

Title	Weak convergence of a sequence of stochastic processes related with U-statistics
Author(s)	Kohatsu-Higa, Arturo
Citation	Osaka Journal of Mathematics. 1990, 27(2), p. 361-371
Version Type	VoR
URL	<a href="https://doi.org/10.18910/11157">https://doi.org/10.18910/11157</a>
rights	
Note	

*Osaka University Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

Osaka University

## WEAK CONVERGENCE OF A SEQUENCE OF STOCHASTIC PROCESSES RELATED WITH U-STATISTICS

ARTURO KOHATSU-HIGA

(Received June 14, 1989)

### 1. Introduction

In Hoeffding's fundamental paper [4], he proved the weak convergence of U-statistics under suitable conditions. Loynes [10] and later, Miller and Sen [12] as well as Mandelbaum and Taqqu [11] respectively considered different types of stochastic processes related with U-statistics and studied their weak convergence. In this paper, we are concerned with a sequence of stochastic processes which are similar to those developed by Mandelbaum and Taqqu [11]. We intend to show a deeper analysis of the weak convergence of the processes. This is achieved by using the martingale approach, a method used extensively by Khmaladze [8], [9]. (See Rao [13] for a survey on Martingale approach to Statistical Inference). Under this martingale approach we intend to find natural expressions for the limits of the martingale part and compensator of the processes associated with a sequence of U-statistics. These limits can be expressed by using multiple Wiener integrals.

Let  $F$  be a distribution function on  $\mathbf{R}$  and  $X_1, \dots, X_n$ , independent observations on  $F$ . Consider a parametric function  $\theta = \theta(F)$ , for which there exists an unbiased estimator. That is,  $\theta(F)$  may be expressed as  $\theta(F) = E_F(h(X_1, \dots, X_m))$  for some function  $h: \mathbf{R}^m \rightarrow \mathbf{R}$ , called a "kernel", where  $h$  can be assumed to be symmetric.

Let's define:

$$h_k(x_1, \dots, x_k) := E(h(X_1, \dots, X_m) | X_1 = x_1, \dots, X_k = x_k) \quad \text{and} \\ \zeta_k := \text{Var}(h_k(X_1, \dots, X_k)) \quad \text{for } k = 1, \dots, m,$$

under the assumption

$$(1) \quad E(h^2(X_1, \dots, X_m)) < \infty.$$

The  $U$ -statistic for estimation of  $\theta$  based on the sample  $X_1, \dots, X_n$  of size  $n \geq m$  is

$$U_n := \frac{1}{\binom{n}{m}} \sum_{C_m^n} h(X_{i_1}, \dots, X_{i_m}),$$

where  $C_m^n = \{(i_1, \dots, i_m) \in \{1, \dots, n\}^m / 1 \leq i_1 < \dots < i_m \leq n\}$ .

The study of the weak convergence of U-statistics have been extensively studied in Hoeffding [4], Gregory [3], Eagleson [6], Serfling [15], Rubin and Vitale [14], Dynkin and Mandelbaum [2], Mandelbaum and Taqqu [11] among others. Results and definitions about U-statistics can be found in Serfling [15].

### 2. Weak convergence of the U-processes

DEFINITION. The stochastic process associated with a U-statistic of size  $n$  (simply, U-process), is defined by:

$$U_n(t) := \frac{1}{\binom{n}{m}} \sum_{C_m^{\lfloor nt \rfloor}} h(X_{i_1}, \dots, X_{i_m}), \text{ for } 0 \leq t \leq 1$$

where  $\lfloor \cdot \rfloor$  is the greatest integer function,  $(i_1, \dots, i_m) \in C_m^{\lfloor nt \rfloor}$  and  $U_n(t) = 0$  for  $t < \frac{m}{n}$ .

This process is adapted to the right continuous  $\sigma$ -field  $\mathcal{F}_n(t) = \sigma[X_1, \dots, X_{\lfloor nt \rfloor}]$ . From now on, we will assume that  $\theta(F) \equiv 0$  (if not, we can always take  $h - \theta$  instead of  $h$ ).

