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Author(s)	Yosimura, Zen-ichi
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A NOTE ON THE RELATION OF Z_2 -GRADED COMPLEX COBORDISM TO COMPLEX K-THEORY

ZEN-ICHI YOSIMURA

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Let $MU^*()$ and $K^*()$ denote the Z_2 -graded complex cobordism theory and the complex K -theory respectively. The Thom homomorphism $\mu_*: \pi_0(MU) \rightarrow \pi_0(K)$ on coefficient groups is identified (up to sign) with the classical Todd genus $Td: \Lambda \rightarrow Z$. We denote by I the ideal of Λ to be the kernel of $Td: \Lambda \rightarrow Z$. Wolff [7] proved that the decreasing filtration $\{I^q MU^*()\}$ of $MU^*()$ consists of cohomology theories defined on the category of based finite CW -complexes, and the associated quotients $I^q MU^*()/I^{q+1} MU^*()$ are determined by the complex K -theories $KG_q^*()$ with coefficients $G_q = I^q/I^{q+1}$.

The purpose of this note is to extend the Wolff's result to the category of based CW -complexes. Let $F_q MU$ be the CW -spectrum associated with the cohomology theory $I^q MU^*()$, i.e., $\{Y, F_q MU\}^* \cong I^q MU^*(Y)$ for any based finite CW -complex (or finite CW -spectrum). We show that $\{F_q MU^*()\}$ is a decreasing filtration of $MU^*()$ consisting of Λ -modules so that the associated quotients are equal to $KG_q^*()$, and in addition that $F_{q+1} MU^*()$ is a direct summand of $F_q MU^*()$.

Moreover we give a tower

$$MU \rightarrow \cdots \rightarrow Q_q MU \rightarrow Q_{q-1} MU \rightarrow \cdots \rightarrow Q_0 MU = K$$

of MU -module spectra such that $KG_q \rightarrow Q_q MU \rightarrow Q_{q-1} MU$ is a cofiber sequence of MU -module spectra, which factorizes the Thom map $\mu_*: MU \rightarrow K$.

Baas [3] constructed a tower of CW -spectra

$$MU \rightarrow \cdots \rightarrow MU\langle n \rangle \rightarrow MU\langle n-1 \rangle \rightarrow \cdots \rightarrow MU\langle 0 \rangle = H$$

factorizing the Thom map $\mu: MU \rightarrow H$. In appendix we show that the tower is of MU -module spectra and the sequence $\Sigma^{2n} MU\langle n \rangle \xrightarrow{m_{x_n}} MU\langle n \rangle \rightarrow MU\langle n-1 \rangle$ is a cofiber sequence where m_{x_n} is the multiplication by x_n a ring generator of Λ with degree $2n$.

1. Decreasing filtration of $MU_*()$

1.1. A pair (E, ρ) is called a Z_2 -graded CW -spectrum if E is a CW -spectrum

and $\rho: \Sigma^2 E \rightarrow E$ is a homotopy equivalence. Such a pair (E, ρ) gives rise to natural isomorphisms

$$\rho_*: E_*(X) \rightarrow E_{*+2}(X), \quad \rho^*: E^{*+2}(X) \rightarrow E^*(X)$$

for any CW -spectrum X . So we can define Z_2 -graded homology and cohomology theories $E_*()$, $E^*()$ by putting

$$E_*(X) = E_0(X) \oplus E_1(X), \quad E^*(X) = E^0(X) \oplus E^1(X).$$

For a CW -spectrum E we put

$$\mathbf{E} = \bigvee_n \Sigma^{2n} E, \quad \bar{\mathbf{E}} = \prod_n \Sigma^{2n} E.$$

Taking the canonical identifications $\rho: \Sigma^2 \mathbf{E} \rightarrow \mathbf{E}$ and $\bar{\rho}: \Sigma^2 \bar{\mathbf{E}} \rightarrow \bar{\mathbf{E}}$ as structure morphisms \mathbf{E} and $\bar{\mathbf{E}}$ admit structures of Z_2 -graded CW -spectra respectively. From definition it follows that

$$E_0(X) \cong \sum_n E_{2n}(X), \quad E_1(X) \cong \sum_n E_{2n+1}(X),$$

$$\bar{E}^0(X) \cong \prod_n E^{2n}(X), \quad \bar{E}^1(X) \cong \prod_n E^{2n+1}(X)$$

for all CW -spectra X . In particular, the canonical morphism $\mathbf{H} \rightarrow \bar{\mathbf{H}}$ becomes a homotopy equivalence for the Eilenberg-MacLane spectrum H .

The BU -spectrum K may be regarded as a Z_2 -graded CW -spectrum because it possesses the Bott map $\beta: \Sigma^2 K \rightarrow K$ which is a homotopy equivalence.

Denote by F_n the direct sum of n -copies of the integers Z and by F the direct limit of F_n , i.e., F is a free abelian group with countably many factors. Putting

$$BU_{F_n} = BU \times \cdots \times BU, \text{ the product of } n\text{-copies of } BU,$$

$$BU_F = \bigcup_n BU_{F_n}, \text{ the union of } BU_{F_n},$$

we obtain

Proposition 1. *There exists a natural isomorphism*

$$[X, BU_F] \rightarrow KF^0(X)$$

for any based connected CW-complex X .

