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A NOTE ON THE RELATION OF $\mathbb{Z}_2$-GRADED COMPLEX COBORDISM TO COMPLEX K-THEORY

Zen-ichi Yosimura

(Received December 16, 1974)

Let $MU^*(\ )$ and $K^*(\ )$ denote the $\mathbb{Z}_2$-graded complex cobordism theory and the complex $K$-theory respectively. The Thom homomorphism $\mu_\ast: \pi_0(MU) \to \pi_0(K)$ on coefficient groups is identified (up to sign) with the classical Todd genus $Td: \Lambda \to \mathbb{Z}$. We denote by $I$ the ideal of $\Lambda$ to be the kernel of $Td: \Lambda \to \mathbb{Z}$. Wolff [7] proved that the decreasing filtration $\{I^qMU^*(\ )\}$ of $MU^*(\ )$ consists of cohomology theories defined on the category of based finite $CW$-complexes, and the associated quotients $I^qMU^*(\ )/I^{q+1}MU^*(\ )$ are determined by the complex $K$-theories $KG^q(\ )$ with coefficients $G_q = I^q/I^{q+1}$.

The purpose of this note is to extend the Wolff's result to the category of based $CW$-complexes. Let $F_qMU$ be the $CW$-spectrum associated with the cohomology theory $I^qMU^*(\ )$, i.e., $\{Y, F_qMU\}^* \simeq I^qMU^*(Y)$ for any based finite $CW$-complex (or finite $CW$-spectrum). We show that $\{F_qMU^*(\ )\}$ is a decreasing filtration of $MU^*(\ )$ consisting of $\Lambda$-modules so that the associated quotients are equal to $KG^q(\ )$, and in addition that $F_{q+1}MU^*(\ )$ is a direct summand of $F_qMU^*(\ )$.

Moreover we give a tower

$$MU \to \cdots \to Q_qMU \to Q_{q-1}MU \to \cdots \to Q_0MU = K$$

of $MU$-module spectra such that $KG_q \to Q_qMU \to Q_{q-1}MU$ is a cofiber sequence of $MU$-module spectra, which factorizes the Thom map $\mu_\ast: MU \to K$.

Baas [3] constructed a tower of $CW$-spectra

$$MU \to \cdots \to MU<n> \to MU<n-1> \to \cdots \to MU<0> = H$$

factorizing the Thom map $\mu_\ast: MU \to H$. In appendix we show that the tower is of $MU$-module spectra and the sequence $\Sigma^sMU<n>\xrightarrow{m_s}MU<n>\xrightarrow{x_n}MU<n-1>$ is a cofiber sequence where $m_s$ is the multiplication by $x_n$ a ring generator of $\Lambda$ with degree $2n$.

1. Decreasing filtration of $MU^*(\ )$

1.1. A pair $(E, \rho)$ is called a $\mathbb{Z}_2$-graded $CW$-spectrum if $E$ is a $CW$-spectrum.
and $\rho: \Sigma^2 E \rightarrow E$ is a homotopy equivalence. Such a pair $(E, \rho)$ gives rise to natural isomorphisms

$$\rho_*: E_*(X) \rightarrow E_{*+2}(X), \quad \rho^*: E^{*+2}(X) \rightarrow E^*(X)$$

for any CW-spectrum $X$. So we can define $\mathbb{Z}_2$-graded homology and cohomology theories $E_*(\_), E^*(\_)$ by putting

$$E_0(X) = E_0(X) \oplus E_1(X), \quad E^0(X) = E^0(X) \oplus E^1(X).$$

For a CW-spectrum $E$ we put

$$E = \bigvee \Sigma^{2n} E, \quad \overline{E} = \prod \Sigma^{2n} E.$$

Taking the canonical identifications $\rho: \Sigma^2 E \rightarrow E$ and $\overline{\rho}: \Sigma^2 \overline{E} \rightarrow \overline{E}$ as structure morphisms $E$ and $\overline{E}$ admit structures of $\mathbb{Z}_2$-graded CW-spectra respectively. From definition it follows that

$$E_0(X) \cong \bigvee \Sigma^{2n} E_{2n}(X), \quad E_1(X) \cong \bigvee \Sigma^{2n+2} E_{2n+2}(X),$$

$$\overline{E}^0(X) \cong \prod E^{2n}(X), \quad \overline{E}^1(X) \cong \prod E^{2n+2}(X)$$

for all CW-spectra $X$. In particular, the canonical morphism $H \rightarrow \overline{H}$ becomes a homotopy equivalence for the Eilenberg-MacLane spectrum $H$.

The $BU$-spectrum $K$ may be regarded as a $\mathbb{Z}_2$-graded CW-spectrum because it possesses the Bott map $\beta: \Sigma^2 K \rightarrow K$ which is a homotopy equivalence.