**Lemma 1.**  $U_n(t)$  is a semimartingale, its martingale part is given by:

$$X_n(t) := \frac{1}{\binom{n}{m}} \sum_{C_m^{\lfloor nt \rfloor}} \{h(X_{i_1}, \dots, X_{i_m}) - h_{m-1}(X_{i_1}, \dots, X_{i_{m-1}})\}.$$

Following Hoeffding [4],  $U_n(t)$  can also be expressed as:

$$(2) \quad U_n(t) = \frac{1}{\binom{n}{m}} \sum_{k=1}^m \binom{\lfloor nt \rfloor - k}{m - k} U_{n,k}(t),$$

where

$$(3) \quad U_{n,k}(t) = \sum_{C_k^{\lfloor nt \rfloor}} \int h_k(x_1, \dots, x_k) \prod_{j=1}^k d(I_{\{x_{i_j} \leq x_j\}} - F(x_j)).$$

Here,

$$I_{\{x \leq y\}} = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{if } x > y. \end{cases}$$

Similarly, we have

$$(4) \quad X_n(t) = \frac{1}{\binom{n}{m}} \sum_{k=1}^m \sum_{C_k^{[nt]}} \left[ \binom{i_k - k}{m - k} \int h_k(x_1, \dots, x_k) \prod_{j=1}^k d(I_{\{x_{i_j} \leq x_j\}} - F(x_j)) \right]$$

which can be obtained by using methods similar to the ones showed at the end of the proof of Lemma 4.

**Lemma 2.**

- (i)  $U_{n,k}(t)$  is a martingale.
- (ii)  $E(U_{n,k}^2(1)) \leq \binom{n}{k} 2^k \zeta_k$ .
- (iii)  $P(\max_t |n^r U_{n,k}(t)| > \varepsilon) \rightarrow 0$  as  $n \rightarrow \infty$  for  $r < -\frac{k}{2}$ .

Proof. (i) is easily proved by noticing that

$$E(d(I_{\{x_j \leq x\}} - F(x)) | \mathcal{F}_n(s)) = 0 \quad \text{if } [ns] < j \leq [nt].$$

$$(ii) \quad E(U_{n,k}^2(1)) = E\left(\left[\sum_{C_k^n} \int h_k(x_1, \dots, x_k) \prod_{j=1}^k d(I_{\{x_{i_j} \leq x_j\}} - F(x_j))\right]^2\right)$$

because, if  $(i_1, \dots, i_k) \neq (l_1, \dots, l_k)$  then

$$E\left(\prod_{j=1}^k [d(I_{\{x_{i_j} \leq x_j\}} - F(x_j)) d(I_{\{x_{l_j} \leq y_j\}} - F(y_j))]\right) = 0.$$

It will be enough to prove that

$$E\left(\int h_k(x_1, \dots, x_k) h_k(x_{k+1}, \dots, x_{2k}) \prod_{j=1}^k d(I_{\{x_{i_j} \leq x_j\}} - F(x_j)) d(I_{\{x_{i_j} \leq x_{j+k}\}} - F(x_{j+k}))\right)$$

is bounded by  $2^k \zeta_k$ . As

$$\begin{aligned} & E(d(I_{\{x_{i_j} \leq x_j\}} - F(x_j)) d(I_{\{x_{i_j} \leq x_{j+k}\}} - F(x_{j+k}))) \\ &= I_{\{x_j = x_{j+k}\}} dF(x_j) - dF(x_j) dF(x_{j+k}), \end{aligned}$$

then, the above expectation becomes:

$$\begin{aligned} & \int h_k(x_1, \dots, x_k) h_k(x_{k+1}, \dots, x_{2k}) \prod_{j=1}^k \{I_{\{x_j = x_{j+k}\}} dF(x_j) - dF(x_j) dF(x_{j+k})\} \\ &= \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} h_k(x_1, \dots, x_k) h_k(x_1, \dots, x_i, x_{k+1}, \dots, x_{2k-i}) dF(x_1) \dots dF(x_{2k-i}). \end{aligned}$$

By the Cauchy-Schwartz inequality:

$$\left| \int h_k(x_1, \dots, x_k) h_k(x_1, \dots, x_i, x_{k+1}, \dots, x_{2k-i}) dF(x_1) \dots dF(x_{2k-i}) \right| \leq \zeta_k,$$

from which we get the announced boundedness.

