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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 12(3) P.583-P.595</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1975</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/11134">https://doi.org/10.18910/11134</a></td>
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<td>DOI</td>
<td>10.18910/11134</td>
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Osaka University
A NOTE ON THE RELATION OF $\mathbb{Z}_2$-GRADED COMPLEX COBORDISM TO COMPLEX K-THEORY

ZEN-ICHI YOSIMURA

(Received December 16, 1974)

Let $MU^*(\ )$ and $K^*(\ )$ denote the $\mathbb{Z}_2$-graded complex cobordism theory and the complex $K$-theory respectively. The Thom homomorphism $\mu_c: \pi_0(MU) \to \pi_0(K)$ on coefficient groups is identified (up to sign) with the classical Todd genus $Td: \Lambda \to \mathbb{Z}$. We denote by $I$ the ideal of $\Lambda$ to be the kernel of $Td: \Lambda \to \mathbb{Z}$.

Wolff [7] proved that the decreasing filtration $\{I^qMU^*(\ )\}$ of $MU^*(\ )$ consists of cohomology theories defined on the category of based finite CW-complexes, and the associated quotients $I^qMU^*(\ )/I^{q+1}MU^*(\ )$ are determined by the complex $K$-theories $KG^*_q(\ )$ with coefficients $G_q=I^q/I^{q+1}$.

The purpose of this note is to extend the Wolff's result to the category of based CW-complexes. Let $F_qMU$ be the CW-spectrum associated with the cohomology theory $I^qMU^*(\ )$, i.e., $\{Y, F_qMU\}^\ast\simeq I^qMU^*(Y)$ for any based finite CW-complex (or finite CW-spectrum). We show that $\{F_qMU^*(\ )\}$ is a decreasing filtration of $MU^*(\ )$ consisting of $\Lambda$-modules so that the associated quotients are equal to $KG^*_q(\ )$, and in addition that $F_{q+1}MU^*(\ )$ is a direct summand of $F_qMU^*(\ )$.

Moreover we give a tower

$$MU \to \cdots \to Q_qMU \to Q_{q-1}MU \to \cdots \to Q_0MU = K$$

of $MU$-module spectra such that $KG_q \to Q_qMU \to Q_{q-1}MU$ is a cofiber sequence of $MU$-module spectra, which factorizes the Thom map $\mu_c: MU \to K$.

Baas [3] constructed a tower of CW-spectra

$$MU \to \cdots \to MU\langle n \rangle \to MU\langle n-1 \rangle \to \cdots \to MU\langle 0 \rangle = H$$

factorizing the Thom map $\mu: MU \to H$. In appendix we show that the tower is of $MU$-module spectra and the sequence $\Sigma^2MU\langle n \rangle \xrightarrow{m_{2n}} MU\langle n \rangle \to MU\langle n-1 \rangle$ is a cofiber sequence where $m_{2n}$ is the multiplication by $x_n$ a ring generator of $\Lambda$ with degree $2n$.

1. Decreasing filtration of $MU^*(\ )$

1.1. A pair $(E, \rho)$ is called a $\mathbb{Z}_2$-graded CW-spectrum if $E$ is a CW-spectrum
and $\rho : \Sigma^2 E \to E$ is a homotopy equivalence. Such a pair $(E, \rho)$ gives rise to natural isomorphisms

$$\rho_* : E_*(X) \to E_{*+2}(X), \quad \rho^* : E^{*+2}(X) \to E^*(X)$$

for any CW-spectrum $X$. So we can define $\mathbb{Z}_2$-graded homology and cohomology theories $E_*(\ ), E^*(\ )$ by putting

$$E_4(X) = E_0(X) \oplus E_2(X), \quad E^q(X) = E^0(X) \oplus E^2(X).$$

For a CW-spectrum $E$ we put

$$E = \bigvee \Sigma^{2n} E, \quad \bar{E} = \prod \Sigma^{2n} E.$$ 

Taking the canonical identifications $\rho : \Sigma^2 E \to E$ and $\bar{\rho} : \Sigma^2 \bar{E} \to \bar{E}$ as structure morphisms $E$ and $\bar{E}$ admit structures of $\mathbb{Z}_2$-graded CW-spectra respectively. From definition it follows that

$$E_0(X) \cong \bigoplus \Sigma^{2n} E_{2n}(X), \quad E_1(X) \cong \bigoplus \Sigma^{2n+1} E_{2n+1}(X),$$

$$\bar{E}^0(X) \cong \prod E^{2n}(X), \quad \bar{E}^1(X) \cong \prod E^{2n+1}(X)$$

for all CW-spectra $X$. In particular, the canonical morphism $H \to \bar{H}$ becomes a homotopy equivalence for the Eilenberg-MacLane spectrum $H$.

The $BU$-spectrum $K$ may be regarded as a $\mathbb{Z}_2$-graded CW-spectrum because it possesses the Bott map $\beta : \Sigma^2 K \to K$ which is a homotopy equivalence.

