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BLOCK INTERSECTION NUMBERS OF BLOCK DESIGNS

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1. Introduction

Let t, v, k and λ be positive integers with $v\geqslant k\geqslant t$. A $t-(v, k, \lambda)$ design is a pair consisting of a v-set Ω and a family $\boldsymbol{B}$ of k-subsets of Ω, such that each t-subset of Ω is contained in λ elements of $\boldsymbol{B}$. Elements of Ω and $\boldsymbol{B}$ are called points and blocks, respectively. A $t-(v, k, \lambda)$ design is called nontrivial provided $\boldsymbol{B}$ is a proper subfamily of the family of all k-subsets of Ω, then $t<k<v$. In this paper, we assume that all designs are nontrivial. For a $t-(v, k, \lambda)$ design $\boldsymbol{D}$ we use λ_i $(0\leqslant i\leqslant t)$ to represent the number of blocks which contain a given set of i points of $\boldsymbol{D}$. Then we have

$$\lambda_i = \frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}}\lambda = \frac{(v-i)(v-i-l)\cdots(v-t+l)}{(k-i)(k-i-l)\cdots(k-t+l)}\lambda \qquad (0\leqslant i\leqslant t)\,.$$

A $t-(v, k, \lambda)$ design $\boldsymbol{D}$ is called block-schematic if the blocks of $\boldsymbol{D}$ form an association scheme with the relations determined by size of intersection (cf. [3]). In §2, we prove the following theorem which extends the result in [1].

Theorem 1. (a) *For each $n\geqslant 1$ and $\lambda\geqslant 1$, there exist at most finitely many block-schematic $t-(v, k, \lambda)$ designs with $k-t=n$ and $t\geqslant 3$.*

(b) *For each $n\geqslant 1$ and $\lambda\geqslant 2$, there exist at most finitely many block-schematic $t-(v, k, \lambda)$ designs with $k-t=n$ and $t\geqslant 2$.*

REMARK. Since there exist infinitely many $2-(v, 3, 1)$ designs and since every $2-(v, k, 1)$ design is block-schematic (cf. [2]), Theorem 1 does not hold for $\lambda=1$ and $t=2$.

For a block B of a $t-(v, k, \lambda)$ design $\boldsymbol{D}$ we use $x_i(B)$ $(0\leqslant i\leqslant k)$ to denote the number of blocks each of which has exactly i points in common with B. If, for each i $(i=0, \cdots, k)$, $x_i(B)$ is the same for every block B, we say that $\boldsymbol{D}$ is block-regular and we write x_i instead of $x_i(B)$. We remark that if a $t-(v, k, \lambda)$ design $\boldsymbol{D}$ is block-schematic then $\boldsymbol{D}$ is block-regular. For any $t-(v, k, 1)$ design or any $t-(v, t+1, \lambda)$ design, either of which is block-regular (cf. Lemma 1), every x_i depends only on i, t, v, k or i, t, v, λ respectively (cf. Lemma 1). And Gross [5] and Dehon [4] respectively classified the $t-(v, k, 1)$ designs and the $t-(v, t+1, \lambda)$ designs both of which satisfy $x_i=0$. But for a block-regular $t-(v, k, \lambda)$ design, x_i depends not only on i, t, v, k, λ but also on others in general (cf. Lemma 1). In §3, we prove the following theorem.

Theorem 2. *Let c be a real number with $c>2$. Then for each $n\geqslant 1$ and $l\geqslant 0$, there exist at most finitely many block-regular $t-(v, k, \lambda)$ designs with $k-t=n$, $v\geqslant ct$ and $x_i\leqslant l$ for some i $(0\leqslant i\leqslant t-1)$.*

The author thanks Professor H. Enomoto for giving the direct proof of Lemma 5.

2. Proof of Theorem 1

Lemma 1. *Let $\boldsymbol{D}$ be a block-regular $t-(v, k, \lambda)$ design. Then the following equality holds for $i=0, \cdots, k-1$.*

$$x_i=\sum_{j=i}^{t-1}\binom{j}{i}(\lambda_j-1)\binom{k}{j}(-1)^{i+j}+\sum_{j=t}^{k-1}\binom{j}{i}w_j(-1)^{i+j},$$

where $x_j\leqslant w_j\leqslant(\lambda-1)\binom{k}{j}$ $(t\leqslant j\leqslant k-1)$ and $w_t=(\lambda-1)\binom{k}{t}$.

