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COMPACT SIMPLE LIE ALGEBRAS WITH TWO INVOLUTIONS AND SUBMANIFOLDS OF COMPACT SYMMETRIC SPACES II

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Introduction. This is a continuation of Part I, which appears in the same Journal.

In the previous paper we take a Grassmann bundle $G_s(TM)$ over a compact simply connected irreducible riemannian symmetric space M and consider a G-orbit $\mathcal{V}$ in $G_s(TM)$ by the isometry group G of M. For each $\mathcal{V}$ we can define a class of submanifolds in M, so is called, a *$\mathcal{V}$-geometry.* We moreover assume that $\mathcal{V}$ is a G-orbit which contains an s-dimensional strongly curvature-invariant subspace. Then $\mathcal{V}$ corresponds to a PSLA $(\mathfrak{g}, \sigma, \tau)$ of compact semisimple Lie algebra $\mathfrak{g}$ and two commutative involutions σ, τ. PSLA's are algebraicaly divided into those of inner type and those of outer type.

Our aim in this article is to prove the following

Main Theorem. *Let M be an irreducible compact simply connected riemannian symmetric space and $\mathcal{V}$ a G-orbit of inner type. Then the Lie algebra $\mathfrak{g}$ of Killing vector fields on M is compact simple and the following hold for $\mathfrak{g}$ of classical type:*

(1) *Let $\mathfrak{g}$ be the Lie algebra of type A_l, $l \geq 1$. In this case the $\mathcal{V}$-geometry admits non-totally geodesic $\mathcal{V}$-submanifolds if and only if it is one of the $\mathcal{V}$-geometries in Example* 2, (1).

(2) *Let $\mathfrak{g}$ be the Lie algebra of type B_l, $l \geq 2$. In this case the $\mathcal{V}$-geometry admits non-totally geodesic $\mathcal{V}$-submanifolds if and only if it is one of the $\mathcal{V}$-geometries in Example* 1 *(m : even and r : even).*

(3) *Let $\mathfrak{g}$ be a Lie algebra of type C_l, $l \geq 3$. In this case the $\mathcal{V}$-geometry admits non-totally geodesic $\mathcal{V}$-submanifolds if and only if it is one of the $\mathcal{V}$-geometries in Example* 3, (2).

(4) *Let $\mathfrak{g}$ be the Lie algebra of type D_l, $l \geq 4$. In this case the $\mathcal{V}$-geometry does not admit non-totally geodesic $\mathcal{V}$-submanifolds.*

Examples appeared here are known ones as $\mathcal{V}$-geometries in rank one sym-

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metric spaces of classical type: The $\mathcal{V}$-geometries in Example 2, (1) are the classes of complex submanifolds in the complex projective space; The $\mathcal{V}$-geometries in Example 1 are the classes of even-dimensional submanifolds in the even-dimensional sphere; The $\mathcal{V}$-geometries in Example 3, (2) are the classes of half-dimensional totally complex submanifolds in the quaternion projective space. (For details see Part I.)

The claims (1), (2) have been proved in Part I and the claims (3), (4) will be proved in the present Part II. The procedure is similarly done to Part I; In these cases we first classify the PSLA's of inner type and then for each PSLA we apply the representation-theoretic method which is prepared in §§1, 2, Part I.

We retain the definitions and notations in Part I. Main notations are here described:

(1) $\mathfrak{k}$, $\mathfrak{p}$ mean the (± 1)-eigenspaces by σ and $\mathfrak{k}_{\pm}$(resp. $\mathfrak{p}_{\pm}$) mean the (± 1)-eigenspaces in $\mathfrak{k}$ (resp. $\mathfrak{p}$) by τ;

(2) Take a suitable maximal abelian subspace $\mathfrak{h}$ in $\mathfrak{k}_{+}$. Then $\mathfrak{h}^{C}$ is a Cartan subalgebra of Lie algebras $\mathfrak{k}_{+}^{C}$, $\mathfrak{g}^{C}$. The set Δ (resp. $\Delta_{\mathfrak{k}_{+}}$) means the set of roots for $\mathfrak{g}^{C}$ (resp. $\mathfrak{k}_{+}^{C}$) and the sets $\Delta_{\mathfrak{k}_{-}}$, $\Delta_{\mathfrak{p}_{\pm}}$ mean the sets of weights for $\mathfrak{k}_{+}$-modules $\mathfrak{k}_{-}^{C}$, $\mathfrak{p}_{\pm}^{C}$;

(3) Π (resp. Π_s) means a fundamental root system for $\mathfrak{g}^{C}$ (resp. the semisimple part of $\mathfrak{k}_{+}^{C}$). The vectors $\{H_i\}$ (resp. $\{K_i\}$) mean the dual vectors of Π (resp. Π_s). The notations θ_i, θ_{jk} mean the following involutions:

$$\theta_i = \exp \operatorname{ad} (\sqrt{-1}\pi H_i)\,, \quad \theta_{jk} = \exp \operatorname{ad} (\sqrt{-1}\pi(H_j+H_k))\,.$$

Moreover compare §1, Part I for the homomorphism ρ associated with a PSLA, §2, Part I for the notion "*decomposable*", and §3, Part I for the notion "*the equivalence of first* or *second type*".

5. The PSLA's with Lie algebra $\mathfrak{g}$ of type C_l

Let $\mathfrak{g}$ be the Lie algebra of type C_l, $l\geq 3$, that is, the Lie algebra $\mathfrak{sp}(l)$ of skew symmetric matrices of degree l over quaternions. Then the Dynkin diagram of the fundamental root system Π is given as follows:

$$\begin{array}{ccccccc} \bigcirc & - & \bigcirc & -\cdots- & \bigcirc & \Leftarrow & \bigcirc \\ \alpha_1 & & \alpha_2 & & \alpha_{l-1} & & \alpha_l \end{array} \qquad \begin{array}{l} -\alpha_0 = 2\alpha_1+2\alpha_2+\cdots+2\alpha_{l-1} \\ \qquad +\alpha_l \end{array}$$

Put θ_i, θ_{jk} as in §3 and let C_j, $1\leq j<l$, C_{ij}, $1\leq j<i<l$, $C_{i\,;\,jk}$, $1\leq j<i<k<l$, be the families which contain the PSLA's $(\mathfrak{g}, \theta_l, \theta_j)$, $(\mathfrak{g}, \theta_i, \theta_j)$, $(\mathfrak{g}, \theta_i, \theta_{jk})$, respectively.

Lemma 5.1. *A PSLA* $(\mathfrak{g}, \sigma, \tau)$ *of inner type is equivalent to a PSLA which belongs to one of the families* C_j, C_{ij} *or* $C_{i\,;\,jk}$, *by an inner automorphism of* $\mathfrak{g}$.

Proof. We may assume that $\sigma=\theta_i$. We divide into the following cases:

(1) $1 \leq i < l$, (2) $i = l$.

Case (1): $1 \leq i < l$. Then $\mathfrak{k} = \mathfrak{k}_s$ and the Dynkin diagram of Π_s is given as follows:

$$\begin{array}{ccccccccccc} \bigcirc & \Rightarrow & \bigcirc & -\cdots- & \bigcirc & & \bigcirc & - & \bigcirc & -\cdots- & \bigcirc & \Leftarrow & \bigcirc \\ \alpha_0 & & \alpha_1 & & \alpha_{i-1} & & \alpha_{i+1} & & \alpha_{i+2} & & \alpha_{l-1} & & \alpha_l \end{array}$$

If we put $\bar{\tau} = \exp \operatorname{ad}(\sqrt{-1}\pi K)$, the following cases are considerable: (1) $K = K_j$, $0 \leq j \leq i-1$, (2) $K = K_k$, $i+1 \leq k \leq l$, (3) $K = K_j + K_k$, $0 \leq j \leq i-1$, $i+1 \leq k \leq l$. By Lemma 1.2 (1) we may moreover suppose the following: $j \neq 0$ for (1); $k \neq l$ for (2); (a) $j = 0$, $k = l$ or (b) $j \neq 0$, $k \neq l$ for (3). As above, we represent the vectors K_r by the vectors $H_1, \cdots, H_l$. For Case (1) it follows that $K_j = -H_i + H_j$ and thus the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{C}_{ij}$. For Case (2) it follows that $K_k = H_k - H_i$ and thus the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{C}_{ki}$. For Case (3) (a), it follows that $K_0 + K_l = -H_i + H_l$ and thus the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{C}_i$. For Case (3) (b), it follows that $K_j + K_k = H_j - 2H_i + H_k$ and thus the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{C}_{i\,;\,jk}$.

Case (2): $i = l$. Then $\mathfrak{k} = \mathfrak{c} \oplus \mathfrak{k}_s$ and the Dynkin diagram of Π_s is given as follows:

$$\begin{array}{ccccc} \bigcirc & - & \bigcirc & -\cdots- & \bigcirc \\ \alpha_1 & & \alpha_2 & & \alpha_{l-1} \end{array}$$

Put $\bar{\tau} = \exp \operatorname{ad}(\sqrt{-1}\pi K_j)$, $1 \leq j < l$, and represent the vectors K_j by $H_1, \cdots, H_l$. Then $K_j = H_j + aH_l$ for some $a \in \boldsymbol{R}$ and thus the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{C}_j$. $\square$

From the above proof, we can see that the subalgebras $\mathfrak{k}_+$ for $\mathcal{C}_i, \mathcal{C}_{ij}, \mathcal{C}_{i\,;\,jk}$ are different and thus these families are never equivalent to each other.

We first see the equivalences among the families $\mathcal{C}_{ij}$ and the equivalences among the PSLA's which belong to each $\mathcal{C}_{ij}$.

Put $V = \sqrt{-1}\mathfrak{h}$ and take an orthonormal basis $\{e_1, \cdots, e_l\}$ which satisfies that $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq l-1$, and $\alpha_l = 2e_l$. Then it holds that $H_i = e_1 + \cdots + e_i$ for $1 \leq i < l$ and $H_l = (1/2)(e_1 + \cdots + e_l)$. The Weyl group $W(\Delta)$ is generated by the permutations of $e_1, \cdots, e_l$ and the mappings $w_{\bar{i}}$, $1 \leq i \leq l$: $w_{\bar{i}}(e_i) = -e_i$ and $w_{\bar{i}}(e_j) = e_j$ for $j \neq i$. Define elements $w_0^k (1 \leq k \leq l)$ and $w_1^{jk}(j, k \geq 1,\ j+k \leq l)$ in $W(\Delta)$ in the same way as in §3. Then it similarly follows that

$$w_0^k(H_i) = \begin{cases} H_k - H_{k-i} & (1 \leq i < k < l)\,, \\ 2H_l - H_{l-i} & (i < k = l)\,, \\ H_i & (k \leq i \leq l)\,, \end{cases}$$

$$w_1^{jk}(H_i) = \begin{cases} H_{j+k} - H_k & (i = j,\ j+k < l)\,, \\ 2H_l - H_k & (i = j,\ j+k = l)\,, \\ H_i & (j+k \leq i \leq l)\,. \end{cases}$$

Let φ_0^k, φ_1^{jk}, φ_i^- be inner automorphisms of $\mathfrak{g}$ induced by w_0^k, w_1^{jk}, w_i^-, respectively.

For a family C_{ij} put $i=j+k$ and $l=i+r$. Then $j, k, r\geq 1$ and the following holds.

Proposition 5.2. *Two families C_{ij}, $C_{i'j'}$ are equivalent to each other if and only if the triples (j, k, r), (j', k', r') coincide except order.*

By virtue of this proposition we may consider only the families C_{ij} with triple (j, k, r) such that $j\leq k\leq r$. Such a family is said to be a ***proper family of type CI*** and a family without the above condition is said to be simply a family of type CI.

Proposition 5.3. *Let C_{ij} be a proper family of type CI with triple (j, k, r) and set $(\mathfrak{g}, \sigma, \tau)=(\mathfrak{g}, \theta_i, \theta_j)$. Then the following hold:*

(1) *If $j<k<r$, all the PSLA's in C_{ij} are non-equivalent to each other;*

(2) *If $j=k<r$, only the equivalences of first type hold;*

(3) *If $j<k=r$, only the equivalences of second type hold;*

(4) *If $j=k=r$, all the PSLA's in C_{ij} are equivalent to each other.*

Proposition 5.2, 5.3 can be proved in the same way as Propositions 3.2, 3.3.

We next see the equivalences among families $C_{i\,;\,jk}$ and the equivalences among the PSLA's which belong to each $C_{i\,;\,jk}$.

For a family $C_{i\,;\,jk}$ put $j=a, i=j+b, k=i+c, l=k+d$. Then $a, b, c, d\geq 1$ and the following hold.

Proposition 5.4. *Two families $C_{i\,;\,jk}$, $C_{i'\,;\,j'k'}$ are equivalent to each other if and only if the quadruples (a, b, c, d), $(a', b', c' d')$ coincide except order.*

By virtue of this proposition we may consider only the families $C_{i\,;\,jk}$ with quadruple (a, b, c, d) such that $a\leq b\leq c\leq d$. Such a family is said to be a ***proper family of type CII*** and a family without the above condition is said to be simply a family of type CII.

Proposition 5.5. *Let $C_{i\,;\,jk}$ be a proper family of type CII with quadruple (a, b, c, d) and set $(\mathfrak{g}, \sigma, \tau)=(\mathfrak{g}, \theta_i, \theta_{jk})$. Then the following hold:*

(1) *If $a<b<c<d$, all the PSLA's in $C_{i\,;\,jk}$ are non-equivalent to each other;*

(2) *If $a=b<c\leq d$ or $a\leq b<c=d$, only the equivalences of first type hold;*

(3) *If $a<b=c<d$, only the equivalences of second type hold;*

(4) *If $a=b=c<d$, $a<b=c=d$, or $a=b=c=d$, all the PSLA's in $C_{i\,;\,jk}$ are equivalent to each other.*

Propositions 5.4, 5.5 can be proved in the same way as Propositions 3.4, 3.5.

We last see the equivalences among families C_j and the equivalences among the PSLA's which belong to each C_j.

For a family C_j put $l=j+k$. Then $j, k\geq 1$ and the following holds.

Proposition 5.6. *Two families $C_j, C_{j'}$ are equivalent to each other if and only if the pairs $(j, k), (j', k')$ coincide except order.*

Proof. Consider the PSLA's $(\mathfrak{g}, \theta_l, \theta_j)$, $(\mathfrak{g}, \theta_l, \theta_{j'})$. Then the semisimple part of $\mathfrak{k}_+$(resp. $\mathfrak{k}'_+$) is the sum of Lie algebras of type A_{j-1} (resp. $A_{j'-1}$) and type A_{k-1} (resp. $A_{k'-1}$).

Suppose that C_j is equivalent to $C_{j'}$. Since $\mathfrak{k}_+$ is isomorphic to $\mathfrak{k}'_+$, it follows that pairs $(j, k), (j', k')$ coincide except order.

To prove the converse we may prove the following equivalence: $C_j \simeq C_k$ where C_k has the pair (k, j). This is given by φ_0^l. $\square$

By virtue of this proposition we may consider only the families C_j with pair (j, k) such that $j \leq k$. Such a family is said to be a *proper family of type CIII* and a family without the above condition is said to be simply a family of type *CIII*.

Proposition 5.7. *Let C_j be a proper family of type CIII with pair (j, k) and set $(\mathfrak{g}, \sigma, \tau) = (\mathfrak{g}, \theta_l, \theta_j)$. Then only the equivalences of second type hold.*

Proof. The equivalences of second type are obtained by the following inner automorphism: $\varphi = \varphi_{\overline{j+1}} \cdots \varphi_{\overline{l}}$. We next note that

$$\mathfrak{k}_- = \mathfrak{sp}(j)/\mathfrak{u}(j) \oplus \mathfrak{sp}(k)/\mathfrak{u}(k)\,, \quad \mathfrak{p}_\pm = \mathfrak{su}(l)/\mathfrak{s}(\mathfrak{u}(j) \oplus \mathfrak{u}(k))\,.$$

Hence, as $\mathfrak{k}_+$-modules, $\mathfrak{k}_-$ is not isomorphic to $\mathfrak{p}_\pm$. This implies the non-equivalences of the other pairs. $\square$

We now see the injectivity of the $\mathfrak{k}_+$-homomorphism ρ for each PSLA in the families of types CI, CII, CIII.

Similarly to in §3, fix a positive integer r and set

$$\begin{aligned}
R_1 &= \{\pm(\overbrace{0\cdots0}^{a}\,\overbrace{1\cdots1}^{b}\,\overbrace{0\cdots0}^{c}) \in \boldsymbol{Z}^r;\ a \geq 0, b \geq 0, c \geq 0\}\,,\\
R_2 &= \{\pm(\overbrace{0\cdots0}^{a}\,\overbrace{1\cdots1}^{b}\,\overbrace{2\cdots2}^{c}) \in \boldsymbol{Z}^r;\ a \geq 0, b \geq 0, c > 0\}\,,\\
R'_2 &= \{\pm(\overbrace{0\cdots0}^{a}\,\overbrace{1\cdots1}^{b}\,\overbrace{2\cdots2}^{c}1) \in \boldsymbol{Z}^r;\ a \geq 0, b \geq 0, c > 0\}\,,\\
R &= R_1 \cup R_2\,, \quad R' = R_1 \cup R'_2\,,\\
R^2 &= \{\tbinom{\alpha}{\beta};\ \alpha, \beta \in R\}\,, \quad R'^2 = \{\tbinom{\alpha}{\beta};\ \alpha, \beta \in R'\}\,.
\end{aligned}$$

Moreover let $R^2[\binom{a}{b}_i]$, $R^2[\binom{a}{b}_i, \binom{c}{d}_j]$, $R^2_\lambda[*]$ be subsets of R^2 defined as in §3. The subsets $R'^2[\binom{a}{b}_i]$, $R'^2[\binom{a}{b}_i, \binom{c}{d}_j]$, $R'^2_\lambda[*]$ may be also defined similarly. Then we can check the following lemma by a usual argument.