Denote by $F_n$ the direct sum of $n$-copies of the integers $\mathbb{Z}$ and by $F$ the direct limit of $F_n$, i.e., $F$ is a free abelian group with countably many factors. Putting

$$BU_{F_n} = BU \times \cdots \times BU, \text{ the product of } n\text{-copies of } BU,$$

$$BU_F = \bigcup_n BU_{F_n}, \text{ the union of } BU_{F_n},$$

we obtain

**Proposition 1.** There exists a natural isomorphism

$$[X, BU_F] \to KF^0(X)$$

for any based connected CW-complex $X$.

Proof. Let $Y$ be a based connected finite CW-complex. Then we have a sequence of natural isomorphisms

$$[Y, BU_F] \leftarrow \lim [Y, BU_{F_n}] \leftarrow \lim [Y, BU] \otimes F_n \rightarrow \lim K^0(Y) \otimes F_n \rightarrow K^0(Y) \otimes F \rightarrow KF^0(Y).$$

Therefore the contravariant functor $KF^0$ defined on the category of based connected CW-complexes is represented by $BU_F$ (use [1, Addendum 1.5]).
Proposition 1 implies that $BU_F$ is homotopy equivalent to $\Omega^2 BU_F$ where $\Omega^2$ means the component of the base point in the double loop space. Hence we have

(1.1) \textit{in the $BU$-spectrum $KF$ with the coefficients $F$ every even term is the based CW-complex $BU_F$.}

1.2. Let us denote by $MU$ the unitary Thom spectrum and by $\mu_c: MU \to K$ the Thom map which is a ring morphism. The composition

$$\mu_c: MU \to K \to K$$

of $\vee \Sigma^n \mu_c$ and $\vee \beta^n$ is a morphism of $Z_2$-graded ring-spectra, called the $Z_2$-graded Thom map. As is well known, it is characterized by the coefficient homomorphism $\mu_c: \pi_\ast(MU) \to \pi_\ast(K)$ which coincides (up to sign) with the classical Todd genus $Td$. Putting $\Lambda = \pi_\ast(MU)$, $\pi_\ast(K) = Z$ is viewed as a $Z_2$-graded $\Lambda$-module via $\mu_c = Td$ and it is written $Z\tau_d$ for emphasis.

Using the kernel $I$ of $Td: \Lambda \to Z$ we define a decreasing filtration $\{I^q\}_{q \geq 0}$ consisting of ideals of $\Lambda$. Denoting by $G_q$ the associated $Z_2$-graded $\Lambda$-module $I^q/I^{q+1}$, we see easily [7, Satz 3.8] that

(1.2) $G_0 \cong Z_{Td}$ and $G_q$ is a free abelian group with countably many factors for $q \geq 1$.

For a $Z_2$-graded $\Lambda$-module $A$ we have a decreasing filtration $\{I^qA\}_{q \geq 0}$ consisting of submodules of $A$, whose associated $Z_2$-graded $\Lambda$-module $I^q/I^{q+1}A$ is written $G_q(A)$. Applying the commutative diagram

\[
0 \to \text{Tor}_1^\Lambda(Z_{Td}, A) \to I^\Lambda A \to A \to Z_{Td}^\Lambda A \to 0 \\
0 \to IA \to A \to G_q(A) \to 0
\]

with exact rows, we get an isomorphism

(1.3) $G_q^\Lambda A \cong G_q^\Lambda(Z_{Td}^\Lambda A) \cong G_q^\Lambda G_\delta(A)$

by means of "4 lemma".

Proposition 2. Let $A$ be a $\Lambda$-module with $\text{Tor}_k^\Lambda(Z_{Td}, A) = 0$ for all $k \geq 1$. Then, for every $q \geq 0$ both $I^qA \to I^qA$ and $G_q^\Lambda A \to G_q^\Lambda(A)$ are isomorphisms and $\text{Tor}_k^\Lambda(I^q, A) = \text{Tor}_k^\Lambda(G_q, A) = 0$ for all $k \geq 1$.

Proof. Choose a free $\Lambda$-module $F$ such that $A$ is isomorphic to a quotient $F/B$. By induction on $q$ we shall show that the sequences

$$0 \to I^qB \to I^qF \to I^qA \to 0, \quad 0 \to G_q(B) \to G_q(F) \to G_q(A) \to 0$$

are exact. The $q=0$ case is evident because of (1.3). Applying induction
hypotesis and "3 x 3 lemma" we find easily that $0 \to I^qB \to I^qF \to I^qA \to 0$ is exact. So we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & G_q \otimes G_0(B) & \to & G_q \otimes G_0(F) & \to & G_q \otimes G_0(A) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & G_0(I^qB) & \to & G_0(I^qF) & \to & G_0(I^qA) & \to & 0
\end{array}
\]

with exact rows. Since all vertical arrows are epimorphisms and in particular the central one is an isomorphism, all vertical arrows become isomorphisms. Consequently we get that $0 \to G_0(B) \to G_0(F) \to G_0(A) \to 0$ is exact.

