Title: Block intersection numbers of block designs

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1. Introduction

Let $t, v, k$ and $\lambda$ be positive integers with $v \geq k \geq t$. A $t-(v, k, \lambda)$ design is a pair consisting of a $v$-set $\Omega$ and a family $B$ of $k$-subsets of $\Omega$, such that each $t$-subset of $\Omega$ is contained in $\lambda$ elements of $B$. Elements of $\Omega$ and $B$ are called points and blocks, respectively. A $t-(v, k, \lambda)$ design is called nontrivial provided $B$ is a proper subfamily of the family of all $k$-subsets of $\Omega$, then $t < k < v$. In this paper, we assume that all designs are nontrivial. For a $t-(v, k, \lambda)$ design $D$ we use $\lambda_i (0 \leq i \leq t)$ to represent the number of blocks which contain a given set of $i$ points of $D$. Then we have

$$ \lambda_i = \binom{v-i}{t-i} \binom{v-i}{k-i} \binom{v-i}{k-t+i} \binom{k-i}{k-t+i} \lambda \quad (0 \leq i \leq t). $$

A $t-(v, k, \lambda)$ design $D$ is called block-schematic if the blocks of $D$ form an association scheme with the relations determined by size of intersection (cf. [3]). In §2, we prove the following theorem which extends the result in [1].

**Theorem 1.** (a) For each $n \geq 1$ and $\lambda \geq 1$, there exist at most finitely many block-schematic $t-(v, k, \lambda)$ designs with $k-t=n$ and $t \geq 3$.
(b) For each $n \geq 1$ and $\lambda \geq 2$, there exist at most finitely many block-schematic $t-(v, k, \lambda)$ designs with $k-t=n$ and $t \geq 2$.

**Remark.** Since there exist infinitely many $2-(v, 3, 1)$ designs and since every $2-(v, k, 1)$ design is block-schematic (cf. [2]), Theorem 1 does not hold for $\lambda=1$ and $t=2$.

For a block $B$ of a $t-(v, k, \lambda)$ design $D$ we use $x_i(B) (0 \leq i \leq k)$ to denote the number of blocks each of which has exactly $i$ points in common with $B$. If, for each $i$ ($i=0, \ldots, k$), $x_i(B)$ is the same for every block $B$, we say that $D$ is block-regular and we write $x_i$ instead of $x_i(B)$. We remark that if a $t-(v, k, \lambda)$ design $D$ is block-schematic then $D$ is block-regular. For any $t-(v, k, 1)$ design or any $t-(v, t+1, \lambda)$ design, either of which is block-regular (cf. Lemma 1),
every \( x_i \) depends only on \( i, t, v, k \) or \( i, t, v, \lambda \) respectively (cf. Lemma 1). And Gross [5] and Dehon [4] respectively classified the \( t-(v, k, 1) \) designs and the \( t-(v, t+1, \lambda) \) designs both of which satisfy \( x_i=0 \). But for a block-regular \( t-(v, k, \lambda) \) design, \( x_i \) depends not only on \( i, t, v, k, \lambda \) but also on others in general (cf. Lemma 1). In §3, we prove the following theorem.

**Theorem 2.** Let \( c \) be a real number with \( c>2 \). Then for each \( n \geq 1 \) and \( l \geq 0 \), there exist at most finitely many block-regular \( t-(v, k, \lambda) \) designs with \( k-t=n, v=ct \) and \( x_i \leq l \) for some \( i \) \( (0 \leq i \leq t-1) \).

The author thanks Professor H. Enomoto for giving the direct proof of Lemma 5.

2. **Proof of Theorem 1**

**Lemma 1.** Let \( D \) be a block-regular \( t-(v, k, \lambda) \) design. Then the following equality holds for \( i=0, \ldots, k-1 \).

\[
x_i = \sum_{j=0}^{i} \binom{i}{j}(\lambda_j-1)\binom{k}{j}(-1)^{i+j} + \sum_{j=0}^{i+1} \binom{i}{j}w_j(-1)^{i+j},
\]

where \( x_j \leq w_j \leq (\lambda-1)\binom{k}{j} \) \( (t \leq j \leq k-1) \) and \( w_i=(\lambda-1)\binom{k}{i} \).