Lemma 5.8. *Let λ be an r-tuples in $\boldsymbol{Z}^r$. Then the following hold:*

(1) *The following each set has at most 2 elements:*

$R_\lambda^2[\binom{-1}{1}_r]$, $R_\lambda^2[\binom{-1}{1}_1, \binom{0}{0}_r]$, $R_\lambda^2([\binom{-1}{1}_1, \binom{-2}{2}_r]$, $R_\lambda^2[\binom{0}{1}_1, \binom{-1}{2}_r]$,
$R_\lambda^2[\binom{0}{1}_1, \binom{1}{0}_r]$, $R_\lambda^2[\binom{1}{1}_1, \binom{0}{2}_r]$, $R_\lambda'^2[\binom{1}{1}_1, \binom{1}{0}_r]$;

(2) *For the following sets Lemma* 4.6 (2) *through* (7) *hold:*

$R_\lambda^2[\binom{1}{1}_r]$, $R_\lambda^2[\binom{0}{1}_1, \binom{-1}{0}_r]$, $R_\lambda^2[\binom{1}{1}_1, \binom{0}{0}_r]$, $R_\lambda^2[\binom{1}{1}_1, \binom{2}{2}_r]$,
$R_\lambda^2[\binom{-1}{1}_1, \binom{-2}{0}_r]$, $R_\lambda^2[\binom{1}{2}_r]$, $R_\lambda^2[\binom{1}{1}_1]$, $R_\lambda^2[\binom{-1}{1}_1]$;

(3) *The set* $R_\lambda'^2[\binom{-1}{1}_1, \binom{-1}{0}_r]$ *has at most* 1 *elememt if* $\lambda \neq (2\cdots21)$, *and has just* $r-1$ *elements with form*

$$\begin{pmatrix} -1 & \cdots & -1 & -2 & \cdots & -2 & -1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

if $\lambda=(2\cdots21)$;

(4) *The set* $R_\lambda'^2[\binom{-1}{0}_1, \binom{-1}{-1}_r]$ *has at most* 1 *elememt if* $\lambda \neq (\overbrace{1\cdots1}^{a}\,\overbrace{0\cdots0}^{b})$ $(a>0,$ $b>0)$, *and has just* $r-1$ *elements with forms*

$$\begin{pmatrix} \overbrace{\begin{matrix} -1 & \cdots & -1 & -2 & \cdots & -2 \\ 0 & \cdots & 0 & -1 & \cdots & -1 \end{matrix}}^{a} & \overbrace{\begin{matrix} -2 & \cdots & -2 & -1 \\ -2 & \cdots & -2 & -1 \end{matrix}}^{b} \end{pmatrix},$$

$$\begin{pmatrix} \overbrace{\begin{matrix} -1 & \cdots & -1 \\ 0 & \cdots & 0 \end{matrix}}^{a} & \overbrace{\begin{matrix} -1 & \cdots & -1 & -2 & \cdots & -2 & -1 \\ -1 & \cdots & -1 & -2 & \cdots & -2 & -1 \end{matrix}}^{b} \end{pmatrix}$$

if $\lambda=(\overbrace{1\cdots1}^{a}\,\overbrace{0\cdots0}^{b})$;

(5) *The set* $R_\lambda'^2[\binom{-1}{0}_1, \binom{0}{1}_r]$ *has at most* 1 *element if* $\lambda \neq (\overbrace{1\cdots1}^{a}\,\overbrace{2\cdots21}^{b})$ $(a>0,$ $b>0)$, *and has just* $r-1$ *elements with forms*

$$\begin{pmatrix} \overbrace{\begin{matrix} -1 & \cdots & -1 \\ 0 & \cdots & 0 \end{matrix}}^{a} & \overbrace{\begin{matrix} -1 & \cdots & -1 & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 1 & 2 & \cdots & 2 & 1 \end{matrix}}^{b} \end{pmatrix},$$

$$\begin{pmatrix} \overbrace{\begin{matrix} -1 & \cdots & -1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{matrix}}^{a} & \overbrace{\begin{matrix} 0 & \cdots & 0 & 0 \\ 2 & \cdots & 2 & 1 \end{matrix}}^{b} \end{pmatrix}$$

if $\lambda=(\overbrace{1\cdots1}^{a}\,\overbrace{2\cdots21}^{b})$;

(6) *The set* $R_\lambda'^2[\binom{1}{1}_1]$ *has at most* 2 *elements if* $\lambda \neq (0\cdots0)$, *and has just* $2r-2$ *elements with forms*

$$\begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \cdots & 1 & 2 & \cdots & 2 & 1 \\ 1 & \cdots & 1 & 2 & \cdots & 2 & 1 \end{pmatrix}$$

if $\lambda=(0\cdots0)$ *and* $r \neq 1$;

(7) *The set* $R_\lambda'^2[\binom{-1}{1}_1]$ *has at most* 2 *elements if* $\lambda \neq (2\cdots21)$, *and has just* $2r-2$ *elements with forms*

$$\begin{pmatrix} -1 & \cdots & -1 & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 1 & 2 & \cdots & 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & \cdots & -1 & -2 & \cdots & -2 & -1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

if $\lambda=(2\cdots 21)$;

(8) *The set* $R_\lambda^2[\binom{-1}{2}_r]$ *has at most* 2 *elements if* $\lambda \neq (\overbrace{0\cdots 0}^{a}\,\overbrace{1\cdots 1}^{b}\,\overbrace{2\cdots 2}^{c}\,\overbrace{3\cdots 3}^{d})$ $(a\geq 0, b>0, c>0, d>0)$, *and has just* 3 *elements with form*

$$\begin{pmatrix} \overbrace{0 \cdots 0}^{a} & \overbrace{0 \cdots 0}^{b} & \overbrace{0 \cdots 0}^{c} & \overbrace{-1 \cdots -1}^{d} \\ 0 \cdots 0 & 1 \cdots 1 & 2 \cdots 2 & 2 \cdots 2 \end{pmatrix},$$

$$\begin{pmatrix} \overbrace{0 \cdots 0}^{a} & \overbrace{0 \cdots 0}^{b} & \overbrace{-1 \cdots -1}^{c} & \overbrace{-1 \cdots -1}^{d} \\ 0 \cdots 0 & 1 \cdots 1 & 1 \cdots 1 & 2 \cdots 2 \end{pmatrix},$$

$$\begin{pmatrix} \overbrace{0 \cdots 0}^{a} & \overbrace{-1 \cdots -1}^{b} & \overbrace{-1 \cdots -1}^{c} & \overbrace{-1 \cdots -1}^{d} \\ 0 \cdots 0 & 0 \cdots 0 & 1 \cdots 1 & 2 \cdots 2 \end{pmatrix}$$

if $\lambda=(\overbrace{0\cdots 0}^{a}\,\overbrace{1\cdots 1}^{b}\,\overbrace{2\cdots 2}^{c}\,\overbrace{3\cdots 3}^{d})$;

(9) *The set* $R_\lambda'^2[\binom{-1}{0}_1, \binom{-1}{1}_r]$ *has at most* 2 *elements if*

$$\lambda \neq (\overbrace{1\cdots 1}^{a}\,\overbrace{2\cdots 2}^{b}\,\overbrace{3\cdots 3}^{c}\,\overbrace{4\cdots 4}^{d}2) \quad (a>0, b>0, c>0, d>0),$$

and has just 3 *elements with forms*

$$\begin{pmatrix} \overbrace{-1 \cdots -1}^{a} & \overbrace{-1 \cdots -1}^{b} & \overbrace{-1 \cdots -1}^{c} & \overbrace{-2 \cdots -2}^{d} & -1 \\ 0 \cdots 0 & 1 \cdots 1 & 2 \cdots 2 & 2 \cdots 2 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \overbrace{-1 \cdots -1}^{a} & \overbrace{-1 \cdots -1}^{b} & \overbrace{-2 \cdots -2}^{c} & \overbrace{-2 \cdots -2}^{d} & -1 \\ 0 \cdots 0 & 1 \cdots 1 & 1 \cdots 1 & 2 \cdots 2 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \overbrace{-1 \cdots -1}^{a} & \overbrace{-2 \cdots -2}^{b} & \overbrace{-2 \cdots -2}^{c} & \overbrace{-2 \cdots -2}^{d} & -1 \\ 0 \cdots 0 & 0 \cdots 0 & 1 \cdots 1 & 2 \cdots 2 & 1 \end{pmatrix}$$

if $\lambda=(\overbrace{1\cdots 1}^{a}\,\overbrace{2\cdots 2}^{b}\,\overbrace{3\cdots 3}^{c}\,\overbrace{4\cdots 4}^{d}2)$;

(10) *The set* $R_\lambda'^2[\binom{-1}{0}_1, \binom{0}{-1}_r]$ *has at most* 2 *elements if*

$$\lambda \neq (\overbrace{1\cdots 1}^{a}\,\overbrace{0\cdots 0}^{b}\,\overbrace{-1\cdots -1}^{c}\,\overbrace{-2\cdots -2}^{d}-1) \quad (a>0, b>0, c>0, d>0),$$

and has just 3 *elements with forms*

$$\begin{pmatrix} \overbrace{-1 \cdots -1}^{a} & \overbrace{0 \cdots 0}^{b} & \overbrace{0 \cdots 0}^{c} & \overbrace{0 \cdots 0}^{d} & 0 \\ 0 \cdots 0 & 0 \cdots 0 & -1 \cdots -1 & -2 \cdots -2 & -1 \end{pmatrix},$$

$$\begin{pmatrix} \overbrace{-1 \cdots -1}^{a} & \overbrace{-1 \cdots -1}^{b} & \overbrace{0 \cdots 0}^{c} & \overbrace{0 \cdots 0}^{d} & 0 \\ 0 \cdots 0 & -1 \cdots -1 & -1 \cdots -1 & -2 \cdots -2 & -1 \end{pmatrix},$$

$$\begin{pmatrix} \overbrace{-1 \cdots -1}^{a} & \overbrace{-1 \cdots -1}^{b} & \overbrace{-1 \cdots -1}^{c} & \overbrace{0 \;\cdots\; 0 \quad 0}^{d} \\ 0 \;\cdots\; 0 & -1 \cdots -1 & -2 \cdots -2 & -2 \cdots -2 \;\; -1 \end{pmatrix}$$

if $\lambda=(\overbrace{1\cdots 1}^{a}\,\overbrace{0\cdots 0}^{b}\,\overbrace{-1\cdots -1}^{c}\,\overbrace{-2\cdots -2-1}^{d})$.

In the following we represent a root of type C_l by a linear combination of the fundamental root system Π and identify it with an l-tuple of coefficients.

Case CI: The families C_{ij} with triple (j, k, r)

Put $\sigma=\theta_i$ and $\tau=\theta_j$. Then, for each PSLA in C_{ij}, the corresponding symmetric space M and the totally geodesic $\mathcal{V}$-submanifold N are given as follows: (N is locally described.)

(a) $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$: $M=Sp(l)/Sp(j+k)\times Sp(r)$. In this case $N=\mathfrak{sp}(j+r)/\mathfrak{sp}(j)\oplus\mathfrak{sp}(r)$;

(b) $\mathcal{V}=(\mathfrak{g}, \sigma, \sigma\tau)$: $M=Sp(l)/Sp(j+k)\times Sp(r)$. In this case $N=\mathfrak{sp}(k+r)/\mathfrak{sp}(k)\oplus\mathfrak{sp}(r)$;

(c) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma)$: $M=Sp(l)/Sp(j)\times Sp(k+r)$. In this case $N=\mathfrak{sp}(j+r)/\mathfrak{sp}(j)\oplus\mathfrak{sp}(r)$;

(d) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma\tau)$: $M=Sp(l)/Sp(j)\times Sp(k+r)$. In this case $N=\mathfrak{sp}(j+k)/\mathfrak{sp}(j)\oplus\mathfrak{sp}(k)$;

(e) $\mathcal{V}=(\mathfrak{g}, \sigma\tau, \sigma)$: $M=Sp(l)/Sp(k)\times Sp(j+r)$. In this case $N=\mathfrak{sp}(k+r)/\mathfrak{sp}(k)\oplus\mathfrak{sp}(r)$;

(f) $\mathcal{V}=(\mathfrak{g}, \sigma\tau, \tau)$: $M=Sp(l)/Sp(k)\times Sp(j+r)$. In this case $N=\mathfrak{sp}(j+k)/\mathfrak{sp}(j)\oplus\mathfrak{sp}(k)$.

For the PSLA $(\mathfrak{g}, \sigma, \tau)$, the subsets $\Delta^+_{\mathfrak{k}_+}$, $\Delta^+_{\mathfrak{k}_-}$, $\Delta^+_{\mathfrak{p}_+}$, $\Delta^+_{\mathfrak{p}_-}$ of Δ^+ are given as follows:

(5.1)
$$\Delta^+_{\mathfrak{k}_+} = \{\delta\in\Delta^+;\ \delta_i = \delta_j = 0, 2\}$$
$$= \left\{ \delta\in\Delta^+;\ \delta = \begin{matrix} (0\cdots 01\cdots 10\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots 0) \\ (0\cdots \overset{j}{0}\cdots 01\cdots 10\cdots \overset{i}{0}\cdots 0) \\ (0\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots 01\cdots 10\cdots 0) \\ (0\cdots 01\cdots 12\cdots \overset{j}{2}\cdots \overset{i}{2}\cdots 21) \\ (0\cdots \overset{j}{0}\cdots 01\cdots 12\cdots \overset{i}{2}\cdots 21) \\ (0\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots 01\cdots 12\cdots 21) \end{matrix} \right\},$$

$$\Delta^+_{\mathfrak{k}_-} = \{\delta\in\Delta^+;\ \delta_i = 0, 2,\ \delta_j = 1\}$$
$$= \left\{ \delta\in\Delta^+;\ \delta = \begin{matrix} (0\cdots 01\cdots \overset{j}{1}\cdots 10\cdots \overset{i}{0}\cdots 0) \\ (0\cdots 01\cdots \overset{j}{1}\cdots 12\cdots \overset{i}{2}\cdots 21) \end{matrix} \right\},$$

$$\Delta^+_{\mathfrak{p}_+} = \{\delta\in\Delta^+;\ \delta_i = 1,\ \delta_j = 0, 2\}$$
$$= \left\{ \delta\in\Delta^+;\ \delta = \begin{matrix} (0\cdots \overset{j}{0}\cdots 01\cdots \overset{i}{1}\cdots 10\cdots 0) \\ (0\cdots \overset{j}{0}\cdots 01\cdots \overset{i}{1}\cdots 12\cdots 21) \end{matrix} \right\},$$

$$\Delta_{\mathfrak{p}_-}^+ = \{\delta \in \Delta^+; \delta_i = \delta_j = 1\}$$
$$= \left\{ \delta \in \Delta^+; \delta = \begin{array}{l} (0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots10\cdots0) \\ (0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots12\cdots21) \end{array} \right\}.$$

Moreover the dominant weights in $\Delta_{\mathfrak{k}_-}$, $\Delta_{\mathfrak{p}_+}$, $\Delta_{\mathfrak{p}_-}$ are given by (5.2), (5.3), (5.4), respectively:

$$(1\cdots\overset{j}{1}2\cdots\overset{i}{2}\cdots21), \tag{5.2}$$
$$(0\cdots\overset{j}{0}1\cdots\overset{i}{1}2\cdots21), \tag{5.3}$$
$$(1\cdots\overset{j}{1}\cdots\overset{i}{1}2\cdots21). \tag{5.4}$$

We now see the injectivity of ρ for Case (a): $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$. Then ρ is a homomorphism of $(\mathfrak{p}_-^{\mathbf{C}})^* \otimes \mathfrak{k}_-^{\mathbf{C}}$ to $\wedge^2(\mathfrak{p}_-^{\mathbf{C}})^* \otimes \mathfrak{p}_+^{\mathbf{C}}$. The minus multiple of dominant weight in $\Delta_{\mathfrak{p}_-}$ and the dominant weight in $\Delta_{\mathfrak{k}_-}$ are given by $(\alpha 1)$, $(\beta 1)$, respectively:

$$(\alpha 1) \quad -(1\cdots\overset{j}{1}\cdots\overset{i}{1}2\cdots21), \qquad (\beta 1) \quad (1\cdots\overset{j}{1}2\cdots\overset{i}{2}\cdots21).$$

Case (1): $l(u)=1$. Represent u as follows: $u=a\,\omega_\alpha \otimes X_\beta$. Then the pair (α, β) is given by $((\alpha 1), (\beta 1))$. Applying Lemma 2.3, we obtain that $\rho(u) \neq 0$.

Case (2): $l(u)=2$. In this case there exists no decomposable u and thus we suppose that u is indecomposable. We consider such the triples $(\alpha, \beta'; \mu)$ as in §3, Case (2). Consider the following elements in $\Delta_{\mathfrak{k}_+}$:

$$(\mu 1) \quad (10\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots0), \qquad (\mu 2) \quad (\overset{j}{2}\cdots\overset{i}{2}\cdots21)\ (j=1).$$

Then such the triples are given in the following:

$$(1) \quad ((\alpha 1), (\beta 1); (\mu 1)), j \geq 2, \qquad (2) \quad ((\alpha 1), (\beta 1); (\mu 2)), j=1.$$

Lemma 2.4 is available for (1) and Lemma 2.2 is available for (2). Hence it follows that $\rho(u) \neq 0$.

Case (3): $l(u) \geq 3$. Note that $i \neq l$. Then, by the same way as **Case (3)** for Case BI §4, we see that $\rho(u) \neq 0$.

Summing up the above arguments, we have the following result for the PSLA of Case (a); the homomorphism ρ is always injective. Similarly for the other cases ρ is always injective.

Theorem 5.9. *Let $\mathcal{V}$ be the G-orbit which corresponds to a PSLA in a family of type CI. Then the $\mathcal{V}$-geometry does not admit non-totally geodesic $\mathcal{V}$-submanifolds.*

Case CII: The families $C_{i;jk}$ with quadruple (a, b, c, d)

Put $\sigma=\theta_i$ and $\tau=\theta_{jk}$. Then, for each PSLA in $C_{i;jk}$ the corresponding symmetric space M and the totally geodesic $\mathcal{V}$-submanifold N are given in the following: (N is locally described.)

