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Osaka University
ON CERTAIN COHOMOLOGY GROUPS ATTACHED TO HERMITIAN SYMMETRIC SPACES (II)

Yozō MATSUSHIMA AND SHINGO MURAKAMI

(Received June 28, 1968)

Introduction

Let $X = G/K$ be a bounded symmetric domain in $\mathbb{C}^N$, where $G$ is a semi-simple Lie group with finite center and $K$ is a maximal compact subgroup of $G$. An automorphic factor $j$ on $X$ is a $C^\infty$-mapping $j : G \times X \to GL(S)$, $S$ being a finite dimensional complex vector space, which satisfies the conditions:

1) $j(t, x)$ is holomorphic in $x \in X$ for each $t \in G$;
2) $j(ts', x) = j(s, ts')j(s', x)$ for $x \in X$ and $s, s' \in G$.

Let $x_0$ be the point of $X = G/K$ represented by the coset $K$. An automorphic factor $j$ defines a representation $\tau$ of the group $K$ by the formula $\tau(t) = j(t, x_0)$ for $t \in K$ and we say that $j$ is a prolongation of the representation $\tau$ of $K$. We know that, given a representation $\tau$ of $K$ in a complex vector space $S$, there exists an automorphic factor $j_\tau : G \times X \to GL(S)$ which is a prolongation of $\tau$ and which we call the canonical automorphic factor of type $\tau$ [4, 6, Part II]. Moreover, if $\tau$ is an irreducible representation of $K$, then the automorphic factors which are prolongations of $\tau$ are equivalent to each other [6, Appendix].

Let $j$ be an automorphic factor on $X$. Then $G$ acts on $X \times S$ as a group of holomorphic transformations if we define the action of $s \in G$ by putting

$$s(x, u) = (sx, j(s, x)u)$$

for $(x, u) \in X \times S$.

Now let $\Gamma$ be a discrete subgroup of $G$. Then $\Gamma$ acts on $X$ properly and discontinuously. In the following we assume that the quotient space $\Gamma \backslash X$ is compact and that $\Gamma$ acts freely on $X$ and let $M = \Gamma \backslash X$. Then $M$ is a compact complex Kähler manifold. Moreover, $\Gamma$ acts on $X \times S$ and let $E(j)$ be the quotient of $X \times S$ by the action of $\Gamma$: $E(j) = \Gamma \backslash (X \times S)$. Then $E(j)$ is a holomorphic vector bundle over $M$ with typical fibre $S$. In this paper we consider exclusively the case where $j$ is a canonical automorphic factor $J_\tau$. In this case the vector bundle $E(J_\tau)$ may be interpreted in the following way [6]. Let $X_u$ be the hermitian symmetric space of compact type associated with $X$. The repre-
sentation \( \tau \) of \( K \) defines a "homogeneous" vector bundle \( E_u(\tau) \) over \( X_u \). Now \( X \) is imbedded in \( X_u \) as an open submanifold and \( \Gamma \) acts on \( E_u(\tau) \mid X \) as a group of bundle automorphisms, where \( E_u(\tau) \mid X \) denotes the portion of \( E_u(\tau) \) over \( X \). Then the quotient of \( E_u(\tau) \mid X \) by the action of \( \Gamma \) is a holomorphic vector bundle over \( M \) which is isomorphic to \( E(J) \).

Let \( E(J) \) denote the sheaf of germs of holomorphic sections of \( E(J) \). Each cohomology class in \( H^q(M, E(J)) \) is represented by an \( E(J) \)-valued harmonic \( q \)-form which we shall call an automorphic harmonic \( q \)-form of type \( J \). In particular, for \( q = 0 \) an automorphic harmonic 0-form is nothing but a holomorphic automorphic form of type \( J \) in the usual sense.

In Part I of this paper we shall show that an automorphic harmonic \( q \)-form \( \eta \) of type \( J \) is identified with a set \( \{ f_S \} \) of holomorphic automorphic forms \( f_S \) of type \( J_{\tau_S} \) provided that the highest weight \( \Lambda \) of the representation \( \tau \) of \( K \) satisfies a certain condition; here \( W^r_{\Lambda} \) denotes a subset of the Weyl group of the Lie algebra \( g^c \) uniquely determined by \( \Lambda \) and \( \tau_S \) is a representation of \( K \) determined by \( \Lambda \) and \( S \) in a certain way. We shall show also that, for \( q = q_\Lambda \), where \( q_\Lambda \) is a number uniquely determined by \( \Lambda \), the set \( W_{\Lambda}(q) \) consists of a single element; thus every automorphic harmonic \( q \)-form \( \eta \) of type \( J \), is identified with a holomorphic automorphic form of type \( J_{\tau_S} \) for \( q = q_\Lambda \).

In Part II of this paper we shall prove a formula which expresses the dimension of the space of "automorphic forms" in terms of the unitary representation of \( G \) in \( L^2(\Gamma \backslash G) \). Combined with the results obtained in Part I we obtain a formula on the dimension of the space of automorphic harmonic forms which might be interesting in view of a conjecture stated by Langlands in [3].

**PART I**

1. We retain the notation introduced in the introduction. We have a decomposition of the Lie algebra

\[
g^c = n^+ \oplus n^- \oplus f^c,
\]

where \( f^c \) is the complexification of the subalgebra \( f \) of \( g \) corresponding \( K \) and \( n^\pm \) are abelian subalgebras of \( g^c \) such that

\[
[f^c, n^\pm] \subset n^\pm, [n^+, n^-] \subset f^c.
\]

Moreover, \( f \) contains a Cartan subalgebra \( h \) of \( g \) and \( n^+ \) and \( n^- \) are spanned by root vectors corresponding to the roots of \( g^c \) (with respect to the Cartan subalgebra \( h \)) (see [6, Part II]). We denote by \( \Psi \) the set of all roots \( \alpha \) such that the root vector \( X_\alpha \) belongs to \( n^+ \). Then

\[
n^+ = \sum_{\alpha \in \Psi} CX_\alpha \quad \text{and} \quad n^- = \sum_{\alpha \in \Psi} CX_{-\alpha}
\]
with $X_{-a} = X_{a}$, where $\cdot \cdot$ denotes the conjugation of $g^c$ with respect to the real form $g$. We choose $X_{a}$ in such a way that
\[ \varphi(X_{a}, X_{-a}) = 1, \]
where $\varphi$ denotes the Killing form of $g^c$.

We know that there exists an ordering of the roots such that the roots in $\Psi$ are all positive. We fix once and for all such an ordering of roots. We denote by $\Theta$ the set of all positive roots not belonging to $\Psi$. Then the root vector of $\beta \in \Theta$ belongs to $\mathfrak{t}^c$. We call a root $\alpha$ belonging to $\Psi$ (resp. $\Theta$) as a non-compact (resp. compact) positive root. We shall denote by $\Sigma$ the set of all roots and by $\Sigma^+$ (resp. $\Sigma^-$) the set of all positive (resp. negative) roots.

Let $W$ be the Weyl group of $g^c$. $W$ is a group of linear transformations of the dual space $\mathfrak{h}_0^*$ of the real vector space $\mathfrak{h}_0 = \sqrt{-1} \mathfrak{h}$ in $g^c$ and $W$ is generated by the reflections $S_{\lambda}(\alpha \in \Sigma^+)$ with respect to the hyperplanes $P_{\alpha} = \{ \alpha \mid (\alpha, \lambda) = 0 \}$.

We shall denote by $W_1$ the subgroup of $W$ generated by $S_{\beta}$ with $\beta \in \Theta$. $W_1$ is isomorphic to the Weyl group of $\mathfrak{t}^c$. For $T \in W$, let
\[ \Phi_T = T(\Sigma^-) \cap \Sigma^+ \]
and let
\[ n(T) = \text{the number of roots in } \Phi_T. \]

Let $W^1$ be the subset of the Weyl group $W$ consisting of all $T \in W$ such that $\Phi_T \subset \Psi$. It is easy to see that $T$ belongs to $W^1$ if and only if $T^{-1}(\Theta) \subset \Sigma^+$.