Proof. Let \( \delta \) be a block of \( D \). Counting in two ways the number of the following set

\[
\{(B', \{\alpha_i, \ldots, \alpha_i\}) \mid B' \text{ a block (} \neq B, B' \cap B \supseteq \alpha_i, \ldots, \alpha_i, \alpha_j \neq \alpha_{j'} \text{ if } j \neq j' \}\}
\]

gives

\[
x_i + \binom{i+1}{i}x_{i+1} + \cdots + \binom{t}{i}x_t + \cdots + \binom{k-1}{i}x_{k-1}=(\lambda_i-1)\binom{k}{i}
\]

for \( i=0, \ldots, t-1 \), and

\[
x_i + \binom{i+1}{i}x_{i+1} + \cdots + \binom{k-1}{i}x_{k-1} \leq (\lambda_i-1)\binom{k}{i}
\]

for \( i=t, \ldots, k-1 \). Let \( w_t(t \leq i \leq k-1) \) be the left hand of the above inequality, where \( w_i=(\lambda-1)\binom{k}{i} \). Let \( A=(a_{ij}) \) be the square matrix with \( a_{ij}=\binom{i}{j} \) \( (0 \leq i, j \leq k-1) \). Then we have

\[
A \begin{pmatrix} x_0 \\ \vdots \\ x_{i-1} \\ x_t \\ \vdots \\ x_{k-1} \end{pmatrix} = \begin{pmatrix} (\lambda_0-1)\binom{k}{0} \\ \vdots \\ (\lambda_{t-1}-1)\binom{k}{t-1} \\ w_t \\ \vdots \\ w_{k-1} \end{pmatrix}.
\]

Let us set \( A^{-1}=(b_{ij}) \) \( (0 \leq i, j \leq k-1) \). Since \( \sum_{j=m}^{n} (-1)^{j+m} \binom{n}{j} \binom{m}{j} = \delta_{mn} \), we have
\[ b_{ij} = \binom{j}{i} (-1)^{i+j}. \] Hence we get the desired result.

**Lemma 2.** Let \( D \) be a \( t-(v,k,\lambda) \) design with \( t, \lambda \geq 2 \). If \( v \geq k^3 \), then there exist three blocks \( B_1, B_2, B_3 \) of \( D \) such that \( |B_1 \cap B_2| = t-1 \), \( |B_2 \cap B_3| \geq t \) and \( |B_1 \cap B_3| = t-2 \).

**Proof.** Let \( B \) be a block of \( D \). Counting in two ways the number of the following set
\[ \{ (B', \alpha_1, \ldots, \alpha_t) \mid B' \text{ a block (not } B \text{), } B' \cap B \ni \alpha_1, \ldots, \alpha_t, \alpha_j \neq \alpha_{j'} \text{ if } j \neq j' \} \]
gives
\[ x_t(B) + \binom{t+1}{t} x_{t+1}(B) + \cdots + \binom{k-1}{t} x_{k-1}(B) = (\lambda-1) \binom{k}{t}. \]
Since \( \lambda \geq 2 \), there is an integer \( q \) \((t \leq q \leq k-1)\) with \( x_t(B) = q \). Hence, we may assume that there exist two blocks \( B_2, B_3 \) such that \( t \leq |B_2 \cap B_3| = q \). Let \( \alpha_1 \) be a point of \( B_2 - B_3 \) and \( \alpha_2, \ldots, \alpha_t-1 \) be \( t-2 \) points of \( B_2 \cap B_3 \). Set \( S = \{ B \mid B \text{ a block, } B \ni \{ \alpha_1, \ldots, \alpha_t-1 \} \} \), where \( |S| = k \). Then we have
\[ \left| \{ B \in S \mid |B \cap B_2| \geq t \text{ or } |B \cap B_3| \geq t-1 \} \right| \leq \lambda(k-t+1)+\lambda(k-t+2). \]
Hence, if \( \frac{v-t+1}{k-t+1} > \lambda(k-t+1)+\lambda(k-t+2) \), then there exists a block \( B_1 \) in \( S \) such that \( |B_1 \cap B_2| = t-1 \) and \( |B_1 \cap B_3| = t-2 \). On the other hand, \( \frac{v-t+1}{k-t+1} > (k-t+1)+(k-t+2) \) holds if \( v \geq k^3 \). So, the proof of Lemma 2 is completed.

**Proposition.** Let \( D \) be a block-schematic \( t-(v,k,\lambda) \) design with \( t, \lambda \geq 2 \). Then \( v < \lambda k^3 \left( \binom{k}{2} \right)^2 \) holds.