(a) $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$: $M=Sp(l)/Sp(a+b)\times Sp(c+d)$.
In this case $N=\mathfrak{sp}(a+c)/\mathfrak{sp}(a)\oplus\mathfrak{sp}(c) \oplus \mathfrak{sp}(b+d)/\mathfrak{sp}(b)\oplus\mathfrak{sp}(d)$;

(b) $\mathcal{V}=(\mathfrak{g}, \sigma, \sigma\tau)$: $M=Sp(l)/Sp(a+b)\times Sp(c+d)$.
In this case $N=\mathfrak{sp}(b+c)/\mathfrak{sp}(b)\oplus\mathfrak{sp}(c) \oplus \mathfrak{sp}(a+d)/\mathfrak{sp}(a)\oplus\mathfrak{sp}(d)$;

(c) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma)$: $M=Sp(l)/Sp(b+c)\times Sp(a+d)$.
In this case $N=\mathfrak{sp}(a+c)/\mathfrak{sp}(a)\oplus\mathfrak{sp}(c) \oplus \mathfrak{sp}(b+d)/\mathfrak{sp}(b)\oplus\mathfrak{sp}(d)$;

(d) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma\tau)$: $M=Sp(l)/Sp(b+c)\times Sp(a+d)$.
In this case $N=\mathfrak{sp}(a+b)/\mathfrak{sp}(a)\oplus\mathfrak{sp}(b) \oplus \mathfrak{sp}(c+d)/\mathfrak{sp}(c)\oplus\mathfrak{sp}(d)$;

(e) $\mathcal{V}=(\mathfrak{g}, \sigma\tau, \sigma)$: $M=Sp(l)/Sp(a+c)\times Sp(b+d)$.
In this case $N=\mathfrak{sp}(b+c)/\mathfrak{sp}(b)\oplus\mathfrak{sp}(c) \oplus \mathfrak{sp}(a+d)/\mathfrak{sp}(a)\oplus\mathfrak{sp}(d)$;

(f) $\mathcal{V}=(\mathfrak{g}, \sigma\tau, \tau)$: $M=Sp(l)/Sp(a+c)\times Sp(b+d)$.
In this case $N=\mathfrak{sp}(a+b)/\mathfrak{sp}(a)\oplus\mathfrak{sp}(b) \oplus \mathfrak{sp}(c+d)/\mathfrak{sp}(c)\oplus\mathfrak{sp}(d)$.

For the PSLA $(\mathfrak{g}, \sigma, \tau)$, the subsets $\Delta^+_{\mathfrak{k}\pm}$, $\Delta^+_{\mathfrak{p}\pm}$ of Δ^+ are given as follows.

(5.5)
$$\Delta^+_{\mathfrak{k}+} = \{\delta\in\Delta^+;\ \delta_i = 0, 2,\ (\delta_j, \delta_k) = (0, 0), (0, 2), (2, 0), (1, 1), (2, 2)\}$$
$$= \left\{\delta\in\Delta^+;\ \delta = \begin{array}{c} (0\cdots 01\cdots 10\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots \overset{k}{0}\cdots 0) \\ (0\cdots \overset{j}{0}\cdots 01\cdots 10\cdots \overset{i}{0}\cdots \overset{k}{0}\cdots 0) \\ (0\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots 01\cdots 10\cdots \overset{k}{0}\cdots 0) \\ (0\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots \overset{k}{0}\cdots 01\cdots 10\cdots 0) \\ (0\cdots 01\cdots 12\cdots \overset{j}{2}\cdots \overset{i}{2}\cdots \overset{k}{2}\cdots 21) \\ (0\cdots \overset{j}{0}\cdots 01\cdots 12\cdots \overset{i}{2}\cdots \overset{k}{2}\cdots 21) \\ (0\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots 01\cdots 12\cdots \overset{k}{2}\cdots 21) \\ (0\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots \overset{k}{0}\cdots 01\cdots 12\cdots 21) \end{array}\right\},$$

$$\Delta^+_{\mathfrak{k}-} = \{\delta\in\Delta^+;\ \delta_i = 0, 2,\ (\delta_j, \delta_k) = (0, 1), (1, 0), (1, 2), (2, 1)\}$$
$$= \left\{\delta\in\Delta^+;\ \delta = \begin{array}{c} (0\cdots 01\cdots \overset{j}{1}\cdots 10\cdots \overset{i}{0}\cdots \overset{k}{0}\cdots 0) \\ (0\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots 01\cdots \overset{k}{1}\cdots 10\cdots 0) \\ (0\cdots 01\cdots \overset{j}{1}\cdots 12\cdots \overset{i}{2}\cdots \overset{k}{2}\cdots 21) \\ (0\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots 01\cdots \overset{k}{1}\cdots 12\cdots 21) \end{array}\right\},$$

$$\Delta^+_{\mathfrak{p}+} = \{\delta\in\Delta^+;\ \delta_i = 1,\ (\delta_j, \delta_k) = (0, 0), (0, 2), (2, 0), (1, 1), (2, 2)\}$$
$$= \left\{\delta\in\Delta^+;\ \delta = \begin{array}{c} (0\cdots \overset{j}{0}\cdots 01\cdots \overset{i}{1}\cdots 10\cdots \overset{k}{0}\cdots 0) \\ (0\cdots 01\cdots \overset{j}{1}\cdots \overset{i}{1}\cdots \overset{k}{1}\cdots 10\cdots 0) \\ (0\cdots \overset{j}{0}\cdots 01\cdots \overset{i}{1}\cdots 12\cdots \overset{k}{2}\cdots 21) \\ (0\cdots 01\cdots \overset{j}{1}\cdots \overset{i}{1}\cdots \overset{k}{1}\cdots 12\cdots 21) \end{array}\right\},$$

$$\Delta^+_{\mathfrak{p}-} = \{\delta\in\Delta^+;\ \delta_i = 1,\ (\delta_j, \delta_k) = (0, 1), (1, 0), (1, 2), (2, 1)\}$$

$$=\left\{\delta\in\Delta^+;\ \delta=\begin{matrix}(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots10\cdots\overset{k}{0}\cdots0)\\ (0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots10\cdots0)\\ (0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots12\cdots\overset{k}{2}\cdots21)\\ (0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots12\cdots21)\end{matrix}\right\}.$$

Moreover the dominant weights in $\Delta_{\mathfrak{k}_-}$, $\Delta_{\mathfrak{p}_+}$, $\Delta_{\mathfrak{p}_-}$ are given by (5.6), (5.7), (5.8), respectively:

$$(0\cdots\overset{j}{0}\cdots\overset{i}{0}1\cdots\overset{k}{1}2\cdots21),\ (1\cdots\overset{j}{1}2\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots21)\,, \tag{5.6}$$

$$(0\cdots\overset{j}{0}1\cdots\overset{i}{1}2\cdots\overset{k}{2}\cdots21),\ (1\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}2\cdots21)\,, \tag{5.7}$$

$$(0\cdots\overset{j}{0}1\cdots\overset{i}{1}\cdots\overset{k}{1}2\cdots21),\ (1\cdots\overset{j}{1}\cdots\overset{i}{1}2\cdots\overset{k}{2}\cdots21)\,. \tag{5.8}$$

We now see the injectivity of ρ for Case (a): $\mathcal{V}=(\mathfrak{g},\sigma,\tau)$. Then ρ is a homomorphism of $(\mathfrak{p}_-^C)^*\otimes\mathfrak{k}_-^C$ to $\wedge^2(\mathfrak{p}_-^C)^*\otimes\mathfrak{p}_+^C$.

The minus multiple of dominant weights in $\Delta_{\mathfrak{p}_-}$ are given by (α1), (α2) and the dominant weights in $\Delta_{\mathfrak{k}_-}$ are given by (β1), (β2):

$$(\alpha1)-(0\cdots\overset{j}{0}1\cdots\overset{i}{1}\cdots\overset{k}{1}2\cdots21),\quad (\alpha2)-(1\cdots\overset{j}{1}\cdots\overset{i}{1}2\cdots\overset{k}{2}\cdots21)\,,$$
$$(\beta1)\ (0\cdots\overset{j}{0}\cdots\overset{i}{0}1\cdots\overset{k}{1}2\cdots21)\,,\quad (\beta2)\ (1\cdots\overset{j}{1}2\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots21)\,.$$

Case (1): $l(u)=1$. Represent u as follows: $u=a\,\omega_\alpha\otimes X_\beta$. Then the pair (α,β) is one of the following pairs: ((αr)), (βs)), where $r, s=1, 2$. Applying Lemma 2.3, we obtain that $\rho(u)\neq0$ for all the pairs.

Case (2): $l(u)=2$. In this case there exists no decomposable u and thus we suppose that u is indecomposable. Consider the following elements in $\Delta_{\mathfrak{k}_+}$:

$(\mu1)$ $(0\cdots\overset{j}{0}10\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0)$, $(\mu2)$ $(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}10\cdots0)$,
$(\mu3)$ $(0\cdots\overset{j}{0}\overset{i}{2}\cdots\overset{k}{2}\cdots21)$ $(i=j+1)$, $(\mu4)$ $(10\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0)$,
$(\mu5)$ $(0\cdots\overset{j}{0}\cdots\overset{i}{0}1\cdots\overset{k}{0}\cdots0)$, $(\mu6)$ $(0\cdots\overset{j}{0}\cdots\overset{i}{0}\overset{k}{2}\cdots21)$ $(k=i+1)$,
$(\mu7)$ $(\overset{j}{2}\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots21)$ $(j=1)$.

Then such the triples $(\alpha,\beta';\mu)$ as in §3, Case (2) are given in the following:

(1) ((α1), (β2); (μ1)), $i-j\geq2$, (2) ((α1), (β1); (μ2)),
(3) ((α1), (β2); (μ3)), $i=j+1$, (4) ((α2), (β2); (μ4)), $j\geq2$,
(5) ((α2), (β1); (μ5)), $k-i\geq2$, (6) ((α2), (β1); (μ6)), $k=i+1$,
(7) ((α2), (β2); (μ7)), $j=1$.

Lemma 2.4 is available for all cases and thus it follows that $\rho(u)\neq0$.

Case (3): $l(u)\geq3$. By the same way as **Case (3)** for Case BII §4, we see that $\rho(u)\neq 0$.

Summing up the above arguments, we have the following result for the PSLA of Case (a); the homomorphism ρ is always injective. Similarly for the other cases ρ is injective.

Theorem 5.10. *Let $\mathcal{V}$ be the G-orbit which corresponds to a PSLA in a family of type CII. Then the $\mathcal{V}$-geometry does not admit non-totally geodesic $\mathcal{V}$-submanifolds.*

Case CIII: The families C_j with pair (j, k)

Put $\sigma=\theta_l$ and $\tau=\theta_j$. Then, for each PSLA in C_j, the corresponding symmetric space M and the totally geodesic $\mathcal{V}$-submanifold N are given as follows: (N is locally described.)

(a) $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$: $M=Sp(l)/U(l)$. In this case $N=\mathfrak{su}(l)/\mathfrak{s}(\mathfrak{u}(j)\oplus\mathfrak{u}(k))$;

(b) $\mathcal{V}=(\mathfrak{g}, \sigma, \sigma\tau)$: $M=Sp(l)/U(l)$. In this case $N=\mathfrak{sp}(j)/\mathfrak{u}(j) \oplus \mathfrak{sp}(k)/\mathfrak{u}(k)$;

(c) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma)$: $M=Sp(l)/Sp(j)\times Sp(k)$. In this case $N=\mathfrak{su}(l)/\mathfrak{s}(\mathfrak{u}(j)\oplus\mathfrak{u}(k))$.

For the the PSLA $(\mathfrak{g}, \sigma, \tau)$, the subsets $\Delta^+_{\mathfrak{k}+}, \Delta^+_{\mathfrak{k}-}, \Delta^+_{\mathfrak{p}+}, \Delta^+_{\mathfrak{p}-}$ of Δ^+ are given as follows:

(5.9)
$$\begin{aligned}
\Delta^+_{\mathfrak{k}+} &= \{\delta\in\Delta^+;\ \delta_l = 0,\ \delta_j = 0, 2\} \\
&= \left\{\delta\in\Delta^+;\ \delta = \begin{matrix}(0\cdots 01\cdots 10\cdots \overset{j}{0}\cdots \overset{l}{0}) \\ (0\cdots \overset{j}{0}\cdots 01\cdots 10\cdots \overset{l}{0})\end{matrix}\right\}, \\
\Delta^+_{\mathfrak{k}-} &= \{\delta\in\Delta^+;\ \delta_l = 0,\ \delta_j = 1\} \\
&= \left\{\delta\in\Delta^+;\ \delta = (0\cdots 01\cdots \overset{j}{1}\cdots 10\cdots \overset{l}{0})\right\}, \\
\Delta^+_{\mathfrak{p}+} &= \{\delta\in\Delta^+;\ \delta_l = 1,\ \delta_j=0, 2\} \\
&= \left\{\delta\in\Delta^+;\ \delta = \begin{matrix}(0\cdots \overset{j}{0}\cdots 01\cdots 12\cdots 2\overset{l}{1}) \\ (0\cdots 01\cdots 12\cdots \overset{j}{2}\cdots 2\overset{l}{1})\end{matrix}\right\}, \\
\Delta^+_{\mathfrak{p}-} &= \{\delta\in\Delta^+;\ \delta_l = \delta_j = 1\} \\
&= \left\{\delta\in\Delta^+;\ \delta = (0\cdots 01\cdots \overset{j}{1}\cdots 12\cdots 2\overset{l}{1})\right\}.
\end{aligned}$$

Moreover the dominant weights in $\Delta_{\mathfrak{k}-}, \Delta_{\mathfrak{p}+}, \Delta_{\mathfrak{p}-}$ are given by (5.10), (5.11), (5.12), respectively:

(5.10)
$$(1\cdots \overset{j}{1}\cdots 1\overset{l}{0}), \quad -(0\cdots 0\overset{j}{1}0\cdots \overset{l}{0}),$$

(5.11)
$$\begin{aligned}
&(0\cdots \overset{j}{0}2\cdots 2\overset{l}{1}), \quad (2\cdots \overset{j}{2}\cdots 2\overset{l}{1}), \\
&-(0\cdots \overset{j}{0}\cdots 0\overset{l}{1}), \quad -(0\cdots 0\overset{j}{2}\cdots 2\overset{l}{1}),
\end{aligned}$$

$$(5.12) \qquad (1\cdots\overset{j}{1}2\cdots2\overset{l}{1}), \quad -(0\cdots0\overset{j}{1}\cdots\overset{l}{1}).$$

We first see the injectivity of ρ for Case (a): $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$. Then ρ is a homomorphism of $(\mathfrak{p}_-^C)^*\otimes\mathfrak{k}_-^C$ to $\wedge^2(\mathfrak{p}_-^C)^*\otimes\mathfrak{p}_+^C$. The minus multiple of dominant weights in $\Delta_{\mathfrak{p}_-}$ are given by $(\alpha 1)$, $(\alpha 2)$ and the dominant weights in $\Delta_{\mathfrak{k}_-}$ are given by $(\beta 1)$, $(\beta 2)$:

$$(\alpha 1)-(1\cdots\overset{j}{1}2\cdots2\overset{l}{1}), \qquad (\alpha 2)\ (0\cdots0\overset{j}{1}\cdots\overset{l}{1}),$$
$$(\beta 1)\ (1\cdots\overset{j}{1}\cdots1\overset{l}{0}), \qquad (\beta 2)-(0\cdots0\overset{j}{1}0\cdots\overset{l}{0}).$$

Case (1): $l(u)=1$. Represent u as follows: $u=a\,\omega_\alpha\otimes X_\beta$. Then the pair (α, β) is one of $((\alpha r), (\beta s))$, where $r, s=1, 2$. Applying Lemma 2.3 for each pair, we obtain that $\rho(u)\neq 0$.

Case (2): $l(u)=2$. In this case there exists no decomposable u and thus we suppose that u is indecomposable. Consider the following elements in $\Delta_{\mathfrak{k}+}$:

$$(\mu 1)\ (10\cdots\overset{j}{0}\cdots\overset{l}{0}), \qquad (\mu 2)\ (0\cdots0\overset{j}{1}0\cdots\overset{l}{0}),$$
$$(\mu 3)\ (0\cdots01\overset{j}{0}\cdots\overset{l}{0}), \qquad (\mu 4)\ (0\cdots\overset{j}{0}\cdots01\overset{l}{0}).$$

Then such the triples $(\alpha, \beta'; \mu)$ as in §3, Case (2) are given in the following:

(1) $((\alpha 1), (\beta 1); (\mu 1))$, $j\geq 2$, (2) $((\alpha 1), (\beta 2); (\mu 1))$, $j=2$,
(3) $((\alpha 1), (\beta 1); (\mu 2))$, $l-j=2$, (4) $((\alpha 1), (\beta 2); (\mu 2))$, $l\geq 2$,
(5) $((\alpha 2), (\beta 1); (\mu 3))$, $j=2$, (6) $((\alpha 2), (\beta 2); (\mu 3))$, $j\geq 2$,
(7) $((\alpha 2), (\beta 1); (\mu 4))$, $l-j\geq 2$, (8) $((\alpha 2), (\beta 2); (\mu 4))$, $l-j=2$.

Lemma 2.4 is available for the cases (1), (4), (6), (7) and Lemma 2.2 is available for the other cases. Hence it follows that $\rho(u)\neq 0$.

Case (3): $l(u)\geq 3$. We see the weight spaces with dim≥ 3. Let λ be a weight in Λ and let α, β be weights such that $\lambda=-\alpha+\beta$, where $\alpha\in\Delta_{\mathfrak{p}_-}$ and $\beta\in\Delta_{\mathfrak{k}_-}$. Denote by a_k, b_k, λ_k the k-th components of α, β, λ, respectively. Since $a_l=\pm 1$ and $b_l=0$, it follows by (5.9) that $\lambda_l=\pm 1$.

Consider the case that $\lambda_l=1$. (For the case that $\lambda_l=-1$ we can similarly do the argument mentioned below.) Then it follows by (5.9) that $\lambda_j=0, 2$.

Suppose that $\lambda_j=0$ (resp. $\lambda_j=2$). Then the pair $\binom{\alpha}{\beta}$ has the form: $\begin{array}{c} \quad j \quad\ \ l \\ \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix} \end{array}$ (resp. $\begin{array}{c} \quad j \quad\ \ l \\ \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \end{array}$). If the weight space for λ has the dimension more than 3, it follows by Lemma 5.8 that

$$\lambda=(0\cdots\overset{j}{0}\cdots01\cdots12\cdots2\overset{l}{1}) \text{ (resp. } \lambda=(0\cdots01\cdots12\cdots\overset{j}{2}\cdots2\overset{l}{1}))$$

and the pair $\binom{\alpha}{\beta}$ has either of the forms

(5.13)

$$\begin{array}{c}\begin{array}{ccccccccccccccc} & & & & j & & & & & & & & & & l\end{array}\\ \left(\begin{array}{ccccccccccccccc} 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & -1 & \cdots & -1 & -2 & \cdots & -2 & -1 \\ 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{array}\right),\end{array}$$

$$\begin{array}{c}\begin{array}{ccccccccccccccc} & & & & j & & & & & & & & & & l\end{array}\\ \left(\begin{array}{ccccccccccccccc} 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & -2 & \cdots & -2 & -2 & \cdots & -2 & -1 \\ 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & -2 & \cdots & -1 & 0 & \cdots & 0 & 0 \end{array}\right)\end{array}$$

$$\left(\begin{array}{l} \text{resp.} \\ \begin{array}{c}\begin{array}{ccccccccccccccccc} & & & & & & & & j & & & & & & & & l\end{array}\\ \left(\begin{array}{ccccccccccccccccc} 0 & \cdots & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & \cdots & -1 & -2 & \cdots & -2 & -1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{array}\right),\end{array} \\ \begin{array}{c}\begin{array}{ccccccccccccccccc} & & & & & & & & j & & & & & & & & l\end{array}\\ \left(\begin{array}{ccccccccccccccccc} 0 & \cdots & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & \cdots & -1 & -2 & \cdots & -2 & -1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{array}\right)\end{array} \end{array}\right).$$

Hence for a maximal vector u in this weight space, it follows by Lemma 2.2 that $\rho(u)\neq 0$.