Denote by $F_n$ the direct sum of $n$-copies of the integers $\mathbb{Z}$ and by $F$ the direct limit of $F_n$, i.e., $F$ is a free abelian group with countably many factors. Putting

$$BU_{F_n} = BU \times \cdots \times BU,$$

the product of $n$-copies of $BU$, $BU_F = \bigcup_n BU_{F_n},$ the union of $BU_{F_n},$

we obtain

**Proposition 1.** There exists a natural isomorphism

$$[X, BU_F] \rightarrow KF^0(X)$$

for any based connected CW-complex $X$.

**Proof.** Let $Y$ be a based connected finite CW-complex. Then we have a sequence of natural isomorphisms

$$[Y, BU_F] \leftarrow \lim [Y, BU_{F_n}] \leftarrow \lim [Y, BU] \otimes F_n \rightarrow \lim K^0(\ Y) \otimes F_n \rightarrow K^0(Y) \otimes F \rightarrow KF^0(Y).$$

Therefore the contravariant functor $KF^0$ defined on the category of based connected CW-complexes is represented by $BU_F$ (use [1, Addendum 1.5]).
Proposition 1 implies that \( BU_F \) is homotopy equivalent to \( \Omega^3 BU_F \) where \( \Omega^3 \) means the component of the base point in the double loop space. Hence we have

\[
\text{in the \( BU \)-spectrum } KF \text{ with the coefficients } F \text{ every even term is the based } CW\text{-complex } BU_F.
\]

1.2. Let us denote by \( MU \) the unitary Thom spectrum and by \( \mu_c: MU \to K \) the Thom map which is a ring morphism. The composition

\[
\mu_c: MU \to K \to K
\]

of \( \sqrt{\Sigma}_n \mu_c \) and \( \sqrt{\beta} \) is a morphism of \( \mathbb{Z}_2 \)-graded ring-spectra, called the \( \mathbb{Z}_2 \)-graded Thom map. As is well known, it is characterized by the coefficient homomorphism \( H_\Sigma: \pi_\bullet(MU) \to \pi_\bullet(K) \) which coincides (up to sign) with the classical Todd genus \( Td \). Putting \( \Lambda = \pi_\bullet(MU), \pi_\bullet(K) = \mathbb{Z} \) is viewed as a \( \mathbb{Z}_2 \)-graded \( \Lambda \)-module via \( \mu_c = Td \) and it is written \( \mathbb{Z} Td \) for emphasis.

Using the kernel \( I \) of \( Td: \Lambda \to \mathbb{Z} \) we define a decreasing filtration \( \{I^q\}_{q \geq 0} \) consisting of ideals of \( \Lambda \). Denoting by \( G_q \) the associated \( \mathbb{Z}_2 \)-graded \( \Lambda \)-module \( I^q/I^{q+1} \), we see easily [7, Satz 3.8] that

\[
G_0 \cong Z_{Td} \text{ and } G_q \text{ is a free abelian group with countably many factors for } q \geq 1.
\]

For a \( \mathbb{Z}_2 \)-graded \( \Lambda \)-module \( A \) we have a decreasing filtration \( \{I^q A\}_{q \geq 0} \) consisting of submodules of \( A \), whose associated \( \mathbb{Z}_2 \)-graded \( \Lambda \)-module \( I^q A/I^{q+1} A \) is written \( G_q(A) \). Applying the commutative diagram

\[
0 \to \text{Tor}_\Lambda^1(Z_{Td}, A) \to I \otimes A \to A \to Z_{Td} \otimes A \to 0
\]

with exact rows, we get an isomorphism

\[
G_q \otimes A \cong G_q \otimes (Z_{Td} \otimes A) \cong G_q \otimes G_0(A)
\]

by means of "4 lemma".

**Proposition 2.** Let \( A \) be a \( \Lambda \)-module with \( \text{Tor}_\Lambda^k(Z_{Td}, A) = 0 \) for all \( k \geq 1 \). Then, for every \( q \geq 0 \) both \( I^q \otimes A \to I_q A \) and \( G_q \otimes A \to G_q(A) \) are isomorphisms and \( \text{Tor}_\Lambda^k(I^q, A) = \text{Tor}_\Lambda^k(G_q, A) = 0 \) for all \( k \geq 1 \).

Proof. Choose a free \( \Lambda \)-module \( F \) such that \( A \) is isomorphic to a quotient \( F/B \). By induction on \( q \) we shall show that the sequences

\[
0 \to I^q B \to I^q F \to I^q A \to 0, \quad 0 \to G_q(B) \to G_q(F) \to G_q(A) \to 0
\]

are exact. The \( q=0 \) case is evident because of (1.3). Applying induction
hypotesis and "3×3 lemma" we find easily that \(0 \to I^qB \to I^qF \to I^qA \to 0\) is exact. So we have a commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
G_\delta(I^qB) \\
\downarrow \\
G_\delta(I^qA)
\end{array}
\quad \begin{array}{c}
G_\delta \otimes G_\delta(B) \\
\downarrow \\
G_\delta \otimes G_\delta(F) \\
\downarrow \\
G_\delta \otimes G_\delta(A)
\end{array}
\quad \begin{array}{c}
G_\delta(B) \\
\downarrow \\
G_\delta(F) \\
\downarrow \\
G_\delta(A)
\end{array}
\]

with exact rows. Since all vertical arrows are epimorphisms and in particular the central one is an isomorphism, all vertical arrows become isomorphisms. Consequently we get that \(0 \to G_\delta(B) \to G_\delta(F) \to G_\delta(A) \to 0\) is exact.