Now let $\tau$ be an irreducible representation of $K$ in a complex vector space $S$. Then $\tau$ defines an irreducible representation of the complex reductive Lie algebra $\mathfrak{t}^c$ which we shall denote by the same letter $\tau$. Let $\Lambda$ be the highest weight of $\tau$. Then we have
\[ (\Lambda, \beta) \geq 0 \quad \text{for all } \beta \in \Theta. \]
We shall assume that $\tau$ satisfies the following condition:
\[ \ast \quad (\Lambda, \alpha) \geq 0 \quad \text{for all } \alpha \in \Psi. \]
Then $(\Lambda, \gamma) \geq 0$ for all $\gamma \in \Sigma^+$ and hence there exists an irreducible representation of $g^c$ whose highest weight is $\Lambda$. We shall denote this representation of $g^c$ by $\rho$ and by $\Lambda'$ the lowest weight of $\rho$.

Now put
\[ W_{\Lambda} = \{ T \in W^1 \mid T \Lambda' = R \Lambda \}, \]
where $R_{\Lambda}$ is the unique element of $W_1$ such that
\[ R_{\Lambda}(\Theta) = -\Theta. \]
Further we let $W^1(\Lambda)(q) = \{ T \in W^1, n(T) = q \}$.

For $T \in W^1$ we shall denote by $\tau_T$ the irreducible representation of $\mathfrak{f}^c$ whose highest weight is $-\xi_T$, where

$$\xi_T = T\Lambda' + \langle \Phi_T \rangle = T(\Lambda' - \delta) + \delta;$$

here, for any subset $\Phi$ of $\Sigma$, $\langle \Phi \rangle$ denotes the sum of the roots belonging to $\Phi$ and $\delta = \langle \Sigma^+ \rangle / 2$.

Now we can state

**Theorem 1.** In the notation introduced above, to each automorphic harmonic $q$-form $\eta$ of type $J_\tau$, where $\tau$ satisfies the assumption (*), we can associate uniquely a set $(\psi)_{s_{00}}$ of holomorphic automorphic forms of type $J_{s_{00}}$ and each of such a set corresponds to an automorphic harmonic $q$-form $\eta$ of type $J_\tau$. In other words we have the following isomorphism of the cohomology groups:

$$H^q(M, E(J_\tau)) \cong \sum_{s_{00} \in W^1(\psi)} H^q(M, E(J_{s_{00}})).$$

**Remark 1.** If $q = N = \dim_c M$, this theorem reduces to the duality theorem of Serre as we shall see later.

**Remark 2.** Actually we shall prove Theorem 1 without the assumption that $\Gamma$ acts freely on $M$. See Theorem 1' in § 2.

Under the same assumption (*) on $\Lambda$, let $q_\Lambda$ be the number of positive roots $\alpha \in \Psi$ such that $\langle \Lambda, \alpha \rangle > 0$.

Then we have

**Theorem 2.** If $q < q_\Lambda$, then $W^1(\Lambda)(q)$ is empty. Moreover, $W^1(\Lambda)(q)$ consists of a single element $T_0$ such that

$$\Phi_{T_0} = \{ \alpha \in \Psi | (R_\Lambda, \alpha) > 0 \}.$$  

Thus we have

$$H^q(M, E(J_\tau)) = 0 \quad \text{for} \quad q < q_\Lambda;$$

$$H^q_{\Lambda}(M, E(J_\tau)) \cong H^q(M, E(J_{\tau_{T_0}})),$$

where the highest weight of the irreducible representation $\tau_{T_0}$ of $\mathfrak{f}^c$ is

$$-\xi_{T_0} = -T_0\Lambda' - \langle \Phi_{T_0} \rangle.$$

**Remark 3.** The vanishing of the cohomology groups $H^q(M, E(J_\tau))$ for $q < q_\Lambda$ has been already proved in [5]. We also remark that $R_\Lambda$ is the lowest weight of the representation $\tau$ of $\mathfrak{f}^c$.

The proof of these theorems will be given in the following sections.
2. We shall recall here some results proved in [4] and [5]. Let \( \rho \) be an irreducible representation of \( g^c \) in a complex vector space \( F \) with highest weight \( \Lambda \).

A*) Restricting the representation \( \rho \) of \( g^c \) to \( n^- \), we may consider \( F \) as an \( n^- \)-module. Let \( C(n^-, F) = \bigoplus C^q(n^-, F) \) be the cochain complex of the abelian Lie algebra \( n^- \) with coefficients in \( F \), where \( C^q(n^-, F) \) is the vector space of all \( q \)-linear alternating maps of \( n^- \) into \( F \). By the Killing form \( \varphi \) of \( g^c \) we can identify \( n^+ \) with the dual space of \( n^- \) and hence \( C(n^-, F) \) with \( F \otimes \wedge n^+ \) and \( C^q(n^-, F) \) with \( F \otimes \wedge n^+ \). Let \( (\cdot, \cdot)_F \) be the inner product in \( F \) such that \( (\rho(x)u, v)_F = (u, \rho(x^*)v)_F \) for all \( x \in n^+ \otimes n^- \) and \( (\rho(y)u, v)_F = -(u, \rho(y^*)v)_F \) for all \( y \in \mathfrak{k}^c \), where \( - \) denotes the conjugation of \( g^c \) with respect to \( g \). The Killing form \( \varphi \) of \( g^c \) defines an inner product in \( n^+ \) such that \( \{X_\alpha | \alpha \in \Psi \} \) is an orthonormal basis. Using these inner products in \( F \) and \( n^+ \) we can define an inner product in \( C(n^-, F) \) which we denote by \( (\cdot, \cdot)_F \). Let \( d^- \) be the coboundary operator. Then there exists an operator \( \delta^- \) of degree \(-1\) such that \( (d^- c, c') = (c, \delta^- c') \) for all \( c, c' \in C(n^-, F) \). Let \( \Delta^- = -d^- \delta^- + \delta^- d^- \). An element \( c \in C^q(n^-, F) \) is called a harmonic \( q \)-cocycle if \( \Delta^- c = 0 \). Every cohomology class of \( H^q(n^-, F) \) is represented by a unique harmonic cocycle. Let \( \mathcal{H}^q \) be the space of all harmonic \( q \)-cocycles.

Now \( C^q(n^-, F) = F \otimes \wedge n^+ \) is a \( \mathfrak{k}^c \)-module, where \( y \in \mathfrak{k}^c \) operates on \( F \) and \( n^+ \) respectively by \( \rho(y) \) and \( \text{ad}(y) \). Let

\[
F \otimes \wedge n^+ = \bigoplus U_{\underline{\nu}} = \bigoplus \sum_{r \in W^1} U_{\underline{\nu}}
\]

be a decomposition of \( F \otimes \wedge n^+ \) into direct sum of irreducible \( \mathfrak{k}^c \)-modules \( U_{\underline{\nu}} \), where \( \underline{\nu} \) denotes the lowest weight of the irreducible representation \( \tau_{\underline{\nu}} \) of \( \mathfrak{k}^c \) in \( U_{\underline{\nu}} \) and \( m_{\underline{\nu}} \) denotes the multiplicity of \( \tau_{\underline{\nu}} \).

For \( T \in W^1 \), let

\[
\xi_T = T \Lambda' + \langle \Phi_T \rangle.
\]

Then the mapping \( T \mapsto \xi_T \) is an injection of \( W^1 \) into the set \( \{\xi'\} \) of lowest weights appearing in (1) and we have:

\[
\mathcal{H}^q = \sum_{T \in W^1} U_{\xi_T}, \quad m_{\xi_T} = 1,
\]

where \( W^1(q) = \{T \in W^1 | n(T) = q\} \). This result is due to Kostant [2].