**Proof.** By Lemma 1, we have
\[ x_{t-2} > (\lambda t-2) \binom{k}{t-2} - (t-1) (\lambda t-1) \binom{k}{t-1} - (k-t) (\lambda-1) \binom{k}{2}. \]
So,
\[ x_{t-2} > \frac{(v-t+2)(v-t+1)}{(k-t+2)(k-t+1)} \lambda \binom{k}{t-2} - (t-1) \frac{v-t+1}{k-t+1} \lambda \binom{k}{t-1} - (k-t) \lambda \binom{k}{2}, \]
and
\[ x_{t-2} > \frac{(v-k)^2}{k^2} \lambda - (t-1) v \lambda \binom{k}{2} - k \lambda \binom{k}{2}. \]
Hence we have
\[ x_{t-2} > \frac{v^2}{k^2} \lambda - k v \lambda \binom{k}{2} - k \lambda \binom{k}{2}. \] (1)
Again by Lemma 1, we have
\[ x_{t-1} < \lambda x_{t-1} \left( \binom{k}{t-1} + (k-t-1) \left( \binom{k}{2} \right)^2 \right). \]

So,
\[ x_{t-1} < \frac{\nu}{2} \lambda \left( \binom{k}{2} \right) +(k-1) \lambda \left( \binom{k}{2} \right)^2. \] (2)

From now on, we may assume that \( v \geq k^3 \). By Lemma 2, there exist three blocks \( B_1, B_2, B_3 \) of \( D \) such that \( |B_1 \cap B_2| = t-1, |B_2 \cap B_3| = q (t \leq q \leq k-1) \), and \( |B_1 \cap B_3| = t-2 \). By Lemma 1, we have
\[ x_q \leq (\lambda-1) \left( \binom{k}{q} \right)^2. \] (3)

Hence, by (1), (2) and (3), we have
\[ x_{t-2} - x_{t-1} x_q > \frac{\nu^2}{k^2} \lambda - k\lambda \left( \binom{k}{2} \right)^2 - k\lambda^2 \left( \binom{k}{2} \right)^2 \frac{\nu}{2} + (k-1) \left( \binom{k}{2} \right)^2. \]

Thus, we have that
\[ x_{t-2} - x_{t-1} x_q > \frac{\nu^2}{k^2} \lambda - \lambda^2 \left( \binom{k}{2} \right)^2 \left( \binom{k}{2} \right)^2. \]

Hence, \( x_{t-2} - x_{t-1} x_q > 0 \) holds if \( v \geq k^3 \left( \binom{k}{2} \right)^2 \). (4)

Let \( B_1, B_2, B_3, \ldots, B_n \) be the blocks of \( D \). Let \( A_h \) \((0 \leq h \leq k)\) be the \( h \)-adjacency matrix of \( D \) of degree \( \lambda_0 \) defined by
\[ A_h(i, j) = \begin{cases} 1 & \text{if } |B_i \cap B_j| = h, \\ 0 & \text{otherwise}. \end{cases} \]

Since \( D \) is block-schematic, we have
\[ A_i A_j = \sum_{h=0}^{k} \mu(i, j, h) A_h \quad (0 \leq i, j \leq k), \]
where \( \mu(i, j, h) \) is a non-negative integer. Let \( a \) be the all-1 vector of degree \( \lambda_0 \). Then,
\[ A_i A_j a = \sum_{h=0}^{k} \mu(i, j, h) A_h a. \]

Hence we have \( x_i x_j = \sum_{h=0}^{k} \mu(i, j, h) x_h. \) In particular,
\[ x_{t-1} = \sum_{k=0}^{t} \mu(t-1, q, h) x_k, \]  
where \( \mu(t-1, q, t-2) \) is a positive integer, because \( |B_1 \cap B_2| = t-1, |B_2 \cap B_3| = q \) and \( |B_1 \cap B_3| = t-2 \). Hence, by (4) and (5), we have \( v < k^2 \left( \frac{k^2}{k^2} \right)^{\lambda} \).

**Lemma 3.** For each \( n \geq 1 \), there is a positive integer \( N_i(n) \) satisfying the following: If \( D \) is a \( t-(v, k, \lambda) \) design with \( k-t=n \) and \( t \geq N_i(n) \), then there exist two blocks \( B_1 \) and \( B_2 \) of \( D \) such that \( |B_1 \cap B_2| = t-1 \).