Next, we consider the commutative diagrams

\[
\begin{array}{c}
0 \\
\downarrow \\
G_\beta(\beta^q) \\
\downarrow \\
G_\beta(\beta^q)
\end{array}
\quad \begin{array}{c}
0 \\
\downarrow \\
G_\beta \otimes G_\delta(B) \\
\downarrow \\
G_\beta \otimes G_\delta(A)
\end{array}
\quad \begin{array}{c}
0 \\
\downarrow \\
G_\beta \otimes G_\delta(B) \\
\downarrow \\
G_\beta \otimes G_\delta(A)
\end{array}
\quad \begin{array}{c}
0 \\
\downarrow \\
G_\beta \otimes G_\delta(F) \\
\downarrow \\
G_\beta \otimes G_\delta(A)
\end{array}
\]

with exact rows. Remark that \(\text{Tor}^\Lambda(I^q, A) = 0\) for all \(k \geq 1\). By use of "4 lemma" and (1.3) we see that all vertical arrows are isomorphisms, and hence we obtain the required results.

For a \(\Lambda\)-module \(A\) we put \(J_\delta(A) = A/\langle I^q+1 \rangle A\) and abbreviate \(J_\delta = J_\delta(\Lambda)\) when \(A = \Lambda\). As an immediate corollary of Proposition 2 we have

**Corollary 3.** Let \(A\) be a \(\Lambda\)-module with \(\text{Tor}^\Lambda(I^q, A) = 0\) for all \(k \geq 1\). Then \(J_\delta \otimes A \to J_\delta(A)\) is an isomorphism and \(\text{Tor}^\Lambda(J_\delta, A) = 0\) for all \(k \geq 1\).

1.3. Let \(\mathcal{MU}\) denote the category of comodules over \(MU_\ast(MU)\) which are finitely presented as \(\Lambda\)-modules. Notice that \(\mathcal{MU}\) is an abelian category which has enough projectives, and also that \(MU_\ast(Y)\) lies in the category \(\mathcal{MU}\) whenever \(Y\) is a finite CW-spectrum. Since the functor \(M \to Z_T \otimes M\) is exact on \(\mathcal{MU}\) [5, Example 3.3] it follows immediately that

\[
\text{Tor}^\Lambda(Z_T, M) = 0 \quad \text{for all } k \geq 1 \text{ if } M \text{ lies in } \mathcal{MU}.
\]

Proposition 1 and Corollary 2 combined with (1.4) say that

\[
\text{the functors } M \to I^qM = I^q \otimes M, M \to G_\delta(M) = G_\delta \otimes M \text{ and } M \to J_\delta(M) = J_\delta \otimes M \text{ on } \mathcal{MU} \text{ are exact}.
\]

**Theorem 1** (Wolff [7]). i) Both \(I^qMU_\ast(\ )\) and \(MU_\ast(\ )/I^{q+1}MU_\ast(\ )\) are
homology theories defined on the category of CW-spectra, so that $I^q \otimes \mathcal{M}U_*(X) \rightarrow I^q \mathcal{M}U_*(X)$ and $\Lambda_\mathcal{M}U_*(X) \otimes \mathcal{M}U_*(X) \rightarrow \mathcal{M}U_*(X)$ are natural isomorphisms for all CW-spectra $X$.

ii) $I^q \mathcal{M}U_*(X) / I^{q+1} \mathcal{M}U_*(X)$ is a homology theory defined on the category of CW-spectra such that there exists a natural isomorphism $I^q \mathcal{M}U_*(X) / I^{q+1} \mathcal{M}U_*(X) \rightarrow K_G^*(X)$ for any CW-spectrum $X$ which is induced by the $Z_2$-graded Thom map $\mu_c$.

Proof. i) and the first half of ii) are immediate from (1.5). The latter half of ii) is also valid because we have a natural isomorphism

$$G_q(\mathcal{M}U_*(X)) \overset{\sim}{\rightarrow} G_q \otimes \mathcal{M}U_*(X) \overset{\sim}{\rightarrow} G_q \otimes (Z_{Tq} \otimes \mathcal{M}U_*(X)) \rightarrow G_q \otimes K_*^*(X) \rightarrow K_G^*(X).$$

Let $\phi: E_*(X) \rightarrow F_*(X)$ be a natural transformation for any CW-spectrum $X$. According to [1, Addendum 1.5] there exists a morphism $f: E \rightarrow F$ inducing $\phi$, and it is unique up to weak homotopy. The proof in [1] is actually given for the category of based connected CW-complexes, but it is easily extended to that of CW-spectra. Such a morphism $f$ is uniquely chosen (up to homotopy) under the assumption that $F_0(E)$ is Hausdorff.

