( E \xrightarrow{f} F \xrightarrow{g} G \) be a sequence of CW-spectra such that the composition \( g \circ f \) is homotopic to the zero. If \( 0 \rightarrow \pi_\ast(E) \rightarrow \pi_\ast(F) \rightarrow \pi_\ast(G) \rightarrow 0 \) is exact, then \( E \rightarrow F \rightarrow G \) is a cofiber sequence. (Cf., Lemma 4).

**Proof.** Let \( C_f \) be the mapping cone of \( f: E \rightarrow F \). Then \( g: F \rightarrow G \) admits a factorization \( F \rightarrow C_f \rightarrow G \). Considering the commutative diagram

\[
0 \rightarrow \pi_\ast(E) \rightarrow \pi_\ast(F) \rightarrow \pi_\ast(C_f) \rightarrow 0
\]

\[
0 \rightarrow \pi_\ast(E) \rightarrow \pi_\ast(F) \rightarrow \pi_\ast(G) \rightarrow 0
\]

with exact rows, we see easily that \( h: C_f \rightarrow G \) is a homotopy equivalence.

Using (A.1) the composition \( \tau_n \cdot m_{x_n} \) becomes homotopic to the zero. We get therefore that

\[
(A.3) \quad \Sigma^m MU \langle n \rangle \xrightarrow{m_{x_n}} MU \langle n \rangle \xrightarrow{\tau_n} MU \langle n-1 \rangle \text{ is a cofiber sequence.}
\]

OSAKA CITY UNIVERSITY

**References**