Proof. Let B be a block of $\boldsymbol{D}$. Counting in two ways the number of the following set

$\{(B', \{\alpha_1,\cdots,\alpha_i\})\,|\,B'$ a block $(\neq B), B'\cap B\ni\alpha_1,\cdots,\alpha_i, \alpha_j\neq\alpha_{j'}$ if $j\neq j'\}$ gives $x_i+\binom{i+1}{i}x_{i+1}+\cdots+\binom{t}{i}x_t+\cdots+\binom{k-1}{i}x_{k-1}=(\lambda_i-1)\binom{k}{i}$ for $i=0, \cdots, t-1$, and $x_i+\binom{i+1}{i}x_{i+1}+\cdots+\binom{k-1}{i}x_{k-1}\leqslant(\lambda-1)\binom{k}{i}$ for $i=t, \cdots, k-1$. Let $w_i(t\leqslant i\leqslant k-1)$ be the left hand of the above inequality, where $w_t=(\lambda-1)\binom{k}{t}$. Let $A=(a_{ij})$ be the square matrix with $a_{ij}=\binom{j}{i}$ $(0\leqslant i,j\leqslant k-1)$. Then we have

$$A\begin{pmatrix} x_0\\ \vdots\\ x_{t-1}\\ x_t\\ \vdots\\ x_{k-1}\end{pmatrix}=\begin{pmatrix}(\lambda_0-1)\binom{k}{0}\\ \vdots\\ (\lambda_{t-1}-1)\binom{k}{t-1}\\ w_t\\ \vdots\\ w_{k-1}\end{pmatrix}.$$

Let us set $A^{-1}=(b_{ij})$ $(0\leqslant i,j\leqslant k-1)$. Since $\sum_{j=m}^{n}(-1)^{j+m}\binom{n}{j}\binom{j}{m}=\delta_{mn}$, we have

$b_{ij}=\binom{j}{i}(-1)^{i+j}$. Hence we get the desired result.

Lemma 2. *Let $\boldsymbol{D}$ be a $t-(v, k, \lambda)$ design with $t, \lambda \geqslant 2$. If $v \geqslant k^3$, then there exist three blocks B_1, B_2, B_3 of $\boldsymbol{D}$ such that $|B_1 \cap B_2|=t-1$, $|B_2 \cap B_3| \geqslant t$ and $|B_1 \cap B_3|=t-2$.*

Proof. Let B be a block of $\boldsymbol{D}$. Counting in two ways the number of the following set $\{(B', \alpha_1, \cdots, \alpha_t)\} \mid B'$ a block $(\neq B)$, $B' \cap B \ni \alpha_1, \cdots, \alpha_t$, $\alpha_j \neq \alpha_{j'}$ if $j \neq j'\}$ gives $x_t(B)+\binom{t+1}{t}x_{t+1}(B)+\cdots+\binom{k-1}{t}x_{k-1}(B)=(\lambda-1)\binom{k}{t}$. Since $\lambda \geqslant -2$, there is an integer q $(t \leqslant q \leqslant k-1)$ with $x_q(B) \neq 0$. Hence, we may assume that there exist two blocks B_2, B_3 such that $t \leqslant |B_2 \cap B_3|=q$. Let α_1 be a point of B_2-B_3 and $\alpha_2, \cdots, \alpha_{t-1}$ be $t-2$ points of $B_2 \cap B_3$. Set $\boldsymbol{S}=\{B \mid B$ a block, $B \supseteq \{\alpha_1, \cdots, \alpha_{t-1}\}\}$, where $|\boldsymbol{S}|=\dfrac{v-t+1}{k-t+1}\lambda$. Then we have

$$|\{B \in \boldsymbol{S} \mid\ |B \cap B_2| \geqslant t \text{ or } |B \cap B_3| \geqslant t-1\}| \leqslant \lambda(k-t+1)+\lambda(k-t+2)\,.$$

Hence, if $\dfrac{v-t+1}{k-t+1}\lambda>\lambda(k-t+1)+\lambda(k-t+2)$, then there exists a block B_1 in $\boldsymbol{S}$ such that $|B_1 \cap B_2|=t-1$ and $|B_1 \cap B_3|=t-2$. On the other hand, $\dfrac{v-t+1}{k-t+1}>(k-t+1)+(k-t+2)$ holds if $v \geqslant k^3$. So, the proof of Lemma 2 is completed.

Proposition. *Let $\boldsymbol{D}$ be a block-schematic $t-(v, k, \lambda)$ design with $t, \lambda \geqslant 2$. Then $v<\lambda k^3\binom{k}{\left[\frac{k}{2}\right]}^2$ holds.*

Proof. By Lemma 1, we have

$$x_{t-2}>(\lambda_{t-2}-1)\binom{k}{t-2}-(t-1)(\lambda_{t-1}-1)\binom{k}{t-1}-(k-t)(\lambda-1)\binom{k}{\left[\frac{k}{2}\right]}^2.$$

So, $x_{t-2}>\dfrac{(v-t+2)(v-t+1)}{(k-t+2)(k-t+1)}\lambda\dbinom{k}{t-2}-(t-1)\dfrac{v-t+1}{k-t+1}\lambda\dbinom{k}{t-1}-(k-t)\lambda\dbinom{k}{\left[\frac{k}{2}\right]}^2$,

and

$$x_{t-2}>\frac{(v-k)^2}{k^2}\lambda-(t-1)v\lambda\binom{k}{\left[\frac{k}{2}\right]}-k\lambda\binom{k}{\left[\frac{k}{2}\right]}^2.$$

Hence we have

$$x_{t-2}>\frac{v^2}{k^2}\lambda-kv\lambda\binom{k}{\left[\frac{k}{2}\right]}-k\lambda\binom{k}{\left[\frac{k}{2}\right]}^2. \tag{1}$$

