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BLOCK INTERSECTION NUMBERS OF BLOCK DESIGNS

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1. Introduction

Let t , v , k and λ be positive integers with $v \geq k \geq t$. A t — (v, k, λ) design is a pair consisting of a v -set Ω and a family \mathbf{B} of k -subsets of Ω , such that each t -subset of Ω is contained in λ elements of \mathbf{B} . Elements of Ω and \mathbf{B} are called points and blocks, respectively. A t — (v, k, λ) design is called nontrivial provided \mathbf{B} is a proper subfamily of the family of all k -subsets of Ω , then $t < k < v$. In this paper, we assume that all designs are nontrivial. For a t — (v, k, λ) design \mathbf{D} we use λ_i ($0 \leq i \leq t$) to represent the number of blocks which contain a given set of i points of \mathbf{D} . Then we have

$$\lambda_i = \frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}} \lambda = \frac{(v-i)(v-i-1) \cdots (v-t+1)}{(k-i)(k-i-1) \cdots (k-t+1)} \lambda \quad (0 \leq i \leq t).$$

A t — (v, k, λ) design \mathbf{D} is called block-schematic if the blocks of \mathbf{D} form an association scheme with the relations determined by size of intersection (cf. [3]). In §2, we prove the following theorem which extends the result in [1].

Theorem 1. (a) *For each $n \geq 1$ and $\lambda \geq 1$, there exist at most finitely many block-schematic t — (v, k, λ) designs with $k-t=n$ and $t \geq 3$.*

(b) *For each $n \geq 1$ and $\lambda \geq 2$, there exist at most finitely many block-schematic t — (v, k, λ) designs with $k-t=n$ and $t \geq 2$.*

REMARK. Since there exist infinitely many 2 — $(v, 3, 1)$ designs and since every 2 — $(v, k, 1)$ design is block-schematic (cf. [2]), Theorem 1 does not hold for $\lambda=1$ and $t=2$.

For a block B of a t — (v, k, λ) design \mathbf{D} we use $x_i(B)$ ($0 \leq i \leq k$) to denote the number of blocks each of which has exactly i points in common with B . If, for each i ($i=0, \dots, k$), $x_i(B)$ is the same for every block B , we say that \mathbf{D} is block-regular and we write x_i instead of $x_i(B)$. We remark that if a t — (v, k, λ) design \mathbf{D} is block-schematic then \mathbf{D} is block-regular. For any t — $(v, k, 1)$ design or any t — $(v, t+1, \lambda)$ design, either of which is block-regular (cf. Lemma 1),

every x_i depends only on i, t, v, k or i, t, v, λ respectively (cf. Lemma 1). And Gross [5] and Dehon [4] respectively classified the $t-(v, k, 1)$ designs and the $t-(v, t+1, \lambda)$ designs both of which satisfy $x_i=0$. But for a block-regular $t-(v, k, \lambda)$ design, x_i depends not only on i, t, v, k, λ but also on others in general (cf. Lemma 1). In §3, we prove the following theorem.

Theorem 2. *Let c be a real number with $c > 2$. Then for each $n \geq 1$ and $l \geq 0$, there exist at most finitely many block-regular $t-(v, k, \lambda)$ designs with $k-t = n, v \geq ct$ and $x_i \leq l$ for some i ($0 \leq i \leq t-1$).*

The author thanks Professor H. Enomoto for giving the direct proof of Lemma 5.

2. Proof of Theorem 1

Lemma 1. *Let D be a block-regular $t-(v, k, \lambda)$ design. Then the following equality holds for $i=0, \dots, k-1$.*

$$x_i = \sum_{j=i}^{t-1} \binom{j}{i} (\lambda_j - 1) \binom{k}{j} (-1)^{i+j} + \sum_{j=t}^{k-1} \binom{j}{i} w_j (-1)^{i+j},$$

where $x_j \leq w_j \leq (\lambda - 1) \binom{k}{j}$ ($t \leq j \leq k-1$) and $w_t = (\lambda - 1) \binom{k}{t}$.

Proof. Let B be a block of D . Counting in two ways the number of the following set

$\{(B', \{\alpha_1, \dots, \alpha_i\}) \mid B' \text{ a block } (\neq B), B' \cap B \ni \alpha_1, \dots, \alpha_i, \alpha_j \neq \alpha_{j'} \text{ if } j \neq j'\}$ gives $x_i + \binom{i+1}{i} x_{i+1} + \dots + \binom{t}{i} x_t + \dots + \binom{k-1}{i} x_{k-1} = (\lambda_i - 1) \binom{k}{i}$ for $i = 0, \dots, t-1$,

and $x_i + \binom{i+1}{i} x_{i+1} + \dots + \binom{k-1}{i} x_{k-1} \leq (\lambda - 1) \binom{k}{i}$ for $i = t, \dots, k-1$. Let w_i ($t \leq i \leq k-1$) be the left hand of the above inequality, where $w_t = (\lambda - 1) \binom{k}{t}$. Let