Let $E_*(\ )$ be a $Z_2$-graded homology theory defined on the category of CW-spectra, i.e., a homology theory equipped with a natural isomorphism $E_*(X) \rightarrow E_*(+X)$ for any CW-spectrum $X$. Then it gives $E$ a structure of $Z_2$-graded CW-spectrum. In particular, the induced structure is unique if $E_0(E)$ is Hausdorff.

Recall that the $Z_2$-graded CW-spectrum $\mathcal{M}U$ is equipped with the canonical identification $p: \Sigma \mathcal{M}U \rightarrow \mathcal{M}U$ as structure morphism. Since $\mathcal{M}U^*(\mathcal{M}U)$ is Hausdorff (use Proposition 6 below), the $Z_2$-graded CW-spectrum $(\mathcal{M}U, p)$ is characterized only by the $Z_2$-graded homology theory $\mathcal{M}U_*(\ )$.

Denote by $F_q \mathcal{M}U$ and $Q_q \mathcal{M}U$ the representing spectra of the new homology theories $I^q \mathcal{M}U_*(\ )$ and $\mathcal{M}U_*(\ ) / I^{q+1} \mathcal{M}U_*(\ )$ respectively, i.e.,

$$I^q \mathcal{M}U_*(X) \simeq \{\Sigma^*, F_q \mathcal{M}U \wedge X\}, \quad \mathcal{M}U_*(X) / I^{q+1} \mathcal{M}U_*(X) \simeq \{\Sigma^*, Q_q \mathcal{M}U \wedge X\}$$

for any CW-spectrum $X$. Of course, they are both $Z_2$-graded CW-spectra. Then there exist morphisms

$$i_q: F_q +_1 \mathcal{M}U \rightarrow F_q \mathcal{M}U, \quad j_q: Q_q \mathcal{M}U \rightarrow Q_{q-1} \mathcal{M}U, \quad i_q: F_q +_1 \mathcal{M}U \rightarrow \mathcal{M}U, \quad \pi_q: \mathcal{M}U \rightarrow Q_q \mathcal{M}U$$

which induce the canonical morphisms in homology groups, and moreover we have morphisms
\[ \mu_q: F_qMU \to KG_q, \quad \nu_q: KG_q \to Q_qMU \]
such that \( \mu_q: F_qMU \otimes (X) \to KG_q \otimes (X) \) and \( \nu_q: KG_q \otimes (X) \to Q_qMU \otimes (X) \) in homology groups are natural homomorphisms induced by the \( \mathbb{Z}_2 \)-graded Thom map \( \mu_c \).

**Lemma 4.** Let \( E \to F \to G \) be a sequence which satisfies the property that \( 0 \to E_\ast(X) \to F_\ast(X) \to G_\ast(X) \to 0 \) is a short exact sequence for every CW-spectrum \( X \). Then it is a cofiber sequence.

**Proof.** Let \( C_f \) be the mapping cone of \( f \), i.e., \( E \to F \to C_f \) a cofiber sequence. Then for any CW-spectrum \( X \) we have a commutative diagram

\[
\begin{array}{ccc}
0 & \to & E_\ast(X) \\
\downarrow & & \downarrow \phi = h_\ast \\
0 & \to & F_\ast(X) \to G_\ast(X) \\
\end{array}
\]

with exact rows. Clearly \( h: C_f \to G \) which induces \( \phi \) is a homotopy equivalence.

By virtue of Lemma 4 we verify that

\[ (1.6) \quad F_{q+1}MU \to MU \to Q_qMU, \quad F_{q+1}MU \to F_qMU \to KG_q \text{ and } KG_q \to Q_qMU \to Q_{q-1}MU \text{ are all cofiber sequences.} \]

**2. \( \mathbb{Z}_2 \)-graded \( MU \)-module spectra**

**2.1.** The inclusion \( \mathbb{Z} \subset Q \) induces a natural transformation \( ch: E^\ast(X) \to EQ^\ast(X) \) for any CW-spectrum \( X \), called the Chern-Dold character.

**Proposition 5.** If \( ch: E^\ast(X) \to EQ^\ast(X) \) is a monomorphism, then \( E^\ast(X) \) is Hausdorff.

**Proof.** Since \( EQ^\ast(X) \) is always Hausdorff [8, Proposition 4], the result is immediate.

Let \( W \) be a connective CW-spectrum with \( H_\ast(W) \) free and assume that \( \pi_\ast(E) \) is torsion free. Then \( H^\ast(W; \pi_\ast(E)) \to H^\ast(W; \pi_\ast(E) \oplus Q) \) is a monomorphism, and hence the Atiyah-Hirzebruch spectral sequences for \( E^\ast(W) \) and \( EQ^\ast(W) \) collapse. Therefore we get that

\[ (2.1) \quad ch: E^\ast(W) \to EQ^\ast(W) \text{ is a monomorphism. (Cf., [8, Lemma 11]).} \]

Applying Proposition 5 we obtain

**Proposition 6.** Let \( W \) be a connective CW-spectrum with \( H_\ast(W) \) free. If \( \pi_\ast(E) \) is torsion free, then \( E^\ast(W) \) is Hausdorff.