Proof. Let \( D \) be a \( t-(v, k, \lambda) \) design with \( k-t=n \). Let \( B \) be a block of \( D \). Counting in two ways the number of the following set 
\{ \( (B', \{ \alpha_1, \ldots, \alpha_t \}) \mid B' \text{ a block ( } \neq B \), \( B' \cap B \ni \alpha_1, \ldots, \alpha_t, \alpha_j \neq \alpha_j \), if \( j \neq j' \) \} gives 
\[ x_{t-1}(B) + \binom{t+1}{t} x_{t+1}(B) + \cdots + \binom{k-1}{t} x_{k-1}(B) = (\lambda-1) \binom{k}{t}. \]  
Since \( \binom{t+i}{t} = \frac{t}{t+i} \binom{t+i-1}{i+1} \) \((i \geq 0)\), we have 
\[ \binom{t+i}{t} = \frac{t}{t+i} \binom{t+i-1}{i+1} \] \((i \geq 0)\) 
\[ \binom{t}{t-1} x_{t-1}(B) + \binom{t+1}{t-1} x_{t+1}(B) + \cdots + \binom{k-1}{t-1} x_{k-1}(B) \leq t(\lambda-1) \binom{k}{t}. \]  
Counting in two ways the number of the following set 
\{ \( (B', \{ \alpha_1, \ldots, \alpha_{t-1} \}) \mid B' \text{ a block ( } \neq B \), \( B' \cap B \ni \alpha_1, \ldots, \alpha_{t-1}, \alpha_j \neq \alpha_j \), if \( j \neq j' \) \} gives 
\[ x_{t-1}(B) + \binom{t}{t-1} x_{t}(B) + \binom{t+1}{t} x_{t+1}(B) + \cdots + \binom{k-1}{t} x_{k-1}(B) \] \[ = (\lambda-1) \binom{k}{t-1}. \]  
By (6) and (7), we have 
\[ x_{t-1}(B) \geq (\lambda-1) \binom{k}{t-1} - t(\lambda-1) \binom{k}{t}, \]  
and 
\[ x_{t-1}(B) \geq \frac{v-t+1}{n+1} \lambda \frac{(n+t) \cdots t}{(n+1)!} \frac{(n+t) \cdots t}{(n+1)!} - \frac{(\lambda-1) (n+t) \cdots t}{n!}. \]  
Since \( D \) is a nontrivial design, \( v \geq k+t \geq 2t+n \). Hence we have 
\[ x_{t-1}(B) \geq \frac{(t+n+1) \cdots t}{(n+2)!} \frac{(t+n+1) \cdots t}{n!} \lambda. \]  
Set \( f(t) := \frac{(t+n+1) \cdots t}{(n+2)!} \frac{(t+n+1) \cdots t}{n!} \). Then there is a positive integer \( N_i(n) \) such that \( f(t) \geq 0 \) holds if \( t \geq N_i(n) \). Hence, the proof of Lemma 3 is completed.

**Lemma 4.** For each \( n \geq 1 \), there is a positive integer \( N_2(n) \) satisfying the
following: If \( D \) is a \( t-(v, k, \lambda) \) design with \( k-t=n \) and \( t \geq N_2(n) \), then there exist three blocks \( B_1, B_2, B_3 \) of \( D \) such that \( |B_1 \cap B_2|=t-1 \), \( |B_2 \cap B_3|=t-1 \) and \( |B_1 \cap B_3|=t-n-2 \).

Proof. Let \( D \) be a \( t-(v, k, \lambda) \) design with \( k-t=n \). We may assume \( t \geq N_1(n) \), where \( N_1(n) \) is a positive integer obtained in Lemma 3. Therefore, there exist two blocks \( B_2 \) and \( B_3 \) of \( D \) with \( |B_2 \cap B_3|=t-1 \). Let \( \alpha_1, \alpha_2, \ldots, \alpha_{t-1} \) be \( n-1 \) distinct points of \( B_2 \cup B_3 \) and \( \alpha_{n+1}, \ldots, \alpha_{t-1} \) be \( t-n-2 \) distinct points of \( B_1 \). Set \( S=\{B|B \text{ a block}, B \supseteq \{\alpha_1, \ldots, \alpha_{t-1}\}\} \), where \( |S|=(v-t+1)/(k-t+1) \). Then we have

\[
|\{B \in S| |B \cap B_2| \geq t \text{ or } |B \cap B_3| \geq t-n-1\}| \leq (k-t+1)+\lambda(k-t+n+2).
\]

Hence, if \( v-t+1/(k-t+1) > \lambda(n+1)+\lambda(2n+2) \), then there exists a block \( B_1 \) in \( S \) such that \( |B_1 \cap B_2|=t-1 \) and \( |B_1 \cap B_3|=t-n-2 \). On the other hand, since \( v>k+t-2t+n \), we have that \( v-t+1/(n+1)+(2n+2) \) holds if \( t \geq 3(n+1)^2 \). Thus, Lemma 4 holds if \( N_2(n)=\max\{N_1(n), 3(n+1)^2\} \).