$A = (a_{ij})$ be the square matrix with $a_{ij} = \binom{j}{i}$ ($0 \leq i, j \leq k-1$). Then we have

$$A \begin{pmatrix} x_0 \\ \vdots \\ x_{t-1} \\ x_t \\ \vdots \\ x_{k-1} \end{pmatrix} = \begin{pmatrix} (\lambda_0 - 1) \binom{k}{0} \\ \vdots \\ (\lambda_{t-1} - 1) \binom{k}{t-1} \\ w_t \\ \vdots \\ w_{k-1} \end{pmatrix}.$$

Let us set $A^{-1} = (b_{ij})$ ($0 \leq i, j \leq k-1$). Since $\sum_{j=m}^n (-1)^{j+m} \binom{n}{j} \binom{j}{m} = \delta_{mn}$, we have

$b_{ij} = \binom{j}{i}(-1)^{i+j}$. Hence we get the desired result.

Lemma 2. *Let D be a $t-(v, k, \lambda)$ design with $t, \lambda \geq 2$. If $v \geq k^3$, then there exist three blocks B_1, B_2, B_3 of D such that $|B_1 \cap B_2| = t-1$, $|B_2 \cap B_3| \geq t$ and $|B_1 \cap B_3| = t-2$.*

Proof. Let B be a block of D . Counting in two ways the number of the following set

$\{(B', \alpha_1, \dots, \alpha_t) \mid B' \text{ a block } (\neq B), B' \cap B \ni \alpha_1, \dots, \alpha_t, \alpha_j \neq \alpha_{j'} \text{ if } j \neq j'\}$ gives $x_t(B) + \binom{t+1}{t}x_{t+1}(B) + \dots + \binom{k-1}{t}x_{k-1}(B) = (\lambda-1)\binom{k}{t}$. Since $\lambda \geq -2$, there is an integer q ($t \leq q \leq k-1$) with $x_q(B) \neq 0$. Hence, we may assume that there exist two blocks B_2, B_3 such that $t \leq |B_2 \cap B_3| = q$. Let α_1 be a point of $B_2 - B_3$ and $\alpha_2, \dots, \alpha_{t-1}$ be $t-2$ points of $B_2 \cap B_3$. Set $S = \{B \mid B \text{ a block, } B \ni \{\alpha_1, \dots, \alpha_{t-1}\}\}$, where $|S| = \frac{v-t+1}{k-t+1}\lambda$. Then we have

$$|\{B \in S \mid |B \cap B_2| \geq t \text{ or } |B \cap B_3| \geq t-1\}| \leq \lambda(k-t+1) + \lambda(k-t+2).$$

Hence, if $\frac{v-t+1}{k-t+1}\lambda > \lambda(k-t+1) + \lambda(k-t+2)$, then there exists a block B_1 in S such that $|B_1 \cap B_2| = t-1$ and $|B_1 \cap B_3| = t-2$. On the other hand, $\frac{v-t+1}{k-t+1} > (k-t+1) + (k-t+2)$ holds if $v \geq k^3$. So, the proof of Lemma 2 is completed.

Proposition. *Let D be a block-schematic $t-(v, k, \lambda)$ design with $t, \lambda \geq 2$. Then $v < \lambda k^3 \binom{k}{\lfloor \frac{k}{2} \rfloor}^2$ holds.*

Proof. By Lemma 1, we have

$$x_{t-2} > (\lambda_{t-2} - 1) \binom{k}{t-2} - (t-1)(\lambda_{t-1} - 1) \binom{k}{t-1} - (k-t)(\lambda-1) \binom{k}{\lfloor \frac{k}{2} \rfloor}^2.$$

So, $x_{t-2} > \frac{(v-t+2)(v-t+1)}{(k-t+2)(k-t+1)} \lambda \binom{k}{t-2} - (t-1) \frac{v-t+1}{k-t+1} \lambda \binom{k}{t-1} - (k-t) \lambda \binom{k}{\lfloor \frac{k}{2} \rfloor}^2$,

and

$$x_{t-2} > \frac{(v-k)^2}{k^2} \lambda - (t-1)v\lambda \binom{k}{\lfloor \frac{k}{2} \rfloor} - k\lambda \binom{k}{\lfloor \frac{k}{2} \rfloor}^2.$$

Hence we have

$$x_{t-2} > \frac{v^2}{k^2} \lambda - kv\lambda \binom{k}{\lfloor \frac{k}{2} \rfloor} - k\lambda \binom{k}{\lfloor \frac{k}{2} \rfloor}^2. \tag{1}$$

Again by Lemma 1, we have

$$x_{t-1} < \lambda_{t-1} \binom{k}{t-1} + (k-t)(\lambda-1) \binom{k}{\lfloor \frac{k}{2} \rfloor}.$$

So,

$$x_{t-1} < \frac{v}{2} \lambda \binom{k}{\lfloor \frac{k}{2} \rfloor} + (k-1) \lambda \binom{k}{\lfloor \frac{k}{2} \rfloor}. \tag{2}$$