Proof. Let Y be a based connected finite CW -complex. Then we have a sequence of natural isomorphisms

$$[Y, BU_F] \leftarrow \varinjlim [Y, BU_{F_n}] \leftarrow \varinjlim [Y, BU] \otimes F_n \rightarrow \varinjlim K^0(Y) \otimes F_n$$

$$\rightarrow K^0(Y) \otimes F \rightarrow KF^0(Y).$$

Therefore the contravariant functor KF^0 defined on the category of based connected CW -complexes is represented by BU_F (use [1, Addendum 1.5]).

Proposition 1 implies that BU_F is homotopy equivalent to $\Omega_0^2 BU_F$ where Ω_0^2 means the component of the base point in the double loop space. Hence we have

(1.1) *in the BU -spectrum KF with the coefficients F every even term is the based CW -complex BU_F .*

1.2. Let us denote by MU the unitary Thom spectrum and by $\mu_c: MU \rightarrow K$ the Thom map which is a ring morphism. The composition

$$\mu_c: MU \rightarrow K \rightarrow K$$

of $\bigvee \Sigma^{2n} \mu_c$ and $\bigvee \beta^n$ is a morphism of Z_2 -graded ring-spectra, called the Z_2 -graded Thom map. As is well known, it is characterized by the coefficient homomorphism $\mu_{c\sharp}: \pi_{\sharp}(MU) \rightarrow \pi_{\sharp}(K)$ which coincides (up to sign) with the classical Todd genus Td . Putting $\Lambda = \pi_{\sharp}(MU)$, $\pi_{\sharp}(K) = Z$ is viewed as a Z_2 -graded Λ -module via $\mu_{c\sharp} = Td$ and it is written Z_{Td} for emphasis.

Using the kernel I of $Td: \Lambda \rightarrow Z$ we define a decreasing filtration $\{I^q\}_{q \geq 0}$ consisting of ideals of Λ . Denoting by G_q the associated Z_2 -graded Λ -module I^q/I^{q+1} , we see easily [7, Satz 3.8] that

(1.2) $G_0 \cong Z_{Td}$ and G_q is a free abelian group with countably many factors for $q \geq 1$.

For a Z_2 -graded Λ -module A we have a decreasing filtration $\{I^q A\}_{q \geq 0}$ consisting of submodules of A , whose associated Z_2 -graded Λ -module $I^q A/I^{q+1} A$ is written $G_q(A)$. Applying the commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow \text{Tor}_1^{\Lambda}(Z_{Td}, A) & \rightarrow & I \otimes A & \rightarrow & A & \rightarrow & Z_{Td} \otimes A \rightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 \rightarrow IA & \rightarrow & A & \rightarrow & G_0(A) & \rightarrow & 0 \end{array}$$

with exact rows, we get an isomorphism

$$(1.3) \quad G_q \otimes_{\Lambda} A \xrightarrow{\cong} G_q \otimes_{\mathbb{Z}} (Z_{Td} \otimes_{\Lambda} A) \xrightarrow{\cong} G_q \otimes_{\mathbb{Z}} G_0(A)$$

by means of "4 lemma".

Proposition 2. *Let A be a Λ -module with $\text{Tor}_k^{\Lambda}(Z_{Td}, A) = 0$ for all $k \geq 1$. Then, for every $q \geq 0$ both $I^q \otimes_{\Lambda} A \rightarrow I^q A$ and $G_q \otimes_{\Lambda} A \rightarrow G_q(A)$ are isomorphisms and $\text{Tor}_k^{\Lambda}(I^q, A) = \text{Tor}_k^{\Lambda}(G_q, A) = 0$ for all $k \geq 1$.*

Proof. Choose a free Λ -module F such that A is isomorphic to a quotient F/B . By induction on q we shall show that the sequences

$$0 \rightarrow I^q B \rightarrow I^q F \rightarrow I^q A \rightarrow 0, \quad 0 \rightarrow G_q(B) \rightarrow G_q(F) \rightarrow G_q(A) \rightarrow 0$$

are exact. The $q=0$ case is evident because of (1.3). Applying induction

hypotesis and “ 3×3 lemma” we find easily that $0 \rightarrow I^q B \rightarrow I^q F \rightarrow I^q A \rightarrow 0$ is exact. So we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & G_q \otimes G_0(B) & \rightarrow & G_q \otimes G_0(F) & \rightarrow & G_q \otimes G_0(A) \rightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ & & G_0(I^q B) & \rightarrow & G_0(I^q F) & \rightarrow & G_0(I^q A) \rightarrow 0 \end{array}$$

with exact rows. Since all vertical arrows are epimorphisms and in particular the central one is an isomorphism, all vertical arrows become isomorphisms. Consequently we get that $0 \rightarrow G_q(B) \rightarrow G_q(F) \rightarrow G_q(A) \rightarrow 0$ is exact.