We next see the injectivity of ρ for Case (c): $\mathcal{V}=(\mathfrak{g}, \tau, \sigma)$. Note that in this case ρ is a homomorphism of $(\mathfrak{p}_-^{\mathbf{C}})^*\otimes\mathfrak{p}_+^{\mathbf{C}}$ to $\wedge^2(\mathfrak{p}_-^{\mathbf{C}})^*\otimes\mathfrak{k}_-^{\mathbf{C}}$. The minus multiple of dominant weights in $\Delta_{\mathfrak{p}_-}$ are given by $(\alpha 1)$, $(\alpha 2)$ and the dominant weights in $\Delta_{\mathfrak{p}+}$ are given by $(\beta 1)\sim(\beta 4)$:

$$\begin{array}{llll} (\alpha 1) & -(1\cdots\overset{j}{1}2\cdots2\overset{l}{1}), & (\alpha 2) & (0\cdots0\overset{j}{1}\cdots\overset{l}{1}), \\ (\beta 1) & (0\cdots\overset{j}{0}2\cdots2\overset{l}{1}), & (\beta 2) & (2\cdots\overset{j}{2}\cdots2\overset{l}{1}), \\ (\beta 3) & -(0\cdots\overset{j}{0}\cdots0\overset{l}{1}), & (\beta 4) & -(0\cdots0\overset{j}{2}\cdots2\overset{l}{1}). \end{array}$$

Case (1): $l(u)=1$. Represent u as follows: $u=a\,\omega_\alpha\otimes X_\beta$. Then the pair (α, β) is one of $((\alpha r), (\beta s))$, where $r=1, 2$ and $s=1, 2, 3, 4$. Applying Lemma 2.3 for each pair, we obtain that $\rho(u)=0$ only for the following cases: $((\alpha 1), (\beta 1))$ $(j=1)$, $((\alpha 1), (\beta 2))$ $(j=l-1)$, $((\alpha 2), (\beta 3))$ $(j=1)$, $((\alpha 2), (\beta 4))$ $(j=l-1)$.

Case (2): $l(u)=2$. In this case there exists no decomposable u and thus we suppose that u is indecomposable. Consider the following elements in $\Delta_{\mathfrak{k}+}$:

$$\begin{array}{llll} (\mu 1) & (10\cdots\overset{j}{0}\cdots\overset{l}{0}), & (\mu 2) & (0\cdots\overset{j}{0}10\cdots\overset{l}{0}), \\ (\mu 3) & (0\cdots01\overset{j}{0}\cdots\overset{l}{0}), & (\mu 4) & (0\cdots\overset{j}{0}\cdots01\overset{l}{0}). \end{array}$$

Then such the triples $(\alpha, \beta'; \mu)$ as in §3 (Case (2) of type AI) are given in the following:

(1) $((\alpha 1), (\beta 2); (\mu 1))$, $j\geq 2$, (2) $((\alpha 1), (\beta 4); (\mu 1))$, $j=2$,
(3) $((\alpha 1), (\beta 1); (\mu 2))$, $l-j\geq 2$, (4) $((\alpha 1), (\beta 3); (\mu 2))$, $l-j=2$,
(5) $((\alpha 2), (\beta 2); (\mu 3))$, $j=2$, (6) $((\alpha 2), (\beta 4); (\mu 3))$, $j\geq 2$,
(7) $((\alpha 2), (\beta 1); (\mu 4))$, $l-j=2$, (8) $((\alpha 2), (\beta 3); (\mu 4))$, $l-j\geq 2$.

Lemma 2.2 is available for the cases (2), (4), (5), (7), while Lemmas 2.2 and 2.4 are not available for the other cases. But for the cases that $j\neq 1$ in (3), (8) and the cases that $j\neq l-1$ in (1), (6), we see that Propostion 2.1 (1) does not hold and thus $\rho(u)\neq 0$. By virtue of Case (1), we do not need to see the remaining cases that $j=1, l-1$.

Case (3): $l(u)\geq 3$. We see the weight spaces with $\dim\geq 3$. Let λ be a weight in Λ and let α, β be weights such that $\lambda=-\alpha+\beta$, where $\alpha\in\Delta_{\mathfrak{p}_-}$ and $\beta\in\Delta_{\mathfrak{p}_+}$. Since $a_l=\pm 1$ and $b_l=\pm 1$, it follows by (5.9) that $\lambda_l=0, \pm 2$.

We first consider the case that $\lambda_l=2$. (For the case that $\lambda_l=-2$ we can similarly do the argument mentioned below.) Then it follows by (5.9) that $\lambda_j=1, 3$.

Suppose that $\lambda_j=1$ (resp. $\lambda_j=3$). Then the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has the form $\begin{pmatrix}& \overset{j}{-1} & \overset{l}{-1}\\ 0\cdots & 0 & 1\end{pmatrix}$ (resp. $\left(\begin{pmatrix}\overset{j}{-1} & & \overset{l}{-1}\\ 2 & \cdots 2 & 1\end{pmatrix}\right)$). If the weight space for λ has the dimension more than 3, it follows by Lemma 5.8 that

$$\lambda=(0\cdots 01\cdots\overset{j}{1}\cdots 1\underbrace{2\cdots 2}_{a}\underbrace{3\cdots 3}_{b}4\cdots 4\overset{l}{2})$$

$$\left(\text{resp. } \lambda=(0\cdots 0\underbrace{1\cdots 1}_{a}\underbrace{2\cdots 2}_{b}3\cdots\overset{j}{3}\cdots 34\cdots 4\overset{l}{2})\right)$$

where $a>0, b>0$, and the weight space has just dimension 3. Then the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has one of the following forms:

(5.14)
$$\begin{pmatrix}0\cdots 0 & -1\cdots\overset{j}{-1}\cdots -1 & -1\cdots -1 & -1\cdots -1 & -2\cdots -2 & \overset{l}{-1}\\ 0\cdots 0 & 0\cdots 0\cdots 0 & 1\cdots 1 & 2\cdots 2 & 2\cdots 2 & 1\end{pmatrix},$$
$$\begin{pmatrix}0\cdots 0 & -1\cdots\overset{j}{-1}\cdots -1 & -1\cdots -1 & -2\cdots -2 & -2\cdots -2 & \overset{l}{-1}\\ 0\cdots 0 & 0\cdots 0\cdots 0 & 1\cdots 1 & 1\cdots 1 & 2\cdots 2 & 1\end{pmatrix},$$
$$\begin{pmatrix}0\cdots 0 & -1\cdots\overset{j}{-1}\cdots -1 & -2\cdots -2 & -2\cdots -2 & -2\cdots -2 & \overset{l}{-1}\\ 0\cdots 0 & 0\cdots 0\cdots 0 & 0\cdots 0 & 1\cdots 1 & 2\cdots 2 & 1\end{pmatrix}$$
$$\left(\begin{array}{l} \begin{pmatrix}0\cdots 0 & 0\cdots 0 & 0\cdots 0 & -1\cdots\overset{j}{-1}\cdots -1 & -2\cdots -2 & \overset{l}{-1}\\ 0\cdots 0 & 1\cdots 1 & 2\cdots 2 & 2\cdots 2\cdots 2 & 2\cdots 2 & 1\end{pmatrix}, \\ \text{resp. } \begin{pmatrix}0\cdots 0 & 0\cdots 0 & -1\cdots -1 & -1\cdots\overset{j}{-1}\cdots -1 & -2\cdots -2 & \overset{l}{-1}\\ 0\cdots 0 & 1\cdots 1 & 1\cdots 1 & 2\cdots 2\cdots 2 & 2\cdots 2 & 1\end{pmatrix}, \\ \begin{pmatrix}0\cdots 0 & -1\cdots -1 & -1\cdots -1 & -1\cdots\overset{j}{-1}\cdots -1 & -2\cdots -2 & \overset{l}{-1}\\ 0\cdots 0 & 0\cdots 0 & 1\cdots 1 & 2\cdots 2\cdots 2 & 2\cdots 2 & 1\end{pmatrix} \end{array}\right).$$

Hence for a maximal vector u in this weight space, it follows by Lemma 2.4 that $\rho(u)\neq 0$.

We next consider the case that $\lambda_l=0$. Then it follows by (5.9) that $\lambda_j=\pm 1$.

Suppose that $\lambda_j=1$. (For the case that $\lambda_j=-1$ we can similarly do the argument mentioned below.) Then the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has either of the forms

$\begin{array}{ccc} & j & l \\ \left(\begin{matrix}\\ 0\cdots\end{matrix}\right. & \begin{matrix}-1\\ 0\end{matrix} & \left.\begin{matrix}-1\\ -1\end{matrix}\right)\end{array}, \begin{array}{cc} j & l \\ \left(\begin{matrix}1\\ 2\cdots 2\end{matrix}\right. & \left.\begin{matrix}1\\ 1\end{matrix}\right)\end{array}$. If the weight space for λ has the dimension more than 3, it follows by Lemma 5.8 that

$$\lambda = (0\cdots 01\cdots \overset{j}{1}\cdots 10\cdots \overset{l}{0})$$

and the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has one of the following forms:

(5.15)

$$\begin{array}{cccccccc} & & j & & & & & l \\ \left(\begin{matrix}0\cdots\cdots 0\\ 0\cdots\cdots 0\end{matrix}\right. & \begin{matrix}-1\cdots\cdots\\ 0\cdots\cdots\end{matrix} & \begin{matrix}-1\\ 0\end{matrix} & \begin{matrix}\cdots -1\\ \cdots\ 0\end{matrix} & \begin{matrix}-2\cdots -2\\ -1\cdots -1\end{matrix} & \begin{matrix}-2\cdots -2\\ -2\cdots -2\end{matrix} & \begin{matrix}-2\cdots -2\\ -2\cdots -2\end{matrix} & \left.\begin{matrix}-1\\ -1\end{matrix}\right)\end{array},$$

$$\begin{array}{cccccccc} & & j & & & & & l \\ \left(\begin{matrix}0\cdots\cdots 0\\ 0\cdots\cdots 0\end{matrix}\right. & \begin{matrix}-1\cdots\cdots\\ 0\cdots\cdots\end{matrix} & \begin{matrix}-1\\ 0\end{matrix} & \begin{matrix}\cdots -1\\ \cdots\ 0\end{matrix} & \begin{matrix}-1\cdots -1\\ 0\cdots\ 0\end{matrix} & \begin{matrix}-1\cdots -1\\ -1\cdots -1\end{matrix} & \begin{matrix}-2\cdots -2\\ -2\cdots -2\end{matrix} & \left.\begin{matrix}-1\\ -1\end{matrix}\right)\end{array},$$

$$\begin{array}{cccc} & j & & l \\ \left(\begin{matrix}0\cdots 0\ 0\cdots 0\ 0\cdots 0\ 1\cdots\\ 0\cdots 0\ 0\cdots 0\ 1\cdots 1\ 2\cdots\end{matrix}\right. & \begin{matrix}1\\ 2\end{matrix} & \begin{matrix}\cdots\cdots 1\ 2\cdots\cdots 2\\ \cdots\cdots 2\ 2\cdots\cdots 2\end{matrix} & \left.\begin{matrix}1\\ 1\end{matrix}\right)\end{array},$$

$$\begin{array}{cccc} & j & & l \\ \left(\begin{matrix}0\cdots 0\ 1\cdots 1\ 1\cdots 1\ 1\cdots\\ 0\cdots 0\ 1\cdots 1\ 2\cdots 2\ 2\cdots\end{matrix}\right. & \begin{matrix}1\\ 2\end{matrix} & \begin{matrix}\cdots\cdots 1\ 2\cdots\cdots 2\\ \cdots\cdots 2\ 2\cdots\cdots 2\end{matrix} & \left.\begin{matrix}1\\ 1\end{matrix}\right)\end{array}.$$

Hence for a maximal vector u in this weight space, it follows by Lemma 2.2 that $\rho(u)\neq 0$.

We last see the injectivity of ρ for Case (b): $\mathcal{V}=(\mathfrak{g}, \sigma, \sigma\tau)$. Note that in this case ρ is a homomorphism of $(\mathfrak{p}_+^C)^*\otimes\mathfrak{p}_-^C$ to $\wedge^2(\mathfrak{p}_+^C)^*\otimes\mathfrak{k}_-^C$. Hence we may regard roots α, β in this case as roots $-\beta$, $-\alpha$ in Case (c), respectively. We retain the notations in Case (c).

Case (1): $l(u)=1$. The pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ is one of $(-(\beta\, s), -(\alpha\, r))$, where $s=1, 2, 3, 4$ and $r=1, 2$. By Lemma 2.3 it follows that $\rho(u)\neq 0$ for all cases.

Case (2): $l(u)=2$. In this case there eixsts no decomposable u. Suppose that u is indecomposable. Then the triples $(\alpha, \beta'; \mu)$ are given as follows:

$$\begin{array}{ll} (1)\ (-(\beta 2), -(\alpha 1); (\mu 1)), & (2)\ (-(\beta 4), -(\alpha 1); (\mu 1)), \\ (3)\ (-(\beta 1), -(\alpha 1); (\mu 2)), & (4)\ (-(\beta 3), -(\beta 1); (\mu 2)), \\ (5)\ (-(\beta 2), -(\alpha 2); (\mu 3)), & (6)\ (-(\beta 4), -(\alpha 2); (\mu 3)), \\ (7)\ (-(\beta 1), -(\alpha 2); (\mu 4)), & (8)\ (-(\beta 3), -(\alpha 2); (\mu 4)). \end{array}$$

Lemma 2.2 is available for the cases (2), (4), (5), (7) and Lemma 2.4 is available for the other cases. Hence it follows that $\rho(u)\neq 0$.

Case (3): $l(u)\geq 3$. Similarly to Case (3) for Case (c), we have the cases which correspond to (5.14), (5.15). Lemma 2.4 is available for the former case and Lemma 2.2 is available for the latter case. Hence it follows that $\rho(u)\neq 0$.

Summing up the above arguments, we have the following result for PSLA's in C_j; the homomorphism ρ is not injective only for Case (c), $j=1, l-1$. These cases imply the cases of Example 3, (2) in §1.

Theorem 5.11. *Let $\mathcal{V}$ be the G-orbit which corresponds to a PSLA in a family of type CIII. Then the $\mathcal{V}$-geometry admits non-totally geodesic $\mathcal{V}$-submanifolds if and only if it is one of the $\mathcal{V}$-geometries in Example* 3, (2).

6. The PSLA's with Lie algebra $\mathfrak{g}$ of type D_l

Let $\mathfrak{g}$ be the Lie algebra of type D_l, $l\geq 4$, that is, the Lie algebra $\mathfrak{so}(2l)$ of real skew symmetric matrices of degree $2l$. Then the Dynkin diagram of the fundamental root system Π is given as follows:

$$\begin{array}{ccccccc} \bigcirc & - & \bigcirc & -\cdots- & \bigcirc & - & \bigcirc \\ \alpha_1 & & \alpha_2 & & \alpha_{l-2}\,| & & \alpha_{l-1} \\ & & & & \bigcirc & & \\ & & & & \alpha_l & & \end{array} \qquad \begin{array}{l} -\alpha_0=\alpha_1+2\alpha_2+\cdots+2\alpha_{l-2} \\ \quad +\alpha_{l-1}+\alpha_l \end{array}$$

Put θ_i, θ_{jk} as in §3 and moreover put

$$\theta_{jkr}=\exp \operatorname{ad}(\sqrt{-1}\pi(H_j+H_k+H_r))$$

for $1\leq j\leq k\leq r\leq l$. Let $\mathcal{D}_{ij}(1\leq j<i\leq l)$, $\mathcal{D}_{i\,;\,jk}(1\leq j<i<k\leq l)$, $\mathcal{D}_{l-2\,;\,j,l-1,l}$ $(1\leq j\leq l-3)$, $\mathcal{D}_{l-2\,;\,l-1,l}$ be the families which contain the PSLA's $(\mathfrak{g}, \theta_i, \theta_j)$, $(\mathfrak{g}, \theta_i, \theta_{jk})$, $(\mathfrak{g}, \theta_{l-2}, \theta_{j,l-1,l})$, $(\mathfrak{g}, \theta_{l-2}, \theta_{l-1,l})$, respectively.

Lemma 6.1. *A PSLA $(\mathfrak{g}, \sigma, \tau)$ of inner type is equivalent to a PSLA which belongs to one of the following families, by an inner automorphism of* $\mathfrak{g}$:

(1) $\mathcal{D}_{ij}$, $1\leq j<i\leq l-2$, (2) $\mathcal{D}_{l-1,j}$, $1\leq j\leq l-2$,
(3) $\mathcal{D}_{lj}$, $1\leq j\leq l-2$, (4) $\mathcal{D}_{l,l-1}$,
(5) $\mathcal{D}_{i\,;\,jk}$, $2\leq j<i<k\leq l-2$, (6) $\mathcal{D}_{i\,;\,1,l-1}$, $2\leq i\leq l-2$,
(7) $\mathcal{D}_{i\,;\,1,l}$, $2\leq i\leq l-2$, (8) $\mathcal{D}_{2\,;\,1,k}$, $3\leq k\leq l-2$,
(9) $\mathcal{D}_{l-2\,;\,j,l-1,l}$, $2\leq j\leq l-3$, (10) $\mathcal{D}_{2\,;\,134}$, $l=4$,
(11) $\mathcal{D}_{l-2\,;\,l-1,l}$.

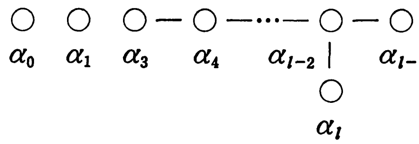
Proof. We may assume that $\sigma=\theta_i$. We divide into the following cases: (1) $i=1$; (2) $i=2$ $(l\geq 5)$; (3) $2<i<l-2$ $(l\geq 6)$; (4) $i=l-2$ $(l\geq 5)$; (5) $i=l-1$; (6) $i=l$; (7) $i=2$ $(l=4)$.