From now on, we may assume that $v \geq k^3$. By Lemma 2, there exist three blocks B_1, B_2, B_3 of \mathbf{D} such that $|B_1 \cap B_2| = t-1$, $|B_2 \cap B_3| = q$ ($t \leq q \leq k-1$), and $|B_1 \cap B_3| = t-2$. By Lemma 1, we have

$$x_q \leq (\lambda-1) \binom{k}{q} < \lambda \binom{k}{\lfloor \frac{k}{2} \rfloor}. \tag{3}$$

Hence, by (1), (2) and (3), we have

$$x_{t-2} - x_{t-1} x_q > \frac{v^2}{k^2} \lambda - kv \lambda \binom{k}{\lfloor \frac{k}{2} \rfloor} - k \lambda \binom{k}{\lfloor \frac{k}{2} \rfloor} - \lambda^2 \binom{k}{\lfloor \frac{k}{2} \rfloor}^2 \left\{ \frac{v}{2} + (k-1) \binom{k}{\lfloor \frac{k}{2} \rfloor} \right\}.$$

Thus, we have that

$$x_{t-2} - x_{t-1} x_q > \frac{v^2}{k^2} \lambda - \lambda^2 \binom{k}{\lfloor \frac{k}{2} \rfloor}^2 v - k \lambda^2 \binom{k}{\lfloor \frac{k}{2} \rfloor}^3.$$

Hence, $x_{t-2} - x_{t-1} x_q > 0$ holds if $v \geq k^3 \binom{k}{\lfloor \frac{k}{2} \rfloor}^2 \lambda$. (4)

Let $B_1, B_2, B_3, \dots, B_{\lambda_0}$ be the blocks of \mathbf{D} . Let A_h ($0 \leq h \leq k$) be the h -adjacency matrix of \mathbf{D} of degree λ_0 defined by

$$A_h(i, j) = \begin{cases} 1 & \text{if } |B_i \cap B_j| = h, \\ 0 & \text{otherwise.} \end{cases}$$

Since \mathbf{D} is block-schematic, we have

$$A_i A_j = \sum_{h=0}^k \mu(i, j, h) A_h \quad (0 \leq i, j \leq k),$$

where $\mu(i, j, h)$ is a non-negative integer. Let \mathbf{a} be the all-1 vector of degree λ_0 . Then,

$$A_i A_j \mathbf{a} = \sum_{h=0}^k \mu(i, j, h) A_h \mathbf{a}.$$

Hence we have $x_i x_j = \sum_{h=0}^k \mu(i, j, h) x_h$. In particular,

$$x_{t-1}x_q = \sum_{h=0}^k \mu(t-1, q, h)x_h, \tag{5}$$

where $\mu(t-1, q, t-2)$ is a positive integer, because $|B_1 \cap B_2| = t-1, |B_2 \cap B_3| = q$ and $|B_1 \cap B_3| = t-2$. Hence, by (4) and (5), we have $v < k^3 \left(\begin{smallmatrix} k \\ 2 \end{smallmatrix} \right)^2 \lambda$.

Lemma 3. *For each $n \geq 1$, there is a positive integer $N_1(n)$ satisfying the following: If D is a $t-(v, k, \lambda)$ design with $k-t=n$ and $t \geq N_1(n)$, then there exist two blocks B_1 and B_2 of D such that $|B_1 \cap B_2| = t-1$.*

Proof. Let D be a $t-(v, k, \lambda)$ design with $k-t=n$. Let B be a block of D . Counting in two ways the number of the following set $\{(B', \{\alpha_1, \dots, \alpha_i\}) \mid B' \text{ a block } (\neq B), B' \cap B \ni \alpha_1, \dots, \alpha_i, \alpha_j \neq \alpha_{j'} \text{ if } j \neq j'\}$ gives $x_i(B) + \binom{t+1}{t} x_{t+1}(B) + \dots + \binom{k-1}{t} x_{k-1}(B) = (\lambda-1) \binom{k}{t}$.

Since $\frac{\binom{t+i}{t-1}}{\binom{t+i}{t}} = \frac{t}{i+1}$ ($i \geq 0$), we have

$$\binom{t}{t-1} x_t(B) + \binom{t+1}{t-1} x_{t+1}(B) + \dots + \binom{k-1}{t-1} x_{k-1}(B) \leq t(\lambda-1) \binom{k}{t}. \tag{6}$$

Counting in two ways the number of the following set

$$\{(B', \{\alpha_1, \dots, \alpha_{t-1}\}) \mid B' \text{ a block } (\neq B), B' \cap B \ni \alpha_1, \dots, \alpha_{t-1}, \alpha_j \neq \alpha_{j'} \text{ if } j \neq j'\}$$

gives $x_{t-1}(B) + \binom{t}{t-1} x_t(B) + \binom{t+1}{t-1} x_{t+1}(B) + \dots + \binom{k-1}{t-1} x_{k-1}(B) = (\lambda_{t-1}-1) \binom{k}{t-1}$. (7)