B. The \( g^c \)-module \( F \) decomposes into sum \( F = S_1 + S_2 + \cdots + S_m \) of mutually orthogonal \( k^c \)-submodules such that

* See [4] §§8, 10 or [6].
1) \( \rho(X)S_t \subseteq S_{t+1} \) for \( X \in \mathbb{n}^+ \) and \( \rho(Y)S_t \subseteq S_{t+1} \) for \( Y \in \mathbb{n}^- \) for \( t=1, 2, \ldots, m \), where \( S_0=S_{m+1}=(0) \);

2) \( S_t \) and \( S_m \) are simple \( \mathfrak{f}^c \)-modules and the highest weight of the representation of \( \mathfrak{f}^c \) in \( S_t \) is \( \Lambda \). Moreover

\[
S_t = \{ u \in F | \rho(X)u = 0 \text{ for all } X \in \mathbb{n}^+ \}, \\
S_m = \{ u \in F | \rho(Y)u = 0 \text{ for all } Y \in \mathbb{n}^- \}.
\]

(See [4, Lemma 5.2] or [6, Lemma 6.1].)

Let

\[
\mathcal{H}_{0,q} = (S_t \otimes \mathfrak{g}^+ ) \cap \mathcal{H}^q.
\]

Then

(2)

\[
\mathcal{H}_{0,q} = \sum_{\tau \in W^1_{\Lambda}(q)} U_{\psi_{\tau}}
\]

where, as in \( \S 1 \), \( W^1_{\Lambda}(q) = \{ T \in W^1 | T \Lambda = R \Lambda, n(T) = q \} \).

**Proof.** Since \( \mathcal{H}_{0,q} \) is a \( \mathfrak{f}^c \)-submodule of \( \mathcal{H}^q \) and \( \mathcal{H}^q \) is a direct sum of simple \( \mathfrak{f}^c \)-modules \( U_{\psi_{\tau}} \) which are not isomorphic to each other, we have

\[
\mathcal{H}_{0,q} = \sum_{\tau \in A} U_{\psi_{\tau}}
\]

where \( A \) is a subset of \( W^1(q) \). Then \( U_{\psi_{\tau}} \) is contained in \( S_t \otimes \mathfrak{g}^+ \) with multiplicity 1 for \( T \in A \).

Now let

\[
F = \sum_{\mu} n_{\mu} F_{\mu'}
\]

be the decomposition of \( F \) into direct sum of simple \( \mathfrak{f}^c \)-submodules \( F_{\mu'} \) with lowest weight \( \mu' \) and with multiplicity \( n_{\mu'} \). Then \( S_t = F_{R \Lambda} \) and \( n_{R \Lambda} = 1 \). For any \( T \in W^1, T \Lambda \) appears as one of \( \mu' \) with \( n_{T \Lambda'} = 1 \). Indeed, as \( T^{-1}(\Theta) \subseteq \Sigma^+ \), \( \langle T \Lambda', \beta \rangle = \langle \Lambda', T^{-1} \beta \rangle \leq 0 \) for all \( \beta \in \Theta \) and therefore \( T \Lambda' \) is the lowest weight of an irreducible representation of \( \mathfrak{f}^c \) which is contained in \( F \) and the eigenspace for the weight \( T \Lambda' \) is of dimension 1. Now let \( T \in A \). Then \( \xi_{\tau} = T \Lambda' + \langle \Phi_{\tau} \rangle \) and hence the 1-dimensional eigenspace for the weight \( \xi_{\tau} \) is contained in \( F_{T \Lambda'} \otimes \mathfrak{g}^+ \). On the other hand, it is contained in \( S_t \otimes \mathfrak{g}^+ \). Therefore we should have \( S_t = F_{T \Lambda'} \) and hence \( R \Lambda = T \Lambda' \). Thus \( T \in W^1_{\Lambda}(q) \). Let, conversely, \( T \in W^1_{\Lambda}(q) \). Then \( F_{T \Lambda'} = S_t \), because \( T \Lambda' = R \Lambda \) and \( F_{T \Lambda'} \otimes \mathfrak{g}^+ \supseteq U_{\psi_{\tau}} \) and therefore \( T \in A \). Thus we have proved that \( A = W^1_{\Lambda}(q) \) and hence our assertion.

**C.** Now let \( \tau \) be a representation of \( K \) in a complex vector space \( S \) and let \( \Gamma \) be a discrete subgroup of \( G \) such that \( \Gamma \backslash G \) is compact. We don’t need to
assume that $\Gamma$ acts freely on $X = G/K$. Let $A^{q}(\Gamma, X, J_{r})$ denote the vector space of all $S$-valued differential forms of type $(0, q)$ on $X$ such that

$$(\gamma \circ L_{\gamma})_{x} = J_{r}(\gamma, x)\eta_{x}$$

for all $\gamma \in \Gamma$ and $x \in X$, where $L_{\gamma}$ denotes the transformation of $X$ by $\gamma$. Then $\Sigma_{r} A^{q}(X, \Gamma, J_{r})$ is a complex with coboundary operator $d''$ and we denote by $H^{q}(X, \Gamma, J_{r})$ the $q$-th cohomology group of this complex. Each cohomology class of $H^{q}(X, \Gamma, J_{r})$ is represented by a harmonic form which we shall call an automorphic harmonic $q$-form of type $J_{r}$. In the case $q = 0$, an automorphic harmonic form is a holomorphic function $f$ on $X$ such that $f(\gamma x) = J_{r}(\gamma, x) f(x)$ for all $\gamma \in \Gamma$ and $x \in X$, i.e. a holomorphic automorphic form of type $\tau$.

If $\Gamma$ acts freely on $X$, then the cohomology group $H^{q}(X, \Gamma, J_{r})$ is isomorphic to $H^{q}(M, E(J_{r}))$.

D. From now on we assume that $\tau$ is an irreducible representation of $K$ such that the highest weight $\Lambda$ of $\tau$ satisfies the condition (*) in §1, i.e. $(\Lambda, \alpha) \geq 0$ for all roots $\alpha \in \Psi$. There exists then an irreducible representation $\rho$ of $\mathfrak{g}^{c}$ in a complex vector space $F$ whose highest weight is $\Lambda$. Then we have a decomposition $F = S_{1} + S_{2} + \cdots + S_{m}$ of $F$ into direct sum of $\mathfrak{g}^{c}$-submodules such that the representation space $S$ of $\rho$ is isomorphic to $S_{1}$ as $\mathfrak{g}^{c}$-module.

We assume that the representation $\rho$ of $\mathfrak{g}^{c}$ is induced from a representation $\rho'$ of the group $G$. Let $A^{q}(X, \Gamma, \rho)$ be the vector space of all $F$-valued $r$-forms $\omega$ on $X$ such that $\omega \circ L_{\gamma} = \rho(\gamma)\omega$ for all $\gamma \in \Gamma$. Then $\Sigma_{r} A^{q}(X, \Gamma, \rho)$ is a complex with coboundary operator $d$ ( $d$ being the operator of exterior differentiation) and each element of the cohomology group $H^{q}(X, \Gamma, \rho)$ is represented by a unique harmonic form; moreover $H^{q}(X, \Gamma, \rho) = \sum_{r \geq 0} H^{q}(X, \Gamma, \rho)$, where $H^{q}(X, \Gamma, \rho)$ denotes the cohomology classes represented by harmonic forms of type $(p, q)$ (see [4], [5], [6]). We have proved in [5] the following results:

a) $H^{0, q}(X, \Gamma, J_{r}) = H^{0, q}(X, \Gamma, \rho);$  

b) The space of harmonic forms of type $(0, q)$ in $A^{q}(X, \Gamma, \rho)$ is identified with the space of all $F \otimes \Lambda^{n^{+}}$ valued smooth functions $f$ on $\Gamma \backslash G$ satisfying the following condition:

(i) $Yf = - (\rho \otimes \text{ad} \eta)(Y) f$ for all $Y \in \mathfrak{g}$.

(ii) For every point $x \in \Gamma \backslash G$, the value $f(x) \in F \otimes \Lambda^{n^{+}}$ is a harmonic cocycle of $C^{\ast}(n^{-}, F) = F \otimes \Lambda^{n^{+}}$ and moreover $f(x) \in S_{1} \otimes \Lambda^{n^{+}}$.