Again by Lemma 1, we have

$$x_{t-1}<\lambda_{t-1}\binom{k}{t-1}+(k-t)(\lambda-1)\binom{k}{\left[\frac{k}{2}\right]}^2.$$

So,

$$x_{t-1}<\frac{v}{2}\lambda\binom{k}{\left[\frac{k}{2}\right]}+(k-1)\lambda\binom{k}{\left[\frac{k}{2}\right]}^2. \tag{2}$$

From now on, we may assume that $v\geqslant k^3$. By Lemma 2, there exist three blocks B_1, B_2, B_3 of $\boldsymbol{D}$ such that $|B_1\cap B_2|=t-1$, $|B_2\cap B_3|=q$ $(t\leqslant q\leqslant k-1)$, and $|B_1\cap B_3|=t-2$. By Lemma 1, we have

$$x_q\leqslant(\lambda-1)\binom{k}{q}<\lambda\binom{k}{\left[\frac{k}{2}\right]}. \tag{3}$$

Hence, by (1), (2) and (3), we have

$$x_{t-2}-x_{t-1}x_q>\frac{v^2}{k^2}\lambda-kv\lambda\binom{k}{\left[\frac{k}{2}\right]}-k\lambda\binom{k}{\left[\frac{k}{2}\right]}^2-\lambda^2\binom{k}{\left[\frac{k}{2}\right]}^2\left\{\frac{v}{2}+(k-1)\binom{k}{\left[\frac{k}{2}\right]}\right\}.$$

Thus, we have that

$$x_{t-2}-x_{t-1}x_q>\frac{v^2}{k^2}\lambda-\lambda^2\binom{k}{\left[\frac{k}{2}\right]}^2 v-k\lambda^2\binom{k}{\left[\frac{k}{2}\right]}^3.$$

Hence, $x_{t-2}-x_{t-1}x_q>0$ holds if $v\geqslant k^3\binom{k}{\left[\frac{k}{2}\right]}^2\lambda$. (4)

Let $B_1, B_2, B_3, \cdots, B_{\lambda_0}$ be the blocks of $\boldsymbol{D}$. Let A_h $(0\leqslant h\leqslant k)$ be the h-adjacency matrix of $\boldsymbol{D}$ of degree λ_0 defined by

$$A_h(i,j)=\begin{cases}1 & \text{if } |B_i\cap B_j|=h,\\ 0 & \text{otherwise.}\end{cases}$$

Since $\boldsymbol{D}$ is block-schematic, we have

$$A_iA_j=\sum_{h=0}^{k}\mu(i,j,h)A_h \quad (0\leqslant i,j\leqslant k),$$

where $\mu(i, j, h)$ is a non-negative integer. Let $\boldsymbol{a}$ be the all-1 vector of degree λ_0. Then,

$$A_iA_j\boldsymbol{a}=\sum_{h=0}^{k}\mu(i,j,h)A_h\boldsymbol{a}.$$

Hence we have $x_ix_j=\sum_{h=0}^{k}\mu(i,j,h)x_h$. In particular,

$$x_{t-1}x_q=\sum_{h=0}^{k}\mu(t-1,q,h)x_h\,, \tag{5}$$

where $\mu(t-1, q, t-2)$ is a positive integer, because $|B_1\cap B_2|=t-1$, $|B_2\cap B_3|=q$ and $|B_1\cap B_3|=t-2$. Hence, by (4) and (5), we have $v<k^3\binom{k}{\left[\frac{k}{2}\right]}^2\lambda$.

Lemma 3. *For each $n\geqslant 1$, there is a positive integer $N_1(n)$ satisfying the following: If $\boldsymbol{D}$ is a $t-(v,k,\lambda)$ design with $k-t=n$ and $t\geqslant N_1(n)$, then there exist two blocks B_1 and B_2 of $\boldsymbol{D}$ such that $|B_1\cap B_2|=t-1$.*

Proof. Let $\boldsymbol{D}$ be a $t-(v,k,\lambda)$ design with $k-t=n$. Let B be a block of $\boldsymbol{D}$. Counting in two ways the number of the following set
$\{(B', \{\alpha_1,\cdots,\alpha_t\})\,|\,B'$ a block $(\neq B)$, $B'\cap B\ni\alpha_1,\cdots,\alpha_t$, $\alpha_j\neq\alpha_{j'}$ if $j\neq j'\}$
gives $x_t(B)+\binom{t+1}{t}x_{t+1}(B)+\cdots+\binom{k-1}{t}x_{k-1}(B)=(\lambda-1)\binom{k}{t}$.