Next, we consider the commutative diagrams

\[
\begin{array}{cccccc}
0 & \to & \text{Tor}_k^I(I^q, A) & \to & I^q \otimes B & \to & I^q \otimes F & \to & I^q \otimes A & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & I^q B & \to & I^q F & \to & I^q A & \to & 0
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \to & \text{Tor}_k^I(G_q, A) & \to & G_q \otimes B & \to & G_q \otimes F & \to & G_q \otimes A & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & G_q \otimes G_0(B) & \to & G_q \otimes G_0(F) & \to & G_q \otimes G_0(A) & \to & 0
\end{array}
\]

with exact rows. Remark that $\text{Tor}_k^I(Z_{T_d}, B)=0$ for all $k \geq 1$. By use of "4 lemma" and (1.3) we see that all vertical arrows are isomorphisms, and hence we obtain the required results.

For a $\Lambda$-module $A$ we put $J_q(A)=A/I^{q+1}A$ and abbreviate $J_q=J_q(\Lambda)$ when $A=\Lambda$. As an immediate corollary of Proposition 2 we have

**Corollary 3.** Let $A$ be a $\Lambda$-module with $\text{Tor}_k^I(Z_{T_d}, A)=0$ for all $k \geq 1$. Then $J_q \otimes A \to J_q(A)$ is an isomorphism and $\text{Tor}_k^I(J_q, A)=0$ for all $k \geq 1$.

1.3. Let $\mathcal{HA}$ denote the category of comodules over $MU_*(MU)$ which are finitely presented as $\Lambda$-modules. Notice that $\mathcal{HA}$ is an abelian category which has enough projectives, and also that $MU_*(Y)$ lies in the category $\mathcal{HA}$ whenever $Y$ is a finite CW-spectrum. Since the functor $M \to Z_{T_d} \otimes M$ is exact on $\mathcal{HA}$ it follows immediately that

\[(1.4) \quad \text{Tor}_k^I(Z_{T_d}, M) = 0 \quad \text{for all } k \geq 1 \text{ if } M \text{ lies in } \mathcal{HA}.\]

Proposition 1 and Corollary 2 combined with (1.4) say that

\[(1.5) \quad \text{the functors } M \to I^qM \simeq I^q \otimes M, \quad M \to G_q(M) \simeq G_q \otimes M \text{ and } M \to J_q(M) \simeq J_q \otimes M \text{ on } \mathcal{HA} \text{ are exact.}\]

**Theorem 1** (Wolff [7]). i) Both $I^qMU_*( )$ and $MU_*( )/I^{q+1}MU_*( )$ are
homology theories defined on the category of CW-spectra, so that \( I^q \otimes \mu_*(X) \rightarrow I^q \mu_*(X) \) and \( \Lambda / I^{q+1} \otimes \mu_*(X) \rightarrow \mu_*(X) / I^{q+1} \mu_*(X) \) are natural isomorphisms for all CW-spectra \( X \).

ii) \( I^q \mu_*(X) / I^{q+1} \mu_*(X) \) is a homology theory defined on the category of CW-spectra such that there exists a natural isomorphism \( I^q \mu_*(X) / I^{q+1} \mu_*(X) \rightarrow KG_q(X) \) for any CW-spectrum \( X \) which is induced by the \( Z_2 \)-graded Thom map \( \mu_c \).

Proof. i) and the first half of ii) are immediate from (1.5). The latter half of ii) is also valid because we have a natural isomorphism

\[
G_q(\mu_*(X)) \rightarrow G_q \otimes \mu_*(X) \rightarrow G_q \otimes (Z_2 \otimes \mu_*(X)) \rightarrow G_q \otimes K_*(X) \rightarrow KG_q(X).
\]

Let \( \phi: E_\#(X) \rightarrow F_\#(X) \) be a natural transformation for any CW-spectrum \( X \). According to [1, Addendum 1.5] there exists a morphism \( f: E \rightarrow F \) inducing \( \phi \), and it is unique up to weak homotopy. The proof in [1] is actually given for the category of based connected CW-complexes, but it is easily extended to that of CW-spectra. Such a morphism \( f \) is uniquely chosen (up to homotopy) under the assumption that \( F \circ (E) \) is Hausdorff.

Let \( E_\#(X) \) be a \( Z_2 \)-graded homology theory defined on the category of CW-spectra, i.e., a homology theory equipped with a natural isomorphism \( E_\#(X) \rightarrow E_{\#+1}(X) \) for any CW-spectrum \( X \). Then it gives \( E \) a structure of \( Z_2 \)-graded CW-spectrum. In particular, the induced structure is unique if \( E \circ (E) \) is Hausdorff.