By (6) and (7), we have

$$x_{t-1}(B) \geq (\lambda_{t-1}-1) \binom{k}{t-1} - t(\lambda-1) \binom{k}{t}, \text{ and}$$

$$x_{t-1}(B) \geq \frac{v-t+1}{n+1} \lambda \frac{(n+t) \cdots t}{(n+1)!} - (\lambda-1) \frac{(n+t) \cdots t}{n!}.$$

Since D is a nontrivial design, $v > k+t \geq 2t+n$. Hence we have

$$x_{t-1}(B) > \left(\frac{(t+n+1) \cdots t}{(n+2)!} - \frac{(t+n) \cdots t}{n!} \right) \lambda.$$

Set $f(t) = \frac{(t+n+1) \cdots t}{(n+2)!} - \frac{(t+n) \cdots t}{n!}$. Then there is a positive integer $N_1(n)$ such that $f(t) \geq 0$ holds if $t \geq N_1(n)$. Hence, the proof of Lemma 3 is completed.

Lemma 4. *For each $n \geq 1$, there is a positive integer $N_2(n)$ satisfying the*

following: If D is a $t-(v, k, \lambda)$ design with $k-t=n$ and $t \geq N_2(n)$, then there exist three blocks B_1, B_2, B_3 of D such that $|B_1 \cap B_2|=t-1$, $|B_2 \cap B_3|=t-1$ and $|B_1 \cap B_3|=t-n-2$.

Proof. Let D be a $t-(v, k, \lambda)$ design with $k-t=n$. We may assume $t \geq N_1(n)$, where $N_1(n)$ is a positive integer obtained in Lemma 3. Therefore, there exist two blocks B_2 and B_3 of D with $|B_2 \cap B_3|=t-1$. Let $\alpha_1, \dots, \alpha_{n+1}$ be $n+1$ points of B_2-B_3 and $\alpha_{n+2}, \dots, \alpha_{t-1}$ be $t-n-2$ points of $B_2 \cap B_3$. Set $S = \{B \mid B \text{ a block, } B \supseteq \{\alpha_1, \dots, \alpha_{t-1}\}\}$, where $|S| = \frac{v-t+1}{k-t+1} \lambda$. Then we have

$$|\{B \in S \mid |B_2 \cap B| \geq t \text{ or } |B_3 \cap B| \geq t-n-1\}| \leq \lambda(k-t+1) + \lambda(k-t+n+2).$$

Hence, if $\frac{v-t+1}{k-t+1} \lambda > \lambda(n+1) + \lambda(2n+2)$, then there exists a block B_1 in S such that $|B_1 \cap B_2|=t-1$ and $|B_1 \cap B_3|=t-n-2$. On the other hand, since $v > k+t = 2t+n$, we have that $\frac{v-t+1}{n+1} > (n+1) + (2n+2)$ holds if $t \geq 3(n+1)^2$. Thus, Lemma 4 holds if $N_2(n) = \max\{N_1(n), 3(n+1)^2\}$.

Proof of Theorem 1. First, let us suppose that D is a block-schematic $t-(v, k, \lambda)$ design with $k-t=n$ and $t, \lambda \geq 2$. By Proposition, we may assume that $t \geq N_2(n)$, where $N_2(n)$ is a positive integer obtained in Lemma 4. By Lemma 1 we have

$$x_{t-n-2} > \lambda_{t-n-2} \binom{t+n}{t-n-2} - \sum_{j=t-n-1}^{t-1} \binom{j}{t-n-2} \lambda_j \binom{t+n}{j} - \sum_{j=t}^{k-1} \binom{j}{t-n-2} \lambda \binom{t+n}{j},$$

where $\lambda_{t-n-2} \binom{t+n}{t-n-2} = \frac{(v-t+n+2) \cdots (v-t+1)}{(n+n+2) \cdots (n+1)} \lambda \cdot \frac{(t+n) \cdots (t-n-1)}{(2n+2)!}$,

$$\begin{aligned} \sum_{j=t-n-1}^{t-1} \binom{j}{t-n-2} \lambda_j \binom{t+n}{j} &< (n+1) \lambda_{t-n-1} \frac{(t+n)!}{(t-n-2)!} \\ &= (n+1) \frac{(v-t+n+1) \cdots (v-t+1)}{(n+n+1) \cdots (n+1)} \frac{(t+n)!}{(t-n-2)!} \lambda, \end{aligned}$$

and $\sum_{j=t}^{k-1} \binom{j}{t-n-2} \lambda \binom{t+n}{j} < n \frac{(t+n)!}{(t-n-2)!} \lambda$.