Case (1): $i=1$. Then $\mathfrak{k}=\mathfrak{c}+\mathfrak{k}_s$ and the Dynkin diagram of Π_s is given as follows:

$$\begin{array}{ccccccc} \bigcirc & - & \bigcirc & -\cdots- & \bigcirc & - & \bigcirc \\ \alpha_2 & & \alpha_3 & & \alpha_{l-2}\,| & & \alpha_{l-1} \\ & & & & \bigcirc & & \\ & & & & \alpha_l & & \end{array}$$

Hence we may assume that the restriction $\bar{\tau}$ of τ is given as follows: $\bar{\tau}=\exp\operatorname{ad}(\sqrt{-1}\pi K_j)$, where $2\leq j\leq l$. Then it follows that $K_j=aH_1+H_j$ for some $a\in\boldsymbol{R}$, and thus the PSLA $(\mathfrak{g},\sigma,\tau)$ belongs to $\mathcal{D}_{j1}$.

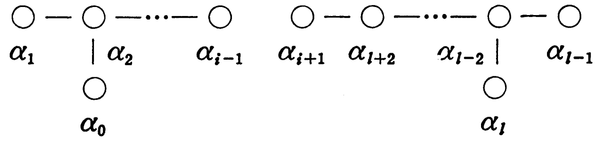
Case (2): $i=2$ ($l\geq 5$). Then $\mathfrak{k}=\mathfrak{k}_s$ and the Dynkin diagram of Π_s is given as follows:



If we put $\bar{\tau}=\exp\operatorname{ad}(\sqrt{-1}\pi K)$, the following cases are considerable: $K=K_0$; $K=K_1$; $K=K_k$, $3\leq k\leq l$; $K=K_0+K_1$; $K=K_0+K_k$, $3\leq k\leq l$; $K=K_1+K_k$, $3\leq k\leq l$; $K=K_0+K_1+K_k$, $3\leq k\leq l$. By Lemma 1.2 (1), the following cases moreover have involutive extensions of $\bar{\tau}$: (i) $K=K_k$, $3\leq k\leq l-2$; (ii) $K=K_0+K_1$; (iii) $K=K_0+K_{l-1}$; (iv) $K=K_0+K_l$; (v) $K=K_1+K_{l-1}$; (vi) $K=K_1+K_l$; (vii) $K=K_0+K_1+K_k$, $3\leq k\leq l-2$. We represent the vectors K_r by the vectors $H_1,\cdots,H_l$.

For Case (i) it follows that $K_k=-H_2+H_k$ and thus the PSLA $(\mathfrak{g},\sigma,\tau)$ belongs to $\mathcal{D}_{k2}$. For Case (ii) it follows that $K_0+K_1=H_1-H_2$ and thus the PSLA $(\mathfrak{g},\sigma,\tau)$ belongs to $\mathcal{D}_{21}$. For Case (iii) it follows that $K_0+K_{l-1}=H_{l-1}-H_2$ and thus the PSLA $(\mathfrak{g},\sigma,\tau)$ belongs to $\mathcal{D}_{l-1,2}$. For Case (iv) it follows that $K_0+K_l=H_l-H_2$ and thus the PSLA $(\mathfrak{g},\sigma,\tau)$ belongs to $\mathcal{D}_{l2}$. For Case (v) it follows that $K_1+K_{l-1}=H_1+H_{l-1}-H_2$ and thus the PSLA $(\mathfrak{g},\sigma,\tau)$ belongs to $\mathcal{D}_{2;1,l-1}$. For Case (vi) it follows that $K_1+K_l=H_1+H_l-H_2$ and thus the PSLA $(\mathfrak{g},\sigma,\tau)$ belongs to $\mathcal{D}_{2,1l}$. For Case (vii) it follows that $K_0+K_1+K_k=\boldsymbol{H}_1-H_2+H_k$ and thus the PSLA $(\mathfrak{g},\sigma,\tau)$ belongs to $\mathcal{D}_{2;1k}$.

Case (3): $2<i<l-2$ ($l\geq 6$). Then $\mathfrak{k}=\mathfrak{k}_s$ and the Dynkin diagram of Π_s is given as follows:

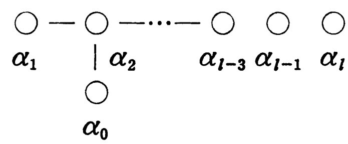


If we put $\bar{\tau}=\exp\operatorname{ad}(\sqrt{-1}\pi K)$, the following cases are considerable: $K=K_j$, $0\leq j\leq i-1$; $K=K_k$, $i+1\leq k\leq l$; $K=K_j+K_k$, $0\leq j\leq i-1$, $i+1\leq k\leq l$. By Lemma 1.2 (1), the following cases moreover have involutive extensions of $\bar{\tau}$: (i) $K=K_j$, $2\leq j\leq i-1$; (ii) $K=K_k$, $i+1\leq k\leq l-2$; (iii) $K=K_0+K_{l-1}$; (iv) $K=K_0+K_l$; (v) $K=K_1+K_{l-1}$; (vi) $K=K_1+K_l$; (vii) $K=K_j+K_k$, $2\leq j\leq i-1$, $i+1\leq k\leq l-2$. We represent the vectors K_r by the vectors $H_1,\cdots,H_l$.

For Case (i) follows that $K_j=-H_i+H_j$ and thus the PSLA $(\mathfrak{g},\sigma,\tau)$ belongs to $\mathcal{D}_{ij}$. For Case (ii) it follows that $K_k=-H_i+H_k$ and thus the PSLA $(\mathfrak{g},\sigma,\tau)$

belongs to $\mathscr{D}_{ki}$. For Case (iii) it follows that $K_0+K_{l-1}=H_{l-1}-H_i$ and thus the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{i,l-1}$. For Case (iv) it follows that $K_0+K_l=H_l-H_i$ and thus the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{il}$. For Case (v) it follows that $K_1+K_{l-1}=H_1+H_{l-1}-H_i$ and thus the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{i\,;\,1,l-1}$. For Case (vi) it follows that $K_1+K_l=H_1+H_l-H_i$ and thus the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{i\,;\,1l}$. For Case (vii) it follows that $K_j+K_k=H_j-2H_i+H_k$ and thus thePSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{i\,;\,jk}$.

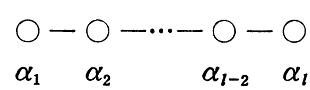
Case (4): $i=l-2$ $(l\geq 5)$. Then $\mathfrak{k}=\mathfrak{k}_s$ and the Dynkin diagram of Π_s is given as follows:



Put $\bar{\tau}=\exp \mathrm{ad}(\sqrt{-1}\pi K)$. Similarly to Case (2), the following cases have involutive extensions of $\bar{\tau}$: (i) $K=K_j$, $2\leq j\leq l-3$; (ii) $K=K_0+K_{l-1}$; (iii) $K=K_0+K_l$; (iv) $K=K_1+K_{l-1}$; (v) $K=K_1+K_l$; (vi) $K=K_{l-1}+K_l$; (vii) $K=K_{l-1}+K_l+K_j$, $2\leq j\leq l-3$.

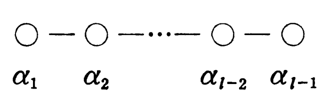
For Case (i) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{l-2,j}$. For Case (ii) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{l-1,l-2}$. For Case (iii) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{l,l-2}$. For Case (iv) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{l-2\,;\,1,l-1}$. For Case (v) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{l-2\,;\,1,l}$. For Case (vi) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{l-2\,;\,l-1,l}$. For Case (vii) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{l-2\,;\,l-1,l}$.

Case (5): $i=l-1$. Then $\mathfrak{k}=\mathfrak{c}+\mathfrak{k}_s$ and the Dynkin diagram of Π_s is given as follows:



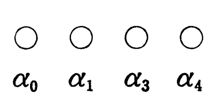
Put $\bar{\tau}=\exp \mathrm{ad}(\sqrt{-1}\pi K_j)$, where $j=1, \cdots, l-2, l$. Similarly to Case (1), the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{l-1,j}$ $(1\leq j\leq l-2)$ or $\mathscr{D}_{l,l-1}$.

Case (6): $i=l$. Then $\mathfrak{k}=\mathfrak{c}+\mathfrak{k}_s$ and the Dynkin diagram of Π_s is given as follows:



Put $\bar{\tau}=\exp \mathrm{ad}(\sqrt{-1}\pi K_j)$, where $1\leq j\leq l-1$. Similarly to Case (1), the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathscr{D}_{l,j}$.

Case (7): $i=2$ $(l=4)$. Then $\mathfrak{k}=\mathfrak{k}_s$ and the Dynkin diagram of Π_s is given as follows:



Put $\bar{\tau}=\exp \operatorname{ad}(\sqrt{-1}\pi K)$. Similarly to Case (2), the following cases have involutive extensions of $\bar{\tau}$: (i) $K=K_0+K_1$; (ii) $K=K_0+K_3$; (iii) $K=K_0+K_4$; (iv) $K=K_1+K_3$; (v) $K=K_1+K_4$; (vi) $K=K_3+K_4$; (vii) $K=K_0+K_1+K_3+K_4$. We represent the vectors K_r by the vectors $H_1, \cdots, H_4$.

For Case (i) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{D}_{21}$. For Case (ii) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{D}_{32}$. For Case (iii) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{D}_{42}$. For Case (iv) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{D}_{2;13}$. For Case (v) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{D}_{2;14}$. For Case (vi) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{D}_{2;34}$. For Case (vii) the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{D}_{2;134}$. $\square$

Pur $V=\sqrt{-1}\mathfrak{h}$ and take an orthonormal basis $\{e_1, \cdots, e_l\}$ which satisfies that $\alpha_i=e_i-e_{i+1}$ for $1\leq i\leq l-1$, and $\alpha_l=e_{l-1}+e_l$. Then it holds that

$$\begin{aligned} H_i &= e_1+\cdots+e_i \qquad \text{for} \quad 1\leq i\leq l-2\,, \\ H_{l-1} &= (1/2)\,(e_1+\cdots+e_{l-2}+e_{l-1}-e_l)\,, \\ H_l &= (1/2)\,(e_1+\cdots+e_{l-2}+e_{l-1}+e_l)\,. \end{aligned}$$

The Weyl group $W(\Delta)$ is generated by the permutations of $e_1, \cdots, e_l$ and the following mappings w_ε^-: Let $\varepsilon=(\varepsilon(1), \cdots, \varepsilon(l))$, where $\varepsilon(i)=\pm 1$ and $\Pi_{i=1}^l \varepsilon(i)=1$. Then $w_\varepsilon^-(e_i)=\varepsilon(i)\, e_i$ for all i.

Define elements $w_0^k(1\leq k\leq l)$ and $w_1^{jk}(j, k\geq 1, j+k\leq l)$ in $W(\Delta)$ in the same way as in §3. Then the following similarly hold:

$$w_0^k(H_i) = \begin{cases} H_k-H_{k-i} & (1\leq i<k<l-1)\,, \\ H_{l-1}+H_l-H_{l-1-i} & (1\leq i<k=l-1)\,, \\ 2H_l-H_{l-i} & (1<i<k=l)\,, \\ H_l-H_{l-1} & (1=i<k=l)\,, \\ H_i & (1\leq k\leq i\leq l)\,. \end{cases}$$

$$w_1^{jk}(H_i) = \begin{cases} H_{j+k}-H_k & (i=j, j+k<l-1)\,, \\ H_{l-1}+H_l-H_k & (i=j, j+k=l-1)\,, \\ 2H_l-H_k & (i=j, j+k=l, k\leq l-2)\,, \\ H_l-H_{l-1} & (i=j, j+k=l, k=l-1)\,, \\ H_i & (j+k\leq i\leq l)\,. \end{cases}$$

Let φ_0^k, φ_1^{jk} be inner automorphisms of $\mathfrak{g}$ induced by w_0^k, w_1^{jk}, respectively.

Moreover let ψ_0 be an automorphism of $\mathfrak{g}$ induced by the following Dynkin automorphism v_0 of Π: $v_0(\alpha_i)=\alpha_i(1\leq i\leq l-1)$, $v_0(\alpha_{l-1})=\alpha_l$, and $v_0(\alpha_l)=\alpha_{l-1}$, i.e., $v_0(e_i)=e_i(1\leq i\leq l-1)$ and $v_0(e_l)=-e_l$. For $\varepsilon=(1\cdots 1 \overset{i}{-1}\, 1\cdots 1 \overset{l}{-1})$ put $v_i^-=w_\varepsilon^-\, v_0$ and let $\psi_i^-(1\leq i\leq l-1)$ be automorphisms of $\mathfrak{g}$ induced by v_i^-. Then the following equivalences moreover hold:

(1) $\mathcal{D}_{l-1,j}\cong\mathcal{D}_{lj}(1\leq j\leq l-2)$ and $\mathcal{D}_{i;1,l-1}\cong\mathcal{D}_{i;1,l}$.
These are obtained by ψ_0;

(2) $\mathcal{D}_{j;1l} \cong \mathcal{D}_{l,l-j}$ $(2 \leq j \leq l-2)$. This is obtained by $\psi_0 \varphi_0^l$;

(3) $\mathcal{D}_{2;1j} \cong \mathcal{D}_{l-2;l-j,l-1,l}$ $(3 \leq j \leq l-2)$, $\mathcal{D}_{21} \cong \mathcal{D}_{l-2;l-1,l}$ and $\mathcal{D}_{l1} \cong \mathcal{D}_{l,l-1}$. These are obtained by φ_0^l.

Hence we may consider only the families of the following cases:

(1) $\mathcal{D}_{ij}(1 \leq j < i \leq l-2)$; (2) $\mathcal{D}_{lj}(1 \leq j \leq l-2)$;

(3) $\mathcal{D}_{i;jk}(2 \leq j < i < k \leq l-2$ or $1 = j < i = 2 < k \leq l-2)$; (4) $\mathcal{D}_{2;134}$.

From the proof of Lemma 6.1, we can see that the subalgebras $\mathfrak{k}_+$ for Cases (1), (2), (3), (4) are diffeιent from each other. Hence these families are never equivalent to each other.

We first see the equivalences among the families $\mathcal{D}_{ij}(1 \leq j < i \leq l-2)$ and the equivalences among the PSLA's which belong to each family. For a family $\mathcal{D}_{ij}$ put $i=j+k$ and $l=i+r$. Then $j, k \geq 1$, $r \geq 2$ and the following holds.

Proposition 6.2. *Two families $\mathcal{D}_{ij}$, $\mathcal{D}_{i'j'}$ are equivalent to each other if and only if the triples (j, k, r), (j', k', r') coincide except order.*

Proof. The proof is done in the same way as that of Proposition 3.2. Consider the PSLA's $(\mathfrak{g}, \theta_i, \theta_j)$, $(\mathfrak{g}, \theta_{i'}, \theta_{j'})$. Then it follows that $\dim \mathfrak{k}_- = 4jk$, $\dim \mathfrak{p}_+ = 4kr$, $\dim \mathfrak{p}_- = 4jr$, and that $\dim \mathfrak{k}'_- = 4j'k'$, $\dim \mathfrak{p}'_+ = 4k'r'$, $\dim \mathfrak{p}'_- = 4j'r'$. (See (6.1) later.) If $\mathcal{D}_{ij}$ is equivalent to $\mathcal{D}_{i'j'}$, the triples (jk, kr, rj), $(j'k', k'r', r'j')$ coincide except order and so the triples (j, k, r), (j', k', r') coincide except order.

To prove the converse we may recall the proof of Proposition 3.2. Then the following equivalences similarly hold:

(1) $\mathcal{D}_{ij} \cong \mathcal{D}_{ik}$, $j \geq 1, k \geq 1, r \geq 2$; (2) $\mathcal{D}_{ij} \cong \mathcal{D}_{r+k,r}$, $k \geq 1, j \geq 1, r \geq 2$.

Using these equivalences we see that the family $\mathcal{D}_{ij}$ is equivalent to a family with triple to which the triple (j, k, r) is rearranged in smaller order. Hence, if triples (i, j, k), (i', j', k') coincide except order, the family $\mathcal{D}_{ij}$ is equivalent to the family $\mathcal{D}_{i'j'}$. □

By virtue of this proposition we may consider only the families $\mathcal{D}_{ij}$ with triple (j, k, r) such that $j \leq k \leq r$. Such a family is said to be a *proper family of tpye DI* and a family without the above condition is said to be simply a family of type *DI*. The following proposition can be proved in the same way as Proposition 3.3.

Proposition 6.3. *Let $\mathcal{D}_{ij}$ be a proper family of type DI with triple (j, k, r) and set $(\mathfrak{g}, \sigma, \tau) = (\mathfrak{g}, \theta_i, \theta_j)$. Then the following hold:*

(1) *If $j < k < r$, all the PSLA's in $\mathcal{D}_{ij}$ are non-equivalent to each other;*

(2) *If $j = k < r$, only the equivalences of first type hold;*

(3) *If $j < k = r$, only the equivalences of second type hold;*

(4) *If $j = k = r$, all the PSLA's in $\mathcal{D}_{ij}$ are equivalent to each other.*

We next see the equivalences among families $\mathcal{D}_{i\,;\,jk}$ $(1\leq j<i<k\leq l-2)$ and the equivalences among the PSLA's which belong to each family. For a family $\mathcal{D}_{i\,;\,jk}$ put $j=a$, $i=j+b$, $k=i+c$, $l=k+d$. Then $a, b, c\geq 1$, $d\geq 2$ and the following holds.

Proposition 6.4. *Two families $\mathcal{D}_{i\,;\,jk}$, $\mathcal{D}_{i'\,;\,j'k'}$ are equivalent to each other if and only if the quadruples (a, b, c, d), (a', b', c', d') coincide except order.*

Proof. This is done in the same way as that of Proposition 3.4. Consider the PSLA's $(\mathfrak{g}, \theta_i, \theta_{jk})$ $(\mathfrak{g}, \theta_{i'}, \theta_{j'k'})$. Then it follows that $\dim \mathfrak{k}_-=4(ab+cd)$, $\dim \mathfrak{p}_+=4(bc+ad)$, $\dim \mathfrak{p}_-=4(ac+bd)$ and that $\dim \mathfrak{k}'_-=4(a'b'+c'd')$ $\dim \mathfrak{p}'_+=4(b'c'+a'd')$, $\dim \mathfrak{p}'_-=4(a'c'+b'd')$. (See (6.5) later.) If $\mathcal{D}_{i\,;\,jk}$, $\mathcal{D}_{i'\,;\,j'k'}$ are equivalent to each other, the triples $(ab+cd, bc+ad, ac+bd)$, $(a'b'+c'd', b'c'+a'd', a'c'+b'd')$ coincide except order. Noting that $a+b+c+d=a'+b'+c'+d'=l$, we see that the quadruples (a, b, c, d), (a', b', c', d') also coincide except order.