(iii) $X_{\alpha} f = 0$ for all $\alpha \in \Psi$;

here we consider an element $X \in \mathfrak{g}^{c}$ as a complex vector field on $\Gamma \backslash G$ which is a projection of the left invariant complex vector field $X$ on $G$. For the details see [5, Theorem 7.1 and Lemma 6.1].
Thus we may identify an automorphic harmonic $q$-form $\eta$ with an $F \otimes \wedge^n \mathfrak{n}^+$ valued function $f$ on $\Gamma \backslash G$ satisfying the above three conditions. We denote by $\mathcal{L}$ the vector space consisting of all these functions. It follows from (ii) that every $f \in \mathcal{L}$ is actually $\mathfrak{h}^{a\mathfrak{a}}$ valued. $\mathfrak{h}^{a\mathfrak{a}}$ is a direct sum of simple $\mathfrak{f}$-modules $U_{\xi \tau}(T \in W_{\lambda}^p(q))$. To simplify the notation we put $U_T = U_{\xi \tau}$ and let $\tau_T$ denote the representation of $\mathfrak{f}$ in $U_T$. Then $\tau_T(\mathfrak{f}) = (\rho \otimes \text{ad} \xi)(\mathfrak{f})$ on $U_T$ for all $\mathfrak{f} \in \mathfrak{f}$.

Let $f_T(x)$ denote the $U_T$-component of $f(x)$ for $x \in \Gamma \backslash G$. Then $f = \Sigma f_T$ and $\mathcal{L}$ decomposes into direct sum

$$\mathcal{L} = \Sigma T \mathcal{L}_T,$$

where $\mathcal{L}_T$ consists of all $U_T$-valued functions $f_T$ on $\Gamma \backslash G$ satisfying

1) $Yf_T = -\tau_T(Y)f_T$, for all $\mathfrak{f} \in \mathfrak{f}$;

2) $X_\alpha f = 0$ for all $\alpha \in \Psi$.

The inner product in $F \otimes \wedge^n \mathfrak{n}^+$ defines an inner product in $U_T$ such that $(\tau_T(\mathfrak{f})u, v) + (u, \tau_T(\mathfrak{f})v) = 0$ ($\mathfrak{f} \in \mathfrak{f}$). Let $\# $ be the conjugate linear isomorphism of $U_T$ onto the dual space $U_T^\ast$ defined by

$$(\#u)(v) = (v, u) \quad \text{for all} \quad v \in U_T.$$ 

Let $\tau_T$ denote the representation of $\mathfrak{f}$ in $U_T^\ast$ contragredient to $\tau_T$. Then we have

$$\tau_T(\mathfrak{f})\#u = \#(\tau_T(\mathfrak{f})u)$$

for $\mathfrak{f} \in \mathfrak{f}$. For every $U_T$-valued function $f$ on $\Gamma \backslash G$ we define $U_T^\ast$-valued function $\#f$ on $\Gamma \backslash G$ by putting

$$(\#f)(x) = \#f(x).$$

Then, for $X \in \mathfrak{g}$, we have

$$\#(Xf) = X(\#f).$$

Thus $\#$ defines a conjugate linear isomorphism of $\mathcal{L}_T$ onto the complex vector space $\mathcal{L}_T^\#$ consisting of all $U_T^\ast$ valued functions $h$ on $\Gamma \backslash G$ satisfying

1) $Yh = -\tau_T(Y)h$ for all $\mathfrak{f} \in \mathfrak{f}$;

2) $X_\beta h = 0$ for all $\beta \in \Psi$.

On the other hand by [4], a function $h \in \mathcal{L}_T^\#$ is identified with a holomorphic automorphic form of type $J_{\tau_T}$; in fact for a holomorphic automorphic form $a(x)$ on $X$ let $\tilde{h}(s) = J_{\tau_T}(s, \pi(s) \pi(s))$ for $s \in G$, where $\pi : G \to X$ denotes the canonical projection. Then the function $\tilde{h}$ on $G$ is left invariant by $\Gamma$ and hence $\tilde{h}$ defines a function $h$ on $\Gamma \backslash G$ and this function $h$ satisfies the above two conditions.
Thus we have proved the following Theorem 1'.

**Theorem 1'**. Let $\Gamma$ be a discrete subgroup of $G$ such that $\Gamma \backslash G$ is compact. Let $\tau$ be an irreducible representation of $K$ with highest weight $\Lambda$ such that $(\Lambda, \alpha) \geq 0$ for all $\alpha \in \Psi$. Then the space of automorphic harmonic $q$-forms of type $J_\tau$ is isomorphic to the direct sum of the spaces of holomorphic automorphic forms of type $J_{\tau_T}$, where $T$ ranges over the subset $W_0'(q)$ of the Weyl group $W$ of $G^c$ and where $\tau_T$ denotes the irreducible representation of $G^c$ with the highest weight $-T\Lambda' - \langle \Phi_T \rangle$, $\Lambda'$ being the lowest weight of the irreducible representation $\rho$ of $G^c$ with highest weight $\Lambda$.

**Remark 1.** Let $q = N = \dim C X$. There exists a unique element $R \in W$ such that $R(\Sigma^+) = \Sigma^-$. Let $R = R_1^*$. Then $R_i \in W'$ and $\Phi_{R_1} = \Psi$. In fact, $(R_i^{-1})(\Theta) = R_i^{-1}R_i(\Theta) = R_i^{-1}(-\Theta) \subseteq \Sigma^+$ and hence $R_i \in W'$. Moreover since $\Psi$ is the set of weights of the $\mathfrak{t}$-module $\pi^+$ and since $R \in W$, we have $R(\Psi) = \Psi$ and hence $R((R_i^{-1})\Psi) = \Psi \subseteq \Sigma^+$ and hence $(R_i^{-1})\Psi \subseteq \Sigma^-$. Thus $\Psi \subseteq R(\Sigma^-)$ and hence $\Psi = \Phi_{R^*}$. Thus $R_i \in W'(N)$. Next we show that $R_i \in W_0'(\Lambda)$, i.e. $R_i\Lambda' = R_i\Lambda$. In fact, we have $R\Lambda' = \Lambda$, i.e. $R_iR_i\Lambda' = \Lambda$. But $R_i^2 = 1$ and hence $R_i\Lambda' = R_i\Lambda$. Moreover, it is clear that $W_0'(\Lambda)$ consists of a single element and hence $W_0'(\Lambda) = \{R_i\}$. Now we show that

$$\tau_{R_1} = \sigma \otimes \tau^*,$$

where $\sigma$ is the 1-dimensional representation of $\mathfrak{t}$ in $\pi^-$ defined by $\sigma(Y) = \text{tr}(\text{ad}_-Y)$ ($Y \in \mathfrak{t}$) and $\tau^*$ is the contragredient of $\tau$. In fact, the highest weight of $\tau_{R_1}$ is $-R_i\Lambda' - \langle \Phi_{R_1} \rangle = -R_i\Lambda - \langle \Psi \rangle$. On the other hand, the weight of $\sigma$ is $-\langle \Psi \rangle$ and the highest weight of $\tau^*$ is $-R_i\Lambda'$ and hence $\tau_{R_1} = \sigma \otimes \tau^*$. By Theorem 1, we have $H^o(N)(X, \Gamma, J_\tau) \approx H^o(N)(X, \Gamma, J_{\sigma \otimes \tau^*})$. If $\Gamma$ acts freely on $X$, we have $E(J_{\sigma \otimes \tau^*}) = K \otimes E(J_\tau)^*$, where $K$ denotes the canonical bundle of $M = \Gamma \backslash X$ and $E(J_\tau)^*$ is the dual vector bundle of $E(J_\tau)$ and hence the isomorphism $H^o(M, E(J_\tau)^*) \approx H^o(M, K \otimes E(J_\tau)^*)$ and this is a special case of Serre duality theorem.

**Remark 2.** We have $(-\xi_T, -\xi_T + 2\delta) = (\Lambda, \Lambda + 2\delta)$ for any $T$, where $\delta = \sum_{\alpha > 0} (\alpha/2)$. In fact $\xi_T = T(\Lambda' - \delta) + \delta$ and hence $(-\xi_T, -\xi_T + 2\delta) = (T(\Lambda' - \delta), T(\Lambda' - \delta) - (\delta, \delta) = (\Lambda' - \delta, \Lambda' - \delta) - (\delta, \delta) = (\Lambda', \Lambda') - 2(\Lambda', \delta) = (R\Lambda, R\Lambda) - 2(R\Lambda, R(\Lambda^- \delta)) = (\Lambda, \Lambda) - 2(\Lambda, -\delta) = (\Lambda, \Lambda + 2\delta)$.