Next, we consider the commutative diagrams

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Tor}_1^\Lambda(I^q, A) & \rightarrow & I^q \otimes B & \rightarrow & I^q \otimes F \rightarrow I^q \otimes A \rightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \rightarrow & I^q B & \rightarrow & I^q F & \rightarrow & I^q A \rightarrow 0 \\ \\ 0 & \rightarrow & \text{Tor}_1^\Lambda(G_q, A) & \rightarrow & G_q \otimes B & \rightarrow & G_q \otimes F \rightarrow G_q \otimes A \rightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \rightarrow & G_q \otimes G_0(B) & \rightarrow & G_q \otimes G_0(F) & \rightarrow & G_q \otimes G_0(A) \rightarrow 0 \end{array}$$

with exact rows. Remark that $\text{Tor}_k^\Lambda(Z_{Td}, B) = 0$ for all $k \geq 1$. By use of “4 lemma” and (1.3) we see that all vertical arrows are isomorphisms, and hence we obtain the required results.

For a Λ -module A we put $J_q(A) = A/I^{q+1}A$ and abbreviate $J_q = J_q(\Lambda)$ when $A = \Lambda$. As an immediate corollary of Proposition 2 we have

Corollary 3. *Let A be a Λ -module with $\text{Tor}_k^\Lambda(Z_{Td}, A) = 0$ for all $k \geq 1$. Then $J_q \otimes A \rightarrow J_q(A)$ is an isomorphism and $\text{Tor}_k^\Lambda(J_q, A) = 0$ for all $k \geq 1$.*

1.3. Let \mathcal{MU} denote the category of comodules over $MU_*(MU)$ which are finitely presented as Λ -modules. Notice that \mathcal{MU} is an abelian category which has enough projectives, and also that $MU_*(Y)$ lies in the category \mathcal{MU} whenever Y is a finite CW -spectrum. Since the functor $M \rightarrow Z_{Td} \otimes_\Lambda M$ is exact on \mathcal{MU} [5, Example 3.3] it follows immediately that

$$(1.4) \quad \text{Tor}_k^\Lambda(Z_{Td}, M) = 0 \quad \text{for all } k \geq 1 \text{ if } M \text{ lies in } \mathcal{MU}.$$

Proposition 1 and Corollary 2 combined with (1.4) say that

$$(1.5) \quad \text{the functors } M \rightarrow I^q M \cong I^q \otimes_\Lambda M, \quad M \rightarrow G_q(M) \cong G_q \otimes_\Lambda M \text{ and } M \rightarrow J_q(M) \cong J_q \otimes_\Lambda M \text{ on } \mathcal{MU} \text{ are exact.}$$

Theorem 1 (Wolff [7]). i) Both $I^q MU_*()$ and $MU_*()/I^{q+1} MU_*()$ are

homology theories defined on the category of CW -spectra, so that $I^q \otimes_{\Lambda} MU_*(X) \rightarrow I^q MU_*(X)$ and $\Lambda/I^{q+1} \otimes_{\Lambda} MU_*(X) \rightarrow MU_*(X)/I^{q+1} MU_*(X)$ are natural isomorphisms for all CW -spectra X .

ii) $I^q MU_*()/I^{q+1} MU_*()$ is a homology theory defined on the category of CW -spectra such that there exists a natural isomorphism $I^q MU_*(X)/I^{q+1} MU_*(X) \rightarrow KG_{q*}(X)$ for any CW -spectrum X which is induced by the Z_2 -graded Thom map μ_c .

Proof. i) and the first half of ii) are immediate from (1.5). The latter half of ii) is also valid because we have a natural isomorphism

$$\begin{aligned} G_q(MU_*(X)) &\xleftarrow{\cong} G_q \otimes_{\Lambda} MU_*(X) \xrightarrow{\cong} G_q \otimes_{\mathbb{Z}} (Z_{Td} \otimes_{\Lambda} MU_*(X)) \\ &\xrightarrow{\cong} G_q \otimes_{\mathbb{Z}} K_*(X) \xrightarrow{\cong} KG_{q*}(X). \end{aligned}$$

Let $\phi: E_*(X) \rightarrow F_*(X)$ be a natural transformation for any CW -spectrum X . According to [1, Addendum 1.5] there exists a morphism $f: E \rightarrow F$ inducing ϕ , and it is unique up to weak homotopy. The proof in [1] is actually given for the category of based connected CW -complexes, but it is easily extended to that of CW -spectra. Such a morphism f is uniquely chosen (up to homotopy) under the assumption that $F^0(E)$ is Hausdorff.

Let $E_*()$ be a Z_2 -graded homology theory defined on the category of CW -spectra, i.e., a homology theory equipped with a natural isomorphism $E_*(X) \rightarrow E_{*+2}(X)$ for any CW -spectrum X . Then it gives E a structure of Z_2 -graded CW -spectrum. In particular, the induced structure is unique if $E^0(E)$ is Hausdorff.

Recall that the Z_2 -graded CW -spectrum MU is equipped with the canonical identification $\rho: \Sigma^2 MU \rightarrow MU$ as structure morphism. Since $MU^*(MU)$ is Hausdorff (use Proposition 6 below), the Z_2 -graded CW -spectrum (MU, ρ) is characterized only by the Z_2 -graded homology theory $MU_*()$.