Proof of Theorem 1. First, let us suppose that \( D \) is a block-schematic \( t-(v, k, \lambda) \) design with \( k-t=n \) and \( t \geq 2 \). By Proposition, we may assume that \( t \geq N_2(n) \), where \( N_2(n) \) is a positive integer obtained in Lemma 4. By Lemma 1 we have

\[
x_{t-s-2} > \lambda_{t-s-2}(t-n-2) - \sum_{j=1}^{s+1} \lambda_j(t-n-2) \lambda(t-n-1) - \sum_{j=1}^{s+1} \lambda_j(t-n-1),
\]

where \( \lambda_{t-s-2}(t-n-2) = (v-t+n+2) \cdots (v-t+1)/(t-n+1) \cdots (t-n)! \),

\[
\sum_{j=1}^{s+1} \lambda_j(t-n-2) \lambda(t-n-1) < (n+1) \lambda_{t-s-1}(t-n-2) !
\]

\[
= (n+1) (v-t+n+1) \cdots (v-t+1)/(t-n+1) \cdots (t-n-1) ! \lambda(t-n-2) !
\]

and

\[
\sum_{j=1}^{s+1} \lambda_j(t-n-2) \lambda(t-n) < n (t-n+1)/(t-n-2) ! \lambda(t-n-1) !
\]

Hence we have

\[
x_{t-s-2} > (v-t)^{s+1}(t-n-1)^{2n+2} \lambda - (v-t+n+1)^{s+1}(t-n)^{2n+2} \lambda.(8)
\]

Again by Lemma 1, we have

\[
x_{t-1} < \frac{v-t+1}{n+1} \lambda(t-n-1) + \sum_{j=1}^{s+1} \lambda(t-n-1) \lambda(t-n) ,
\]

and
Hence we have
\[ x_{t-1} < (v-t+1) (t+n)^{n+1} n(t+n)^{n+1}. \]

By (8) and (9), we have
\[ x_{t-1}^2 < (v-t+n+1)^2 (t+n)^{2n+2} \lambda^2. \] (9)

By the similar argument as in the proof of Proposition, we have
\[ x_{t-1}^2 = \sum_{h=0}^{k} \mu(t-1, t-1, h) x_h, \] (11)

where \( \mu(t-1, t-1, h) \) is a non-negative integer. Moreover, since \( t \geq N(n, \lambda) \) \( \mu(t-1, t-1, t-n-2) \) is a positive integer by Lemma 4. Hence, by (10) and (11), we have \( t \leq N(n, \lambda) \). Therefore, \( k \leq N(n, \lambda) + n \). Hence by Proposition, the proof of Theorem 1 is completed on condition that \( \lambda \leq 2 \).

Next, let us suppose that \( D \) is a block-schematic \( t-(v, k, l) \) design with \( k-t=n \) and \( t \geq 3 \). (The proof of the case \( \lambda = 1 \) is similar to that of the case \( \lambda \geq 2 \). Then, we give an outline of it.) By Theorem in [1], we may assume that \( t \geq N(n, \lambda) \), where \( N(n, \lambda) \) is a positive integer obtained in Lemma 4. By Lemma 1, we get
\[ x_{t-1}^2 - x_{t-1}^2 > \frac{(v-t)^{n+2} (t-n-1)^{2n+2}}{(2n+2)^2} - 2(v-t+n+1)^{n+1} (t+n)^{2n+2}. \]

Hence, there is a positive integer \( N(n) ( \geq N(n, \lambda)) \) such that \( x_{t-1}^2 - x_{t-1}^2 > 0 \) holds if \( t \geq N(n) \). On the other hand, the following equation holds:
\[ x_{t-1}^2 = \sum_{h=0}^{k} \mu(t-1, t-1, h) x_h, \]

where \( \mu(t-1, t-1, h) \) is a non-negative integer and \( \mu(t-1, t-1, t-n-2) \) is positive. Therefore, we have \( t \leq N(n) \), and so \( k \leq N(n) + n \). Hence by Theorem in [1], the proof of Theorem 1 is completed on condition that \( \lambda = 1 \). Thus, Theorem 1 is proved.
3. Proof of Theorem 2