Since $\dfrac{\binom{t+i}{t-1}}{\binom{t+i}{t}}=\dfrac{t}{i+1}$ $(i\geqslant 0)$, we have

$$\binom{t}{t-1}x_t(B)+\binom{t+1}{t-1}x_{t+1}(B)+\cdots+\binom{k-1}{t-1}x_{k-1}(B)\leqslant t(\lambda-1)\binom{k}{t}. \tag{6}$$

Counting in two ways the number of the following set
$\{(B', \{\alpha_1,\cdots,\alpha_{t-1}\})\,|\,B'$ a block $(\neq B)$, $B'\cap B\ni\alpha_1,\cdots,\alpha_{t-1}$, $\alpha_j\neq\alpha_{j'}$ if $j\neq j'\}$
gives

$$x_{t-1}(B)+\binom{t}{t-1}x_t(B)+\binom{t+1}{t-1}x_{t+1}(B)+\cdots+\binom{k-1}{t-1}x_{k-1}(B)=(\lambda_{t-1}-1)\binom{k}{t-1}. \tag{7}$$

By (6) and (7), we have

$$x_{t-1}(B)\geqslant(\lambda_{t-1}-1)\binom{k}{t-1}-t(\lambda-1)\binom{k}{t},\quad\text{and}$$

$$x_{t-1}(B)\geqslant\frac{v-t+1}{n+1}\lambda\frac{(n+t)\cdots t}{(n+1)!}-\frac{(n+t)\cdots t}{(n+1)!}-(\lambda-1)\frac{(n+t)\cdots t}{n!}.$$

Since $\boldsymbol{D}$ is a nontrivial design, $v>k+t\geqslant 2t+n$. Hence we have

$$x_{t-1}(B)>\left(\frac{(t+n+1)\cdots t}{(n+2)!}-\frac{(t+n)\cdots t}{n!}\right)\lambda.$$

Set $f(t)=\dfrac{(t+n+1)\cdots t}{(n+2)!}-\dfrac{(t+n)\cdots t}{n!}$. Then there is a positive integer $N_1(n)$ such that $f(t)\geqslant 0$ holds if $t\geqslant N_1(n)$. Hence, the proof of Lemma 3 is completed.

Lemma 4. *For each $n\geqslant 1$, there is a positive integer $N_2(n)$ satisfying the* *following: If $\boldsymbol{D}$ is a $t-(v, k, \lambda)$ design with $k-t=n$ and $t \geqslant N_2(n)$, then there exist three blocks B_1, B_2, B_3 of $\boldsymbol{D}$ such that $|B_1 \cap B_2|=t-1$, $|B_2 \cap B_3|=t-1$ and $|B_1 \cap B_3|=t-n-2$.*

Proof. Let $\boldsymbol{D}$ be a $t-(v, k, \lambda)$ design with $k-t=n$. We may assume $t \geqslant N_1(n)$, where $N_1(n)$ is a positive integer obtained in Lemma 3. Therefore, there exist two blocks B_2 and B_3 of $\boldsymbol{D}$ with $|B_2 \cap B_3|=t-1$. Let $\alpha_1, \cdots, \alpha_{n+1}$ be $n+1$ points of B_2-B_3 and $\alpha_{n+2}, \cdots, \alpha_{t-1}$ be $t-n-2$ points of $B_2 \cap B_3$. Set $\boldsymbol{S}=\{B \mid B \text{ a block, } B \supseteq \{\alpha_1, \cdots, \alpha_{t-1}\}\}$, where $|\boldsymbol{S}|=\dfrac{v-t+1}{k-t+1}\lambda$. Then we have

$$|\{B \in \boldsymbol{S} \mid |B_2 \cap B| \geqslant t \text{ or } |B_3 \cap B| \geqslant t-n-1\}| \leqslant \lambda(k-t+1)+\lambda(k-t+n+2)\,.$$

Hence, if $\dfrac{v-t+1}{k-t+1}\lambda>\lambda(n+1)+\lambda(2n+2)$, then there exists a block B_1 in $\boldsymbol{S}$ such that $|B_1 \cap B_2|=t-1$ and $|B_1 \cap B_3|=t-n-2$. On the other hand, since $v>k+t=2t+n$, we have that $\dfrac{v-t+1}{n+1}>(n+1)+(2n+2)$ holds if $t \geqslant 3(n+1)^2$. Thus, Lemma 4 holds if $N_2(n)=\max\{N_1(n), 3(n+1)^2\}$.

Proof of Theorem 1. First, let us suppose that $\boldsymbol{D}$ is a block-schematic $t-(v, k, \lambda)$ design with $k-t=n$ and $t, \lambda \geqslant 2$. By Proposition, we may assume that $t \geqslant N_2(n)$, where $N_2(n)$ is a positive integer obtained in Lemma 4. By Lemma 1 we have

$$x_{t-n-2}>\lambda_{t-n-2}\binom{t+n}{t-n-2}-\sum_{j=t-n-1}^{t-1}\binom{j}{t-n-2}\lambda_j\binom{t+n}{j}-\sum_{j=t}^{k-1}\binom{j}{t-n-2}\lambda\binom{t+n}{j},$$

where $\lambda_{t-n-2}\dbinom{t+n}{t-n-2}=\dfrac{(v-t+n+2)\cdots(v-t+1)}{(n+n+2)\cdots(n+1)}\lambda\cdot\dfrac{(t+n)\cdots(t-n-1)}{(2n+2)!}$,

$$\begin{aligned}\sum_{j=t-n-1}^{t-1}\binom{j}{t-n-2}\lambda_j\binom{t+n}{j}&<(n+1)\lambda_{t-n-1}\frac{(t+n)!}{(t-n-2)!}\\&=(n+1)\frac{(v-t+n+1)\cdots(v-t+1)}{(n+n+1)\cdots(n+1)}\frac{(t+n)!}{(t-n-2)!}\lambda\,,\end{aligned}$$

and
$$\sum_{j=t}^{k-1}\binom{j}{t-n-2}\lambda\binom{t+n}{j}<n\frac{(t+n)!}{(t-n-2)!}\lambda\,.$$