To prove the converse we may recall the proof of Proposition 3.4. Then we similarly have the following equivalences:

(1) $\mathcal{D}_{i\,;\,jk}\cong\mathcal{D}_{i\,;\,bk}$, $1\leq j<i<k\leq l-2$;

(2) $\mathcal{D}_{i\,;\,jk}\cong\mathcal{D}_{k-j\,;\,k-i,k}$, $1\leq j<i<k\leq l-2$;

(3) $\mathcal{D}_{i\,;\,jk}\cong\mathcal{D}_{d+c\,;\,d,d+c+b}$, $2\leq j<i<k\leq l-2$.

Using these equivalences we see that $\mathcal{D}_{i\,;\,jk}$ is equivalent to a family with quadruple to which the quadruple (a, b, c, d) is rearranged in smaller order. Hence, if quadruples (a, b, c, d), (a', b', c', d') coincide except order, the family $\mathcal{D}_{i\,;\,jk}$ is equivalent to the family $\mathcal{D}_{i'\,;\,j'k'}$. $\square$

By virtue of this proposition we may consider only the families $\mathcal{D}_{i\,;\,jk}$ with quadruple (a, b, c, d) such that $a\leq b\leq c\leq d$. Such a family is said to be a *proper family of type DII* and a family without the above condition is said to be simply a family of type *DII*. The following proposition can be proved in the same way as Proposition 3.5.

Proposition 6.5. *Let $\mathcal{D}_{i\,;\,jk}$ be a proper family of type DII with quadruple (a, b, c, d) and set $(\mathfrak{g}, \sigma, \tau)=(\mathfrak{g}, \theta_i, \theta_{jk})$. Then the following hold:*

(1) *If $a<b<c<d$, all the PSLA's in $\mathcal{D}_{i\,;\,jk}$ are non-equivalent to each other;*

(2) *If $a=b<c\leq d$ or $a\leq b<c=d$, only the equivalences of first type hold;*

(3) *If $a<b=c<d$, only the equivalences of second type hold;*

(4) *If $a=b=c<d$, $a<b=c=d$, or $a=b=c=d$, all the PSLA's in $\mathcal{D}_{i\,;\,jk}$ are equivalent to each other.*

We next see the equivalenecs among families $\mathcal{D}_{lj}$ $(1\leq j\leq l-2)$ and the equivalences among the PSLA's which belong to each family. In the following the families $\mathcal{D}_{lj}$ are denoted by $\mathcal{D}_j$. For a family $\mathcal{D}_j$ put $l=j+k$. Then $j\geq 1$,

$k \geq 2$ and the following holds.

Proposition 6.6. *Two families $\mathcal{D}_j$, $\mathcal{D}_{j'}$ are eqivalent to each other if and only if the pairs (j, k), (j', k') coincide except order.*

Proof. This is done in the same way as that of Propostion 5.6. Consider the PSLA's $(\mathfrak{g}, \theta_l, \theta_j)$, $(\mathfrak{g}, \theta_l, \theta_{j'})$. Then the semisimple part of $\mathfrak{k}_+$(resp. $\mathfrak{k}'_+$) is the sum of Lie algebras of types A_{j-1} (resp. $A_{j'-1}$) and A_{k-1} (resp. $A_{k'-1}$).

Suppose that $\mathcal{D}_j$ is equivalent to $\mathcal{D}_{j'}$. Since $\mathfrak{k}_+$ is isomorphic to $\mathfrak{k}'_+$, it follows that pairs (j, k), (j', k') coincide except order.

To prove the converse we may recall the proof of Propostion 5.6. Then the following equivalence similarly holds: $\mathcal{D}_j \cong \mathcal{D}_k$, $2 \leq j \leq l-2$. Using this equivalence we see that $\mathcal{D}_j$ is equivalent to a family with pair to which the pair (j, k) is rearranged in smaller order. Hence, if pairs (j, k), (j', k') coincide except order, the family $\mathcal{D}_j$ is equivalent to the family $\mathcal{D}_{j'}$. $\square$

By virtue of this proposition we may consider only the families $\mathcal{D}_j$ with pair (j, k) such that $j \leq k$. Such a family is said to be a *proper family of type DIII* and a family without the above condition is said to be simply a family of type *DIII*.

Proposition 6.7. *Let $\mathcal{D}_j$ be a proper family of type DIII with pair (j, k) and set $(\mathfrak{g}, \sigma, \tau) = (\mathfrak{g}, \theta_l, \theta_j)$. Then the following hold:*

(1) *If $l \geq 5$ or $l=4, j=2$, only the equivalences of second type hold;*

(2) *If $l=4, j=1$, all PSLA's in $\mathcal{D}_1$ are equivalent to each other.*

Proof. For general l the equivalences of second type are obtained by $\psi_{\bar{j}+1} \cdots \psi_{\bar{l}-1} \psi_0$. (See the proof of Proposition 5.7.) We also note that

$$\mathfrak{k}_- = \mathfrak{su}(l-1)/\mathfrak{s}(\mathfrak{u}(j) \oplus \mathfrak{u}(k-1)), \quad \mathfrak{p}_\pm = \mathfrak{so}(2j)/\mathfrak{u}(j) \oplus \mathfrak{so}(2k)/\mathfrak{u}(k).$$

(1) In this case, as $\mathfrak{k}_+$-modules, $\mathfrak{k}_-$ is not isomorphic to $\mathfrak{p}_\pm$. This implies the non-equivalence of the other pairs.

(2) In this case, $\mathfrak{k}_-$, $\mathfrak{p}_\pm$ are isomorphic to each other as $\mathfrak{k}_+$-modules. We may show the equivalence: $(\mathfrak{g}, \sigma, \tau) \cong (\mathfrak{g}, \tau, \sigma)$. Since $l=4$, we moreover have the following Dynkin automorphism v_{14} of Π; $v_{14}(\alpha_1) = \alpha_4$, $v_{14}(\alpha_4) = \alpha_1$, and $v_{14}(\alpha_i) = \alpha_i$ for $i=2, 3,$. Let ψ_{14} be an automorphism of $\mathfrak{g}$ induced by v_{14}. Then the equivalence is given by ψ_{14}. $\square$

We last see the equivalences among PSLA's which belong to $\mathcal{D}_{2;134}$. This is said to be the family *of type D_0*.

Proposition 6.8. *All PSLA's in the family of type $\mathcal{D}_0$ are equivlent to each other.*

Proof. Set $(\mathfrak{g}, \sigma, \tau)=(\mathfrak{g}, \theta_2, \theta_{134})$. We may show the following equivalences:

(1) $(\mathfrak{g}, \sigma, \tau) \cong (\mathfrak{g}, \sigma, \sigma\tau)$ and (2) $(\mathfrak{g}, \sigma, \tau) \cong (\mathfrak{g}, \sigma\tau, \tau)$.

The equivalences (1) and (2) are obtained by φ_1^{11} and $(\varphi_1^{11})^{-1}\varphi_0^3\varphi_1^{11}$, respectively. $\square$

We now see the injectivity of the $\mathfrak{k}_+$-homomorphism ρ for each PSLA in the families of types DI, DII, $DIII$, D_0. Similarly to in §3, fix a positive integer r and set

$$
\begin{aligned}
R_1 &= \{\pm(\overbrace{0\cdots0}^{a}\ \overbrace{1\cdots1}^{b}\ \overbrace{0\cdots0}^{c})\in \boldsymbol{Z}^r;\ a\geq 0, b\geq 0, c\geq 0\}\,,\\
R_1'' &= \{\pm(\overbrace{0\cdots0}^{a}\ \overbrace{1\cdots1}^{b}\ \overbrace{0\cdots0}^{c}\,|\,\overset{r-1}{0}\ \overset{r}{0})\in \boldsymbol{Z}^r;\ a\geq 0, b\geq 0, c\geq 0\}\\
&\quad\cup \{\pm(\overbrace{0\cdots0}^{a}\ \overbrace{1\cdots1}^{b}\,|\,\overset{r-1}{1}\ \overset{r}{0})\in \boldsymbol{Z}^r;\ a\geq 0, b\geq 0\}\\
&\quad\cup \{\pm(\overbrace{0\cdots0}^{a}\ \overbrace{1\cdots1}^{b}\,|\,\overset{r-1}{0}\ \overset{r}{1})\in \boldsymbol{Z}^r;\ a\geq 0, b\geq 0\}\\
&\quad\cup \{\pm(\overbrace{0\cdots0}^{a}\ \overbrace{1\cdots1}^{b}\,|\,\overset{r-1}{1}\ \overset{r}{1})\in \boldsymbol{Z}^r;\ a\geq 0, b\geq 0\}\,,\\
R_2 &= \{\pm(\overbrace{0\cdots0}^{a}\ \overbrace{1\cdots1}^{b}\ \overbrace{2\cdots2}^{c})\in \boldsymbol{Z}^r;\ a\geq 0, b\geq 0, c>0\}\,,\\
R_2'' &= \{\pm(\overbrace{0\cdots0}^{a}\ \overbrace{1\cdots1}^{b}\ \overbrace{2\cdots2}^{c}\,|\,\overset{r-1}{1}\ \overset{r}{1})\in \boldsymbol{Z}^r;\ a\geq 0, b\geq 0, c>0\}\,,\\
R &= R_1\cup R_2\,,\quad R'' = R_1''\cup R_2''\,,\\
R^2 &= \{\tbinom{\alpha}{\beta};\ \alpha, \beta\in R\}\,,\quad R''^2 = \{\tbinom{\alpha}{\beta};\ \alpha, \beta\in R''\}\,.
\end{aligned}
$$

Moreover let $R^2[\tbinom{a}{b}_i]$, $R^2[\tbinom{a}{b}_i, \tbinom{c}{d}_j]$, $R^2_\lambda[*]$ be subsets of R^2 defined as in §3. The subsets $R''^2[\tbinom{a}{b}_i]$, $R''^2[\tbinom{a}{b}_i, \tbinom{c}{d}_j]$, $R''^2_\lambda[*]$ may be also defined similarly. Then we can check the following lemma by a usual argument.

Lemma 6.9. *Let λ be an r-tuples in $\boldsymbol{Z}^r$. Then the following hold:*

(1) *The following each set has at most* 2 *elements:*

$$
\begin{aligned}
&R^2_\lambda[\tbinom{-1}{1}_r]\,,\quad R^2_\lambda[\tbinom{-1}{1}_1, \tbinom{0}{0}_r]\,,\quad R^2_\lambda[\tbinom{-1}{1}_1, \tbinom{-2}{2}_r]\,,\quad R^2_\lambda[\tbinom{0}{1}_1, \tbinom{-1}{2}_r]\,,\\
&R^2_\lambda[\tbinom{0}{1}_1, \tbinom{1}{0}_r]\,,\quad R^2_\lambda[\tbinom{1}{1}_1, \tbinom{0}{2}_r]\,,\quad R''^2_\lambda[\tbinom{1}{1}_1, \tbinom{1}{0}_r]\,,\quad R''^2_\lambda[\tbinom{-1}{0}_{r-1}, \tbinom{0}{1}_r]\,;
\end{aligned}
$$

(2) *For the following sets Lemma* 4.6 ((2) *through* (7)) *and Lemma* 5.8, (8) *hold:*

$$
\begin{aligned}
&R^2_\lambda[\tbinom{1}{1}_r]\,,\quad R^2_\lambda[\tbinom{0}{1}_1, \tbinom{-1}{0}_r]\,,\quad R^2_\lambda[\tbinom{1}{1}_1, \tbinom{0}{0}_r]\,,\quad R^2_\lambda[\tbinom{1}{1}_1, \tbinom{2}{2}_r]\,,\\
&R^2_\lambda[\tbinom{-1}{1}_1, \tbinom{-2}{0}_r]\,,\quad R^2_\lambda[\tbinom{1}{2}_r]\,,\quad R^2_\lambda[\tbinom{1}{1}_1]\,,\quad R^2_\lambda[\tbinom{-1}{1}_1]\,,\\
&R^2_\lambda[\tbinom{-1}{2}_r]\,;
\end{aligned}
$$

(3) *The set $R''^2_\lambda[\tbinom{-1}{0}_1, \tbinom{-1}{-1}_r]$* (resp. $R''^2_\lambda[\tbinom{-1}{0}_1, \tbinom{0}{1}_r]$) *has at most* 1 *element if* λ

is none of r-tuples

$$(1\cdots1\,|\,10),\quad (\overbrace{1\cdots1}^{a}\,\overbrace{0\cdots0}^{b}\,|\,00)$$

$$(\textit{resp.}\ (1\cdots1\,|\,01),\quad (\overbrace{1\cdots1}^{a}\,\overbrace{2\cdots2}^{b}\,|\,11))$$

where $a>0, b\geq 0$.

If $\lambda=(1\cdots1\,|\,10)$ (*resp.* $(1\cdots1\,|\,01)$), *the set has just* $r-2$ *elements with form*

$$\begin{pmatrix} -1 & \cdots & -1 & -2 & \cdots & -2 & | & -1 & -1 \\ 0 & \cdots & 0 & -1 & \cdots & -1 & | & 0 & -1 \end{pmatrix}$$

$$\left(\textit{resp.}\ \begin{pmatrix} -1 & \cdots & -1 & 0 & \cdots & 0 & | & 0 & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & | & 0 & 1 \end{pmatrix}\right).$$

If $\lambda=(\overbrace{1\cdots1}^{a}\,\overbrace{0\cdots0}^{b}\,|\,00)$ (*resp.* $(\overbrace{1\cdots1}^{a}\,\overbrace{2\cdots2}^{b}\,|\,11)$), *the set has just* $r-1$ *elements with forms*

$$\begin{pmatrix} \overbrace{-1\cdots-1\quad-2\cdots-2}^{a} & \overbrace{-2\cdots-2}^{b} & | & -1 & -1 \\ 0\cdots0\quad-1\cdots-1 & -2\cdots-2 & | & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} \overbrace{-1\cdots-1}^{a} & \overbrace{-1\cdots-1\quad-2\cdots-2}^{b} & | & -1 & -1 \\ 0\cdots0 & -1\cdots-1\quad-2\cdots-2 & | & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} \overbrace{-1\cdots-1}^{a} & \overbrace{-1\cdots-1}^{b} & | & 0 & -1 \\ 0\cdots0 & -1\cdots-1 & | & 0 & -1 \end{pmatrix}$$

$$\left(\begin{array}{l} \begin{pmatrix} \overbrace{-1\cdots-1\quad0\cdots0}^{a} & \overbrace{0\cdots0}^{b} & | & 0 & 0 \\ 0\cdots0\quad1\cdots1 & 2\cdots2 & | & 1 & 1 \end{pmatrix}, \\ \textit{resp.}\ \begin{pmatrix} \overbrace{-1\cdots-1}^{a} & \overbrace{-1\cdots-1\quad0\cdots0}^{b} & | & 0 & 0 \\ 0\cdots0 & 1\cdots1\quad2\cdots2 & | & 1 & 1 \end{pmatrix}, \\ \begin{pmatrix} \overbrace{-1\cdots-1}^{a} & \overbrace{-1\cdots-1}^{b} & | & -1 & 0 \\ 0\cdots0 & 1\cdots1 & | & 0 & 1 \end{pmatrix} \end{array}\right);$$

(4) *The set* $R_\lambda''^2[(\binom{-1}{0})_{r-1},(\binom{-1}{-1})_r]$ (*resp.* $R_\lambda''^2[(\binom{1}{0})_{r-1},(\binom{0}{1})_r]$) *has at most* 1 *element if* λ *is none of the r-tuples*

$$(\overbrace{0\cdots0}^{a}\,\overbrace{1\cdots1}^{b}\,|\,10)\quad(\textit{resp.}\ (0\cdots0\,|\,-1\ 1))$$

where $a\geq 0, b\geq 0$.

If $\lambda=(\overbrace{0\cdots0}^{a}\,\overbrace{1\cdots1}^{b}\,|\,10)$ (*resp.* $(0\cdots0\,|\,-1\ 1)$), *the set has just* $r-1$ *elements with forms*

$$\begin{pmatrix} \overbrace{0\cdots0 \quad -1\cdots-1}^{a} & \overbrace{-2\cdots-2}^{b} & | & -1 & -1 \\ 0\cdots0 \quad -1\cdots-1 & -1\cdots-1 & | & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} \overbrace{0\cdots0}^{a} & \overbrace{-1\cdots-1 \quad -2\cdots-2}^{b} & | & -1 & -1 \\ 0\cdots0 & 0 \cdots 0 \quad -1\cdots-1 & | & 0 & -1 \end{pmatrix}$$

$$\left(resp.\ \begin{pmatrix} 0\cdots0 & 1\cdots1 & | & 1 & 0 \\ 0\cdots0 & 1\cdots1 & | & 0 & 1 \end{pmatrix}\right);$$

(5) *The set* $R'^{\prime 2}_{\lambda}[\binom{-1}{1}_1, \binom{-1}{0}_{r-1}, \binom{-1}{0}_r]$ *has at most* 1 *element if* λ *is not the r-tuple* $(2\cdots2|11)$.