Denote by $F_q MU$ and $Q_q MU$ the representing spectra of the new homology theories $I^q MU_*()$ and $MU_*()/I^{q+1} MU_*()$ respectively, i.e.,

$$I^q MU_*(X) \cong \{ \Sigma^*, F_q MU \wedge X \}, \quad MU_*(X)/I^{q+1} MU_*(X) \cong \{ \Sigma^*, Q_q MU \wedge X \}$$

for any CW -spectrum X . Of course, they are both Z_2 -graded CW -spectra. Then there exist morphisms

$$\begin{aligned} i_q: F_{q+1} MU &\rightarrow F_q MU, \quad j_q: Q_q MU \rightarrow Q_{q-1} MU, \\ \iota_q: F_{q+1} MU &\rightarrow MU, \quad \pi_q: MU \rightarrow Q_q MU \end{aligned}$$

which induce the canonical morphisms in homology groups, and moreover we have morphisms

$$\mu_q: F_q \mathbf{MU} \rightarrow KG_q, \quad \nu_q: KG_q \rightarrow Q_q \mathbf{MU}$$

such that $\mu_{q*}: F_q \mathbf{MU}_*(X) \rightarrow KG_{q*}(X)$ and $\nu_{q*}: KG_{q*}(X) \rightarrow Q_q \mathbf{MU}_*(X)$ in homology groups are natural homomorphisms induced by the Z_2 -graded Thom map μ_c .

Lemma 4. *Let $E \xrightarrow{f} F \xrightarrow{g} G$ be a sequence which satisfies the property that $0 \rightarrow E_*(X) \rightarrow F_*(X) \rightarrow G_*(X) \rightarrow 0$ is a short exact sequence for every CW -spectrum X . Then it is a cofiber sequence.*

Proof. Let C_f be the mapping cone of f , i.e., $E \rightarrow F \rightarrow C_f$ a cofiber sequence. Then for any CW -spectrum X we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & E_*(X) & \rightarrow & F_*(X) & \rightarrow & C_{f*}(X) \rightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \phi = h_* \\ 0 & \rightarrow & E_*(X) & \rightarrow & F_*(X) & \rightarrow & G_*(X) \rightarrow 0 \end{array}$$

with exact rows. Clearly $h: C_f \rightarrow G$ which induces ϕ is a homotopy equivalence.

By virtue of Lemma 4 we verify that

(1.6) $F_{q+1} \mathbf{MU} \rightarrow \mathbf{MU} \rightarrow Q_q \mathbf{MU}$, $F_{q+1} \mathbf{MU} \rightarrow F_q \mathbf{MU} \rightarrow KG_q$ and $KG_q \rightarrow Q_q \mathbf{MU} \rightarrow Q_{q-1} \mathbf{MU}$ are all cofiber sequences.

2. Z_2 -graded \mathbf{MU} -module spectra

2.1. The inclusion $Z \subset Q$ induces a natural transformation $ch: E^*(X) \rightarrow EQ^*(X)$ for any CW -spectrum X , called the Chern-Dold character.

Proposition 5. *If $ch: E^*(X) \rightarrow EQ^*(X)$ is a monomorphism, then $E^*(X)$ is Hausdorff.*

Proof. Since $EQ^*(X)$ is always Hausdorff [8, Proposition 4], the result is immediate.

Let W be a connective CW -spectrum with $H_*(W)$ free and assume that $\pi_*(E)$ is torsion free. Then $H^*(W; \pi_*(E)) \rightarrow H^*(W; \pi_*(E) \otimes Q)$ is a monomorphism, and hence the Atiyah-Hirzebruch spectral sequences for $E^*(W)$ and $EQ^*(W)$ collapse. Therefore we get that

(2.1) $ch: E^*(W) \rightarrow EQ^*(W)$ is a monomorphism. (Cf., [8, Lemma 11]).

Applying Proposition 5 we obtain

Proposition 6. *Let W be a connective CW -spectrum with $H_*(W)$ free. If $\pi_*(E)$ is torsion free, then $E^*(W)$ is Hausdorff.*

By means of Proposition 5 we get the following lemmas.

Lemma 7. Assume that $\pi_0(E)$ is torsion free and $\pi_1(E)=0$. Then $E^0(KG_q \wedge MU \wedge \cdots \wedge MU)$ and $E^0(Q_q MU \wedge MU \wedge \cdots \wedge MU)$ are Hausdorff and $E^1(F_q MU \wedge MU \wedge \cdots \wedge MU)=0$.