3. In this section we shall prove Theorem 2. Let $R$ and $R_1$ be the elements in $W$ and $W_1$, respectively such that $R(\Sigma^+) = -\Sigma^+$ and $R_1(\Theta) = -\Theta$. Then $R^2 = R_1^2 = 1$. Let

$$V_\Lambda = \{ T \in W \mid T\Lambda = \Lambda \}.$$

Then we see easily that $R_1V_\Lambda = \{ T \in W \mid T\Lambda' = R_1\Lambda \}$ and hence
W_1 = W^1 \cap R_i V_\Lambda R.

On the other hand we have

W^1 = R_i W^1 R.

In fact, let \( T \in W^1 \) and \( \alpha \in \Theta \). Then \((R_iTR)^{-1}\alpha = RT^{-1}R_i\alpha \in RT^{-1}(\Theta) \subset -\Sigma^+ = \Sigma^+\) and hence \((R_i TR)^{-1}\Theta \subset \Sigma^+\) which shows that \( R_iTR \) is in \( W^1 \). Thus \( R_i W^1 R \subset W^1 \). But we have \( R_i^2 = R^2 = 1 \) and hence we get \( W^1 \subset R_i W^1 R \) and hence (2). From (1) and (2) we obtain:

W_1^{\Lambda} = R_i(W^1 \cap V_\Lambda) R.

Let

\((W^1 \cap V_\Lambda)(q) = \{T \in W^1 \cap V_\Lambda | n(T) = q\}\).

For a subset \( \Phi \) of \( \Sigma^+ \), \( \Phi^c \) will denote the complement of \( \Phi \) in \( \Sigma^+ \).

**Lemma 1.** Let \( T \in W^1 \). Then \( \Phi_{R_iTR} = R_i(\Psi \cap \Phi^c_T) \) and hence \( n(R_iTR) = N - n(T) \), where \( N \) denotes the number of roots in \( \Psi \).

**Proof.** We first remark that \( R_i(\Psi) = \Psi \) and that, as \( R_iTR \) and \( T \) are in \( W^1 \), \( \Phi_{R_iTR} \) and \( \Phi_T \) are subsets of \( \Psi \). \( \Phi_{R_iTR} \) consists of all \( \alpha \in \Psi \) such that \((R_iTR)^{-1}\alpha < 0\). Now \((R_iTR)^{-1}\alpha = RT^{-1}R_i\alpha (\alpha \in \Psi) \) is negative if and only if \( T^{-1}R_i\alpha \) is positive. But \( R_i\alpha \in \Psi \) and hence \( T^{-1}R_i\alpha \) is positive if and only if \( R_i\alpha \in \Phi_T \) and this proves Lemma 1.

From (3) and Lemma 1 we get

\( W_1^{\Lambda}(q) = R_i(W^1 \cap V_\Lambda)(N - q) R. \)

Let now

\( \Psi_\Lambda^q = \{\alpha \in \Psi | (\Lambda, \alpha) = 0\} \).

Then the number of roots in \( \Psi_\Lambda^q \) is \( N - q_\Lambda \).

**Lemma 2.** If \( T \in W^1 \cap V_\Lambda \), then \( \Phi_T \subset \Psi^q_\Lambda \).

**Proof.** Let \( \alpha \in \Phi_T \). Then \( \alpha \in \Psi \) and \( T^{-1}\alpha < 0 \) and hence \((\Lambda, T^{-1}\alpha) = (T\Lambda, \alpha) \leq 0\). Since \( T \in V_\Lambda \), we have \( T\Lambda = \Lambda \) and hence \((\Lambda, \alpha) \leq 0\). By our assumption on \( \Lambda \), \( (\Lambda, \alpha) \geq 0 \) and therefore \((\Lambda, \alpha) = 0\) and this shows that \( \alpha \in \Psi^q_\Lambda \).

From Lemma 1 and 2 we see that, if \( T \in W_\Lambda^1 \), then \( n(T) \geq q_\Lambda \) which proves the first part of Theorem 2.

Now we are going to show that there exists a unique elements \( S \in W^2 \cap V_\Lambda \) such that \( \Phi_S = \Phi_\Lambda^q \). Then \( T_0 = R_iSR \) will be the unique element in \( W_\Lambda^1(q_\Lambda) \) and \( \Phi_{T_\Lambda} = \{R_i\alpha | \alpha \in \Psi, (\Lambda, \alpha) > 0\} = \{\alpha | \alpha \in \Psi, (R_i\Lambda, \alpha) > 0\} \).
First we remark that the uniqueness of $T$ such that $\Phi_T = \Psi^T$ follows from a results of Kostant [2, Prop. 5.10]*: The mapping $T \mapsto \Phi_T$ ($T \in W$) defines an injection of $W$ into the set of subsets of $\Sigma^+$.

Now let

$$g_u = \sqrt{-1}m + \mathfrak{t},$$

where $g = m + \mathfrak{t}$ is the Cartan decomposition of $g$ so that $m^c = n^+ + n^-$. Then $g_u$ is a compact real form of $g^c$. Let $H_0$ be the element in $\mathfrak{h}$ such that $\varphi(H, H_0) = \sqrt{-1} \Lambda(H)$ for all $H \in \mathfrak{h}$. Then $[H_0, X_\alpha] = \sqrt{-1} (\Lambda, \alpha) X_\alpha$ for all root $\alpha$.

Let $I$ be the centralizer of $H_0$ in $g_u$. Then $I$ is the Lie algebra of a compact Lie group and $I^c$ is identified with the centralizer of $H_0$ in $g^c$. We see easily that

$$I^c = \mathfrak{h}^c + \sum_{\alpha \in \Xi} (g_\alpha + g_{-\alpha}),$$

where $\Xi = \{ \alpha \in \Sigma^+ | (\Lambda, \alpha) = 0 \}$ and $g_\alpha$ denotes the 1-dimensional eigenspace for the root $\alpha$.

Let $G_u$ be the adjoint group of $g_u$ and $L$ the subgroup of $G_u$ consisting of all $\sigma \in G_u$ such that $\sigma(H_0) = H_0$. Then the Lie algebra of $L$ is $I$ and, $L$ being the centralizer of a 1-parameter subgroup, $L$ is connected. Now let $T \in V_\Lambda$.

We consider $W$ as a group of linear transformations of $\mathfrak{h}$ by identifying roots $\alpha$ with elements $H_\alpha$ in $\mathfrak{h}$ such that $\sqrt{-1} \alpha(H) = \varphi(H, H_\alpha)$ for all $H \in \mathfrak{h}$. Then we know that for $T \in W$ there exists an element $t \in G_u$ such that $t(X) \in T(X)$ for all $X \in \mathfrak{h}$. Then $T \Lambda = \Lambda$ implies $t(H_\alpha) = H_\alpha$ and hence $t$ belongs to $L$. Thus $T$ belongs to the Weyl group of $I^c$. It follows then that $V_\Lambda$ is the subgroup of $W$ generated by $S_\alpha$ with $\alpha \in \Xi$, where $S_\alpha$ denotes the reflection with respect to the hyperplane $\alpha = 0$.

Now let

$$\Omega = \{ \alpha \in \Sigma^+ | (\Lambda, \alpha) > 0 \}.$$

Then $\Xi \cup (-\Xi) \cup \Omega$ is a closed system, i.e. if $\alpha$ and $\beta$ belong to this set of roots and $\alpha + \beta$ is also a root, then $\alpha + \beta$ belongs also to this set. Then

$$\mathfrak{n} = \mathfrak{h}^c + \sum_{\alpha \in \Xi} (g_\alpha + g_{-\alpha}) + \sum_{\alpha \in \Omega} g_\alpha$$

$$= I^c + \sum_{\alpha \in \Omega} g_\alpha$$

is a parabolic subalgebra and $I^c$ and $\sum_{\alpha \in \Omega} g_\alpha$ are the reductive part and the nilpotent part of $\mathfrak{n}$ respectively. Then by a theorem of Kostant [2, Prop. 5.13], every $T \in W$ is written uniquely

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* The existence of $T \in W$ such that $\Phi_T = \Psi^T$ follows also the same proposition, but it is not easy to see that $T \in W_\Lambda$. 