Lemma 5. Let $D$ be a block-regular $t-(v, k, \lambda)$ design. Then the following equality holds for $i=0, \cdots, t-1$.

\[ x_i = -\frac{\lambda \binom{k}{i}}{\binom{v-k}{k-t}} \left( \binom{v-k}{k-i} + \sum_{j=0}^{k-i} \binom{k}{j} \binom{t-i-1+q}{q} \binom{v-k+q}{k-t} \right) \]

\[ + (\lambda-1) \sum_{j=1}^{k-i} \binom{j}{i} \binom{k}{j} \binom{t-j-1+q}{q} \binom{v-k+q}{k-t} \]

where $x_j \leq w_j \leq (\lambda-1) \binom{k}{j}$ ($1 \leq j \leq k-1$) and $w_i = (\lambda-1) \binom{k}{t}$.

(The essential part of Lemma 5 is [5, Lemma 6].)

Proof. In this proof, we use the following three combinatorial identities:

1. \[ (\binom{-a}{b}) = (-1)^a \binom{a+b-1}{b} \]
2. \[ \sum \binom{a}{r} \binom{b+r}{c} (-1)^r = (-1)^a \binom{b}{c-a} (a \geq 0) \]
3. \[ \sum \binom{a}{r} \binom{b}{c-r} = \binom{a+b}{c} (a \geq 0) \]

By Lemma 1, we have

\[ x_i = \sum_{j=1}^{k-i} \binom{j}{i} (\lambda_j-1) \binom{k}{j} (-1)^{i+j} + \sum_{j=1}^{k-i} \binom{j}{i} w_j (-1)^{i+j} \]

where $x_j \leq w_j \leq (\lambda-1) \binom{k}{j}$ ($1 \leq j \leq k-1$).

Then,

\[ x_i = \lambda \sum_{j=1}^{k-i} \binom{j}{i} (\lambda_j-1) \binom{k}{j} (-1)^{i+j} + (\lambda-1) \sum_{j=1}^{k-i} \binom{j}{i} \binom{k}{j} (-1)^{i+j} \]

\[ + \sum_{j=1}^{k-i} \binom{j}{i} w_j (-1)^{i+j} \]

where $\lambda_j = \binom{v-j}{t-j} \binom{v-t}{k-t} \binom{k-j}{t-j} = (0 \leq j \leq t-1)$.

Hence, in order to prove Lemma 5, it is sufficient to show that the following equality holds for $i=0, \cdots, k-1$.

\[ \sum_{j=1}^{k-i} \binom{j}{i} (\lambda_j-1) \binom{k}{j} (-1)^{i+j} \]
First suppose that \( t \leq i \leq k-1 \). Then,

\[
\sum_{t=0}^{k-i-1} \binom{t-i+1+q}{q} \binom{v-k+q}{k-t} = \sum_{t=0}^{k-i-1} \binom{t-i}{q} \binom{v-k+q}{k-t} = (-1)^{i-t} \binom{v-k}{k-1}.
\]

Hence, the right hand of (12) = 0 = the left hand of (12).

Let \( A=(a_{rs}) \) be the square matrix with \( a_{rs}=\binom{s}{r} \) (0 \( r, s \leq k-1 \)). Since \( \det(A) \neq 0 \), \( A^{-1} = \left( \binom{s}{r} \right) (0 \leq r, s \leq k-1) \) and (12) holds for \( i=t, \cdots, k-1 \), we have that (12) holds for \( i=0, \cdots, k-1 \) if the following holds for \( i=0, \cdots, t-1 \).

\[
\sum_{j=1}^{k-i} \binom{j}{i} \binom{k}{i} \binom{v-k}{k-j} = \binom{k-i}{k-j} \binom{v-k+q}{k-t} = (-1)^{t+j+1} \sum_{t=0}^{k-i-1} \binom{t-j+1+q}{q} \binom{v-k+q}{k-t} = (\lambda_i-1) \binom{k}{i}.
\]

Since \( \binom{j}{i} \binom{k}{i} = \binom{k-i}{k-j} \),

the left hand of (13) = \( \binom{k}{i} \sum_{j=1}^{k-i} \binom{k-i}{k-j} \binom{v-k}{k-j} + (-1)^{t+j+1} \sum_{t=0}^{k-i-1} \binom{t-j+1+q}{q} \binom{v-k+q}{k-t} \).