Recall that the \( Z_2 \)-graded CW-spectrum \( \mu_\# \) is equipped with the canonical identification \( \rho: \Sigma^\# \mu_\# \rightarrow \mu_\# \) as structure morphism. Since \( \mu_\#(\mu_\#) \) is Hausdorff (use Proposition 6 below), the \( Z_2 \)-graded CW-spectrum \( (\mu_\#, \rho) \) is characterized only by the \( Z_2 \)-graded homology theory \( \mu_\#(\#) \).

Denote by \( F_q \mu_\# \) and \( Q_q \mu_\# \) the representing spectra of the new homology theories \( I^q \mu_\#(\#) \) and \( \mu_\#(\#) / I^{q+1} \mu_\#(\#) \) respectively, i.e.,

\[
I^q \mu_\#(\#) \cong \{ \Sigma^*, F_q \mu_\# \wedge X \}, \quad \mu_\#(\#) / I^{q+1} \mu_\#(\#) \cong \{ \Sigma^*, Q_q \mu_\# \wedge X \}
\]

for any CW-spectrum \( X \). Of course, they are both \( Z_2 \)-graded CW-spectra. Then there exist morphisms

\[
i_q: F_{q+1} \mu_\# \rightarrow F_q \mu_\#, \quad j_q: Q_q \mu_\# \rightarrow Q_{q-1} \mu_\#, \quad e_q: F_{q+1} \mu_\# \rightarrow \mu_\#, \quad \pi_q: \mu_\# \rightarrow Q_q \mu_\#
\]

which induce the canonical morphisms in homology groups, and moreover we have morphisms
\[ \mu_q : F_q MU \to KG_q, \quad \nu_q : KG_q \to Q_q MU \]
such that \( \mu_q^* : F_q MU_* (X) \to KG_q^*(X) \) and \( \nu_q^* : KG_q^*(X) \to Q_q MU_* (X) \) in homology groups are natural homomorphisms induced by the \( \mathbb{Z}_2 \)-graded Thom map \( \mu_c \).

**Lemma 4.** Let \( E \to F \to G \) be a sequence which satisfies the property that \( 0 \to E_*(X) \to F_*(X) \to G_*(X) \to 0 \) is a short exact sequence for every CW-spectrum \( X \). Then it is a cofiber sequence.

**Proof.** Let \( C_f \) be the mapping cone of \( f \), i.e., \( E \to F \to C_f \) a cofiber sequence. Then for any CW-spectrum \( X \) we have a commutative diagram

\[
\begin{array}{c}
0 \to E_*(X) \to F_*(X) \to C^f_*(X) \to 0 \\
0 \to E_*(X) \to F_*(X) \to G_*(X) \to 0
\end{array}
\]

with exact rows. Clearly \( h : C_f \to G \) which induces \( \phi \) is a homotopy equivalence.

By virtue of Lemma 4 we verify that

(1.6) \( F_{q+1} MU \to MU \to Q_q MU, \ F_{q+1} MU \to F_q MU \to KG_q \) and \( KG_q \to Q_q MU \to Q_{q-1} MU \) are all cofiber sequences.

2. \( \mathbb{Z}_2 \)-graded \( MU \)-module spectra

**2.1.** The inclusion \( Z \subset Q \) induces a natural transformation \( ch : E^*(X) \to EQ^*(X) \) for any CW-spectrum \( X \), called the Chern-Dold character.

**Proposition 5.** If \( ch : E^*(X) \to EQ^*(X) \) is a monomorphism, then \( E^*(X) \) is Hausdorff.

**Proof.** Since \( EQ^*(X) \) is always Hausdorff [8, Proposition 4], the result is immediate.

Let \( W \) be a connective CW-spectrum with \( H_*(W) \) free and assume that \( \pi_*(E) \) is torsion free. Then \( H^*(W; \pi_*(E)) \to H^*(W; \pi_*(E) \otimes Q) \) is a monomorphism, and hence the Atiyah-Hirzebruch spectral sequences for \( E^*(W) \) and \( EQ^*(W) \) collapse. Therefore we get that

(2.1) \( ch : E^*(W) \to EQ^*(W) \) is a monomorphism. (Cf., [8, Lemma 11]).

Applying Proposition 5 we obtain

**Proposition 6.** Let \( W \) be a connective CW-spectrum with \( H_*(W) \) free. If \( \pi_*(E) \) is torsion free, then \( E^*(W) \) is Hausdorff.

By means of Proposition 5 we get the following lemmas.
Lemma 7. Assume that $\pi_0(E)$ is torsion free and $\pi_1(E) = 0$. Then $E^0(KG_q \wedge MU \wedge \cdots \wedge MU)$ and $E^0(Q_qMU \wedge MU \wedge \cdots \wedge MU)$ are Hausdorff and $E^0(F_qMU \wedge MU \wedge \cdots \wedge MU) = 0$.