(iii) As  $U_{n,k}(t)$  is a martingale, by Doob's inequality we have:

$$P(\max_t |n^r U_{n,k}(t)| > \varepsilon) \leq \frac{n^{2r}}{\varepsilon^2} \binom{n}{k} 2^k \zeta_k \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For any kernel  $h$  there exists  $c$  ( $1 \leq c \leq m$ ) such that  $\zeta_1=0, \dots, \zeta_{c-1}=0, \zeta_c>0$ . From now on, we will take  $c$  fixed.

Let's define:

$$Y_{n,c}(t) := \frac{n^{c/2} \binom{[nt]-c}{m-c}}{\binom{n}{m}} U_{n,c}(t).$$

**Lemma 3.**

$$(5) \quad \sup_t |n^{c/2} U_n(t) - Y_{n,c}(t)| \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

*i.e. the process  $U_n$  can be approximated by the  $c$ -th term process appearing in the decomposition (2).*

Proof. As  $\zeta_1=0, \dots, \zeta_{c-1}=0$ , then  $h_1=\dots=h_{c-1}=0$ . Using (2), the left term of (5) can be written as:

$$\sup_t |n^{c/2} \left\{ \frac{1}{\binom{n}{m}} \sum_{k=c+1}^m \binom{[nt]-k}{m-k} U_{n,k}(t) \right\}| \leq \frac{n^{c/2}}{\binom{n}{m}} \sum_{k=c+1}^m \binom{n-k}{m-k} \sup_t |U_{n,k}(t)|,$$

which tends to zero by Lemma 2 (iii).

**Lemma 4.** *The martingale part of the process  $Y_{n,c}(t)$  is given by:*

$$(6) \quad Z_{n,c}(t) := \frac{n^{c/2}}{\binom{n}{m}} \sum_{C_c^{[nt]}} \binom{i_c-c}{m-c} h_c(X_{i_1}, \dots, X_{i_c}),$$

*and the compensator is:*

$$\frac{n^{c/2}}{\binom{n}{m}} \sum_{C_c^{[nt]-1}} [h_c(X_{i_1}, \dots, X_{i_c}) \{ \binom{[nt]-c}{m-c} - \binom{i_c-c}{m-c} \}].$$

Proof. The compensator of  $Y_{n,c}(t)$  is given by:

$$\begin{aligned} & \sum_{k \leq [nt]} \{ E(Y_{n,c}(\frac{k}{n}) | \mathcal{F}_n(\frac{k-1}{n})) - Y_{n,c}(\frac{k-1}{n}) \} \\ &= \sum_{k=m}^{[nt]} \frac{n^{c/2} \binom{k-1-c}{m-1-c}}{\binom{n}{m}} \sum_{C_c^{k-1}} h_c(X_{i_1}, \dots, X_{i_c}) \\ &= \sum_{C_c^{[nt]-1}} h_c(X_{i_1}, \dots, X_{i_c}) \sum_{k=i_c+1}^{[nt]} \frac{n^{c/2} \binom{k-c-1}{m-c-1}}{\binom{n}{m}} \end{aligned}$$

$$= \frac{n^{c/2}}{\binom{n}{m}} \sum_{C_c^{[nt]-1}} [h_c(X_{i_1}, \dots, X_{i_c}) \{ \binom{[nt]-c}{m-c} - \binom{i_c-c}{m-c} \}],$$

where we have used

$$\sum_{n=a}^b \binom{n}{a} = \binom{b+1}{a+1}.$$

Then, as  $C_c^{[nt]} = C_c^{[nt]-1} + C_{c-1}^{[nt]-1} \times \{[nt]\}$ , we have that the martingale part of  $Y_{n,c}(t)$  is:

$$\begin{aligned} Z_{n,c}(t) &= \frac{n^{c/2}}{\binom{n}{m}} \binom{[nt]-c}{m-c} \sum_{C_{c-1}^{[nt]-1}} h_c(X_{i_1}, \dots, X_{i_{c-1}}, X_{[nt]}) \\ &\quad + \frac{n^{c/2}}{\binom{n}{m}} \sum_{C_c^{[nt]-1}} \binom{i_c-c}{m-c} h_c(X_{i_1}, \dots, X_{i_c}) \\ &= \frac{n^{c/2}}{\binom{n}{m}} \sum_{C_c^{[nt]}} \binom{i_c-c}{m-c} h_c(X_{i_1}, \dots, X_{i_c}). \end{aligned}$$