Hence we have

$$x_{t-n-2} > \frac{(v-t)^{n+2} (t-n-1)^{2n+2}}{((2n+2)!)^2} \lambda - (v-t+n+1)^{n+1} (t+n)^{2n+2} \lambda. \tag{8}$$

Again by Lemma 1, we have

$$x_{t-1} < \frac{v-t+1}{n+1} \lambda \binom{t+n}{t-1} + \sum_{j=t}^{k-1} \binom{j}{t-1} \lambda \binom{t+n}{j}, \text{ and}$$

$$x_{t-1} < (v-t+1)(t+n)^{n+1}\lambda + n(t+n)^{n+1}\lambda.$$

Hence we have

$$x_{t-1}^2 < (v-t+n+1)^2(t+n)^{2n+2}\lambda^2. \tag{9}$$

By (8) and (9), we have

$$x_{t-n-2} - x_{t-1}^2 > \frac{(v-t)^{n+2}(t-n-1)^{2n+2}}{((2n+2)!)^2} \lambda - 2(v-t+n+1)^{n+1}(t+n)^{2n+2}\lambda^2.$$

Set $f(t) = \frac{\lambda}{((2n+2)!)^2} t^{n+2} \cdot (t-n-1)^{2n+2} - 2\lambda^2(t+n+1)^{n+1}(t+n)^{2n+2}.$

Then there is a positive integer $N(n, \lambda) (\geq N_2(n))$ such that $f(t) \geq 0$ holds if $t \geq N(n, \lambda)$. Since $v-t > t$, we have that

$$x_{t-n-2} - x_{t-1}^2 > 0 \text{ holds if } t \geq N(n, \lambda). \tag{10}$$

By the similar argument as in the proof of Proposition, we have

$$x_{t-1}^2 = \sum_{h=0}^k \mu(t-1, t-1, h)x_h, \tag{11}$$

where $\mu(t-1, t-1, h)$ is a non-negative integer. Moreover, since $t \geq N_2(n)$ $\mu(t-1, t-1, t-n-2)$ is a positive integer by Lemma 4. Hence, by (10) and (11), we have $t \leq N(n, \lambda)$. Therefore, $k \leq N(n, \lambda) + n$. Hence by Proposition, the proof of Theorem 1 is completed on condition that $\lambda \leq 2$.

Next, let us suppose that D is a block-schematic $t-(v, k, l)$ design with $k-t=n$ and $t \geq 3$. (The proof of the case $\lambda=1$ is similar to that of the case $\lambda \geq 2$. Then, we give an outline of it.) By Theorem in [1], we may assume that $t \geq N_2(n)$, where $N_2(n)$ is a positive integer obtained in Lemma 4. By Lemma 1, we get

$$x_{t-n-2} - x_{t-1}^2 > \frac{(v-t)^{n+2}(t-n-1)^{2n+2}}{((2n+2)!)^2} - 2(v-t+n+1)^{n+1}(t+n)^{2n+2}.$$

Hence, there is a positive integer $N(n) (\geq N_2(n))$ such that $x_{t-n-2} - x_{t-1}^2 > 0$ holds if $t \geq N(n)$. On the other hand, the following equation holds:

$$x_{t-1}^2 = \sum_{h=0}^k \mu(t-1, t-1, h)x_h,$$

where $\mu(t-1, t-1, h)$ is a non-negative integer and $\mu(t-1, t-1, t-n-2)$ is positive. Therefore, we have $t \leq N(n)$, and so $k \leq N(n) + n$. Hence by Theorem in [1], the proof of Theorem 1 is completed on condition that $\lambda=1$. Thus, Theorem 1 is proved.

3. Proof of Theorem 2

Lemma 5. *Let D be a block-regular $t-(v, k, \lambda)$ design. Then the following equality holds for $i=0, \dots, t-1$.*

$$x_i = \frac{\lambda \binom{k}{i}}{\binom{v-t}{k-t}} \left\{ \binom{v-k}{k-i} + (-1)^{t+i+1} \sum_{q=0}^{k-i-1} \binom{t-i-1+q}{q} \binom{v-k+q}{k-t} \right\} \\ + (\lambda-1) \sum_{j=1}^{t-1} \binom{j}{i} \binom{k}{j} (-1)^{i+j} + \sum_{j=i}^{k-1} \binom{j}{i} w_j (-1)^{i+j},$$

where $x_j \leq w_j \leq (\lambda-1) \binom{k}{j}$ ($t \leq j \leq k-1$) and $w_t = (\lambda-1) \binom{k}{t}$.

(The essential part of Lemma 5 is [5, Lemma 6].)

Proof. In this proof, we use the following three combinatorial identities:

- (i) $\binom{-a}{b} = (-1)^b \binom{a+b-1}{b}$,
- (ii) $\sum_r \binom{a}{r} \binom{b+r}{c} (-1)^r = (-1)^a \binom{b}{c-a}$ ($a \geq 0$),
- (iii) $\sum_r \binom{a}{r} \binom{b}{c-r} = \binom{a+b}{c}$ ($a \geq 0$).