Proof. Since $E^{2n-1}(BU_q \wedge MU \wedge \cdots \wedge MU) = EQ^{2n-1}(BU_q \wedge MU \wedge \cdots \wedge MU) = 0$ we have a commutative square

$$
\begin{array}{c}
E^0(KG_q \wedge MU \wedge \cdots \wedge MU) \\
\downarrow \\
E^0(KG_q \wedge MU \wedge \cdots \wedge MU)
\end{array}
\quad \quad \quad \begin{array}{c}
\leftarrow E^{2n}(BU_q \wedge MU \wedge \cdots \wedge MU) \\
\downarrow \\
\leftarrow E^{2n}(BU_q \wedge MU \wedge \cdots \wedge MU)
\end{array}
$$

such that the horizontal arrows are isomorphisms. The left arrow is a monomorphism because so is the right one by use of (2.1). Since Theorem 1 implies that $0 \to EQ^*(Q_qMU \wedge MU \wedge \cdots \wedge MU) \to EQ^*(Q_qMU \wedge MU \wedge \cdots \wedge MU) \to EQ^*(KG_q \wedge MU \wedge \cdots \wedge MU) = 0$ is exact, an induction on $q$ involving "4 lemma" shows that $E^*(Q_qMU \wedge MU \wedge \cdots \wedge MU)$ is a monomorphism. Then we find that $E^*(Q_qMU \wedge MU \wedge \cdots \wedge MU) \to E^*(MU \wedge \cdots \wedge MU)$ is a monomorphism because so is $EQ^*(Q_qMU \wedge MU \wedge \cdots \wedge MU) \to EQ^*(MU \wedge \cdots \wedge MU)$. Therefore $E^*(MU \wedge \cdots \wedge MU) \to E^*(F_{q+1}MU \wedge MU \wedge \cdots \wedge MU)$ is an epimorphism, and hence $E^*(F_{q+1}MU \wedge MU \wedge \cdots \wedge MU) = 0$.

Lemma 8. $KG_p(F_qMU \wedge MU \wedge \cdots \wedge MU)$ and $Q_pMU^0(F_qMU \wedge MU \wedge \cdots \wedge MU)$ are Hausdorff and $KG_p(F_qMU \wedge MU \wedge \cdots \wedge MU) = Q_pMU^0(F_qMU \wedge MU \wedge \cdots \wedge MU) = 0$.

Proof. Putting $X = F_qMU \wedge MU \wedge \cdots \wedge MU$, we note by Theorem 1 i) that $K_d(X)$ is free and $K_0(X) = 0$. Applying the universal coefficient sequence

$$
\begin{array}{l}
0 \to \text{Ext}(K_{d-1}(X), G_p) \to KG_p(X) \to \text{Hom}(K_d(X), G_p) \to 0
\end{array}
$$

for $K$ (see [9, (3.1)]) we get immediately that $ch: KG^0_p(X) \to KG_p \otimes Q^p(X)$ is a monomorphism and $KG^0_p(X) = 0$. By induction on $p$ we obtain that $ch: Q_pMU^0(X) \to Q_pMUQ^0(X)$ is a monomorphism and $Q_pMU^0(X) = 0$ because $0 \to KG_p \otimes Q^p(X) \to Q_pMUQ^0(X) \to Q_{p-1}MUQ^0(X) \to 0$ is exact.

Assume that $\pi_0(E)$ is free and of finite type and put again $X = F_qMU \wedge MU \wedge \cdots \wedge MU$. Using the universal coefficient sequence for $E$ [9, (1.8)] we have a commutative diagram

$$
\begin{array}{c}
0 \to \text{Ext}(\hat{E}_{d-1}(X), Z) \to E^*(X) \to \text{Hom}(\hat{E}_d(X), Z) \to 0 \\
\downarrow \\
0 \to \text{Ext}(\hat{E}_{d-1}(X), Q) \to EQ^*(X) \to \text{Hom}(\hat{E}_d(X), Q) \to 0
\end{array}
$$

with exact rows where $\hat{E}$ is the dual of $E$ constructed in [9]. Note that $\pi_0(\hat{E})$ is free and hence so is $\hat{E}_d(X)$. Then the central arrow becomes a monomorphism. Considering the commutative square
\[ E^t(X) \rightarrow \Pi E^*(X) \]
\[ EQ^t(X) \rightarrow \Pi EQ^*(X) \]

in which the upper arrow is an isomorphism, we find that the left one is a monomorphism. Thus we get that

\[ (2.2) \quad \overline{E^t}(F_q MU \land MU \land \cdots \land MU) \text{ is Hausdorff.} \]

Let \( \overline{MU} \) denote the mapping cone of the canonical morphism \( MU \rightarrow \overline{MU} \).

Since \( \Pi Z/\Sigma Z \rightarrow \Pi Q/\Sigma Q \) is a monomorphism we remark that

\[ (2.3) \quad \pi_0(\overline{MU}) = \prod_n \pi_{2n}(MU)/\Sigma \pi_{2n}(MU) \text{ is torsion free and } \pi_1(\overline{MU}) = 0, \]

(see [4, Exercise IV 20]). Then \( \overline{MU} \) has a unique structure of \( Z_2 \)-graded \( CW \)-spectrum so that the cofiber sequence \( MU \rightarrow \overline{MU} \rightarrow \overline{MU} \) is of \( Z_2 \)-graded \( CW \)-spectra.