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where \( V^\Lambda = \{ T \in W \mid \Phi_T \subset \Omega \} \).

**Lemma 3.** Let \( U \in W^1 \) and let
\[
U = ST, \quad S \in V_\Lambda, \quad T \in V^\Lambda.
\]
Then \( \Phi_S = \Phi_U \cap \Psi_\Lambda^0 \).

**Proof.** Let \( \alpha \in \Omega \). Then \( (\Lambda, \alpha) > 0 \) and \( (\Lambda, \alpha) = (S^{-1} \Lambda, S^{-1} \alpha) = (\Lambda, S^{-1} \alpha) \), because \( S^{-1} \Lambda = \Lambda \). Then \( S^{-1} \alpha \) is positive and belongs to \( \Omega \). Hence \( \Phi_S \cap \Omega = \phi \).

Next let \( \alpha \in \Xi \cap \Theta \). Since \( U \in W^1 \), we have \( U^{-1} \alpha > 0 \). Suppose \( S^{-1} \alpha \) is negative. Then \( -S^{-1} \alpha = T(-U^{-1} \alpha) > 0 \) and \( -U^{-1} \alpha > 0 \) and hence \( T(-U^{-1} \alpha) \) belongs to \( \Phi_T \). Since \( T \in V^\Lambda \), \( \Phi_T \) is contained in \( \Omega \) and hence \( -S^{-1} \alpha \in \Omega \). One the other hand, as \( \alpha \in \Xi \) and \( S \in V_\Lambda \), we have \( (\Lambda, -S^{-1} \alpha) = (S \Lambda, -\alpha) = -(\Lambda, \alpha) = 0 \) and this contradicts the fact \( -S^{-1} \alpha \in \Omega \). Therefore \( S^{-1} \alpha \) must be positive for all \( \alpha \in \Xi \cap \Theta \) and this shows that \( \Phi_S \cap \Xi \cap \Theta = \phi \).

Finally let \( \alpha \in \Xi \cap \Psi \). Suppose \( U^{-1} \alpha < 0 \) and \( S^{-1} \alpha > 0 \). Then \( S^{-1} \alpha = TU^{-1} \alpha \) and hence \( S^{-1} \alpha \in \Xi \cap \Phi_T \subset \Xi \cap \Omega = \phi \) and this is a contradiction. Therefore if \( U^{-1} \alpha < 0 \), then \( S^{-1} \alpha \) must be negative. Analogously we can show that if \( U^{-1} \alpha > 0 \), then \( S^{-1} \alpha > 0 \). These show that
\[
\Phi_S \cap \Xi \cap \Psi = \Phi_U \cap \Xi \cap \Psi = \Phi_U \cap \Psi_\Lambda^0.
\]

On the other hand we have
\[
\Sigma^+ = \Xi \cup \Omega = (\Xi \cap \Psi) \cup (\Xi \cap \Theta) \cup \Omega \text{ (disjoint)}
\]
Therefore we get from what we have proved so far
\[
\Phi_S = \Phi_U \cap \Psi_\Lambda^0.
\]

**Lemma 4.** An element \( U \) of \( W \) belongs to \( W^1 \cap V_\Lambda \) if and only if \( \Phi_U \subset \Psi_\Lambda^0 \).

**Proof.** By Lemma 2, if \( U \in W^1 \cap V_\Lambda \), \( \Phi_U \) is contained \( \Psi_\Lambda^0 \). Conversely, let \( \Phi_U \subset \Psi_\Lambda^0 \) and let \( U = S \cdot T \) as in Lemma 3. Then \( \Phi_S = \Phi_U \) by Lemma 3. But the mapping \( T \rightarrow \Phi_T \) is bijective and hence \( U = S \in V_\Lambda \). But \( \Phi_U \subset \Psi_\Lambda^0 \subset \Psi^0 \) and hence \( U \in W^1 \). Thus \( U \in W^1 \cap V_\Lambda \).

Now let \( R^1 = R \cdot R \). Then \( R^1 \in W^1 \) and \( \Phi_{R^1} = \Psi \) (see Remark 1 in § 2). Let \( R^1 = ST \) as in Lemma 3. Then \( S \) belongs to \( W^1 \cap V_\Lambda \) and \( \Phi_S = \Phi_{R^1} \cap \Psi_\Lambda^0 = \Psi_\Lambda^0 \). Thus \( S \) is the unique element in \( (W^1 \cap W) (N-q_\Lambda) \) and Theorem 2 is proved.

From Theorems 1' and 2 we get the following theorem.

**Theorem 2'.** The assumptions and the notation being as in Theorems 1' and 2, the space of automorphic harmonic \( q_\Lambda \)-forms of type \( j \), is isomorphic to the
space of holomorphic automorphic forms of type $f_{T,T_0}$, where $T_0$ is the unique element in $W^{q}_{1}(\mathcal{A})$. We have

$$\Phi_{T_0} = \{\alpha \in \Psi \mid (R, \Lambda, \alpha) > 0\}.$$ 

The highest weight of $\tau_{T_0}$ is

$$-T_0 R \Lambda - \langle \Phi_{T_0} \rangle.$$ 

\textbf{Part II}

We retain the notation introduced in Part I\textsuperscript{*}. Let $g = m + \mathfrak{I}$ be the Cartan decomposition of $g$ and let \{\(X_1, \ldots, X_m\)\} and \{\(Y_1, \ldots, Y_r\)\} be the bases of $m$ and $\mathfrak{I}$ respectively such that $\varphi(X_i, X_j) = \delta_{ij}$, $\varphi(Y_a, Y_b) = -\delta_{ab}$ ($i, j = 1, \ldots, m$; $a, b = 1, \ldots, r$). Then the Casimir operator $C$ is the differential operator on $G$ given by

$$C = \sum_{i=1}^{m} X_i^2 - \sum_{a=1}^{r} Y_a^2.$$

We denote by $C^\infty(G, V)$ the complex vector space of all $C^\infty$-functions on $G$ with values in a finite dimensional complex vector space $V$.

\textbf{1.} Let $\tau$ be a representation of $K$ in a complex vector space $V$ and let $\Gamma$ be a discrete subgroup of $G$. By an automorphic form of type $(\Gamma, \tau, \lambda)$ we mean a function $f \in C^\infty(G, V)$ satisfying the following three conditions (cf. \cite{[1]}):

1) $f(\gamma g) = \tau(k^{-1})f(g)$, $k \in K$, $g \in G$;
2) $f(\gamma g) = \tau(g)f(\gamma)$, $\gamma \in \Gamma$, $g \in G$;
3) $Cf = \lambda f$, where $\lambda$ is a complex constant depending only on $\tau$.

We denote by $A(\Gamma, \tau, \lambda)$ the vector space of all automorphic forms of type $(\Gamma, \tau, \lambda)$.

\textbf{Proposition 1.} Assume $\Gamma \backslash G$ is compact. Then the dimension of the vector space $A(\Gamma, \tau, \lambda)$ is finite.

Proof. ([1]). It follows from the condition 1) for automorphic forms that $Yf = -\tau(Y)f$ for $Y \in \mathfrak{I}$ and $f \in A(\Gamma, \tau, \lambda)$. Put

$$C' = \sum_{a=1}^{r} Y_a^2.$$ 

Then $C'f = \tau(C')f$, where $\tau(C') = \sum_{a=1}^{r} \tau(Y_a)^2$. Let

$$L = C + 2C' = \sum_{i=1}^{m} X_i^2 + \sum_{a=1}^{r} Y_a^2.$$ 

* In Part II we don’t need to assume that $G/K$ has a $G$-invariant complex structure.
Then \( L \) is a left invariant elliptic differential operator on \( G \). For \( f \in A(\Gamma, \tau, \lambda_\tau) \) we have

\[
Lf = Mf,
\]

where \( M \) denotes the endomorphism of \( V \) defined by

\[
M = \lambda_\tau I + 2\tau(C') .
\]

Let \( F(x) \) be a polynomial such that \( F(M) = 0 \) and let \( P = F(L) \). Then \( P \) is also a left invariant elliptic operator on \( G \) and we have \( Pf = 0 \) for \( f \in A(\Gamma, \tau, \lambda_\tau) \).