**Lemma 5.** *As  $n \rightarrow \infty$ ,*

$$\begin{aligned} &\sup_t \left| \frac{n^{c/2}}{\binom{n}{m}} \left\{ \sum_{C_m^{[nt]}} h_{m-1}(X_{i_1}, \dots, X_{i_{m-1}}) \right. \right. \\ &\quad \left. \left. - \sum_{C_c^{[nt]-1}} h_c(X_{i_1}, \dots, X_{i_c}) \left\{ \binom{[nt]-c}{m-c} - \binom{m-c}{i_c-c} \right\} \right\} \right| \xrightarrow[n \uparrow \infty]{P} 0. \end{aligned}$$

*i.e., the compensator of  $n^{c/2} U_n$  and  $Y_{n,c}(t)$  have the same limit.*

**Proof.** Noting that  $h_1 = \dots = h_{c-1} = 0$ , and using (2) with (4), the compensator of  $U_n(t)$  can be written as:

$$\frac{1}{\binom{n}{m}} \sum_{k=c}^{m-1} \sum_{C_k^{[nt]-1}} \left\{ \binom{[nt]-k}{m-k} - \binom{i_k-k}{m-k} \right\} \int h_k(x_1, \dots, x_k) \prod_{j=1}^k d(I_{\{X_{i_j} \leq x_j\}} - F(x_j)).$$

It is easy to see that the compensator of  $Y_{n,c}$  is  $n^{c/2}$  times the first term of the above summation. Therefore it is enough to prove that the following holds: as  $n \rightarrow \infty$

$$\begin{aligned} (7) \quad &P \left( \sup_t \left| \frac{n^{c/2}}{\binom{n}{m}} \sum_{k=c+1}^{m-1} \sum_{C_k^{[nt]-1}} \binom{[nt]-k}{m-k} \right. \right. \\ &\quad \left. \left. \times \int h_k(x_1, \dots, x_k) \prod_{j=1}^k d(I_{\{X_{i_j} \leq x_j\}} - F(x_j)) \right| > \varepsilon \right) \rightarrow 0. \end{aligned}$$

$$(8) \quad P(\sup_t \left| \frac{n^{c/2}}{\binom{n}{m}} \sum_{k=c+1}^{m-1} \sum_{C_k^{[nt]-1}} \binom{i_k-k}{m-k} \times \int h_k(x_1, \dots, x_k) \prod_{j=1}^k d(I_{(X_i \leq x_j)} - F(x_j)) \right| > \varepsilon) \rightarrow 0.$$

(7) is easily proved as in Lemma 3. (8) is proved as in Lemma 2 using the martingale structure of the process in (8) with respect to  $\mathcal{G}_n(t) = \sigma[X_1, \dots, X_{[nt]-1}]$ .

A consequence of Lemmas 3 and 5 is

**Lemma 6.**  *$n^{c/2} X_n(t)$ , the martingale part of  $n^{c/2} U_n(t)$ , can be approximated by  $Z_{n,c}(t)$ , the martingale part of  $Y_{n,c}(t)$ .*

Now, we make an assumption on the kernel function  $h$ .