By means of Proposition 5 we get the following lemmas.
Lemma 7. Assume that $\pi_0(E)$ is torsion free and $\pi_1(E)=0$. Then $E^\ast(KG_q \wedge MU \wedge \cdots \wedge MU)$ and $E^\ast(Q_q MU \wedge MU \wedge \cdots \wedge MU)$ are Hausdorff and $E^\ast(F_q MU \wedge MU \wedge \cdots \wedge MU)=0$.

Proof. Since $E^{2n-1}(BU_G \wedge MU \wedge \cdots \wedge MU)=EQ^{2n-1}(BU_G \wedge MU \wedge \cdots \wedge MU)=0$ we have a commutative square

$$E^\ast(KG_q \wedge MU \wedge \cdots \wedge MU) \rightarrow \lim E^{2n}(BU_G \wedge MU \wedge \cdots \wedge MU)$$

$$E^\ast(KG_q \wedge MU \wedge \cdots \wedge MU) \rightarrow \lim E^{2n}(BU_G \wedge MU \wedge \cdots \wedge MU)$$

such that the horizontal arrows are isomorphisms. The left arrow is a monomorphism because so is the right one by use of (2.1). Since Theorem 1 implies that $0 \rightarrow EQ^\ast(Q_q MU \wedge MU \wedge \cdots \wedge MU) = EQ^\ast(Q_q MU \wedge MU \wedge \cdots \wedge MU) \rightarrow 0$ is exact, an induction on $q$ involving "4 lemma" shows that $E^\ast(Q_q MU \wedge MU \wedge \cdots \wedge MU)$ is a monomorphism. Then we find that $E^\ast(Q_q MU \wedge MU \wedge \cdots \wedge MU) \rightarrow E^\ast(MU \wedge \cdots \wedge MU)$ is a monomorphism because so is $EQ^\ast(Q_q MU \wedge MU \wedge \cdots \wedge MU) \rightarrow EQ^\ast(MU \wedge \cdots \wedge MU)$. Therefore $E^\ast(MU \wedge \cdots \wedge MU) \rightarrow E^\ast(F_q+1 MU \wedge MU \wedge \cdots \wedge MU)$ is an epimorphism, and hence $E^\ast(F_q+1 MU \wedge MU \wedge \cdots \wedge MU)=0$.

Lemma 8. $KG^\ast(F_q MU \wedge MU \wedge \cdots \wedge MU)$ and $Q_p MU^\ast(F_q MU \wedge MU \wedge \cdots \wedge MU)$ are Hausdorff and $KG^\ast(F_q MU \wedge MU \wedge \cdots \wedge MU)=Q_p MU^\ast(F_q MU \wedge MU \wedge \cdots \wedge MU)=0$.

Proof. Putting $X=F_q MU \wedge MU \wedge \cdots \wedge MU$, we note by Theorem 1 i) that $K_q(X)$ is free and $K_{q}(X)=0$. Applying the universal coefficient sequence

$$0 \rightarrow Ext(K_{q+1}(X), G_p) \rightarrow KG^\ast_p(X) \rightarrow Hom(K_q(X), G_p) \rightarrow 0$$

for $K$ (see [9, (3.1)]) we get immediately that $ch: KG^\ast_p(X) \rightarrow KG^\ast_p(X)$ is a monomorphism and $KG^\ast_p(X)=0$. By induction on $p$ we obtain that $ch: Q_p MU^\ast(X) \rightarrow Q_p MU^\ast(X)$ is a monomorphism and $Q_p MU^\ast(X)=0$ because $0 \rightarrow KG^\ast_p \otimes Q^\ast(X) \rightarrow Q_p MU^\ast(X) \rightarrow 0$ is exact.

Assume that $\pi_0(E)$ is free and of finite type and put again $X=F_q MU \wedge MU \wedge \cdots \wedge MU$. Using the universal coefficient sequence for $E$ [9, (1.8)] we have a commutative diagram

$$0 \rightarrow Ext(\hat{E}_{q+1}(X), Z) \rightarrow E^\ast(X) \rightarrow Hom(\hat{E}_q(X), Z) \rightarrow 0$$

$$0 \rightarrow Ext(\hat{E}_{q+1}(X), Q) \rightarrow EQ^\ast(X) \rightarrow Hom(\hat{E}_q(X), Q) \rightarrow 0$$

with exact rows where $\hat{E}$ is the dual of $E$ constructed in [9]. Note that $\pi_0(\hat{E})$ is free and hence so is $\hat{E}_q(X)$. Then the central arrow becomes a monomorphism. Considering the commutative square
in which the upper arrow is an isomorphism, we find that the left one is a monomorphism. Thus we get that

$$E^t(X) \rightarrow \Pi E^n(X)$$

$$E^Q(X) \rightarrow \Pi E^Q(X)$$

Let $\overline{MU}$ denote the mapping cone of the canonical morphism $MU \rightarrow \overline{MU}$. Since $\Pi Z/\Sigma Z \rightarrow \Pi Q/\Sigma Q$ is a monomorphism we remark that

$$(2.3) \quad \tau_0(\overline{MU}) = \prod \tau_0(MU)/\Sigma \tau_0(MU) \text{ is torsion free and } \tau_1(\overline{MU}) = 0,$$

(see [4, Exercise IV 20]). Then $\overline{MU}$ has a unique structure of $Z_2$-graded CW-spectrum so that the cofiber sequence $MU \rightarrow \overline{MU} \rightarrow \overline{MU}$ is of $Z_2$-graded CW-spectra.