Hence we have

$$x_{t-n-2}>\frac{(v-t)^{n+2}(t-n-1)^{2n+2}}{((2n+2)!)^2}\lambda-(v-t+n+1)^{n+1}(t+n)^{2n+2}\lambda\,. \tag{8}$$

Again by Lemma 1, we have

$$x_{t-1}<\frac{v-t+1}{n+1}\lambda\binom{t+n}{t-1}+\sum_{j=t}^{k-1}\binom{j}{t-1}\lambda\binom{t+n}{j},\text{ and}$$

$$x_{t-1}<(v-t+1)\,(t+n)^{n+1}\lambda+n(t+n)^{n+1}\lambda\,.$$

Hence we have

$$x_{t-1}^2<(v-t+n+1)^2(t+n)^{2n+2}\lambda^2\,. \tag{9}$$

By (8) and (9), we have

$$x_{t-n-2}-x_{t-1}^2>\frac{(v-t)^{n+2}(t-n-1)^{2n+2}}{((2n+2)!)^2}\lambda-2(v-t+n+1)^{n+1}(t+n)^{2n+2}\lambda^2\,.$$

Set $\quad f(t)=\dfrac{\lambda}{((2n+2)!)^2}t^{n+2}\cdot(t-n-1)^{2n+2}-2\lambda^2(t+n+1)^{n+1}(t+n)^{2n+2}\,.$

Then there is a positive integer $N(n, \lambda)$ $(\geqslant N_2(n))$ such that $f(t)\geqslant 0$ holds if $t\geqslant N(n, \lambda)$. Since $v-t>t$, we have that

$$x_{t-n-2}-x_{t-1}^2>0 \text{ holds if } t\geqslant N(n,\lambda)\,. \tag{10}$$

By the similar argument as in the proof of Proposition, we have

$$x_{t-1}^2=\sum_{h=0}^{k}\mu(t-1, t-1, h)x_h\,, \tag{11}$$

where $\mu(t-1,\ t-1,\ h)$ is a non-negative integer. Moreover, since $t\geqslant N_2(n)$ $\mu(t-1,\ t-1,\ t-n-2)$ is a positive integer by Lemma 4. Hence, by (10) and (11), we have $t\leqslant N(n, \lambda)$. Therefore, $k\leqslant N(n, \lambda)+n$. Hence by Proposition, the proof of Theorem 1 is completed on condition that $\lambda\leqslant 2$.

Next, let us suppose that $\boldsymbol{D}$ is a block-schematic $t-(v, k, l)$ design with $k-t=n$ and $t\geqslant 3$. (The proof of the case $\lambda=1$ is similar to that of the case $\lambda\geqslant 2$. Then, we give an outline of it.) By Theorem in [1], we may assume that $t\geqslant N_2(n)$, where $N_2(n)$ is a positive integer obtained in Lemma 4. By Lemma 1, we get

$$x_{t-n-2}-x_{t-1}^2>\frac{(v-t)^{n+2}(t-n-1)^{2n+2}}{((2n+2)!)^2}-2(v-t+n+1)^{n+1}(t+n)^{2n+2}\,.$$

Hence, there is a positive integer $N(n)$ $(\geqslant N_2(n))$ such that $x_{t-n-2}-x_{t-1}^2>0$ holds if $t\geqslant N(n)$. On the other hand, the following equation holds:

$$x_{t-1}^2=\sum_{h=0}^{k}\mu(t-1, t-1, h)x_h\,,$$

where $\mu(t-1,\ t-1,\ h)$ is a non-negative integer and $\mu(t-1,\ t-1,\ t-n-2)$ is positive. Therefore, we have $t\leqslant N(n)$, and so $k\leqslant N(n)+n$. Hence by Theorem in [1], the proof of Theorem 1 is completed on condition that $\lambda=1$. Thus, Theorem 1 is proved.

3. Proof of Theorem 2

Lemma 5. *Let D be a block-regular $t-(v, k, \lambda)$ design. Then the following equality holds for $i=0, \cdots, t-1$.*

$$x_i=\frac{\lambda\binom{k}{i}}{\binom{v-t}{k-t}}\left\{\binom{v-k}{k-i}+(-1)^{t+i+1}\sum_{q=0}^{k-t-1}\binom{t-i-1+q}{q}\binom{v-k+q}{k-t}\right\}$$
$$+(\lambda-1)\sum_{j=1}^{t-1}\binom{j}{i}\binom{k}{j}(-1)^{i+j}+\sum_{j=t}^{k-1}\binom{j}{i}w_j(-1)^{i+j},$$

where $x_j \leqslant w_j \leqslant (\lambda-1)\binom{k}{j}$ $(t \leqslant j \leqslant k-1)$ *and* $w_t=(\lambda-1)\binom{k}{t}$.