(H) There exists some  $\phi \in L^2(\mathbf{R}, F)$  with  $E(\phi(X)) = 0$  and  $E(\phi^2(X)) = 1$  such that:

$$h_c(x_1, \dots, x_c) = \prod_{i=1}^c \phi(x_i)$$

Dynkin and Mandelbaum [2] as well as Mandelbaum and Taqqu [11], work apparently under the same restrictions for  $h$ . From now on, we assume (H). Let

$$S_n(t) := \sum_{i=1}^{[nt]} \frac{\phi(X_i)}{\sqrt{n}},$$

$$f_n(t) := \frac{n^c \binom{[nt]-c}{m-c}}{\binom{n}{m}}.$$

Then, we have

$$(9) \quad Z_{n,c}(t) = \frac{n^{c/2}}{\binom{n}{m}} \sum_{C_c^{[nt]}} \binom{i_c-c}{m-c} h_c(X_{i_1}, \dots, X_{i_c})$$

$$= \frac{n^c}{\binom{n}{m}} \sum_{i_c=c}^{[nt]} \dots \sum_{i_1=1}^{i_2-1} \binom{i_c-c}{m-c} \prod_{j=1}^c \frac{\phi(X_{i_j})}{\sqrt{n}}$$

$$= \int_{[0,t]} f_n(t_c) \int_{[0,t_c]} \dots \int_{[0,t_2]} dS_n(t_1) \dots dS_n(t).$$

Similarly,

$$(10) \quad Y_{n,c}(t) = f_n(t) \int_{[0,t]} \dots \int_{[0,t_2]} dS_n(t_1) \dots dS_n(t_c).$$

In the following we will denote by  $\mathcal{L}(D[0, 1])$ , the weak convergence in  $D[0, 1]$  under the Skorokhod topology.

**Lemma 7.** As  $n \rightarrow \infty$ ,

$$(i) \quad Z_{n,c}(t) \xrightarrow{\mathcal{L}(D[0,1])} \frac{m!}{(m-c)!} \int_0^t t_c^{m-c} \int_0^{t_c} \dots \int_0^{t_2} dW(t_1) \dots dW(t_c).$$

$$(ii) \quad Y_{n,c}(t) \xrightarrow{\mathcal{L}(D[0,1])} \frac{m!}{(m-c)!} t^{m-c} \int_0^t \int_0^{t_c} \dots \int_0^{t_2} dW(t_1) \dots dW(t_c).$$

where  $W$  is the Wiener process.

To prove this lemma we will make extensive use of the Skorokhod representation theorem ([16]) and the following lemma. We denote “in probability in  $D[0, 1]$ ” by  $P(D[0, 1])$ .

**Lemma 8.** Let  $A_n, A_0$  be locally bounded predictable processes in  $D[0, 1]$  and  $M_n, M_0$  be martingales in  $D[0, 1]$  such that:

- (a)  $M_n \rightarrow M_0, A_n \rightarrow A_0$  in  $P(D[0, 1])$ .
- (b)  $M_0, A_0 \in C([0, 1])$ .
- (c)  $\sup_n E(M_n^2(1)) < \infty$ .

Then  $\int_0^\cdot A_n dM_n \rightarrow \int_0^\cdot A_0 dM_0$  in  $P(D[0, 1])$ .

Proof. As  $A_0 \in C[0, 1]$  and, by tightness of  $P A_n^{-1}$  for  $n=0, 1, \dots$  we have:

- ( $\alpha$ )  $\lim_{k \rightarrow \infty} \sup_n P(\sup_t |A_n(t)| > k) = 0$ .
- ( $\beta$ )  $\lim_{\delta \downarrow 0} \overline{\lim}_n P(\sup_{|t-s| < \delta} |A_n(t) - A_n(s)| > \epsilon) = 0$  for  $\forall \epsilon > 0$ .

Let  $\varphi_k(x) = (x+k+1) I_{[-k-1, -k]}(x) + I_{[-k, k]}(x) + (-x+k+1) I_{[k, k+1]}(x)$ ,

$$A_{n,k} = \varphi_k(A_n(t)) A_n(t),$$

$$N_{n,k}(t) = \int_0^t A_{n,k}(s) dM_n(s),$$

$$N_n(t) = \int_0^t A_n(s) dM_n(s).$$

As  $P(d_0(N_{n,k}, N_n) > 0) \leq P(\sup_{0 \leq t \leq 1} |A_n(t)| > k)$ , where  $d_0$  is the metric that defines the Skorokhod topology in  $D[0, 1]$ .