**Lemma 9.** \( F_p MU^n(F_q MU \land MU \land \cdots \land MU) \) is Hausdorff and \( F_p MU^1(F_q MU \land MU \land \cdots \land MU) = 0 \)

Proof. We put \( X = F_q MU \land MU \land \cdots \land MU \). From Lemma 7 it follows that \( F_p MU^1(X) = 0 \). In the sequence

\[ F_p MU^n(X) \rightarrow MU^n(X) \rightarrow \overline{MU}^n(X) \]

the former arrow is a monomorphism because of Lemma 8 and the latter one is so by means of (2.3) and Lemma 7. Thus the above composition is a monomorphism. On the other hand, (2.2) says that \( \overline{MU}^n(X) \) is Hausdorff. So we get the remaining result.

**2.2.** Since \( F_q MU^n(F_q MU) \) and \( Q_q MU^n(Q_q MU) \) are both Hausdorff, we verify that

\[ (2.4) \quad \text{the } Z_2 \text{-graded homology theories } I^g MU_* ( ) \text{ and } MU_* ( )/I^{g+1} MU_* ( ) \text{ give } F_q MU \text{ and } Q_q MU \text{ unique structures of } Z_2 \text{-graded } CW \text{-spectra respectively.} \]

Moreover, by virtue of Lemmas 7, 8 and 9 we see that

\[ (2.5) \quad i_q: F_{q+1} MU \rightarrow F_q MU, \quad \iota_q: F_{q+1} MU \rightarrow MU, \quad \mu_q: F_q MU \rightarrow KG_q \\
j_q: Q_{q+1} MU \rightarrow Q_q MU, \quad \pi_q: MU \rightarrow Q_q MU, \quad \nu_q: KG_q \rightarrow Q_q MU \]

are uniquely determined (up to homotopy), which induce the canonical morphisms in homology groups. In particular, the composition \( \iota_{q-1} \circ i_q \) is homotopic to \( \iota_q \) and \( j_q \circ \pi_q \) is so to \( \pi_{q-1} \).
Consider the diagram

\[
\begin{array}{ccc}
F_{q+1}MU & \rightarrow & F_qMU \\
& \searrow & \downarrow \\
KG_q & \rightarrow & Q_qMU \\
& \rightarrow & \rightarrow \\
\Sigma F_{q+1}MU & \rightarrow & \Sigma F_qMU \\
\end{array}
\]

consisting of cofiber sequences. With an application of Verdier's lemma (see [2, Lemma 6.8]) we get a cofiber sequence

\[
KG_q \rightarrow Q_qMU \rightarrow Q_{q-1}MU \rightarrow \Sigma KG_q
\]

(denoted by dotted arrows in the above diagram) which makes the diagram homotopy commutative. Clearly this yields the canonical exact sequence

\[
0 \rightarrow KG_q(X) \rightarrow Q_qMU_*(X) \rightarrow Q_{q-1}MU_*(X) \rightarrow 0.
\]

By uniqueness of \(\nu_q, j_q\) the above cofiber sequence coincides with

\[
KG_q \rightarrow Q_qMU \rightarrow Q_{q-1}MU \rightarrow \Sigma KG_q.
\]

The multiplication \(\phi: MU \wedge MU \rightarrow MU\) gives rise to natural \(\mathbb{Z}_2\)-graded homomorphisms

\[
m_q: F_qMU_*(X) \otimes MU_*Y \rightarrow F_qMU_*(X \wedge Y)
\]

\[
m_q: Q_qMU_*(X) \otimes MU_*Y \rightarrow Q_qMU_*(X \wedge Y)
\]

for all CW-spectra \(X\) and \(Y\). By use of Lemmas 7 and 9 there exist unique pairings

\[
\phi_q: F_qMU_\wedge MU \rightarrow F_qMU, \quad \bar{\phi}_q: Q_qMU_\wedge MU \rightarrow Q_qMU
\]

which induce the above \(m_q\) and \(\bar{m}_q\) respectively. Then it follows that

\(2.6\) both \(F_qMU\) and \(Q_qMU\) are (associative) \(\mathbb{Z}_2\)-graded \(MU\)-module spectra.

**Proposition 10.** Let \(M\) be a \(\mathbb{Z}_2\)-graded ring spectrum, \(E, F\) and \(G\) \(\mathbb{Z}_2\)-graded \(M\)-module spectra and \(E \rightarrow F \rightarrow G\) a cofiber sequence. Assume that \(E^n(E), E^n(E \wedge M), G^n(F)\) and \(G^n(F \wedge M)\) are Hausdorff, or that \(F^n(E), F^n(E \wedge M), G^n(G)\) and \(G^n(G \wedge M)\) are Hausdorff. If for any CW-spectrum \(X 0 \rightarrow E_q(X) \rightarrow F_q(X) \rightarrow G_q(X) \rightarrow 0\) is a short exact sequence of \(\mathbb{Z}_2\)-graded \(M_q\) modules, then the cofiber sequence \(E \rightarrow F \rightarrow G\) is of \(\mathbb{Z}_2\)-graded \(M\)-module spectra.