(The essential part of Lemma 5 is [5, Lemma 6].)

Proof. In this proof, we use the following three combinatorial identities:

(i) $\binom{-a}{b}=(-1)^b\binom{a+b-1}{b}$,

(ii) $\sum_r\binom{a}{r}\binom{b+r}{c}(-1)^r=(-1)^a\binom{b}{c-a}$ $(a \geqslant 0)$,

(iii) $\sum_r\binom{a}{r}\binom{b}{c-r}=\binom{a+b}{c}$ $(a \geqslant 0)$.

By Lemma 1, we have

$$x_i=\sum_{j=i}^{t-1}\binom{j}{i}(\lambda_j-1)\binom{k}{j}(-1)^{i+j}+\sum_{j=t}^{k-1}\binom{j}{i}w_j(-1)^{i+j},$$

where $x_j \leqslant w_j \leqslant (\lambda-1)\binom{k}{j}$ $(t \leqslant j \leqslant k-1)$.

Then, $x_i=\lambda\sum_{j=i}^{t-1}\binom{j}{i}(\lambda'_j-1)\binom{k}{j}(-1)^{i+j}+(\lambda-1)\sum_{j=i}^{t-1}\binom{j}{i}\binom{k}{j}(-1)^{i+j}$
$$+\sum_{j=t}^{k-1}\binom{j}{i}w_j(-1)^{i+j},$$

where $\lambda'_j=\dfrac{\binom{v-j}{t-j}}{\binom{k-j}{t-j}}=\dfrac{\binom{v-j}{k-j}}{\binom{v-t}{k-t}}$ $(0 \leqslant j \leqslant t-1)$.

Hence, in order to prove Lemma 5, it is sufficient to show that the following equality holds for $i=0, \cdots, k-1$.

$$\sum_{j=i}^{t-1}\binom{j}{i}(\lambda'_j-1)\binom{k}{j}(-1)^{i+j}$$

$$=\frac{\binom{k}{i}}{\binom{v-t}{k-t}}\left\{\binom{v-k}{k-i}+(-1)^{t+i+1}\sum_{q=0}^{k-t-1}\binom{t-i+1+q}{q}\binom{v-k+q}{k-t}\right\}. \tag{12}$$

First suppose that $t\leqslant i\leqslant k-1$. Then,

$$\sum_{q=0}^{k-t-1}\binom{t-i-1+q}{q}\binom{v-k+q}{k-t}=\sum_{q=0}^{k-t-1}(-1)^q\binom{i-t}{q}\binom{v-k+q}{k-t} \quad \text{(cf. (i))}$$

$$=(-1)^{i-t}\binom{v-k}{k-1}. \quad \text{(cf. (ii))}$$

Hence, the right hand of (12)=0=the left hand of (12).

Let $A=(a_{rs})$ be the square matrix with $a_{rs}=\binom{s}{r}$ $(0\leqslant r,\ s\leqslant k-1)$. Since $\det(A)\neq 0$, $A^{-1}=\left(\binom{s}{r}(-1)^{r+s}\right)(0\leqslant r,\ s\leqslant k-1)$ and (12) holds for $i=t,\cdots,k-1$, we have that (12) holds for $i=0,\cdots,k-1$ if the following holds for $i=0,\cdots,t-1$.

$$\sum_{j=1}^{k-1}\binom{j}{i}\frac{\binom{k}{j}}{\binom{v-t}{k-t}}\left\{\binom{v-k}{k-j}+(-1)^{t+j+1}\sum_{q=0}^{k-t-1}\binom{t-j-1+q}{q}\binom{v-k+q}{k-t}\right\}=(\lambda_i'-1)\binom{k}{i}. \tag{13}$$

Since $\binom{j}{i}\binom{k}{j}=\binom{k}{i}\binom{k-i}{k-j}$,

$$\text{the left hand of (13)}=\frac{\binom{k}{i}}{\binom{v-t}{k-t}}\sum_{j=i}^{k-1}\binom{k-i}{k-j}\left\{\binom{v-k}{k-j}\right.$$

$$\left.+(-1)^{t+j+1}\sum_{q=0}^{k-t-1}\binom{t-j-1-q}{q}\binom{v-k+q}{k-t}\right\}. \tag{14}$$

Now, $$\sum_{j=i}^{k-1}\binom{k-i}{k-j}\binom{v-k}{k-j}=\sum_{j=i}^{k-1}\binom{k-i}{j-i}\binom{v-k}{k-j}$$

$$=\sum_{h=0}^{k-i}\binom{k-i}{h}\binom{v-k}{k-i-h}-1 \quad (h=j-i)$$

$$=\binom{v-i}{k-i}-1. \quad \text{(cf. (iii))} \tag{15}$$

On the other hand,

$$\sum_{j=i}^{k-1}\binom{k-i}{k-j}(-1)^{t+j+1}\sum_{q=0}^{k-t-1}\binom{t-j-1+q}{q}\binom{v-k+q}{k-t}$$

$$=\sum_{q=0}^{k-t-1}(-1)^{t+1}\binom{v-k+q}{k-t}\sum_{j=i}^{k-1}\binom{k-i}{j-i}\binom{t-j-1+q}{q}(-1)^j$$