**Lemma 9.** $F_pMU^n(F_qMU \wedge MU \wedge \cdots \wedge MU)$ is Hausdorff and $F_pMU^n(F_qMU \wedge MU \wedge \cdots \wedge MU) = 0$

Proof. We put $X = F_qMU \wedge MU \wedge \cdots \wedge MU$. From Lemma 7 it follows that $F_pMU^n(X) = 0$. In the sequence $F_pMU^n(X) \rightarrow MU^n(X) \rightarrow \overline{MU}^n(X)$ the former arrow is a monomorphism because of Lemma 8 and the latter one is so by means of (2.3) and Lemma 7. Thus the above composition is a monomorphism. On the other hand, (2.2) says that $\overline{MU}^n(X)$ is Hausdorff. So we get the remaining result.

2.2. Since $F_qMU^n(F_qMU)$ and $Q_qMU^n(Q_qMU)$ are both Hausdorff, we verify that

$$(2.4) \quad \text{the } Z_2\text{-graded homology theories } I^qMU_* ( ) \text{ and } MU_* ( )/I^{q+1}MU_* ( ) \text{ give } F_qMU \text{ and } Q_qMU \text{ unique structures of } Z_2\text{-graded CW-spectra respectively.}$$

Moreover, by virtue of Lemmas 7, 8 and 9 we see that

$$(2.5) \quad i_q : F_{q+1}MU \rightarrow F_qMU, \quad \iota_q : F_{q+1}MU \rightarrow MU, \quad \mu_q : F_qMU \rightarrow KG_q$$

$$j_q : Q_{q+1}MU \rightarrow Q_{q}MU, \quad \pi_q : MU \rightarrow Q_qMU, \quad \nu_q : KG_q \rightarrow Q_qMU$$

are uniquely determined (up to homotopy), which induce the canonical morphisms in homology groups. In particular, the composition $\iota_{q-1} \circ i_q$ is homotopic to $i_q$ and $j_q \circ \pi_q$ is so to $\pi_{q-1}$. 
Consider the diagram

\[ \begin{array}{cccc}
F_{q+1}MU & \to & F_qMU & \to \\
\downarrow & & \downarrow & \\
\Sigma F_{q+1}MU & \to & \Sigma F_qMU & \\
\end{array} \]

consisting of cofiber sequences. With an application of Verdier's lemma (see [2, Lemma 6.8]) we get a cofiber sequence

\[ KG_q \to Q_qMU \to Q_{q-1}MU \to \Sigma KG_q \]

(denoted by dotted arrows in the above diagram) which makes the diagram homotopy commutative. Clearly this yields the canonical exact sequence

\[ 0 \to KG_q(X) \to Q_qMU(Y) \to Q_{q-1}MU(Y) \to 0. \]

By uniqueness of \( \nu_q, j_q \) the above cofiber sequence coincides with

\[ KG_q \to Q_qMU \to Q_{q-1}MU \to \Sigma KG_q. \]

The multiplication \( \phi : MU \wedge MU \to MU \) gives rise to natural \( Z_2 \)-graded homomorphisms

\[ m_q : F_qMU(X) \otimes MU(Y) \to F_qMU(X \wedge Y) \]
\[ \bar{m}_q : Q_qMU(X) \otimes MU(Y) \to Q_qMU(X \wedge Y) \]

for all CW-spectra \( X \) and \( Y \). By use of Lemmas 7 and 9 there exist unique pairings

\[ \phi_q : F_qMU \wedge MU \to F_qMU, \quad \overline{\phi}_q : Q_qMU \wedge MU \to Q_qMU \]

which induce the above \( m_q \) and \( \bar{m}_q \) respectively. Then it follows that

\( (2.6) \) both \( F_qMU \) and \( Q_qMU \) are (associative) \( Z_2 \)-graded \( MU \)-module spectra.

**Proposition 10.** Let \( M \) be a \( Z_2 \)-graded ring spectrum, \( E, F \) and \( G \) \( Z_2 \)-graded \( M \)-module spectra and \( E \to F \to G \) a cofiber sequence. Assume that \( E^\circ(E) \), \( E^\circ(E \wedge M) \), \( G^\circ(F) \) and \( G^\circ(F \wedge M) \) are Hausdorff, or that \( F^\circ(E), F^\circ(E \wedge M), G^\circ(G) \) and \( G^\circ(G \wedge M) \) are Hausdorff. If for any CW-spectrum \( X \to E(X) \to F(X) \to G(X) \to 0 \) is a short exact sequence of \( Z_2 \)-graded \( M \)-modules, then the cofiber sequence \( E \to F \to G \) is of \( Z_2 \)-graded \( M \)-module spectra.