Proof. Assuming that \(F^n(E), F^n(E \wedge M), G^n(G)\) and \(G^n(G \wedge M)\) are Hausdorff, we consider the diagrams

\[
\begin{array}{cccc}
\Sigma^2E & \rightarrow & \Sigma^2F & \rightarrow & \Sigma^2G & \rightarrow & \Sigma^2E \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
E & \rightarrow & F & \rightarrow & G & \rightarrow & \Sigma E \\
\end{array}
\]

\[
\begin{array}{cccc}
E \wedge M & \rightarrow & F \wedge M & \rightarrow & G \wedge M & \rightarrow & \Sigma E \wedge M \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
E \rightarrow & F \rightarrow & G \rightarrow & \Sigma E \\
\end{array}
\]
with cofiber sequences. Under the first two assumptions two left squares become homotopy commutative because they induce the $\mathbb{Z}_2$-graded homomorphism $E_\delta(\ ) \to F_\delta(\ )$ of $M_\delta(\ )$-modules. Therefore there exist morphisms $\Sigma G \to G$ and $G \wedge M \to G$ which make the above diagrams into morphisms of cofiber sequences. As is easily checked, they give $G_\delta(\ )$ a structure of $\mathbb{Z}_2$-graded $M_\delta(\ )$-module, which coincides with the original one. So, using the remaining assumptions again we see that the above morphisms are homotopic to the given ones respectively. Consequently the cofiber sequence $E \to F \to G$ becomes the required one.

Another case is similarly proved.

The ring spectrum $K$ may be regarded as a $\mathbb{Z}_2$-graded $\mathbb{MU}$-module spectrum via the $\mathbb{Z}_2$-graded Thom map $\mu_c : MU \to K$.

Applying Proposition 10 to three cofiber sequences of (1.6) we get

**Theorem 2.** The sequences $F_{q+1}MU \to MU \to Q_qMU$, $F_{q+1}MU \to F_qMU \to KG_q$ and $KG_q \to Q_qMU \to Q_{q-1}MU$ are cofiber sequences of $\mathbb{Z}_2$-graded $\mathbb{MU}$-module spectra.

**Proof.** The assumptions needed in Proposition 10 are satisfied by Lemmas 7, 8 and 9.

As a result we have a tower

$$
\mathbb{MU} \to \cdots \to Q_qMU \to Q_{q-1}MU \to \cdots \to Q_0MU = K
$$

of $\mathbb{Z}_2$-graded $\mathbb{MU}$-module spectra such that $KG_q \to Q_qMU \to Q_{q-1}MU$ is a cofiber sequence, which factorizes the $\mathbb{Z}_2$-graded Thom map $\mu_c : MU \to K$.

**2.3.** Here we extend the Wolff's result to the case of based CW-complexes.

**Proposition 11.** There exists an (unstable) natural homomorphism

$$
\Phi_q : KG_q^\mathbf{a}(X) \to F_qMU^*(X)
$$

for any based CW-complex $X$, which satisfies the equality that $\mu_q \circ \Phi = \text{id}$.

**Proof.** We may assume that $X$ is connected. Let $i : BU_G \to KG_q$ be the inclusion. Then we can choose a morphism $\epsilon_q : BU_G \to F_qMU$ such that $i$ is homotopic to the composition $\mu_q \cdot \epsilon_q$, because $F_{q+1}MU(BU_G) = 0$. In the commutative diagram

$$
[X, BU_G] \xrightarrow{J_q} \{X, BU_G\} \xrightarrow{i_*} \{X, KG_q\} = KG_q^\mathbf{a}(X)
$$

$$
\begin{array}{c}
\epsilon_q \circ \Delta \\
\uparrow \mu_q \\
\{X, F_qMU\} = F_qMU^*(X)
\end{array}
$$

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the composition \( i^* \cdot J^* \) is an isomorphism because of Proposition 2 (see [6, Theorem 14.5]). So we put that \( \Phi^q = c_{q^*} \cdot J^* \cdot (i^* \cdot J^*)^{-1} \).

**Remark.** If \( \Phi^q \) is stable, then we have a natural split exact sequence

\[
0 \rightarrow F_{q+1}MU^q(X) \rightarrow F_qMU^q(X) \rightarrow KG^q(X) \rightarrow 0
\]

for every CW-spectrum \( X \). Therefore \( F_qMU \) becomes homotopy equivalent to the wedge \( F_{q+1}MU \vee KG_q \). However \( H_*(F_qMU) \) is a free abelian group and \( H_*(KG) \) is a \( \mathbb{Q} \)-module. This is a contradiction.