Now, \( \sum_{j=1}^{k-i} \binom{k-i}{k-j} \binom{v-k}{k-j} = \sum_{j=i}^{k-i} \binom{k-i}{j-i} \binom{v-k}{k-j} = \sum_{h=i}^{k-i} \binom{k-i}{h} \binom{v-k}{k-i-h} = (\lambda_i-1) \binom{v-k}{k-i} \).

On the other hand,

\[
\sum_{j=1}^{k-i} \binom{k-i}{k-j} (-1)^{t+j+1} \sum_{t=0}^{k-i-1} \binom{t-j+1+q}{q} \binom{v-k+q}{k-t} = \sum_{t=0}^{k-i-1} \binom{v-k+q}{k-t} \sum_{j=1}^{k-i} \binom{k-i}{j-i} \binom{t-j+1+q}{q} (-1)^j = \sum_{t=0}^{k-i-1} \binom{v-k+q}{k-t} \sum_{j=1}^{k-i} \binom{k-i}{j-i} \binom{t-j}{q} (-1)^{j+q} (\text{cf. (i)})
\]
Hence by (14), (15) and (16), we have that

\[
\text{the left hand of (13)} = \frac{\binom{k}{i}}{\left(\frac{v-t}{k-t}\right)^{i}} \left\{ \binom{v-k-i}{k-t} - 1 + \binom{v-t}{k-t} \right\} = \left\{ \frac{\binom{v-i}{k-i}}{\left(\frac{v-t}{k-t}\right)^{i}} - 1 \right\} \binom{k}{i} = \text{the right hand of (13)}.
\]

Thus, Lemma 5 is proved.

**Lemma 6.** For each \( k \geq 2 \) and \( l \geq 0 \), there exist at most finitely many block-regular \( t-(v, k, \lambda) \) designs with \( x_i \leq l \) for some \( i \) (0 \( \leq i \leq t-1 \)).

**Proof.** In order to prove Lemma 6, it is sufficient to show the following: For each \( k \geq 2 \), \( l \geq 0 \), \( t \) (1 \( \leq t \leq k \)) and \( i \) (0 \( \leq i \leq t \)), there exist at most finitely many block-regular \( t-(v, k, \lambda) \) designs with \( x_i \leq l \).

Let \( k, l, t \) and \( i \) be integers with \( k \geq 2 \), \( l \geq 0 \), 1 \( \leq t \leq k \) and 0 \( \leq i \leq t \), and let \( D \) be a block-regular \( t-(v, k, \lambda) \) design with \( x_i \leq l \). By Lemma 1, we have

\[
x_i = \sum_{j=t}^{k-1} \binom{i}{j} (\lambda_j - 1) \binom{k}{j} (-1)^{i+j} + \sum_{j=t}^{k-1} \binom{i}{j} \omega_j (-1)^{i+j},
\]

where \( x_i \leq \omega_j \leq (\lambda - 1) \binom{k}{j} \) (\( j = t, \ldots, k-1 \)). Therefore,

\[
x_i - l > \binom{v-t}{k-t} \cdots \binom{v-t+1}{k-t} (\lambda - 1) \binom{k}{j} - \sum_{j=t}^{k-1} \binom{i}{j} (\lambda_j - 1) \binom{k}{j} - l.
\]
In the above expression, if we suppose that \( k, l, t \) and \( i \) are constants, and that \( v \) and \( \lambda \) are variables with \( v > k \) and \( \lambda \geq 1 \), then we can obtain the following:

The right hand of the expression \( \lambda \cdot f(v) + \lambda \cdot g(v) + d \), where \( f(v) \) is a polynomial in \( v \) of degree \( t - i \) with the leading coefficient of \( f(v) > 0 \), \( g(v) \) is a polynomial in \( v \) of degree \( t - i - 1 \), and \( d \) is a constant. Hence, there exists a constant \( C(k, l, t, i) > 0 \) such that \( x_i < l \) holds if \( v > C(k, l, t, i) \). Namely, if \( x_i < l \), then \( v < C(k, l, t, i) \).