Proof. Since $E^{2n-1}(BU_{G_q} \wedge MU \wedge \cdots \wedge MU) = EQ^{2n-1}(BU_{G_q} \wedge MU \wedge \cdots \wedge MU)=0$ we have a commutative square

$$\begin{array}{ccc} E^0(KG_q \wedge MU \wedge \cdots \wedge MU) & \xrightarrow{\quad} & \lim_{\leftarrow} E^{2n}(BU_{G_q} \wedge MU \wedge \cdots \wedge MU) \\ \downarrow & & \downarrow \\ EQ^0(KG_q \wedge MU \wedge \cdots \wedge MU) & \xrightarrow{\quad} & \lim_{\leftarrow} EQ^{2n}(BU_{G_q} \wedge MU \wedge \cdots \wedge MU) \end{array}$$

such that the horizontal arrows are isomorphisms. The left arrow is a monomorphism because so is the right one by use of (2.1). Since Theorem 1 implies that $0 \rightarrow EQ^*(Q_{q-1} MU \wedge MU \wedge \cdots \wedge MU) \rightarrow EQ^*(Q_q MU \wedge MU \wedge \cdots \wedge MU) \rightarrow EQ^*(KG_q \wedge MU \wedge \cdots \wedge MU) \rightarrow 0$ is exact, an induction on q involving "4 lemma" shows that $ch: E^0(Q_q MU \wedge MU \wedge \cdots \wedge MU) \rightarrow EQ^0(Q_q MU \wedge MU \wedge \cdots \wedge MU)$ is a monomorphism. Then we find that $E^0(Q_q MU \wedge MU \wedge \cdots \wedge MU) \rightarrow E^0(MU \wedge \cdots \wedge MU)$ is a monomorphism because so is $EQ^0(Q_q MU \wedge MU \wedge \cdots \wedge MU) \rightarrow EQ^0(MU \wedge \cdots \wedge MU)$. Therefore $E^1(MU \wedge \cdots \wedge MU) \rightarrow E^1(F_{q+1} MU \wedge MU \wedge \cdots \wedge MU)$ is an epimorphism, and hence $E^1(F_{q+1} MU \wedge MU \wedge \cdots \wedge MU)=0$.

Lemma 8. $KG_p^0(F_q MU \wedge MU \wedge \cdots \wedge MU)$ and $Q_p MU^0(F_q MU \wedge MU \wedge \cdots \wedge MU)$ are Hausdorff and $KG_p^1(F_q MU \wedge MU \wedge \cdots \wedge MU) = Q_p MU^1(F_q MU \wedge MU \wedge \cdots \wedge MU)=0$.

Proof. Putting $X = F_q MU \wedge MU \wedge \cdots \wedge MU$, we note by Theorem 1 i) that $K_0(X)$ is free and $K_1(X)=0$. Applying the universal coefficient sequence

$$0 \rightarrow \text{Ext}(K_{\#-1}(X), G_p) \rightarrow KG_p^{\sharp}(X) \rightarrow \text{Hom}(K_{\sharp}(X), G_p) \rightarrow 0$$

for K (see [9, (3.1)]) we get immediately that $ch: KG_p^0(X) \rightarrow KG_p \otimes Q^0(X)$ is a monomorphism and $KG_p^1(X)=0$. By induction on p we obtain that $ch: Q_p MU^0(X) \rightarrow Q_p MU Q^0(X)$ is a monomorphism and $Q_p MU^1(X)=0$ because $0 \rightarrow KG_p \otimes Q^*(X) \rightarrow Q_p MU Q^*(X) \rightarrow Q_{p-1} MU Q^*(X) \rightarrow 0$ is exact.

Assume that $\pi_*(E)$ is free and of finite type and put again $X = F_q MU \wedge MU \wedge \cdots \wedge MU$. Using the universal coefficient sequence for E [9, (1.8)] we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Ext}(\hat{E}_{*-1}(X), Z) & \rightarrow & E^*(X) & \rightarrow & \text{Hom}(\hat{E}_*(X), Z) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \text{Ext}(\hat{E}_{*-1}(X), Q) & \rightarrow & EQ^*(X) & \rightarrow & \text{Hom}(\hat{E}_*(X), Q) \rightarrow 0 \end{array}$$

with exact rows where \hat{E} is the dual of E constructed in [9]. Note that $\pi_*(\hat{E})$ is free and hence so is $\hat{E}_*(X)$. Then the central arrow becomes a monomorphism. Considering the commutative square

$$\begin{array}{ccc} \bar{E}^*(X) & \rightarrow & \prod E^n(X) \\ \downarrow & & \downarrow \\ \bar{E}Q^*(X) & \rightarrow & \prod EQ^n(X) \end{array}$$

in which the upper arrow is an isomorphism, we find that the left one is a monomorphism. Thus we get that

(2.2) $\bar{E}^*(F_q\mathbf{MU} \wedge \mathbf{MU} \wedge \cdots \wedge \mathbf{MU})$ is Hausdorff.

Let $\widetilde{\mathbf{MU}}$ denote the mapping cone of the canonical morphism $\mathbf{MU} \rightarrow \overline{\mathbf{MU}}$. Since $\prod Z/\sum Z \rightarrow \prod Q/\sum Q$ is a monomorphism we remark that

$$(2.3) \quad \pi_0(\widetilde{\mathbf{MU}}) = \prod_n \pi_{2n}(\mathbf{MU})/\sum_n \pi_{2n}(\mathbf{MU}) \text{ is torsion free and } \pi_1(\widetilde{\mathbf{MU}}) = 0,$$

(see [4, Exercise IV 20]). Then $\widetilde{\mathbf{MU}}$ has a unique structure of Z_2 -graded CW-spectrum so that the cofiber sequence $\mathbf{MU} \rightarrow \overline{\mathbf{MU}} \rightarrow \widetilde{\mathbf{MU}}$ is of Z_2 -graded CW-spectra.