**Proof.** Assuming that \( F^\circ(E), F^\circ(E \wedge M), G^\circ(G) \) and \( G^\circ(G \wedge M) \) are Hausdorff, we consider the diagrams

\[ \begin{array}{cccc}
\Sigma^2E & \to & \Sigma^2F & \to & \Sigma^2G & \to & \Sigma^2E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E & \to & F & \to & G & \to & \Sigma E \\
\end{array} \]

\[ \begin{array}{cccc}
E \wedge M & \to & F \wedge M & \to & G \wedge M & \to & \Sigma E \wedge M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E & \to & F & \to & G & \to & \Sigma E \\
\end{array} \]

\[ \begin{array}{cccc}
E^\circ(E) & \to & E^\circ(F) & \to & E^\circ(G) & \to & E^\circ(E) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E & \to & F & \to & G & \to & \Sigma E \\
\end{array} \]

\[ \begin{array}{cccc}
E^\circ(E \wedge M) & \to & E^\circ(F \wedge M) & \to & E^\circ(G \wedge M) & \to & E^\circ(E \wedge M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E & \to & F & \to & G & \to & \Sigma E \\
\end{array} \]
with cofiber sequences. Under the first two assumptions two left squares become homotopy commutative because they induce the $Z_2$-graded homomorphism $E_q(\ ) \to F_q(\ )$ of $M_q(\ )$-modules. Therefore there exist morphisms $\Sigma^G \to G$ and $G \wedge M \to G$ which make the above diagrams into morphisms of cofiber sequences. As is easily checked, they give $G_q(\ )$ a structure of $Z_2$-graded $M_q(\ )$-module, which coincides with the original one. So, using the remaining assumptions again we see that the above morphisms are homotopic to the given ones respectively. Consequently the cofiber sequence $E \to F \to G$ becomes the required one.

Another case is similarly proved.

The ring spectrum $K$ may be regarded as a $Z_2$-graded $MU$-module spectrum via the $Z_2$-graded Thom map $\mu_c: MU \to K$.

Applying Proposition 10 to three cofiber sequences of (1.6) we get

**Theorem 2.** The sequences $F_{q+1}MU \to MU \to Q_qMU$, $F_{q+1}MU \to F_qMU \to KG_q$ and $KG_q \to Q_qMU \to q_{-1}MU$ are cofiber sequences of $Z_2$-graded $MU$-module spectra.

**Proof.** The assumptions needed in Proposition 10 are satisfied by Lemmas 7,8 and 9.

As a result we have a tower

$$MU \to \cdots \to Q_qMU \to Q_{q-1}MU \to \cdots \to Q_0MU = K$$

of $Z_2$-graded $MU$-module spectra such that $KG_q \to Q_qMU \to Q_{q-1}MU$ is a cofiber sequence, which factorizes the $Z_2$-graded Thom map $\mu_c: MU \to K$.

2.3. Here we extend the Wolff’s result to the case of based CW-complexes.

**Proposition 11.** There exists an (unstable) natural homomorphism

$$\Phi_q: KG^*_q(X) \to F_qMU^*(X)$$

for any based CW-complex $X$, which satisfies the equality that $\mu_q \cdot \Phi_q = \text{id}$.

**Proof.** We may assume that $X$ is connected. Let $i: BU_G \to KG_G$ be the inclusion. Then we can choose a morphism $c_q: BU_G \to F_qMU$ such that $i$ is homotopic to the composition $\mu_q \cdot c_q$, because $F_{q+1}MU^*(BU_G) = 0$. In the commutative diagram

$$\begin{array}{c}
[X, BU_G] \xrightarrow{J_x} \{X, BU_G\} \xrightarrow{i_*} \{X, KG_G\} = KG^*_x(X) \\
\uparrow \mu_q \circ c_q^* \downarrow \mu_q \circ \Phi_q \\
\{X, F_qMU\} = F_qMU^*(X)
\end{array}$$
the composition $i_\ast \cdot J_\ast$ is an isomorphism because of Proposition 2 (see [6, Theorem 14.5]). So we put that $\Phi_\ast = c_\ast \cdot J_\ast \cdot (i_\ast \cdot J_\ast)^{-1}$.

**Remark.** If $\Phi_\ast$ is stable, then we have a natural split exact sequence

$$0 \to F_{q+1} \text{MU}^* (X) \to F_\ast \text{MU}^* (X) \to KG_\ast (X) \to 0$$

for every CW-spectrum $X$. Therefore $F_\ast \text{MU}$ becomes homotopy equivalent to the wedge $F_{q+1} \text{MU} \vee KG$. However $H_\ast (F_\ast \text{MU})$ is a free abelian group and $H_\ast (KG)$ is a $Q$-module. This is a contradiction.

We now obtain our main result.