Now by the second condition on automorphic forms, we may consider \( f \) as a \( V \)-valued function on \( \Gamma \backslash G \) and by the left invariance of \( P \), we may consider \( P \) as an elliptic operator on \( \Gamma \backslash G \). The manifold \( \Gamma \backslash G \) being compact, the vector space of all \( V \)-valued functions on \( \Gamma \backslash G \) satisfying the equation \( Pf = 0 \) is finite dimensional and in particular \( A(\Gamma, \tau, \lambda_\tau) \) is finite dimensional.

**Example.** Let us assume that \( G/K \) is a bounded symmetric domain in \( \mathbb{C}^N \) as in Part I and let \( J_\tau \) be the canonical automorphic factor of type \( \tau \). Assume \( \tau \) is irreducible and let \( \Lambda \) be the highest weight of \( \tau \). Then the space of all holomorphic automorphic forms of type \( J_\tau \) on \( G/K \) is identified with the space of all automorphic forms of type \( (\Gamma, \tau, \lambda_\tau) \) with

\[
\lambda_\tau = (\Lambda, \Lambda + 2\delta) ,
\]

where

\[
\delta = \sum_{\alpha > 0} \alpha/2 = \langle \Sigma^+ \rangle/2 .
\]

2. Let \( T \) be a unitary representation of \( G \) in a Hilbert space \( H \) and let \( C_\tau \) be the Casimir operator of the representation \( T \). \( C_\tau \) is a self-adjoint operator of \( H \) with a dense domain and if \( T \) is irreducible, there exists a complex number \( \lambda_\tau \) such that \( C_\tau \varphi = \lambda_\tau \varphi \) for all \( \varphi \) in the domain of \( C_\tau \).

Let \( T \) be irreducible and let \( T_K \) be the restriction of \( T \) onto \( K \). Then \( T_K \) is a unitary representation of \( K \) and it is known that \( T_K \) decomposes into a countable sum of irreducible representations of \( K \) and each irreducible representation \( \tau \) of \( K \) enters in \( T_K \) with finite multiplicity which we shall denote by \( (T_K : \tau) \).

Let \( U \) be the unitary representation of \( G \) in the Hilbert space \( L_2(\Gamma \backslash G) \): \( (U(g)f)(x) = f(xg), x \in \Gamma \backslash G, g \in G \). We know that \( U \) decomposes into sum of a countable number of irreducible unitary representations in which each irreducible representation \( T \) enters with a finite multiplicity which we shall denote by \( (U : T) \).

Note that we have \( C_\tau f = C f \) for \( f \in L_2(\Gamma \backslash G) \cap C^\infty(\Gamma \backslash G) \).

**Theorem 3.** Assume that \( \Gamma \backslash G \) is compact and \( \tau \) is irreducible. Then...
\[ \dim A(\Gamma, \tau, \lambda) = \sum_{T \in D_{\lambda\tau}} (U: T)(T_K: \tau^*) , \]

where \( \tau^* \) denotes the irreducible representation of \( K \) contragredient to \( \tau \) and \( D_{\lambda\tau} \) denotes the set of irreducible representations \( T \) of \( G \) such that \( \lambda_T = \lambda \), \( \lambda_T \) being the constant such that \( C_T \varphi = \lambda_T \varphi \) for all \( \varphi \) in the domain of the Casimir operator \( C_T \).

From Theorem 2' and Remark 1 in Part I, §2 and Example in Part II, §1, we obtain the following corollary.

**Corollary.** The notation and the assumptions being as in Theorems 2' and 3, the dimension of the space of automorphic harmonic \( q \)-forms of type \( J \) is equal to

\[ \sum_{T \in D_{\tau P_0}} (U: T)(T_K: \tau_{T_0}) , \]

where \( \tau_{T_0} \) denotes the irreducible representation of \( K \) with lowest weight

\[ \xi_{T_0} = T_0 \Lambda' + \langle \Phi_{T_0} \rangle , \]

where

\[ \Phi_{T_0} = \{ \alpha \in \Psi | (R, \Lambda, \alpha) > 0 \} . \]

3. Proof of Theorem 3. Let

\[ L^2(\Gamma \backslash G) = \bigoplus_{a=1}^{\infty} H_a \]

be the decomposition of \( L^2(\Gamma \backslash G) \) into direct sum of irreducible invariant closed subspaces and let

\[ U_a = U \upharpoonright H_a . \]

Let \( a \) be an index such that \( C_{U_a} \varphi = \lambda \varphi \) for \( \varphi \) in the domain of \( C_{U_a} \) and let

\[ m_a = ((U_a)_K: \tau^*) . \]

Further let

\[ H_a = \bigoplus_{b=1}^{\infty} H_{a,b} \]

be the decomposition of \( H_a \) into direct sum of irreducible \( K \)-invariant subspaces. We may assume that for \( b=1, 2, \ldots, m \), the irreducible representation of \( K \) in \( H_{a,b} \), is equivalent to \( \tau^* \).

Take a basis \( \{v_1, \ldots, v_n\} \) of the representation space \( V \) of \( \tau \) and let

\[ \tau(k)v_{\lambda} = \sum_{p} \tau_{\lambda}^p(k)v_p , \quad k \in K . \]

Then there exists a basis \( \{f_{a,b}, \ldots, f_{a,b}^n\} \) of \( H_{a,b} \) such that
(1) \[ f_{a,b}(xk) = \sum_{\tau} \tau_b^{-1}(k) f_{a,b}(x) \]

for \( k \in K \) and \( x \in \Gamma \backslash G \). Define a \( V \)-valued function \( f_{a,b} \) on \( \Gamma \backslash G \) by putting

\[ f_{a,b}(x) = \sum_{\lambda} f_{a,b}^\lambda(x) \varphi_{\lambda}. \]

Then

\[ f_{a,b}(xk) = \tau(k^{-1}) f_{a,b}(x). \]

Put

\[ \tilde{f}_{a,b} = f_{a,b} \circ \pi, \]

where \( \pi \) denotes the projection of \( G \) onto \( \Gamma \backslash G \). Then \( \tilde{f}_{a,b} \) satisfies the conditions 1) and 2) of automorphic forms of type \( (\Gamma, \tau, \lambda) \). If \( \{g^a_{a,b}, \ldots, g^n_{a,b}\} \) is another basis of \( H_{a,b} \) satisfying the condition (1), then there exists a complex number \( \alpha \) such that \( \alpha g^a_{a,b} = f^a_{a,b} \) for \( \lambda = 1, \ldots, n \) by Schur's Lemma. Therefore the function \( \tilde{f}_{a,b} \) is well defined up to constant multiple.

Let us prove that \( \tilde{f}_{a,b} \) is differentiable and satisfies the equation \( C \tilde{f}_{a,b} = \lambda \tilde{f}_{a,b} \). To show this it is sufficient to show that every function \( \varphi \in H_{a,b} \) is differentiable. In fact, \( \varphi \) is then in the domain of the operator \( C_U = C \) and \( C_U \varphi = C_{Ua} \varphi = \lambda \varphi \) (Remark that \( C \) is a differential operator on \( G \) left invariant by \( G \) and hence we may consider \( C \) as a differential operator on \( \Gamma \backslash G \)). To show the differentiability of \( \varphi \in H_{a,b} \), we remark first that for any \( h \in C^\infty(\Gamma \backslash G) \) and \( \psi \in H_a \) we have

\[ (Ch, \psi) = (h, \lambda, \psi). \]

In fact, let \( \psi \) be an element in the domain of the operator \( C_{Ua} \). Then

\[ (Ch, \psi) = (C_{Ua}h, \psi) = (h, C_{Ua} \varphi) = (h, \lambda, \psi). \]