Proof of Theorem 2. By Lemma 6, we may assume that \( t \geq \frac{2n + ((2n + 2)l)^2}{c-2} + 2n \). Let \( D \) be a block-regular \( t-(v, t+n, \lambda) \) design with \( v > ct, t \geq \frac{2n + ((2n + 2)l)^2}{c-2} + 2n \) and \( x_i < l \) for some \( i \) \((0 \leq i < t-1)\). Set \( v = mt \) \((m \geq c)\), where \( m \) is not always integral. By Lemma 5, we have

\[
x_i = \frac{\lambda(t+n)}{n} \bigg\{ \binom{t+n}{i} \bigg( (m-1)t-n \bigg) + (t-i+1) \sum_{j=0}^{r-1} \binom{t-i-1+q}{j} \binom{m-1}{q} \bigg( t-n+q \bigg) \bigg\}
+ (\lambda-1) \sum_{j=1}^{r-1} \binom{j+n}{j} \binom{t+n}{j} (-1)^{i+j} + \sum_{j=i}^{r-1} \binom{j}{i} x_j (1)^{i+j},
\]

where \( x_j < x_j \leq (\lambda-1) \binom{j+n}{j} \) \((t \leq j < k-1)\).

Now,

\[
(\lambda-1) \sum_{j=1}^{r-1} \binom{j+n}{j} \binom{t+n}{j} (-1)^{i+j} + \sum_{j=i}^{r-1} \binom{j}{i} x_j (1)^{i+j}
= (\lambda-1) \sum_{j=1}^{r-1} \binom{j+n}{j} \binom{t+n}{j} (-1)^{i+j} + \sum_{j=i}^{r-1} \binom{j}{i} x_j (1)^{i+j}
> -2\lambda(n+1) \frac{(t+n)!}{i!(t-i)!}.
\]

On the other hand,

\[
\frac{\lambda(t+n)}{n} \bigg\{ \binom{t+n}{i} \bigg( (m-1)t-n \bigg) + (t-i+1) \sum_{j=0}^{r-1} \binom{t-i-1+q}{j} \binom{m-1}{q} \bigg( t-n+q \bigg) \bigg\}
+ (\lambda-1) \sum_{j=1}^{r-1} \binom{j+n}{j} \binom{t+n}{j} (-1)^{i+j} + \sum_{j=i}^{r-1} \binom{j}{i} x_j (1)^{i+j}
> \frac{\lambda(t+n)!}{i!(t+n-i)!} \frac{(m-1)t-n)!}{(m-1)t-n)!} \frac{(m-1)t-n)!}{(m-2)t-2n+i)!} \frac{\lambda(t+n)!}{i!(t-i)!}.
\]

By (17), (18) and (19), we have
Then since \( \frac{t!}{(t+n)!} \leq \frac{1}{\lambda} < 1 \), we have

\[
x_i > \frac{x_i!(t-i)!}{(t+n)!} \frac{\{(m-1)t-n\}^2(t-i)!}{(t+n-i)!^2((m-1)t)!(m-2)t-2n+i!} - 5n.
\]

Hence, \( x_i \geq \frac{(m-1)t-n}{(m-1)t} \cdot \frac{((m-1)t-n-1) \cdots ((m-1)t-2n+i+1)}{((m-1)t) \cdots ((m-1)t-n+1) \cdot (t+n-i) \cdots (t-i+1)} \cdot \frac{((m-1)t-3n-1) \cdots ((m-2)t-2n+i+2) \cdot (m-2)t-2n+i+1}{(2n+3)!(2n+2)!} - 5n \)

holds if \( t-i \geq n+3 \), and

\[
x_i > ((m-1)t-n) \cdot \frac{(m-1)t-n-1}{(m-1)t} \cdots ((m-1)t-2n) \cdots (m-1)t-n+1 \cdot \frac{(m-1)t-2n-1}{(2n+2)!} - 5n \]

holds if \( 2 \leq t-i \leq n+2 \),

and \( x_i \geq ((m-1)t-n) \cdot \frac{(m-1)t-n-1}{(m-1)t} \cdots ((m-1)t-2n) \frac{1}{(n+1)!} - 5n \)

holds if \( t-i = 1 \).

In any case, since \( t \geq \frac{2n+(2n+2)!}{c-2} \cdot 2n \), we have

\[
x_i > ((m-1)t-n) \cdot \frac{(m-2)t}{(m-1)t} \cdot \frac{1}{((n+1)!)^2} - 5n
\]

\[
> \frac{(c-1)t-n}{(c-1)!} \cdot \frac{(c-2)^n}{(c-1)!} - 5n.
\]

Therefore, there exists a positive integer \( N(c, n, l) \left( \geq \frac{2n+(2n+2)!}{c-2} \cdot 2n \right) \) such that \( x_i - l > 0 \) holds if \( t \geq N(c, n, l) \). Namely, if \( x_i \leq l \), then \( t \leq N(c, n, l) \). Hence by Lemma 6, the proof of Theorem 2 is completed.
References


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