$$=\sum_{q=0}^{k-t-1}(-1)^{t+1}\binom{v-k+q}{k-t}\sum_{j=i}^{k-1}\binom{k-i}{j-i}\binom{j-t}{q}(-1)^{j+q} \quad \text{(cf. (i))}$$

$$= \sum_{q=0}^{k-t-1}(-1)^{t+1}\binom{v-k+q}{k-t}\left\{\sum_{h=0}^{k-i}\binom{k-i}{h}\binom{i-t+h}{q}(-1)^{i+h+q}-\binom{k-t}{q}(-1)^{k+q}\right\}$$
$$(h=j-i)$$
$$= \sum_{q=0}^{k-t-1}(-1)^{t+1}\binom{v-k+q}{k-t}\left\{(-1)^{(k-i)+(i+q)}\binom{i-t}{q-k+i}-\binom{k-t}{q}(-1)^{k+q}\right\} \quad \text{(cf. (ii))}$$
$$= \sum_{q=0}^{k-t-1}(-1)^{t+k+q}\binom{v-k+q}{k-t}\binom{k-t}{q} \qquad (q-k+i<0)$$
$$= (-1)^{k+t}\sum_{q=0}^{k-t}\binom{k-t}{q}\binom{v-k+q}{k-t}(-1)^{q}-\binom{v-t}{k-t}$$
$$= (-1)^{k+t+k-t}\binom{v-k}{k-t-k+t}-\binom{v-t}{k-t} \qquad \text{(cf. (ii))}$$
$$= 1-\binom{v-t}{k-t}. \tag{16}$$

Hence by (14), (15) and (16), we have that

$$\text{the left hand of (13)} = \frac{\binom{k}{i}}{\binom{v-t}{k-t}}\left\{\binom{v-i}{k-i}-1+1-\binom{v-t}{k-t}\right\}$$
$$= \left\{\frac{\binom{v-i}{k-i}}{\binom{v-t}{k-t}}-1\right\}\binom{k}{i} = \text{the right hand of (13)}.$$

Thus, Lemma 5 is proved.

Lemma 6. *For each $k\geqslant 2$ and $l\geqslant 0$, there exist at most finitely many block-regular $t-(v,k,\lambda)$ designs with $x_i\leqslant l$ for some i $(0\leqslant i\leqslant t-1)$.*

Proof. In order to prove Lemma 6, it is sufficient to show the following: For each $k\geqslant 2$, $l\geqslant 0$, t $(1\leqslant t<k)$ and i $(0\leqslant i<t)$, there exist at most finitely many block-regular $t-(v,k,\lambda)$ designs with $x_i\leqslant l$.

Let k, l, t and i be integers with $k\geqslant 2$, $l\geqslant 0$, $1\leqslant t<k$ and $0\leqslant i<t$, and let $\boldsymbol{D}$ be a block-regular $t-(v,k,\lambda)$ design with $x_i\leqslant l$. By Lemma 1, we have

$$x_i=\sum_{j=i}^{t-1}\binom{j}{i}(\lambda_j-1)\binom{k}{j}(-1)^{i+j}+\sum_{j=t}^{k-1}\binom{j}{i}w_j(-1)^{i+j},$$

where $x_j\leqslant w_j\leqslant(\lambda-1)\binom{k}{j}$ $(j=t,\cdots,k-1)$. Therefore,

$$x_i-l>\left(\frac{(v-i)\cdots(v-t+1)}{(k-i)\cdots(k-t+1)}\lambda-1\right)\binom{k}{i}-\sum_{j=i+1}^{t-1}\binom{j}{i}\left(\frac{(v-j)\cdots(v-t+1)}{(k-j)\cdots(k-t+1)}\lambda-1\right)\binom{k}{j}$$
$$-\sum_{j=t}^{k-1}\binom{j}{i}(\lambda-1)\binom{k}{j}-l.$$

In the above expression, if we suppose that k, l, t and i are constants, and that v and λ are variables with $v>k$ and $\lambda\geqslant 1$, then we can obtain the following:

The right hand of the expression $=\lambda\cdot f(v)+\lambda\cdot g(v)+d$, where $f(v)$ is a polynomial in v of degree $t-i$ with the leading coefficient of $f(v)>0$, $g(v)$ is a polynomial in v of degree $t-i-1$, and d is a constant. Hence, there exists a constant $C(k, l, t, i)>0$ such that $x_i-l>0$ holds if $v\geqslant C(k, l, t, i)$. Namely, if $x_i\leqslant l$, then $v<C(k, l, t, i)$.