Lemma 9. $F_p\mathbf{MU}^0(F_q\mathbf{MU} \wedge \mathbf{MU} \wedge \cdots \wedge \mathbf{MU})$ is Hausdorff and $F_p\mathbf{MU}^1(F_q\mathbf{MU} \wedge \mathbf{MU} \wedge \cdots \wedge \mathbf{MU})=0$

Proof. We put $X=F_q\mathbf{MU} \wedge \mathbf{MU} \wedge \cdots \wedge \mathbf{MU}$. From Lemma 7 it follows that $F_p\mathbf{MU}^1(X)=0$. In the sequence

$$F_p\mathbf{MU}^0(X) \rightarrow \mathbf{MU}^0(X) \rightarrow \overline{\mathbf{MU}}^0(X)$$

the former arrow is a monomorphism because of Lemma 8 and the latter one is so by means of (2.3) and Lemma 7. Thus the above composition is a monomorphism. On the other hand, (2.2) says that $\overline{\mathbf{MU}}^0(X)$ is Hausdorff. So we get the remaining result.

2.2. Since $F_q\mathbf{MU}^0(F_q\mathbf{MU})$ and $Q_q\mathbf{MU}^0(Q_q\mathbf{MU})$ are both Hausdorff, we verify that

(2.4) the Z_2 -graded homology theories $I^q\mathbf{MU}_*(\)$ and $\mathbf{MU}_*(\)/I^{q+1}\mathbf{MU}_*(\)$ give $F_q\mathbf{MU}$ and $Q_q\mathbf{MU}$ unique structures of Z_2 -graded CW-spectra respectively.

Moreover, by virtue of Lemmas 7, 8 and 9 we see that

$$(2.5) \quad \begin{array}{llll} i_q: F_{q+1}\mathbf{MU} \rightarrow F_q\mathbf{MU}, & \iota_q: F_{q+1}\mathbf{MU} \rightarrow \mathbf{MU}, & \mu_q: F_q\mathbf{MU} \rightarrow KG_q \\ j_q: Q_q\mathbf{MU} \rightarrow Q_{q-1}\mathbf{MU}, & \pi_q: \mathbf{MU} \rightarrow Q_q\mathbf{MU}, & \nu_q: KG_q \rightarrow Q_q\mathbf{MU} \end{array}$$

are uniquely determined (up to homotopy), which induce the canonical morphisms in homology groups. In particular, the composition $\iota_{q-1} \cdot i_q$ is homotopic to ι_q and $j_q \cdot \pi_q$ is so to π_{q-1} .

Consider the diagram

$$\begin{array}{ccccccc}
 & & F_{q+1}MU & & & & \\
 & \swarrow & & \searrow & & & \\
 & F_qMU & & MU & & & \\
 \swarrow & & & & \searrow & & \\
 KG_q & \cdots \cdots \cdots & Q_qMU & \cdots \cdots \cdots & Q_{q-1}MU & \cdots \cdots \cdots & \Sigma KG_q \\
 \downarrow & & \swarrow & & \searrow & & \\
 \Sigma F_{q+1}MU & \xrightarrow{\quad} & \Sigma F_qMU & & & &
 \end{array}$$

consisting of cofiber sequences. With an application of Verdier's lemma (see [2, Lemma 6.8]) we get a cofiber sequence $KG_q \rightarrow Q_qMU \rightarrow Q_{q-1}MU \rightarrow \Sigma KG_q$ (denoted by dotted arrows in the above diagram) which makes the diagram homotopy commutative. Clearly this yields the canonical exact sequence $0 \rightarrow KG_q(X) \rightarrow Q_qMU_*(X) \rightarrow Q_{q-1}MU_*(X) \rightarrow 0$. By uniqueness of ν_q, j_q the above cofiber sequence coincides with $KG_q \xrightarrow{\nu_q} Q_qMU \xrightarrow{j_q} Q_{q-1}MU \rightarrow \Sigma KG_q$.

The multiplication $\phi: MU \wedge MU \rightarrow MU$ gives rise to natural Z_2 -graded homomorphisms

$$\begin{aligned}
 m_q: F_qMU_*(X) \otimes MU_*(Y) &\rightarrow F_qMU_*(X \wedge Y) \\
 \bar{m}_q: Q_qMU_*(X) \otimes MU_*(Y) &\rightarrow Q_qMU_*(X \wedge Y)
 \end{aligned}$$

for all CW -spectra X and Y . By use of Lemmas 7 and 9 there exist unique pairings

$$\phi_q: F_qMU \wedge MU \rightarrow F_qMU, \quad \bar{\phi}_q: Q_qMU \wedge MU \rightarrow Q_qMU$$

which induce the above m_q and \bar{m}_q respectively. Then it follows that

(2.6) both F_qMU and Q_qMU are (associative) Z_2 -graded MU -module spectra.