By Lemma 1, we have

$$x_i = \sum_{j=i}^{t-1} \binom{j}{i} (\lambda_j - 1) \binom{k}{j} (-1)^{i+j} + \sum_{j=i}^{k-1} \binom{j}{i} w_j (-1)^{i+j},$$

where $x_j \leq w_j \leq (\lambda-1) \binom{k}{j}$ ($t \leq j \leq k-1$).

Then,
$$x_i = \lambda \sum_{j=i}^{t-1} \binom{j}{i} (\lambda'_j - 1) \binom{k}{j} (-1)^{i+j} + (\lambda-1) \sum_{j=i}^{t-1} \binom{j}{i} \binom{k}{j} (-1)^{i+j} \\ + \sum_{j=t}^{k-1} \binom{j}{i} w_j (-1)^{i+j},$$

where
$$\lambda'_j = \frac{\binom{v-j}{t-j}}{\binom{k-j}{t-j}} = \frac{\binom{v-j}{k-j}}{\binom{v-t}{k-t}} \quad (0 \leq j \leq t-1).$$

Hence, in order to prove Lemma 5, it is sufficient to show that the following equality holds for $i=0, \dots, k-1$.

$$\sum_{j=i}^{t-1} \binom{j}{i} (\lambda'_j - 1) \binom{k}{j} (-1)^{i+j}$$

$$= \frac{\binom{k}{i}}{\binom{v-t}{k-t}} \left\{ \binom{v-k}{k-i} + (-1)^{t+i+1} \sum_{q=0}^{k-t-1} \binom{t-i+1+q}{q} \binom{v-k+q}{k-t} \right\}. \tag{12}$$

First suppose that $t \leq i \leq k-1$. Then,

$$\sum_{q=0}^{k-t-1} \binom{t-i-1+q}{q} \binom{v-k+q}{k-t} = \sum_{q=0}^{k-t-1} (-1)^q \binom{i-t}{q} \binom{v-k+q}{k-t} \tag{cf. (i)}$$

$$= (-1)^{i-t} \binom{v-k}{k-1}. \tag{cf. (ii)}$$

Hence, the right hand of (12)=0=the left hand of (12).

Let $A=(a_{rs})$ be the square matrix with $a_{rs}=\binom{s}{r}$ ($0 \leq r, s \leq k-1$). Since $\det(A) \neq 0$, $A^{-1}=\left(\binom{s}{r}(-1)^{r+s}\right)$ ($0 \leq r, s \leq k-1$) and (12) holds for $i=t, \dots, k-1$, we have that (12) holds for $i=0, \dots, k-1$ if the following holds for $i=0, \dots, t-1$.

$$\sum_{j=1}^{k-1} \binom{j}{i} \frac{\binom{k}{j}}{\binom{v-t}{k-t}} \left\{ \binom{v-k}{k-j} + (-1)^{t+j+1} \sum_{q=0}^{k-t-1} \binom{t-j-1+q}{q} \binom{v-k+q}{k-t} \right\} = (\lambda_i - 1) \binom{k}{i}. \tag{13}$$

Since $\binom{j}{i} \binom{k}{j} = \binom{k}{i} \binom{k-i}{k-j}$,

$$\begin{aligned} \text{the left hand of (13)} &= \frac{\binom{k}{i}}{\binom{v-t}{k-t}} \sum_{j=i}^{k-1} \binom{k-i}{k-j} \left\{ \binom{v-k}{k-j} \right. \\ &\quad \left. + (-1)^{t+j+1} \sum_{q=0}^{k-t-1} \binom{t-j-1+q}{q} \binom{v-k+q}{k-t} \right\}. \end{aligned} \tag{14}$$

$$\begin{aligned} \text{Now, } \sum_{j=i}^{k-1} \binom{k-i}{k-j} \binom{v-k}{k-j} &= \sum_{j=i}^{k-1} \binom{k-i}{j-i} \binom{v-k}{k-j} \\ &= \sum_{h=0}^{k-i} \binom{k-i}{h} \binom{v-k}{k-i-h} - 1 \quad (h=j-i) \\ &= \binom{v-i}{k-i} - 1. \quad \text{(cf. (iii))} \end{aligned} \tag{15}$$