Proof of Theorem 2. By Lemma 6, we may assume that $t\geqslant \dfrac{2n+((2n+2)!)^2}{c-2}+2n$. Let $\boldsymbol{D}$ be a block-regular $t-(v, t+n, \lambda)$ design with $v\geqslant ct$, $t\geqslant \dfrac{2n+((2n+2)!)^2}{c-2}+2n$, and $x_i\leqslant l$ for some i $(0\leqslant i\leqslant t-1)$. Set $v=mt$ $(m\geqslant c)$, where m is not always integral. By Lemma 5, we have

$$x_i=\frac{\lambda\binom{t+n}{i}}{\binom{(m-1)t}{n}}\left\{\binom{(m-1)t-n}{t+n-i}+(-1)^{t+i+1}\sum_{q=0}^{n-1}\binom{t-i-1+q}{q}\binom{(m-1)t-n+q}{n}\right\}$$

$$+(\lambda-1)\sum_{j=i}^{t-1}\binom{j}{i}\binom{t+n}{j}(-1)^{i+j}+\sum_{j=t}^{t+n-1}\binom{j}{i}w_j(-1)^{i+j}, \tag{17}$$

where $x_j\leqslant w_j\leqslant(\lambda-1)\binom{t+n}{j}$ $\quad(t\leqslant j\leqslant k-1)$.

$$\text{Now,}\quad (\lambda-1)\sum_{j=i}^{t-1}\binom{j}{i}\binom{t+n}{j}(-1)^{i+j}+\sum_{j=t}^{t+n-1}\binom{j}{i}w_j(-1)^{i+j}$$

$$=-(\lambda-1)\sum_{j=t}^{t+n}\binom{j}{i}\binom{t+n}{j}(-1)^{i+j}+\sum_{j=t}^{t+n-1}\binom{j}{i}w_j(-1)^{i+j}$$

$$>-2\lambda(n+1)\frac{(t+n)!}{i!(t-i)!}\,. \tag{18}$$

On the other hand,

$$\frac{\lambda\binom{t+n}{i}}{\binom{(m-1)t}{n}}\left\{\binom{(m-1)t-n}{t+n-i}+(-1)^{t+i+1}\sum_{q=0}^{n-1}\binom{t-i-1+q}{q}\binom{(m-1)t-n+q}{n}\right\}$$

$$>\frac{\lambda\binom{t+n}{i}}{\binom{(m-1)t}{n}}\left\{\binom{(m-1)t-n}{t+n-i}-n\binom{t+n-i}{n}\binom{(m-1)t}{n}\right\}$$

$$>\frac{\lambda(t+n)!((m-1)t-n)!((m-1)t-n)!}{i!(t+n-i)!((m-1)t)!(t+n-i)!((m-2)t-2n+i)!}-\frac{\lambda(t+n)!}{i!(t-i)!} \tag{19}$$

By (17), (18) and (19), we have

$$\frac{x_i\, i!(t-i)!}{(t+n)!\lambda} > \frac{\{((m-1)t-n)!\}^2(t-i)!}{((t+n-i)!)^2((m-1)t)!((m-2)t-2n+i)!} - 5n\,.$$

Then since $\dfrac{i!(t-i)!}{(t+n)!\lambda} \leqslant \dfrac{t!}{(t+n)!\lambda} < 1$, we have

$$x_i > \frac{((m-1)t-n)\cdots((m-2)t-2n+i+1)}{((m-1)t)\cdots((m-1)t-n+1)\cdot(t+n-i)\cdots(t-i+1)\,(t+n-i)!} - 5n\,.$$

Hence, $$x_i > ((m-1)t-n)\,\frac{((m-1)t-n-1)\cdots((m-1)t-2n)}{((m-1)t)\cdots((m-1)t-n+1)}$$
$$\cdot\frac{((m-1)\ -2n-1)\cdots((m-1)t-3n)}{(t+n-i)\cdots(t-i+1)}$$
$$\cdot\frac{((m-1)t-3n-1)\cdots((m-2)t-2n+i+2)}{(t+n-i)\cdots(2n+3)}\cdot\frac{(m-2)t-2n+i+1}{(2n+2)!} - 5n$$

holds if $t-i \geqslant n+3$, and

$$x_i > ((m-1)t-n)\,\frac{((m-1)t-n-1)\cdots((m-1)t-2n)}{((m-1)t)\cdots((m-1)t-n+1)}$$
$$\cdot\frac{((m-1)t-2n-1)\cdots((m-2)t-2n+i+1)}{((2n+2)!)^2} - 5n$$

holds if $2 \leqslant t-i \leqslant n+2$,

and $$x_i > ((m-1)t-n)\,\frac{((m-1)t-n-1)\cdots((m-1)t-2n)}{((m-1)t)\cdots((m-1)t-n+1)}\,\frac{1}{((n+1)!)^2} - 5n$$

holds if $t-i=1$.

In any case, since $t \geqslant \dfrac{2n+((2n+2)!)^2}{c-2} + 2n$, we have

$$x_i > ((m-1)t-n)\,\frac{((m-2)t)^n}{((m-1)t)^n}\cdot\frac{1}{((n+1)!)^2} - 5n$$
$$> \frac{((c-1)t-n)}{((n+1)!)^2}\left(\frac{c-2}{c-1}\right)^n - 5n\,.$$

Therefore, there exists a positive integer $N(c, n, l)\left(\geqslant \dfrac{2n+((2n+2)!)^2}{c-2} + 2n\right)$ such that $x_i - l > 0$ holds if $t \geqslant N(c, n, l)$. Namely, if $x_i \leqslant l$, then $t \leqslant N(c, n, l)$. Hence by Lemma 6, the proof of Theorem 2 is completed.

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