Then, because of ( $\alpha$ ), in order to prove  $N_n \rightarrow N_0$  in  $P(D[0, 1])$  it is enough to prove  $N_{n,k} \rightarrow N_0$  in  $P(D[0, 1])$ . Therefore, we will assume from now on, that

$$|A_n(t)| \leq C \quad \text{for } \forall n \quad \text{for some } C > 0.$$

Let  $\tau^m = \{t_i = \frac{i}{m} / i=0, \dots, m\}$  a partition of  $[0, 1]$ , and define

$$A_n^m(t) = A_n(0) I_{(0)}(t) + \sum_{i=0}^{k-1} A_n(t_i) I_{]t_i, t_{i+1}[}(t).$$

We obtain the proof in two steps, as follows:



$$\begin{aligned}
 P(d_0(\int_0^{\cdot} A_n dM_n, \int_0^{\cdot} A_0 dM_0) > \varepsilon) &\leq P(d_0(\int_0^{\cdot} A_n dM_n, \int_0^{\cdot} A_n^m dM_n) > \frac{\varepsilon}{3}) \\
 &+ P(d_0(\int_0^{\cdot} A_n^m dM_n, \int_0^{\cdot} A_0^m dM_0) > \frac{\varepsilon}{3}) + P(d_0(\int_0^{\cdot} A_0^m dM_0, \int_0^{\cdot} A_0 dM_0) > \frac{\varepsilon}{3}) \\
 &=: I_1 + I_2 + I_3.
 \end{aligned}$$

We first prove that  $I_1 \rightarrow 0$ .

$$\begin{aligned}
 I_1 &\leq P(\sup_{0 \leq t \leq 1} |\int_0^t (A_n - A_n^m) dM_n| > \frac{\varepsilon}{3}) \\
 &\leq \frac{3}{\varepsilon} E[\sup_{0 \leq t \leq 1} |\int_0^t (A_n - A_n^m) dM_n|].
 \end{aligned}$$

By the Davis inequality,

$$\leq \frac{12}{\varepsilon} E(\sup_{|t-s| < \delta} |A_n(t) - A_n(s)| \langle M_n, M_n \rangle^{1/2}(1)).$$

By the Cauchy-Schwartz inequality,

$$\leq \frac{12}{\varepsilon^2} [E(\sup_{|t-s| < \delta} |A_n(t) - A_n(s)|^2) E(\langle M_n, M_n \rangle(1))]^{1/2},$$

where  $\delta = m^{-1}$ . But,  $\sup_n EM_n^2(1) < \infty$  and  $E(\sup_{|t-s| < \delta} |A_n(t) - A_n(s)|^2) \rightarrow 0$  as  $n \rightarrow \infty$  and  $\delta \downarrow 0$ , therefore,  $I_1 \rightarrow 0$ . The proof of  $I_3 \rightarrow 0$  is analogous.

We next prove that  $I_2 \rightarrow 0$ .

$$\begin{aligned}
 I_2 &\leq P(\sup_{0 \leq t \leq 1} \sum_{i=0}^{m-1} C |M_n(t_{i+1} \wedge t) - M_0(t_{i+1} \wedge t)| + |M_n(t_i \wedge t) - M_0(t_i \wedge t)| > \frac{\varepsilon}{6}) \\
 &+ P(\sup_{0 \leq t \leq 1} |\sum_{i=0}^{m-1} (A_n(t_i) - A_0(t_i))(M_0(t_{i+1} \wedge t) - M_0(t_i \wedge t))| > \frac{\varepsilon}{6}).
 \end{aligned}$$

The terms on the right-hand side tend to zero by assumptions (a) and (b).

Proof of Lemma 7.

By the Skorokhod theorem, it is possible to change the sample space and construct two processes  $S_n^*(t)$  and  $W^*(t)$  on the new sample space, which have the same law as  $S_n(t)$  and  $W(t)$ , such that  $S_n$  converges a.e. to  $W^*$  in  $D[0, 1]$ . As  $E((S_n^*(1))^2) = 1$ , applying Lemma 8 consecutively to obtain

$$\int_{[0, t_c)} \dots \int_{[0, t_2)} dS_n^*(t_1) \dots dS_n^*(t_{c-1}) \xrightarrow{P(D[0, 1])} \int_0^{t_c} \dots \int_0^{t_2} dW(t_1) \dots dW(t_{c-1}).$$

Noticing the uniform convergence of  $f_n(t)$  to  $f(t) = \frac{m!}{(m-c)!} t^{m-c}$  and the representation (9), we apply Lemma 8 again; to obtain (i). At the same time we have already proved (ii).