Proposition 10. Let M be a Z_2 -graded ring spectrum, E, F and G Z_2 -graded M -module spectra and $E \rightarrow F \rightarrow G$ a cofiber sequence. Assume that $E^0(E)$, $E^0(E \wedge M)$, $G^0(F)$ and $G^0(F \wedge M)$ are Hausdorff, or that $F^0(E)$, $F^0(E \wedge M)$, $G^0(G)$ and $G^0(G \wedge M)$ are Hausdorff. If for any CW -spectrum X $0 \rightarrow E_*(X) \rightarrow F_*(X) \rightarrow G_*(X) \rightarrow 0$ is a short exact sequence of Z_2 -graded $M_*(\)$ -modules, then the cofiber sequence $E \rightarrow F \rightarrow G$ is of Z_2 -graded M -module spectra.

Proof. Assuming that $F^0(E)$, $F^0(E \wedge M)$, $G^0(G)$ and $G^0(G \wedge M)$ are Hausdorff, we consider the diagrams

$$\begin{array}{ccccccc}
 \Sigma^2 E & \rightarrow & \Sigma^2 F & \rightarrow & \Sigma^2 G & \rightarrow & \Sigma^3 E \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 E & \rightarrow & F & \rightarrow & G & \rightarrow & \Sigma E
 \end{array}
 \quad
 \begin{array}{ccccccc}
 E \wedge M & \rightarrow & F \wedge M & \rightarrow & G \wedge M & \rightarrow & \Sigma E \wedge M \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 E & \rightarrow & F & \rightarrow & G & \rightarrow & \Sigma E
 \end{array}$$

with cofiber sequences. Under the first two assumptions two left squares become homotopy commutative because they induce the Z_2 -graded homomorphism $E_*() \rightarrow F_*()$ of $M_*()$ -modules. Therefore there exist morphisms $\Sigma^2 G \rightarrow G$ and $G \wedge M \rightarrow G$ which make the above diagrams into morphisms of cofiber sequences. As is easily checked, they give $G_*()$ a structure of Z_2 -graded $M_*()$ -module, which coincides with the original one. So, using the remaining assumptions again we see that the above morphisms are homotopic to the given ones respectively. Consequently the cofiber sequence $E \rightarrow F \rightarrow G$ becomes the required one.

Another case is similarly proved.

The ring spectrum K may be regarded as a Z_2 -graded MU -module spectrum via the Z_2 -graded Thom map $\mu_c: MU \rightarrow K$.

Applying Proposition 10 to three cofiber sequences of (1.6) we get

Theorem 2. *The sequences $F_{q+1}MU \rightarrow MU \rightarrow Q_qMU$, $F_{q+1}MU \rightarrow F_qMU \rightarrow KG_q$ and $KG_q \rightarrow Q_qMU \rightarrow Q_{q-1}MU$ are cofiber sequences of Z_2 -graded MU -module spectra.*

Proof. The assumptions needed in Proposition 10 are satisfied by Lemmas 7, 8 and 9.

As a result we have a tower

$$(2.7) \quad MU \rightarrow \cdots \rightarrow Q_qMU \rightarrow Q_{q-1}MU \rightarrow \cdots \rightarrow Q_0MU = K$$

of Z_2 -graded MU -module spectra such that $KG_q \rightarrow Q_qMU \rightarrow Q_{q-1}MU$ is a cofiber sequence, which factorizes the Z_2 -graded Thom map $\mu_c: MU \rightarrow K$.

2.3. Here we extend the Wolff's result to the case of based CW -complexes.

Proposition 11. *There exists an (unstable) natural homomorphism*

$$\Phi_q: KG_q^*(X) \rightarrow F_qMU^*(X)$$

for any based CW -complex X , which satisfies the equality that $\mu_{q*} \cdot \Phi_q = \text{id}$.

Proof. We may assume that X is connected. Let $i: BU_{G_q} \rightarrow KG_q$ be the inclusion. Then we can choose a morphism $c_q: BU_{G_q} \rightarrow F_qMU$ such that i is homotopic to the composition $\mu_q \cdot c_q$, because $F_{q+1}MU^1(BU_{G_q}) = 0$. In the commutative diagram

$$\begin{array}{ccc} [X, BU_{G_q}] \xrightarrow{J_0} \{X, BU_{G_q}\} & \xrightarrow{i_*} & \{X, KG_q\} = KG_q^0(X) \\ & \searrow c_{q*} & \uparrow \mu_{q*} \\ & & \{X, F_qMU\} = F_qMU^0(X) \end{array}$$

the composition $i_* \cdot J_0$ is an isomorphism because of Proposition 2 (see [6, Theorem 14.5]). So we put that $\Phi_q = c_{q*} \cdot J_0 \cdot (i_* \cdot J_0)^{-1}$.