**Theorem 3.** For any based CW-complex $X$ the natural sequences

$$0 \to F_{q+1} \text{MU}^* (X) \to F_\ast \text{MU}^* (X) \to KG_\ast (X) \to 0$$

$$0 \to KG_\ast (X) \to Q_\ast \text{MU}^* (X) \to Q_{q-1} \text{MU}^* (X) \to 0$$

of $Z_\ast$-graded $\Lambda$-modules are split exact.

**Proof.** The first case is immediate from Proposition 11. On the other hand, a diagram chase shows that the second sequence is exact for any based CW-complex $X$ and hence it is split.

**Appendix**

Recall that $\text{MU}$ is a ring spectrum with coefficients $\Lambda_\ast = \mathbb{Z}[x_1, \ldots, x_n, \ldots]$ where $\deg x_n = 2n$. By killing certain bordism classes Baas [3] constructed homology theories $\text{MU} \langle n \rangle_\ast$ with coefficient $\pi_\ast (\text{MU} \langle n \rangle) = \Lambda_\ast / (x_{n+1}, \ldots)$, whose representing spectrum we denote by $\text{MU} \langle n \rangle$. $\text{MU} \langle n \rangle_\ast$ is an (associative) $\text{MU}_\ast$-module, thus there exists a natural homomorphism

$$m_\ast : \text{MU} \langle n \rangle \otimes \text{MU} \langle m \rangle \to \text{MU} \langle n \rangle \otimes \text{MU} \langle m \rangle$$

for any CW-spectra $X$ and $Y$. This gives us a pairing

$$\phi \langle n \rangle : \text{MU} \wedge \text{MU} \langle n \rangle \to \text{MU} \langle n \rangle$$

by which the above $m_\ast$ is induced.

An easy computation shows that $\text{MU} \langle n \rangle \otimes \mathbb{Z}[z] \otimes \mathbb{Z} \langle x \rangle$ is Hausdorff (see also [9, Corollary 13]).

Hence $\phi \langle n \rangle$ is uniquely determined (up to homotopy) and moreover

(A.1) $\text{MU} \langle n \rangle \otimes (\text{MU} \wedge \cdots \wedge \text{MU} \wedge \text{MU} \langle m \rangle)$ is Hausdorff (see also [9, Corollary 13]).

(A.2) $\text{MU} \langle n \rangle$ is an (associative) $\text{MU}$-module spectrum.
An important relationship between $MU\langle n \rangle_\ast(\_)$ and $MU\langle n-1 \rangle_\ast(\_)$ is given in the form of a natural exact sequence

$$
\rightarrow MU\langle n \rangle_\ast-2n(X) \overset{\cdot x_n}{\longrightarrow} MU\langle n \rangle_\ast(X) \overset{t_n}{\longrightarrow} MU\langle n-1 \rangle_\ast(X) \rightarrow MU\langle n \rangle_\ast-2n-1(X) \rightarrow
$$
of $MU_\ast(\_)$-modules where $\cdot x_n$ denotes the multiplication by $x_n$. Because of (A.1) there exists a unique morphism $\tau_n: MU\langle n \rangle \rightarrow MU\langle n-1 \rangle$ of $MU$-module spectra whose induced homomorphism is the above $t_n$. On the other hand, the composition

$$m_{x_n}: \Sigma^{2n}MU\langle n \rangle \overset{x_n \wedge 1}{\longrightarrow} MU \wedge MU\langle n \rangle \overset{\phi(n)}{\longrightarrow} MU\langle n \rangle$$
is characterized by the above multiplication $\cdot x_n$.

**Lemma A.** Let $E \xrightarrow{f} F \xrightarrow{g} G$ be a sequence of CW-spectra such that the composition $g \circ f$ is homotopic to the zero. If $0 \rightarrow \pi_\ast(E) \rightarrow \pi_\ast(F) \rightarrow \pi_\ast(G) \rightarrow 0$ is exact, then $E \rightarrow F \rightarrow G$ is a cofiber sequence. (Cf., Lemma 4).

Proof. Let $C_f$ be the mapping cone of $f: E \rightarrow F$. Then $g: F \rightarrow G$ admits a factorization $F \rightarrow C_f \rightarrow G$. Considering the commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & \pi_\ast(E) \\
\longrightarrow & \pi_\ast(F) & \pi_\ast(C_f) \rightarrow 0 \\
\longrightarrow & \pi_\ast(E) & \pi_\ast(F) \rightarrow \pi_\ast(G) \rightarrow 0
\end{array}
$$

with exact rows, we see easily that $h: C_f \rightarrow G$ is a homotopy equivalence.

Using (A.1) the composition $\tau_n \cdot m_{x_n}$ becomes homotopic to the zero. We get therefore that

$$(A.3) \quad \Sigma^{2n}MU\langle n \rangle \overset{m_{x_n}}{\longrightarrow} MU\langle n \rangle \overset{\tau_n}{\longrightarrow} MU\langle n-1 \rangle$$
is a cofiber sequence.

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**References**