If $\lambda=(2\cdots2|11)$, *the set has just* $r-2$ *elements with form*

$$\begin{pmatrix} -1\cdots-1 & -2\cdots-2 & | & -1 & -1 \\ 1 \cdots 1 & 0 \cdots 0 & | & 0 & 0 \end{pmatrix};$$

(6) *The set* $R'^{\prime 2}_{\lambda}[\binom{-1}{1}_1]$ (*resp.* $R'^{\prime 2}_{\lambda}[\binom{1}{1}_1]$) *has at most* 2 *elements if* λ *is not the r-tuple*

$$(2\cdots2|11) \quad (resp.\ (0\cdots0|00)).$$

If $\lambda=(2\cdots2|11)$ (*resp.* $(0\cdots0|00)$), *the set has just* $2r-2$ *elements with forms*

$$\begin{pmatrix} -1\cdots-1 & 0\cdots0 & | & 0 & 0 \\ 1 \cdots 1 & 2\cdots2 & | & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1\cdots-1 & -2\cdots-2 & | & -1 & -1 \\ 1 \cdots 1 & 0 \cdots 0 & | & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -1\cdots-1 & | & 0 & -1 \\ 1 \cdots 1 & | & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1\cdots-1 & | & -1 & 0 \\ 1 \cdots 1 & | & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{l} resp. \\ \begin{pmatrix} 1\cdots1 & 0\cdots0 & | & 0 & 0 \\ 1\cdots1 & 0\cdots0 & | & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1\cdots1 & 2\cdots2 & | & 1 & 1 \\ 1\cdots1 & 2\cdots2 & | & 1 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1\cdots1 & | & 1 & 0 \\ 1\cdots1 & | & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1\cdots1 & | & 0 & 1 \\ 1\cdots1 & | & 0 & 1 \end{pmatrix} \end{array}\right);$$

(7) *The set* $R^2_{\lambda}[\binom{-1}{0}_1, \binom{-2}{2}_r]$ *has at most* 2 *elements if* λ *is none of the r-tuples*

$$(\overbrace{1\cdots1}^{a}\ \overbrace{2\cdots2}^{b}\ \overbrace{3\cdots3}^{c}\ \overbrace{4\cdots4}^{d}) \quad (a>0, b>0, c>0, d>0),$$

and it has just 3 *elements with forms*

$$\begin{pmatrix} \overbrace{-1\cdots-1}^{a} & \overbrace{-1\cdots-1}^{b} & \overbrace{-1\cdots-1}^{c} & \overbrace{-2\cdots-2}^{d} \\ 0 \cdots 0 & 1 \cdots 1 & 2 \cdots 2 & 2 \cdots 2 \end{pmatrix},$$

$$\begin{pmatrix} \overbrace{-1\cdots-1}^{a} & \overbrace{-1\cdots-1}^{b} & \overbrace{-2\cdots-2}^{c} & \overbrace{-2\cdots-2}^{d} \\ 0 \cdots 0 & 1 \cdots 1 & 1 \cdots 1 & 2 \cdots 2 \end{pmatrix},$$

$$\left(\overbrace{\begin{matrix}-1 & \cdots & -1\\ 0 & \cdots & 0\end{matrix}}^{a}\ \overbrace{\begin{matrix}-2 & \cdots & -2\\ 0 & \cdots & 0\end{matrix}}^{b}\ \overbrace{\begin{matrix}-2 & \cdots & -2\\ 1 & \cdots & 1\end{matrix}}^{c}\ \overbrace{\begin{matrix}-2 & \cdots & -2\\ 2 & \cdots & 2\end{matrix}}^{d}\right)$$

if $\lambda=(\overbrace{1\cdots1}^{a}\ \overbrace{2\cdots2}^{b}\ \overbrace{3\cdots3}^{c}\ \overbrace{4\cdots4}^{d})$;

(8) *The set* $R_\lambda'^{\prime 2}[\binom{-1}{0}_1, \binom{-1}{1}_r]$ (*resp.* $R_\lambda'^{\prime 2}[\binom{1}{0}_1, \binom{0}{1}_r]$) *has at most* 2 *elememts if* λ *is none of r-tuples*

$$\begin{array}{ll} (\overbrace{1\cdots1}^{a}\ \overbrace{2\cdots2}^{b}\ \overbrace{3\cdots3}^{c}\,|\,12) & (a>0, b>0, c>0)\,, \\ (\overbrace{1\cdots1}^{a}\ \overbrace{2\cdots2}^{b}\ \overbrace{3\cdots3}^{c}\ \overbrace{4\cdots4}^{d}\,|\,22) & (a>0, b>0, c>0, d\geq 0) \end{array}$$

$$\left(\textit{resp.}\ \begin{array}{ll} (\overbrace{-1\cdots-1}^{a}\ \overbrace{0\cdots0}^{b}\ \overbrace{1\cdots1}^{c}\,|\,01) & (a>0, b>0, c>0)\,, \\ (\overbrace{-1\cdots-1}^{a}\ \overbrace{0\cdots0}^{b}\ \overbrace{1\cdots1}^{c}\ \overbrace{2\cdots2}^{e}\,|\,11) & (a>0, b>0, c>0, d\geq 0) \end{array}\right)$$

If $\lambda=(\overbrace{1\cdots1}^{a}\ \overbrace{2\cdots2}^{b}\ \overbrace{3\cdots3}^{c}\,|\,12)$ (*resp.* $(\overbrace{-1\cdots-1}^{a}\ \overbrace{0\cdots0}^{b}\ \overbrace{1\cdots1}^{c}\,|\,01)$, *the set has just* 3 *elements with forms*

$$\left(\overbrace{\begin{matrix}-1 & \cdots & -1\\ 0 & \cdots & 0\end{matrix}}^{a}\ \overbrace{\begin{matrix}-1 & \cdots & -1\\ 1 & \cdots & 1\end{matrix}}^{b}\ \overbrace{\begin{matrix}-1 & \cdots & -1\\ 2 & \cdots & 2\end{matrix}}^{c}\ \begin{array}{|cc}0 & -1\\ 1 & 1\end{array}\right),$$

$$\left(\overbrace{\begin{matrix}-1 & \cdots & -1\\ 0 & \cdots & 0\end{matrix}}^{a}\ \overbrace{\begin{matrix}-1 & \cdots & -1\\ 1 & \cdots & 1\end{matrix}}^{b}\ \overbrace{\begin{matrix}-2 & \cdots & -2\\ 1 & \cdots & 1\end{matrix}}^{c}\ \begin{array}{|cc}-1 & -1\\ 0 & 1\end{array}\right),$$

$$\left(\overbrace{\begin{matrix}-1 & \cdots & -1\\ 0 & \cdots & 0\end{matrix}}^{a}\ \overbrace{\begin{matrix}-2 & \cdots & -2\\ 0 & \cdots & 0\end{matrix}}^{b}\ \overbrace{\begin{matrix}-2 & \cdots & -2\\ 1 & \cdots & 1\end{matrix}}^{c}\ \begin{array}{|cc}-1 & -1\\ 0 & 1\end{array}\right)$$

$$\left(\textit{resp.}\ \begin{array}{l} \left(\overbrace{\begin{matrix}1 & \cdots & 1\\ 0 & \cdots & 0\end{matrix}}^{a}\ \overbrace{\begin{matrix}0 & \cdots & 0\\ 0 & \cdots & 0\end{matrix}}^{b}\ \overbrace{\begin{matrix}0 & \cdots & 0\\ 1 & \cdots & 1\end{matrix}}^{c}\ \begin{array}{|cc}0 & 0\\ 0 & 1\end{array}\right), \\ \left(\overbrace{\begin{matrix}1 & \cdots & 1\\ 0 & \cdots & 0\end{matrix}}^{a}\ \overbrace{\begin{matrix}1 & \cdots & 1\\ 1 & \cdots & 1\end{matrix}}^{b}\ \overbrace{\begin{matrix}0 & \cdots & 0\\ 1 & \cdots & 1\end{matrix}}^{c}\ \begin{array}{|cc}0 & 0\\ 0 & 1\end{array}\right), \\ \left(\overbrace{\begin{matrix}1 & \cdots & 1\\ 0 & \cdots & 0\end{matrix}}^{a}\ \overbrace{\begin{matrix}1 & \cdots & 1\\ 1 & \cdots & 1\end{matrix}}^{b}\ \overbrace{\begin{matrix}1 & \cdots & 1\\ 2 & \cdots & 2\end{matrix}}^{c}\ \begin{array}{|cc}1 & 0\\ 1 & 1\end{array}\right) \end{array}\right).$$

If $\lambda=(\overbrace{1\cdots1}^{a}\ \overbrace{2\cdots2}^{b}\ \overbrace{3\cdots3}^{c}\ \overbrace{4\cdots4}^{d}\,|\,/22)$ (*resp.* $(\overbrace{-1\cdots-1}^{a}\ \overbrace{0\cdots0}^{b}\ \overbrace{1\cdots1}^{c}\ \overbrace{2\cdots2}^{d}\,|\,11)$, *the set has just* 3 *elements with forms*

$$\left(\overbrace{\begin{matrix}-1 & \cdots & -1\\ 0 & \cdots & 0\end{matrix}}^{a}\ \overbrace{\begin{matrix}-1 & \cdots & -1\\ 1 & \cdots & 1\end{matrix}}^{b}\ \overbrace{\begin{matrix}-1 & \cdots & -1\\ 2 & \cdots & 2\end{matrix}}^{c}\ \overbrace{\begin{matrix}-2 & \cdots & -2\\ 2 & \cdots & 2\end{matrix}}^{d}\ \begin{array}{|cc}-1 & -1\\ 1 & 1\end{array}\right),$$

$$\left(\begin{array}{cccc|cc}\overbrace{\begin{matrix}-1\cdots-1\\0\ \cdots\ 0\end{matrix}}^{a} & \overbrace{\begin{matrix}-1\cdots-1\\1\ \cdots\ 1\end{matrix}}^{b} & \overbrace{\begin{matrix}-2\cdots-2\\1\ \cdots\ 1\end{matrix}}^{c} & \overbrace{\begin{matrix}-2\cdots-2\\2\ \cdots\ 2\end{matrix}}^{d} & \begin{matrix}-1\\1\end{matrix} & \begin{matrix}-1\\1\end{matrix}\end{array}\right),$$

$$\left(\begin{array}{cccc|cc}\overbrace{\begin{matrix}-1\cdots-1\\0\ \cdots\ 0\end{matrix}}^{a} & \overbrace{\begin{matrix}-2\cdots-2\\0\ \cdots\ 0\end{matrix}}^{b} & \overbrace{\begin{matrix}-2\cdots-2\\1\ \cdots\ 1\end{matrix}}^{c} & \overbrace{\begin{matrix}-2\cdots-2\\2\ \cdots\ 2\end{matrix}}^{d} & \begin{matrix}-1\\1\end{matrix} & \begin{matrix}-1\\1\end{matrix}\end{array}\right)$$

$$\left(\begin{array}{l} \left(\begin{array}{cccc|cc}\overbrace{\begin{matrix}1\cdots1\\0\cdots0\end{matrix}}^{a} & \overbrace{\begin{matrix}0\cdots0\\0\cdots0\end{matrix}}^{b} & \overbrace{\begin{matrix}0\cdots0\\1\cdots1\end{matrix}}^{c} & \overbrace{\begin{matrix}0\cdots0\\2\cdots2\end{matrix}}^{d} & \begin{matrix}0\\1\end{matrix} & \begin{matrix}0\\1\end{matrix}\end{array}\right), \\ \textit{resp.}\ \left(\begin{array}{cccc|cc}\overbrace{\begin{matrix}1\cdots1\\0\cdots0\end{matrix}}^{a} & \overbrace{\begin{matrix}1\cdots1\\1\cdots1\end{matrix}}^{b} & \overbrace{\begin{matrix}0\cdots0\\1\cdots1\end{matrix}}^{c} & \overbrace{\begin{matrix}0\cdots0\\2\cdots2\end{matrix}}^{d} & \begin{matrix}0\\1\end{matrix} & \begin{matrix}0\\1\end{matrix}\end{array}\right), \\ \left(\begin{array}{cccc|cc}\overbrace{\begin{matrix}1\cdots1\\0\cdots0\end{matrix}}^{a} & \overbrace{\begin{matrix}1\cdots1\\1\cdots1\end{matrix}}^{b} & \overbrace{\begin{matrix}1\cdots1\\2\cdots2\end{matrix}}^{c} & \overbrace{\begin{matrix}0\cdots0\\2\cdots2\end{matrix}}^{d} & \begin{matrix}0\\1\end{matrix} & \begin{matrix}0\\1\end{matrix}\end{array}\right) \end{array}\right);$$

(9) *The set* $R_{\lambda}^{\prime\prime 2}[\binom{-1}{0}_{r-1}, \binom{-1}{1}_{r}]$ *has at most* 2 *elements if* λ *is none of the r-tuples*

$$(\overbrace{0\cdots0}^{a}\,\overbrace{1\cdots1}^{b}\,\overbrace{2\cdots2}^{c}\,\overbrace{3\cdots3}^{d}\,|\,12) \quad (a\geq 0, b>0, c>0, d\geq 0).$$

If $\lambda=(\overbrace{0\cdots0}^{a}\,\overbrace{1\cdots1}^{b}\,\overbrace{2\cdots2}^{c}\,\overbrace{3\cdots3}^{d}\,|\,12)$, *the set has just* 3 *elements with forms*

$$\left(\begin{array}{cccc|cc}\overbrace{\begin{matrix}0\cdots0\\0\cdots0\end{matrix}}^{a} & \overbrace{\begin{matrix}0\cdots0\\1\cdots1\end{matrix}}^{b} & \overbrace{\begin{matrix}-1\cdots-1\\1\ \cdots\ 1\end{matrix}}^{c} & \overbrace{\begin{matrix}-2\cdots-2\\1\ \cdots\ 1\end{matrix}}^{d} & \begin{matrix}-1\\0\end{matrix} & \begin{matrix}-1\\1\end{matrix}\end{array}\right),$$

$$\left(\begin{array}{cccc|cc}\overbrace{\begin{matrix}0\cdots0\\0\cdots0\end{matrix}}^{a} & \overbrace{\begin{matrix}-1\cdots-1\\0\ \cdots\ 0\end{matrix}}^{b} & \overbrace{\begin{matrix}-1\cdots-1\\1\ \cdots\ 1\end{matrix}}^{c} & \overbrace{\begin{matrix}-2\cdots-2\\1\ \cdots\ 1\end{matrix}}^{d} & \begin{matrix}-1\\0\end{matrix} & \begin{matrix}-1\\1\end{matrix}\end{array}\right),$$

$$\left(\begin{array}{cccc|cc}\overbrace{\begin{matrix}0\cdots0\\0\cdots0\end{matrix}}^{a} & \overbrace{\begin{matrix}-1\cdots-1\\0\ \cdots\ 0\end{matrix}}^{b} & \overbrace{\begin{matrix}-2\cdots-2\\0\ \cdots\ 0\end{matrix}}^{c} & \overbrace{\begin{matrix}-2\cdots-2\\1\ \cdots\ 1\end{matrix}}^{d} & \begin{matrix}-1\\0\end{matrix} & \begin{matrix}-1\\1\end{matrix}\end{array}\right).$$

In this lemma, if we change a subset $R_{\lambda}''[*, \binom{a}{b}_r]$ for a subset $R_{\lambda}''[*, \binom{a}{b}_{r-1}]$, we can get the elements in $R_{\lambda}''[*, \binom{a}{b}_{r-1}]$ from the elements in $R_{\lambda}''[*, \binom{a}{b}_r]$, by changing the r-term for the $(r-1)$-term.

In the following we represent a root of type D_l by a linear combination of the fundamental root system Π and identify it with an l-tuple of coefficients. Note that the l-tuples $\pm(0\cdots0\,|\,11)$ are not roots.

Case DI: The families $\mathcal{D}_{ij}$ with triple (j, k, r)

Put $\sigma=\theta_i$ and $\tau=\theta_j$. Then, for each PSLA in $\mathcal{D}_{ij}$, the corresponding symmetric space M and the totally geodesic $\mathcal{V}$-submanifold N are given as follows: (N is locally described.)

(a) $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$: $M=SO(2l)/SO(2j+2k)\times SO(2r)$.

In this case $N=\mathfrak{so}(2j+2r)/\mathfrak{so}(2j)\oplus\mathfrak{so}(2r)$;

(b) $\mathcal{V}=(\mathfrak{g}, \sigma, \sigma\tau)$: $M=SO(2l)/SO(2j+2k)\times SO(2r)$.

In this case $N=\mathfrak{so}(2k+2r)/\mathfrak{so}(2k)\oplus\mathfrak{so}(2r)$;

(c) $\mathcal{V}=(\mathfrak{g},\tau,\sigma)$: $M=SO(2l)/SO(2j)\times SO(2k+2r)$.

In this case $N=\mathfrak{so}(2j+2r)/\mathfrak{so}(2j)\oplus\mathfrak{so}(2r)$;

(d) $\mathcal{V}=(\mathfrak{g},\tau,\sigma\tau)$: $M=SO(2l)/SO(2j)\times SO(2k+2r)$.

In this case $N=\mathfrak{so}(2j+2k)/\mathfrak{so}(2j)\oplus\mathfrak{so}(2k)$;

(e) $\mathcal{V}=(\mathfrak{g},\sigma\tau,\sigma)$: $M=SO((2l)/SO(2k)\times SO(2j+2r)$.

In this case $N=\mathfrak{so}(2k+2r)/\mathfrak{so}(2k)\oplus\mathfrak{so}(2r)$;

(f) $\mathcal{V}=(\mathfrak{g},\sigma\tau,\tau)$: $M=SO(2l)/SO(2k)\times SO(2j+2r)$.

In this case $N=\mathfrak{so}(2j+2k)/\mathfrak{so}(2j)\oplus\mathfrak{so}(2k)$.

For the PSLA $(\mathfrak{g},\sigma,\tau)$, the subsets $\Delta^+_{\mathfrak{k}+}$, $\Delta^+_{\mathfrak{k}-}$, $\Delta^+_{\mathfrak{p}+}$, $\Delta^+_{\mathfrak{p}-}$ of Δ^+ are given as follows:

$$
\begin{aligned}
\Delta^+_{\mathfrak{k}+} &= \{\delta\in\Delta^+;\ \delta_i=\delta_j=0,2\} \\
&= \left\{\delta\in\Delta^+;\ \delta=\begin{array}{l}
(0\cdots01\cdots10\cdots0\cdots\overset{j}{0}\cdots\overset{i}{0}\,|\,00)\\
(0\cdots\overset{j}{0}\cdots01\cdots10\cdots\overset{i}{0}\cdots0\,|\,00)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots10\cdots0\,|\,00)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots1\,|\,10)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots1\,|\,01)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots11\,|\,11)\\
(0\cdots01\cdots12\cdots\overset{j}{2}\cdots\overset{i}{2}\cdots2\,|\,11)\\
(0\cdots\overset{j}{0}\cdots01\cdots12\cdots\overset{i}{2}\cdots2\,|\,11)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots12\cdots2\,|\,11)
\end{array}\right\}
\end{aligned}
\tag{6.1}
$$

$$
\begin{aligned}
\Delta^+_{\mathfrak{k}-} &= \{\delta\in\Delta^+;\ \delta_i=0,2,\ \delta_j=1\} \\
&= \left\{\delta\in\Delta^+;\ \delta=\begin{array}{l}
(0\cdots01\cdots\overset{j}{1}\cdots10\cdots\overset{i}{0}\cdots0\,|\,00)\\
(0\cdots01\cdots\overset{j}{1}\cdots12\cdots\overset{i}{2}\cdots2\,|\,11)
\end{array}\right\},\\
\Delta^+_{\mathfrak{p}+} &= \{\delta\in\Delta^+;\ \delta_i=1,\ \delta_j=0,2\} \\
&= \left\{\delta\in\Delta^+;\ \delta=\begin{array}{l}
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots10\cdots0\,|\,00)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots1\,|\,10)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots1\,|\,01)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots11\,|\,11)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots12\cdots2\,|\,11)
\end{array}\right\},\\
\Delta^+_{\mathfrak{p}-} &= \{\delta\in\Delta^+;\ \delta_i=\delta_j=1\} \\
&= \left\{\delta\in\Delta^+;\ \delta=\begin{array}{l}
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots10\cdots0\,|\,00)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots1\,|\,10)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots1\,|\,01)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots11\,|\,11)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots12\cdots2\,|\,11)
\end{array}\right\}.
\end{aligned}
$$

Moreover the dominant weights in $\Delta_{\mathfrak{k}_-}$, $\Delta_{\mathfrak{p}_+}$, $\Delta_{\mathfrak{p}_-}$ are given by (6.2), (6.3), (6.4), respectively:

(6.2) $\quad (1\cdots\overset{j}{1}2\cdots\overset{i}{2}\cdots 2\,|\,11)$, $\quad (1\cdots\overset{j}{1}\overset{i}{0}\cdots 0\,|\,00)$ $(i=j+1)$,

$\quad -(\overset{j}{1}0\cdots\overset{i}{0}\cdots 0\,|\,00)$ $(j=1)$, $\quad -(\overset{j}{1}\overset{i}{2}\cdots 2\,|\,11)$ $(j=1, i=2)$.