We now obtain our main result.

**Theorem 3.** For any based CW-complex \( X \) the natural sequences

\[
0 \rightarrow F_{q+1}MU^q(X) \rightarrow F_qMU^q(X) \rightarrow KG^q(X) \rightarrow 0
\]

\[
0 \rightarrow KG^q(X) \rightarrow Q_qMU^q(X) \rightarrow Q_{q-1}MU^q(X) \rightarrow 0
\]

of \( \mathbb{Z}_2 \)-graded \( \Lambda \)-modules are split exact.

**Proof.** The first case is immediate from Proposition 11. On the other hand, a diagram chase shows that the second sequence is exact for any based CW-complex \( X \) and hence it is split.

**Appendix**

Recall that \( MU \) is a ring spectrum with coefficients \( \Lambda^\ast = \mathbb{Z}[x_1, \ldots, x_n, \ldots] \) where \( \deg x_n = 2n \). By killing certain bordism classes Baas [3] constructed homology theories \( MU^{(n)}(\ ) \) with coefficient \( \pi_{\ast}(MU^{(n)}) = \Lambda^\ast/(x_{n+1}, \ldots) \), whose representing spectrum we denote by \( MU^{(n)} \). \( MU^{(n)}(\ ) \) is an (associative) \( MU^\ast(\ ) \)-module, thus there exists a natural homomorphism

\[
m_{\ast}: MU^\ast(X) \otimes MU^{(n)}(Y) \rightarrow MU^{(n)}(X \wedge Y)
\]

for any CW-spectra \( X \) and \( Y \). This gives us a pairing

\[
\phi^{(n)}: MU \wedge MU^{(n)} \rightarrow MU^{(n)}
\]

by which the above \( m_{\ast} \) is induced.

An easy computation shows that \( MU^{(n)} \otimes \mathbb{Z}^{2k-1}(MU \wedge \cdots \wedge MU \wedge MU^{(n)}) = 0 \) because \( \pi_{2l+1}(MU^{(n)}) = 0 \) for all \( l \). Then [8, Theorem 1] says that

\[
(MU^{(n)}^{(n)}(MU \wedge \cdots \wedge MU \wedge MU^{(n)}) \text{ is Hausdorff} \text{ (see also [9, Corollary 13]).}
\]

Hence \( \phi^{(n)} \) is uniquely determined (up to homotopy) and moreover

\[
(MU^{(n)} \text{ is an (associative) } MU\text{-module spectrum}).
\]
An important relationship between $MU\langle n \rangle_\ast$ and $MU\langle n - 1 \rangle_\ast$ is given in the form of a natural exact sequence

$$
\rightarrow MU\langle n \rangle_\ast - 2n(X) \xrightarrow{x_n^\ast} MU\langle n \rangle_\ast(X) \xrightarrow{t_n} MU\langle n - 1 \rangle_\ast(X) \rightarrow MU\langle n \rangle_\ast - 2n - 1(X) \rightarrow
$$

of $MU_\ast(\cdot)$-modules where $\cdot x_n$ denotes the multiplication by $x_n$. Because of (A.1) there exists a unique morphism $\tau_n: MU\langle n \rangle \rightarrow MU\langle n - 1 \rangle$ of $MU$-module spectra whose induced homomorphism is the above $t_n$. On the other hand, the composition

$$m_{x_n}: \Sigma^{\infty}MU\langle n \rangle x_n^\ast \rightarrow MU_\ast MU\langle n \rangle \xrightarrow{\phi\langle n \rangle} MU\langle n \rangle$$

is characterized by the above multiplication $\cdot x_n$.

**Lemma A.** Let $E \xrightarrow{f} F \xrightarrow{g} G$ be a sequence of CW-spectra such that the composition $g \circ f$ is homotopic to the zero. If $0 \rightarrow \pi_\ast(E) \xrightarrow{\pi_\ast(f)} \pi_\ast(F) \rightarrow \pi_\ast(G) \rightarrow 0$ is exact, then $E \rightarrow F \rightarrow G$ is a cofiber sequence. (Cf., Lemma 4).

Proof. Let $C_f$ be the mapping cone of $f: E \rightarrow F$. Then $g: F \rightarrow G$ admits a factorization $F \rightarrow C_f \rightarrow G$. Considering the commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & \pi_\ast(E) \\
\| & & \| \\
0 & \rightarrow & \pi_\ast(F) \\
\| & & \| \\
0 & \rightarrow & \pi_\ast(C_f) \\
\end{array}
$$

with exact rows, we see easily that $h: C_f \rightarrow G$ is a homotopy equivalence.

Using (A.1) the composition $\tau_n \cdot m_{x_n}$ becomes homotopic to the zero. We get therefore that

$$\Sigma^{\infty}MU\langle n \rangle \xrightarrow{m_{x_n}} MU\langle n \rangle \xrightarrow{\tau_n} MU\langle n - 1 \rangle$$

is a cofiber sequence.

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References