REMARK. If Φ_q is stable, then we have a natural split exact sequence

$$0 \rightarrow F_{q+1}MU^*(X) \rightarrow F_qMU^*(X) \rightarrow KG_q^*(X) \rightarrow 0$$

for every CW -spectrum X . Therefore F_qMU becomes homotopy equivalent to the wedge $F_{q+1}MU \vee KG_q$. However $H_*(F_qMU)$ is a free abelian group and $H_*(KG_q)$ is a \mathbb{Q} -module. This is a contradiction.

We now obtain our main result.

Theorem 3. *For any based CW -complex X the natural sequences*

$$\begin{aligned} 0 \rightarrow F_{q+1}MU^*(X) \rightarrow F_qMU^*(X) \rightarrow KG_q^*(X) \rightarrow 0 \\ 0 \rightarrow KG_q^*(X) \rightarrow Q_qMU^*(X) \rightarrow Q_{q-1}MU^*(X) \rightarrow 0 \end{aligned}$$

of Z_2 -graded Λ -modules are split exact.

Proof. The first case is immediate from Proposition 11. On the other hand, a diagram chase shows that the second sequence is exact for any based CW -complex X and hence it is split.

Appendix

Recall that MU is a ring spectrum with coefficients $\Lambda_* = Z[x_1, \dots, x_n, \dots]$ where $\deg x_n = 2n$. By killing certain bordism classes Baas [3] constructed homology theories $MU\langle n \rangle_*()$ with coefficient $\pi_*(MU\langle n \rangle) = \Lambda_*/(x_{n+1}, \dots)$, whose representing spectrum we denote by $MU\langle n \rangle$. $MU\langle n \rangle_*()$ is an (associative) $MU_*()$ -module, thus there exists a natural homomorphism

$$m_n: MU_*(X) \otimes MU\langle n \rangle_*(Y) \rightarrow MU\langle n \rangle_*(X \wedge Y)$$

for any CW -spectra X and Y . This gives us a pairing

$$\phi\langle n \rangle: MU \wedge MU\langle n \rangle \rightarrow MU\langle n \rangle$$

by which the above m_n is induced.

An easy computation shows that $MU\langle n \rangle \hat{Z}/Z^{2k-1}(MU \wedge \dots \wedge MU \wedge MU\langle n \rangle) = 0$ because $\pi_{2l+1}(MU\langle n \rangle) = 0$ for all l . Then [8, Theorem 1] says that

(A.1) $MU\langle n \rangle^{2k}(MU \wedge \dots \wedge MU \wedge MU\langle m \rangle)$ is Hausdorff (see also [9, Corollary 13]).

Hence $\phi\langle n \rangle$ is uniquely determined (up to homotopy) and moreover

(A.2) $MU\langle n \rangle$ is an (associative) MU -module spectrum.

An important relationship between $MU\langle n\rangle_*()$ and $MU\langle n-1\rangle_*()$ is given in the form of a natural exact sequence

$$\rightarrow MU\langle n\rangle_{*-2n}(X) \xrightarrow{\cdot x_n} MU\langle n\rangle_*(X) \xrightarrow{t_n} MU\langle n-1\rangle_*(X) \rightarrow MU\langle n\rangle_{*-2n-1}(X) \rightarrow$$

of $MU_*()$ -modules where $\cdot x_n$ denotes the multiplication by x_n . Because of (A.1) there exists a unique morphism $\tau_n: MU\langle n\rangle \rightarrow MU\langle n-1\rangle$ of MU -module spectra whose induced homomorphism is the above t_n . On the other hand, the composition

$$m_{x_n}: \Sigma^{2n} MU\langle n\rangle \xrightarrow{x_n \wedge 1} MU \wedge MU\langle n\rangle \xrightarrow{\phi\langle n\rangle} MU\langle n\rangle$$

is characterized by the above multiplication $\cdot x_n$.

Lemma A. *Let $E \xrightarrow{f} F \xrightarrow{g} G$ be a sequence of CW-spectra such that the composition $g \cdot f$ is homotopic to the zero. If $0 \rightarrow \pi_*(E) \rightarrow \pi_*(F) \rightarrow \pi_*(G) \rightarrow 0$ is exact, then $E \rightarrow F \rightarrow G$ is a cofiber sequence. (Cf., Lemma 4).*

Proof. Let C_f be the mapping cone of $f: E \rightarrow F$. Then $g: F \rightarrow G$ admits a factorization $F \rightarrow C_f \xrightarrow{h} G$. Considering the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \pi_*(E) & \rightarrow & \pi_*(F) & \rightarrow & \pi_*(C_f) \rightarrow 0 \\ & & \parallel & & \parallel & & \downarrow h_* \\ 0 & \rightarrow & \pi_*(E) & \rightarrow & \pi_*(F) & \rightarrow & \pi_*(G) \rightarrow 0 \end{array}$$

with exact rows, we see easily that $h: C_f \rightarrow G$ is a homotopy equivalence.

Using (A.1) the composition $\tau_n \cdot m_{x_n}$ becomes homotopic to the zero. We get therefore that

$$(A.3) \quad \Sigma^{2n} MU\langle n\rangle \xrightarrow{m_{x_n}} MU\langle n\rangle \xrightarrow{\tau_n} MU\langle n-1\rangle \text{ is a cofiber sequence.}$$

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