On the other hand,

$$\begin{aligned} &\sum_{j=i}^{k-1} \binom{k-i}{k-j} (-1)^{t+j+1} \sum_{q=0}^{k-t-1} \binom{t-j-1+q}{q} \binom{v-k+q}{k-t} \\ &= \sum_{q=0}^{k-t-1} (-1)^{t+1} \binom{v-k+q}{k-t} \sum_{j=i}^{k-1} \binom{k-i}{j-i} \binom{t-j-1+q}{q} (-1)^j \\ &= \sum_{q=0}^{k-t-1} (-1)^{t+1} \binom{v-k+q}{k-t} \sum_{j=i}^{k-1} \binom{k-i}{j-i} \binom{j-t}{q} (-1)^{j+q} \quad \text{(cf. (i))} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{q=0}^{k-t-1} (-1)^{t+1} \binom{v-k+q}{k-t} \left\{ \sum_{h=0}^{k-i} \binom{k-i}{h} \binom{i-t+h}{q} (-1)^{i+h+q} - \binom{k-t}{q} (-1)^{k+q} \right\} \\
 &\hspace{25em} (h = j-i) \\
 &= \sum_{q=0}^{k-t-1} (-1)^{t+1} \binom{v-k+q}{k-t} \left\{ (-1)^{(k-i)+(i+q)} \binom{i-t}{q-k+i} - \binom{k-t}{q} (-1)^{k+q} \right\} \quad (\text{cf. (ii)}) \\
 &= \sum_{q=0}^{k-t-1} (-1)^{t+k+q} \binom{v-k+q}{k-t} \binom{k-t}{q} \quad (q-k+i < 0) \\
 &= (-1)^{k+t} \sum_{q=0}^{k-t} \binom{k-t}{q} \binom{v-k+q}{k-t} (-1)^q - \binom{v-t}{k-t} \\
 &= (-1)^{k+t+k-t} \binom{v-k}{k-t-k+t} - \binom{v-t}{k-t} \quad (\text{cf. (ii)}) \\
 &= 1 - \binom{v-t}{k-t}. \hspace{15em} (16)
 \end{aligned}$$

Hence by (14), (15) and (16), we have that

$$\begin{aligned}
 \text{the left hand of (13)} &= \frac{\binom{k}{i}}{\binom{v-t}{k-t}} \left\{ \binom{v-i}{k-i} - 1 + 1 - \binom{v-t}{k-t} \right\} \\
 &= \left\{ \frac{\binom{v-i}{k-i}}{\binom{v-t}{k-t}} - 1 \right\} \binom{k}{i} = \text{the right hand of (13)}.
 \end{aligned}$$

Thus, Lemma 5 is proved.

Lemma 6. *For each $k \geq 2$ and $l \geq 0$, there exist at most finitely many block-regular $t-(v, k, \lambda)$ designs with $x_i \leq l$ for some i ($0 \leq i \leq t-1$).*

Proof. In order to prove Lemma 6, it is sufficient to show the following: For each $k \geq 2, l \geq 0, t$ ($1 \leq t < k$) and i ($0 \leq i < t$), there exist at most finitely many block-regular $t-(v, k, \lambda)$ designs with $x_i \leq l$.

Let k, l, t and i be integers with $k \geq 2, l \geq 0, 1 \leq t < k$ and $0 \leq i < t$, and let D be a block-regular $t-(v, k, \lambda)$ design with $x_i \leq l$. By Lemma 1, we have

$$x_i = \sum_{j=i}^{t-1} \binom{j}{i} (\lambda_j - 1) \binom{k}{j} (-1)^{i+j} + \sum_{j=t}^{k-1} \binom{j}{i} w_j (-1)^{i+j},$$

where $x_j \leq w_j \leq (\lambda - 1) \binom{k}{j}$ ($j = t, \dots, k-1$). Therefore,

$$\begin{aligned}
 x_i - l &> \frac{(v-i) \cdots (v-t+1)}{(k-i) \cdots (k-t+1)} (\lambda - 1) \binom{k}{i} - \sum_{j=i+1}^{t-1} \binom{j}{i} \frac{(v-j) \cdots (v-t+1)}{(k-j) \cdots (k-t+1)} (\lambda - 1) \binom{k}{j} \\
 &\quad - \sum_{j=i}^{k-1} \binom{j}{i} (\lambda - 1) \binom{k}{j} - l.
 \end{aligned}$$

In the above expression, if we suppose that k, l, t and i are constants, and that v and λ are variables with $v > k$ and $\lambda \geq 1$, then we can obtain the following:

The right hand of the expression $= \lambda \cdot f(v) + \lambda \cdot g(v) + d$, where $f(v)$ is a polynomial in v of degree $t-i$ with the leading coefficient of $f(v) > 0$, $g(v)$ is a polynomial in v of degree $t-i-1$, and d is a constant. Hence, there exists a constant $C(k, l, t, i) > 0$ such that $x_i - l > 0$ holds if $v \geq C(k, l, t, i)$. Namely, if $x_i \leq l$, then $v < C(k, l, t, i)$.

Proof of Theorem 2. By Lemma 6, we may assume that $t \geq \frac{2n + ((2n+2)!)^2}{c-2} + 2n$. Let D be a block-regular $t-(v, t+n, \lambda)$ design with $v \geq ct, t \geq \frac{2n + ((2n+2)!)^2}{c-2} + 2n$, and $x_i \leq l$ for some $i (0 \leq i \leq t-1)$. Set $v = mt (m \geq c)$, where m is not always integral. By Lemma 5, we have

$$x_i = \frac{\lambda \binom{t+n}{i}}{\binom{(m-1)t}{n}} \left\{ \binom{(m-1)t-n}{t+n-i} + (-1)^{t+i+1} \sum_{q=0}^{n-1} \binom{t-i-1+q}{q} \binom{(m-1)t-n+q}{n} \right\} + (\lambda-1) \sum_{j=i}^{t-1} \binom{j}{i} \binom{t+n}{j} (-1)^{i+j} + \sum_{j=i}^{t+n-1} \binom{j}{i} w_j (-1)^{i+j}, \tag{17}$$

where $x_j \leq w_j \leq (\lambda-1) \binom{t+n}{j} (t \leq j \leq k-1)$.