The next theorem is a consequence of Lemmas 3, 6 and 7.

**Theorem 1.** *Under (H), we have:*

$$(11) \quad n^{c/2} U_n(t) \xrightarrow{\mathcal{L}(D[0, 1])} \frac{m!}{(m-c)!} t^{m-c} \int_0^t \int_0^{t_c} \dots \int_0^{t_2} dW(t_1) \dots dW(t_c).$$

$$(12) \quad n^{c/2} X_n(t) \xrightarrow{\mathcal{L}(D[0, 1])} \frac{m!}{(m-c)!} \int_0^t t_c^{m-c} \int_0^{t_c} \dots \int_0^{t_2} dW(t_1) \dots dW(t_c).$$

$$(13) \quad n^{c/2} (U_n(t) - X_n(t)) \xrightarrow{\mathcal{L}(D[0, 1])} \frac{m!}{(m-c-1)!} \int_0^t s^{m-c-1} \int_0^s \dots \int_0^{t_2} dW(t_1) \dots dW(t_c) ds.$$

Proof of (13) is a direct consequence of (11), (12), Ito's formula and the fact that the limit processes are continuous.

### 3. Remarks

3.1. The result of Theorem 1 is different from the one obtained by Mandelbaum and Taqqu [11], Corollary 1; because we use the usual one dimensional time Wiener process, whereas they use multidimensional Wiener process. Also, the "projecting" functions  $h_n$  are easier to handle in our calculations.

3.2. Although the result of Lemma 8 is known to hold under weaker conditions than (c) (see, for example Jakubowski, et al. [7]), this version is suitable for our application and its proof is relatively easier. In fact, condition (c) can be replaced by the boundness of the jumps of the processes  $M_n$ , but the proof loses its simplicity.

3.3. The proof of Theorem 1 can also be obtained by means of Hermite polynomials, although it requires a rather complicated calculation. For example in the case of  $Z_{n,c}(t)$ , we have from (9) that:

$$(14) \quad Z_{n,c}(t) = f_n(t) F_c(S_n(t), \sqrt{\frac{[nt]}{n}}) - \int_0^t f'_n(s) F_c(S_n(s), \sqrt{\frac{[ns]}{n}}) ds + o_P(1),$$

where

$$f'_n(t) = \frac{n^{c-1} \binom{[nt]-c-1}{m-c-1}}{\binom{n}{m}},$$

$$F_c(a, b) = \frac{b^c}{c!} H_c\left(\frac{a}{b}\right),$$

$H_c$  denotes the Hermite polynomial of degree  $c$  with leading coefficient 1 and the symbol  $_-$  denotes limit from the left with respect of  $t$ .

Actually, the first term on the right of (14) is equal to  $Y_{n,c}(t)$  (see (10)) and the second term is equal to its compensator up to a term  $o_p(1)$ . From here, as the Hermite polynomials are continuous, we get the limit processes in terms of  $H_c$ .

By Ito's formula it can be proved that these expressions coincide with (11), (12) and (13). In the case of  $Z_{n,c}(t)$  we have:

$$\begin{aligned} & \frac{m!}{m-c!} \int_0^t t_c^{m-c} \int_0^{t_c} \dots \int_0^{t_2} dW(t_1) \dots dW(t_c) \\ &= f(t) F_c(W(t), \sqrt{t}) - \int_0^t f'(s) F(W(s), \sqrt{s}) ds. \end{aligned}$$

3.4. Theorem 1 can be easily extended to the case

$$h_c(x_1, \dots, x_c) = \sum_{i=1}^{\infty} \lambda_i \prod_{j=1}^c \phi_i(x_j),$$

where  $\phi_i$  is an orthonormal set of  $L^2(\mathbf{R}, F)$ .