The elements \( \psi \) being dense in \( H_a \), the equality (2) holds for any \( \psi \in H_a \). Now let \( \varphi \in H_{a,b} \). Then for any \( Y \in \mathfrak{t}, \)

\[ \lim_{t \to 0} \frac{1}{t} (U_a(\exp tY)\varphi - \varphi) = U'_a(Y)\varphi \]

exists. In fact, \( U'_a(Y)\varphi = -\tau^*(Y)\varphi \). In particular

\[ U'(C')\varphi = \tau^*(C')\varphi, \]

where, as in the proof of Proposition 1 in §1, we put \( C' = \sum_a Y^2_a \). Let

\[ L = C + 2C'. \]

Then for any \( \varphi \in H_{a,b} \) and \( h \in C^\infty(\Gamma \backslash G) \) we get from (2):

\[ (Lh, \varphi) = (Ch, \varphi) + 2(C'h, \varphi) = (h, \lambda, \varphi) + (h, 2\tau^*(C')\varphi) = (h, B\varphi), \]
where \( B = \lambda, I + 2\tau^*(C') \) is an endomorphism of the finite dimensional vector space \( H_{a,b} \). By induction we get \((L^k h, \varphi) = (h, B^k \varphi)\) for \( k = 1, 2, \ldots \). Let \( F(x) \) be a polynomial such that \( F(B) = 0 \) and let \( P = F(L) \). Then \((Ph, \varphi) = 0\). This shows that the distribution \( D_\Phi \) defined by \( D_\Phi(h) = (h, \varphi) \) satisfies the elliptic equation \( PD_\phi = 0 \). Then \( \varphi \) is differentiable and in fact \( P\varphi = 0 \).

Thus we have shown that for each index \( a \) such that \( U_{a}\in D_{\lambda} \), we get \( m_a \) functions \( \tilde{f}_{a,b}(b = 1, \ldots, m_a) \) belonging to \( A(\Gamma, \tau, \lambda, \tau) \). If \( c \) is another index such that \( U_{c}\in D_{\lambda} \), then we get also functions \( \tilde{f}_{c,d} \) (\( d = 1, 2, \ldots, m_c \); \( m_c = ((U_{c})_K: \tau^*) \)) belonging to \( A(\Gamma, \tau, \lambda, \tau) \) and it is easy to see that \( \tilde{f}_{a,b} \) and \( \tilde{f}_{c,d} \) are linearly independent. By Proposition 1, the dimension \( A(\Gamma, \tau, \lambda, \tau) \) is finite. It follows then that the number of the irreducible unitary representations \( T \) of \( G \) belonging to \( D_{\lambda} \) such that \((U: T) = 0 \) and \((T: \tau^*) = 0 \) is finite. We may therefore assume that \( U_1, U_2, \ldots, U_t \) are these unitary representations. For each \( a, 1 \leq a \leq t \). We get functions \( \tilde{f}_{a,b} \) (\( 1 \leq b \leq ((U_{a})_K: \tau^*) \)) in \( A(\Gamma, \tau, \lambda, \tau) \) and these functions are linearly independent. The number of these functions equals \( \sum_{a=1}^{t} ((U_{a})_K: \tau^*) \). Thus we get

\[
\dim A(\Gamma, \tau, \lambda, \tau) \geq \sum_{a=1}^{t} (U: T)(T_K: \tau^*) .
\]

Now let \( \tilde{f} \in A(\Gamma, \tau, \lambda, \tau) \) and let

\[
\tilde{f}(g) = \sum_{\lambda} \tilde{f}^\lambda(g) \psi_\lambda, \quad g \in G .
\]

Then \( \tilde{f}^\lambda \) is a differentiable function on \( G \) such that \( \tilde{f}^\lambda(\gamma g) = \tilde{f}^\lambda(g) \) for all \( \gamma \in \Gamma \). Then there exists a differentiable function \( f^\lambda \) on \( \Gamma \backslash G \) such that \( f^\lambda = \tilde{f}^\lambda \circ \pi \). Then we have

\[
f^\lambda(xh) = \sum_{\mu} \tau^\mu(h^{-1}) f^\mu(x)
\]

for all \( k \in K \) and \( x \in \Gamma \backslash G \). Moreover

\[
Cf^\lambda = \lambda \cdot f^\lambda .
\]

Let \( P_a \) be the projection of \( L^2(\Gamma \backslash G) \) onto \( H_a \). If \( h \) is differentiable, we have

\[
P_a X h = \lim_{t \to 0} \frac{1}{t} P_a (U(\exp tX)h - h) = \lim_{t \to 0} \frac{1}{t} (U_{a}(\exp tX)P_a h - P_a h) = U_{a}'(X)P_a h .
\]

It follows that \( P_a h \) is in the domain of \( C_{U_{a}} = \sum_{i} U_{a}'(X_i) - \sum_{y} U_{a}'(Y_b) \) and \( P_a Ch = C_{U_{a}} P_a h \). Thus we get:

\[
C_{U_{a}} P_a f^\lambda = \lambda \cdot P_a f^\lambda .
\]

It follows that, if \( U_{a}\in D_{\lambda} \), then \( P_a f^\lambda = 0 \). Therefore we have:

\[
(3) \quad f^\lambda = P_1 f^\lambda + P_2 f^\lambda + \cdots + P_t f^\lambda .
\]
Let \( a \in [1, t] \) and assume \( P_a f^a \neq 0 \) for some \( \lambda \). Let \( F \) be the linear subspace of \( H_\lambda \) spanned by \( \{ P_a f^1, P_a f^2, \ldots, P_a f^m \} \). Then \( F \neq (0) \) and \( F \) is \( K \)-invariant. In fact

\[
U_a(k)P_a f^a = P_a U(k)f^a = P_a \sum_\nu \tau_\nu(k^{-1})f^\nu = \sum_\nu \tau_\nu(k^{-1})P_a f^\nu.
\]

Let \( \{ \xi_1, \xi_2, \ldots, \xi_n \} \) be the basis of \( V^* \) dual to the basis \( \{ v_1, \ldots, v_n \} \) of \( V \). The linear map of \( V^* \) onto \( F \) defined by \( \xi^\nu \rightarrow P_a f^\nu \) is a \( K \)-homomorphism and in fact a \( K \)-isomorphism, because \( V^* \) is irreducible. It follows that \( P_a f^1, \ldots, P_a f^m \) are linearly independent and \( F \) is contained in \( \bigoplus_{b=1}^m H_{a,b} \) because of the orthogonality relation. Then we can write:

\[
P_a f^\lambda = \sum_{b=1}^m \sum_{\mu=1}^n \alpha(b)_{a,b} f^\mu_{a,b},
\]

Then

\[
(P_a f^\lambda)(xk) = \sum_\nu \alpha(b)^\mu_{a,b} f^\nu_{a,b}(xk) = \sum_\nu \alpha(b)^\mu_{a,b} \tau_\nu(k^{-1}) f^\nu_{a,b}(x).
\]

On the other hand

\[
(P_a f^\lambda)(xk) = \sum_\mu \tau_\mu(k^{-1})(P_a f^a)(x) = \sum_\mu \tau_\mu(k^{-1}) \sum_\nu \alpha(b)^\mu_{a,b} f^\nu_{a,b}(x).
\]

Therefore we get:

\[
\sum_\mu \alpha(b)^\mu_{a,b} \tau_\mu(k^{-1}) = \sum_\mu \tau_\mu(k^{-1}) \alpha(b)^\mu_{a,b} \quad (\lambda, \nu = 1, \ldots, n).
\]

By Schur's Lemma we have \( \alpha(b)^\mu_{a,b} = \alpha_{a,b} \delta^\mu_{a,b} \) where \( \alpha_{a,b} \) is a constant depending on \( a \) and \( b \). Thus

\[
P_a f^\lambda = \sum_{b=1}^m \alpha_{a,b} f^\lambda_{a,b},
\]

whence \( f^\lambda = \sum_a P_a f^a = \sum_{a,b} \alpha_{a,b} f^\lambda_{a,b} \). Therefore \( \sum_\lambda f^\lambda v_\lambda = \sum_{a,b} \alpha_{a,b} f_{a,b} \). It follows then \( f \) is a linear combination of \( f_{a,b} \)'s and therefore \( \{ f_{a,b} \} \) form a basis of the vector space \( A(\Gamma, \tau, \lambda, \nu) \) and the theorem is proved.
Bibliography