$$\begin{aligned} \text{Now, } & (\lambda-1) \sum_{j=i}^{t-1} \binom{j}{i} \binom{t+n}{j} (-1)^{i+j} + \sum_{j=i}^{t+n-1} \binom{j}{i} w_j (-1)^{i+j} \\ &= -(\lambda-1) \sum_{j=i}^{t+n} \binom{j}{i} \binom{t+n}{j} (-1)^{i+j} + \sum_{j=i}^{t+n-1} \binom{j}{i} w_j (-1)^{i+j} \\ &> -2\lambda(n+1) \frac{(t+n)!}{i!(t-i)!}. \end{aligned} \tag{18}$$

On the other hand,

$$\begin{aligned} & \frac{\lambda \binom{t+n}{i}}{\binom{(m-1)t}{n}} \left\{ \binom{(m-1)t-n}{t+n-i} + (-1)^{t+i+1} \sum_{q=0}^{n-1} \binom{t-i-1+q}{q} \binom{(m-1)t-n+q}{n} \right\} \\ &> \frac{\lambda \binom{t+n}{i}}{\binom{(m-1)t}{n}} \left\{ \binom{(m-1)t-n}{t+n-i} - n \binom{t+n-i}{n} \binom{(m-1)t}{n} \right\} \\ &> \frac{\lambda(t+n)!((m-1)t-n)!((m-1)t-n)!}{i!(t+n-i)!((m-1)t)!(t+n-i)!((m-2)t-2n+i)!} - \frac{\lambda(t+n)!}{i!(t-i)!} \end{aligned} \tag{19}$$

By (17), (18) and (19), we have

$$\frac{x_i i!(t-i)!}{(t+n)! \lambda} > \frac{\{((m-1)t-n)!\}^2 (t-i)!}{((t+n-i)!)^2 ((m-1)t)! ((m-2)t-2n+i)!} - 5n.$$

Then since $\frac{i!(t-i)!}{(t+n)! \lambda} \leq \frac{1}{(t+n)! \lambda} < 1$, we have

$$x_i > \frac{((m-1)t-n) \cdots ((m-2)t-2n+i+1)}{((m-1)t) \cdots ((m-1)t-n+1) \cdot (t+n-i) \cdots (t-i+1) (t+n-i)!} - 5n.$$

$$\begin{aligned} \text{Hence, } x_i &> ((m-1)t-n) \frac{((m-1)t-n-1) \cdots ((m-1)t-2n)}{((m-1)t) \cdots ((m-1)t-n+1)} \\ &\quad \cdot \frac{((m-1)t-2n-1) \cdots ((m-1)t-3n)}{(t+n-i) \cdots (t-i+1)} \\ &\quad \cdot \frac{((m-1)t-3n-1) \cdots ((m-2)t-2n+i+2)}{(t+n-i) \cdots (2n+3)} \cdot \frac{(m-2)t-2n+i+1}{(2n+2)!} - 5n \end{aligned}$$

holds if $t-i \geq n+3$, and

$$\begin{aligned} x_i &> ((m-1)t-n) \frac{((m-1)t-n-1) \cdots ((m-1)t-2n)}{((m-1)t) \cdots ((m-1)t-n+1)} \\ &\quad \cdot \frac{((m-1)t-2n-1) \cdots ((m-2)t-2n+i+1)}{(2n+2)!} - 5n \end{aligned}$$

holds if $2 \leq t-i \leq n+2$,

$$\text{and } x_i > ((m-1)t-n) \frac{((m-1)t-n-1) \cdots ((m-1)t-2n)}{((m-1)t) \cdots ((m-1)t-n+1)} \frac{1}{((n+1)!)^2} - 5n$$

holds if $t-i=1$.

In any case, since $t \geq \frac{2n + ((2n+2)!)^2}{c-2} + 2n$, we have

$$\begin{aligned} x_i &> ((m-1)t-n) \frac{((m-2)t)^n}{((m-1)t)^n} \cdot \frac{1}{((n+1)!)^2} - 5n \\ &> \frac{((c-1)t-n) \left(\frac{c-2}{c-1}\right)^n}{((n+1)!)^2} - 5n. \end{aligned}$$

Therefore, there exists a positive integer $N(c, n, l) \left(\geq \frac{2n + ((2n+2)!)^2}{c-2} + 2n \right)$ such that $x_i - l > 0$ holds if $t \geq N(c, n, l)$. Namely, if $x_i \leq l$, then $t \leq N(c, n, l)$. Hence by Lemma 6, the proof of Theorem 2 is completed.

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