This representation covers completely the case  $c=2$  by taking  $\phi_i$  as an orthonormal base of  $L^2(\mathbf{R}, F)$ .

3.5. In relation with U-processes, V-processes can be defined as (here we can't assume  $\theta=0$ )

$$V_n(t) = \frac{1}{n^m} \sum_{i_1=1}^{[nt]} \dots \sum_{i_m=1}^{[nt]} h(X_{i_1}, \dots, X_{i_m}).$$

Under the condition

$$\zeta^*(F) = \max_{1 \leq i_1 \leq \dots \leq i_m \leq m} E(h^2(X_{i_1}, \dots, X_{i_m})) < \infty,$$

instead of (1), analogous lemmas and theorems can be proved. A similar decomposition as in (2) can be obtained, i.e.,

$$\begin{aligned} V_n(t) &= \frac{[nt]^m}{n^m} \sum_{k=0}^m \binom{m}{k} \int h_k(x_1, \dots, x_k) \prod_{i=1}^k d(F_{[nt]}(x_i) - F(x_i)), \\ &\text{where } F_{[nt]}(x) = \sum_{j=1}^{[nt]} \frac{I_{\{X_j \leq x\}}}{[nt]}. \end{aligned}$$

In this case,  $n^{c/2}(V_n(t) - \theta)$ , its martingale and compensator parts have the same limits as  $n^{c/2} U_n(t)$ ,  $n^{c/2} X_n(t)$  and  $n^{c/2}(U_n(t) - X_n(t))$  in Theorem 1.

---

**References**

[1] P. Billingsley: *Convergence of probability measures*, Willey, New York, 1968.  
 [2] E.B. Dynkin and A. Mandelbaum: *Symmetric statistics, Poisson point processes and Multiple Wiener integrals*, Ann. Statist. **11**, N<sup>o</sup>3 (1983), 739-745.

- [3] G.G. Gregory: *Large sample theory for U-statistics and tests of fit*, Ann. Statist. **5**, N°1 (1977), 110–123.
- [4] W. Hoeffding: *A class of statistics with asymptotically normal distribution*, Ann. Math. Stat. **19** (1948), 293–325.
- [5] K. Ito: *Multiple Wiener integral*, J. Math. Soc. Japan **3** (1951), 157–169.
- [6] G.K. Eagleson: *Orthogonal expansions and U-statistics*, Austral. J. Statist. **21** (3) (1979), 221–237.
- [7] A. Jakubowski, J. Mémin and J. Pages: *Convergence en loi des suites d'intégrales stochastiques sur l'espace  $D^1$  de Skorokhod*, Probab. Theory Related Fields **81** (1989), 111–137.
- [8] E.V. Khmaladze: *Some applications of the theory of martingales to statistics*, Russian Math. Surveys **37** (1982), 215–237.
- [9] E.V. Khmaladze: *Martingale limit theorems for divisible statistics*, Theory. Probab. Appl. **28** (1983), 530–548.
- [10] R.M. Loynes: *An invariance principle for reversed martingales*, Proc. Amer. Math. Soc. **25** (1970), 56–64.
- [11] A. Mandelbaum and M.S. Taqqu: *Invariance principle for symmetric statistics*, Ann. Statist. **12**, N°2 (1984), 483–496.
- [12] R.G. Miller Jr. and P.K. Sen: *Weak convergence of U-statistics and Von-Mises differentiable statistical functions*, Ann. Math. Stat. **43** (1972), 31–41.
- [13] P.B.L.S. Rao: *Asymptotic theory of Statistical Inference*, Willey, New York, 1987.
- [14] H. Rubin and R.A. Vitale: *Asymptotic distribution of symmetric statistics*, Ann. Statist. **8** (1980), 165–170.
- [15] R.J. Serfling: *Approximation Theorems of Mathematical Statistics*, Willey, New York, 1980.
- [16] A.V. Skorokhod: *Studies in the Theory of Random Processes*, Dover Publ., New York, 1982.
- [17] H. Teicher: *Distribution and moment convergence of martingales*, Probab. Theory Related Fields **79** (1988), 303–316.

Department of Mathematics  
Faculty of Science  
Osaka University  
Toyonaka, Osaka 560, Japan