(6.3) $\quad (0\cdots\overset{j}{0}1\cdots\overset{i}{1}2\cdots 2\,|\,11)$, $\quad -(0\cdots\overset{j}{0}\overset{i}{1}0\cdots 0\,|\,00)$ $(i=j+1)$.

(6.4) $\quad (1\cdots\overset{j}{1}\cdots\overset{i}{1}2\cdots 2\,|\,11)$, $\quad -(\overset{j}{1}\cdots\overset{i}{1}0\cdots 0\,|\,00)$ $(j=1)$.

We now see the injectivity of ρ for Case (a): $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$. Then ρ is a homomorphism of $(\mathfrak{p}^{C}_{-})^{*}\otimes\mathfrak{k}^{C}_{-}$ to $\wedge^{2}(\mathfrak{p}^{C}_{-})^{*}\otimes\mathfrak{p}^{C}_{+}$. The minus multiple of dominant weights in $\Delta_{\mathfrak{p}_-}$ are given by $(\alpha 1)$, $(\alpha 2)$ and the dominant weights in $\Delta_{\mathfrak{k}_-}$ are given by $(\beta 1)\sim(\beta 4)$:

$(\alpha 1)-(1\cdots\overset{j}{1}\cdots\overset{i}{1}2\cdots 2\,|\,11)$, $\quad (\alpha 2)\ (\overset{j}{1}\cdots\overset{i}{1}0\cdots 0\,|\,00)$ $(j=1)$,

$(\beta 1)\ (1\cdots\overset{j}{1}2\cdots\overset{i}{2}\cdots 2\,|\,11)$, $\quad (\beta 2)\ (1\cdots\overset{j}{1}\overset{i}{0}\cdots 0\,|\,00)$ $(i=j+1)$,

$(\beta 3)-(\overset{j}{1}0\cdots\overset{i}{0}\cdots 0\,|\,00)$ $(j=1)$, $\quad (\beta 4)-(\overset{j}{1}\overset{i}{2}\cdots 2\,|\,11)$ $(j=1, i=2)$.

Case (1): $l(u)=1$. Represent u as fololws: $u=a\,\omega_{\alpha}\otimes X_{\beta}$. Then the pair (α, β) is one of $((\alpha\,\mathrm{r}), (\beta\,\mathrm{s}))$, where $r=1, 2$ and $s=1, 2, 3, 4$. Applying Lemma 2.3 for each pair, we obtain that $\rho(u)\neq 0$.

Case (2): $l(u)=2$. We first suppose that u is indecomposable. Consider the following elements in $\Delta_{\mathfrak{k}_+}$:

$(\mu 1)\ (10\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots 0\,|\,00)$, $\quad (\mu 2)\ (1\overset{j}{2}\cdots\overset{i}{2}\cdots 2\,|\,11)$ $(j=2)$.

Then such the triples $(\alpha, \beta'; \mu)$ as in Case (2) of type AI are given in the following:

(1) $((\alpha 1), (\beta 1); (\mu 1))$, $j\geq 2$, $\quad$ (2) $((\alpha 1), (\beta 2); (\mu 1))$, $j\geq 2, i=j+1$,

(3) $((\alpha 1), (\beta 1); (\mu 2))$, $j=2$, $\quad$ (4) $((\alpha 1), (\beta 2); (\mu 2))$, $j=2, i=3$.

Lemma 2.4 is available for all the cases and thus it follows that $\rho(u)\neq 0$.

We next suppose that u is decomposable. Put $u=a\,\omega_{\alpha_1}\otimes X_{\beta_1}+b\,\omega_{\alpha_2}\otimes X_{\beta_2}$. Then the weights λ are roots and the following cases are possible:

(1) The pairs (α_i, β_i) are $((\alpha 1), (\beta 3))$, $((\alpha 2), (\beta 1))$, where $j=1$ and $\lambda=(\overset{j}{0}1\cdots\overset{i}{1}2\cdots 2\,|\,11)$;

(2) The pairs (α_i, β_i) are $((\alpha 1), (\beta 4))$, $((\alpha 2), (\beta 2))$, where $j=1$, $i=2$ and $\lambda=-(\overset{j}{0}\overset{i}{1}0\cdots 0\,|\,00)$.

Lemma 2.2 is available for these cases and thus it follows that $\rho(u)\neq 0$.

Case (3): $l(u)\geq 3$. Note that $i\leq l-2$. Then, by the same way as **Case (3)** for Case BI §4, we see that $\rho(u)\neq 0$.

Summing up the above arguments, we have the following result for the PSLA of Case (a); the homomorphism ρ is always injective. Similarly for the other cases ρ is always injective.

Theorem 6.10. *Let $\mathcal{V}$ be the G-orbit which corresponds to a PSLA in a family of type DI. Then the $\mathcal{V}$-geometry does not admit non-totally geodesic $\mathcal{V}$-submanifolds.*

Case DII: The families $\mathscr{D}_{i\,;\,jk}$ with quadruple (a, b, c, d)

Put $\sigma=\theta_i$ and $\tau=\theta_{jk}$. Then for each PSLA in $\mathscr{D}_{i\,;\,jk}$, the corresponding symmetric space M and the totally geodesic $\mathcal{V}$-submanifold N are given in the following: (N is locally described.)

(a) $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$: $M=SO(2l)/SO(2a+2b)\times SO(2c+2d)$.
In this case $N=(\mathfrak{so}(2a+2c)/\mathfrak{so}(2a)\oplus\mathfrak{so}(2c))\oplus(\mathfrak{so}(2b+2d)/\mathfrak{so}(2b)\oplus\mathfrak{so}(2d))$;
(b) $\mathcal{V}=(\mathfrak{g}, \sigma, \sigma\tau)$: $M=SO(2l)/SO(2a+2b)\times SO(2c+2d)$.
In this case $N=(\mathfrak{so}(2b+2c)/\mathfrak{so}(2b)\oplus\mathfrak{so}(2c))\oplus(\mathfrak{so}(2a+2d)/\mathfrak{so}(2a)\oplus\mathfrak{so}(2d))$;
(c) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma)$: $M=SO(2l)/SO(2b+2c)\times SO(2a+2d)$.
In this case $N=(\mathfrak{so}(2a+2c)/\mathfrak{so}(2a)\oplus\mathfrak{so}(2c))\oplus(\mathfrak{so}(2b+2d)/\mathfrak{so}(2b)\oplus\mathfrak{so}(2d))$;
(d) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma\tau)$: $M=SO(2l)/SO(2b+2c)\times SO(2a+2d)$.
In this case $N=(\mathfrak{so}(2a+2b)/\mathfrak{so}(2a)\oplus\mathfrak{so}(2b))\oplus(\mathfrak{so}(2c+2d)/\mathfrak{so}(2c)\oplus\mathfrak{so}(2d))$;
(e) $\mathcal{V}=(\mathfrak{g}, \sigma\tau, \sigma)$: $M=SO(2l)/SO(2a+2c)\times SO(2b+2d)$.
In this case $N=(\mathfrak{so}(2b+2c)/\mathfrak{so}(2b)\oplus\mathfrak{so}(2c))\oplus(\mathfrak{so}(2a+2d)/\mathfrak{so}(2a)\oplus\mathfrak{so}(2d))$;
(f) $\mathcal{V}=(\mathfrak{g}, \sigma\tau, \tau)$: $M=SO(2l)/SO(2a+2c)\times SO(2b+2d)$.
In this case $N=(\mathfrak{so}(2a+2b)/\mathfrak{so}(2a)\oplus\mathfrak{so})2b))\oplus(\mathfrak{so}(2c+2d)/\mathfrak{so}(2c)\oplus\mathfrak{so}(2d))$.

For the PSLA $(\mathfrak{g}, \sigma, \tau)$, the subsets $\Delta^+_{\mathfrak{k}\pm}$, $\Delta^+_{\mathfrak{p}\pm}$ of Δ^+ are given as follows:

(6.5) $$\Delta^+_{\mathfrak{k}+} = \{\delta\in\Delta^+;\, \delta_i = 0, 2, (\delta_j, \delta_k) = (0, 0), (0, 2), (2, 0), (1, 1), (2, 2)\}$$

$$= \left\{\delta\in\Delta^+;\, \delta = \begin{array}{l}
(0\cdots01\cdots10\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0\,|\,00)\\
(0\cdots\overset{j}{0}\cdots01\cdots10\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0\,|\,00)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots10\cdots\overset{k}{0}\cdots0\,|\,00)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots01\cdots10\cdots0\,|\,00)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots01\cdots1\,|\,10)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots01\cdots1\,|\,01)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots01\cdots11\,|\,11)\\
(0\cdots01\cdots12\cdots\overset{j}{2}\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots2\,|\,11)\\
(0\cdots\overset{j}{0}\cdots01\cdots12\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots2\,|\,11)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots12\cdots\overset{k}{2}\cdots2\,|\,11)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots01\cdots12\cdots2\,|\,11)
\end{array}\right\},$$

$$\Delta^+_{\mathfrak{k}-} = \{\delta\in\Delta^+;\, \delta_i = 0, 2, (\delta_j, \delta_k) = (0, 1), (1, 0), (1, 2), (2, 1)\}$$

$$
=\left\{\delta\in\Delta^+;\ \delta=\begin{matrix}
(0\cdots01\cdots\overset{j}{1}\cdots10\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0\,|\,00)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots\overset{k}{1}\cdots10\cdots0\,|\,00)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots\overset{k}{1}\cdots1\,|\,10)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots\overset{k}{1}\cdots1\,|\,01)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots\overset{k}{1}\cdots11\,|\,11)\\
(0\cdots01\cdots\overset{j}{1}\cdots12\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots2\,|\,11)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots\overset{k}{1}\cdots12\cdots2\,|\,11)
\end{matrix}\right\},
$$

$$
\Delta^+_{\mathfrak{p}+}=\{\delta\in\Delta^+;\ \delta_i=1,\ (\delta_j,\delta_k)=(0,0),(0,2),(2,0),(1,1),(2,2)\}
$$

$$
=\left\{\delta\in\Delta^+;\ \delta=\begin{matrix}
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots10\cdots\overset{k}{0}\cdots0\,|\,00)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots10\cdots0\,|\,00)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots1\,|\,10)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots1\,|\,01)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots11\,|\,11)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots12\cdots\overset{k}{2}\cdots2\,|\,11)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots12\cdots2\,|\,11)
\end{matrix}\right\},
$$

$$
\Delta^+_{\mathfrak{p}-}=\{\delta\in\Delta^+;\ \delta_i=1,\ (\delta_j,\delta_k)=(0,1),(1,0),(1,2),(2,1)\}
$$

$$
=\left\{\delta\in\Delta^+;\ \delta=\begin{matrix}
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots10\cdots\overset{k}{0}\cdots0\,|\,00)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots10\cdots0\,|\,00)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots1\,|\,10)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots1\,|\,01)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots11\,|\,11)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots12\cdots\overset{k}{2}\cdots2\,|\,11)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots12\cdots2\,|\,11)
\end{matrix}\right\}.
$$

Moreover the dominant weights in $\Delta_{\mathfrak{k}-}$, $\Delta_{\mathfrak{p}+}$, $\Delta_{\mathfrak{p}-}$ are given by (6.6), (6.7), (6.8), respectively:

$$
\begin{aligned}
&(0\cdots\overset{j}{0}\cdots\overset{i}{0}1\cdots\overset{k}{1}2\cdots2\,|\,11)\,, && (1\cdots\overset{j}{1}2\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots2\,|\,11)\,,\\
(6.6)\qquad &(1\cdots\overset{j}{1}\overset{i}{0}\cdots\overset{k}{0}\cdots0\,|\,00)\ (i=j+1)\,, && -(0\cdots\overset{j}{0}\cdots0\overset{i}{1}\overset{k}{0}\cdots0\,|\,00)\ (k=i+1)\,,\\
&-(\overset{j}{1}0\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0\,|\,00)\ (j=1)\,, && -(\overset{j}{1}\overset{i}{2}\cdots\overset{k}{2}\cdots2\,|\,11)\ (j=1,i=2)\,.
\end{aligned}
$$

$$
\begin{aligned}
&(0\cdots\overset{j}{0}1\cdots\overset{i}{1}2\cdots\overset{k}{2}\cdots2\,|\,11)\,, && (1\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}2\cdots2\,|\,11)\,,\\
(6.7)\qquad &(0\cdots\overset{j}{0}1\cdots\overset{i}{1}\overset{k}{0}\cdots0\,|\,00)\ (k=i+1)\,, && -(0\cdots\overset{j}{0}\overset{i}{1}0\cdots\overset{k}{0}\cdots0\,|\,00)\ (i=j+1)\,,\\
&-(\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}0\cdots0\,|\,00)\ (j=1)\,, && -(0\cdots\overset{j}{0}\overset{i}{1}\overset{k}{2}\cdots2\,|\,11)\ (i=j+1,k=i+1)\,.
\end{aligned}
$$

$$
\begin{array}{lll}
 & (0\cdots\overset{j}{0}1\cdots\overset{i}{1}\cdots\overset{k}{1}2\cdots2\,|\,11)\,, & (1\cdots\overset{j}{1}\cdots\overset{i}{1}2\cdots\overset{k}{2}\cdots2\,|\,11)\,, \\
(6.8) & (1\cdots\overset{j}{1}\cdots\overset{i}{1}\overset{k}{0}\cdots0\,|\,00)\ (k=i+1)\,, & -(0\cdots\overset{j}{0}\overset{i}{1}\cdots\overset{k}{1}0\cdots0\,|\,00)\ (i=j+1)\,, \\
 & -(\overset{j}{1}\cdots\overset{i}{1}0\cdots\overset{k}{0}\cdots0\,|\,00)\ (j=1)\,, & -(\overset{j}{1}\cdots\overset{i}{1}\overset{k}{2}\cdots2\,|\,11)\ (j=1,\,k=i+1)\,.
\end{array}
$$

We now see the injectivity of ρ for Case (a): $\mathcal{V}=(\mathfrak{g},\sigma,\tau)$. Then ρ is a homomorphism of $(\mathfrak{p}_-^C)^*\otimes\mathfrak{k}_-^C$ to $\wedge^2(\mathfrak{p}_-^C)^*\otimes\mathfrak{p}_+^C$. The minus multiple of dominant weights in $\Delta_{\mathfrak{p}_-}$ are given by $(\alpha1)\sim(\alpha6)$ and the dominant weights in $\Delta_{\mathfrak{k}_-}$ are given by $(\beta1)\sim(\beta6)$:

$$
\begin{array}{ll}
(\alpha1)-(0\cdots\overset{j}{0}1\cdots\overset{i}{1}\cdots\overset{k}{1}2\cdots2\,|\,11)\,, & (\alpha2)-(1\cdots\overset{j}{1}\cdots\overset{i}{1}2\cdots\overset{k}{2}\cdots2\,|\,/11)\,, \\
(\alpha3)-(1\cdots\overset{j}{1}\cdots\overset{i}{1}\overset{k}{0}\cdots0\,|\,00)\ (k=i+1)\,, & (\alpha4)\ \ (0\cdots\overset{j}{0}\overset{i}{1}\cdots\overset{k}{1}0\cdots0\,|\,00)\ (i=j+1)\,, \\
(\alpha5)\ \ (\overset{j}{1}\cdots\overset{i}{1}0\cdots\overset{k}{0}\cdots0\,|\,00)\ (j=1)\,, & (\alpha6)\ \ (\overset{j}{1}\cdots\overset{i}{1}\overset{k}{2}\cdots2\,|\,11)\ (j=1,\,k=i+1)\,, \\
(\beta1)\ \ (0\cdots\overset{j}{0}\cdots\overset{i}{0}1\cdots\overset{k}{1}2\cdots2\,|\,11)\,, & (\beta2)\ \ (1\cdots\overset{j}{1}2\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots2\,|\,11)\,, \\
(\beta3)\ \ (1\cdots\overset{j}{1}\overset{i}{0}\cdots\overset{k}{0}\cdots0\,|\,00)\ (i=j+1)\,, & (\beta4)-(0\cdots\overset{j}{0}\cdots\overset{i}{0}\overset{k}{1}0\cdots0\,|\,00)\ (k=i+1)\,, \\
(\beta5)-(\overset{j}{1}0\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0\,|\,00)\ (j=1)\,, & (\beta6)-(\overset{j}{1}\overset{i}{2}\cdots\overset{k}{2}\cdots2\,|\,11)\ (j=1,\,i=2)\,.
\end{array}
$$

Case (1): $l(u)=1$. Represent u as follows: $u=a\,\omega_\beta\otimes X_\delta$. Then the pair (α,β) is one of the pairs $((\alpha\,\mathrm{r}),(\beta\,\mathrm{s}))$, where $r,s=1,2,3,4,5,6$. Applying Lemma 2.3 for each pair, we obtain that $\rho(u)\neq0$.

Case (2): $l(u)=2$. We first suppose that u is indecomposable. Consider the following elements in $\Delta_{\mathfrak{k}_+}$:

$$
\begin{array}{l}
(\mu1)\ (0\cdots\overset{j}{0}10\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0\,|\,00)\,, \\
(\mu2)\ (0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}10\cdots0\,|\,00)\,,\ (0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\,|\,10)\,,\ (0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\,|\,01)\,, \\
(\mu3)\ (0\cdots\overset{j}{0}1\overset{i}{2}\cdots\overset{k}{2}\cdots2\,|\,11)\ (i=j+2)\,,\quad (\mu4)\ (10\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0\,|\,00)\,, \\
(\mu5)\ (0\cdots\overset{j}{0}\cdots\overset{i}{0}10\cdots\overset{k}{0}\cdots0\,|\,00)\,,\qquad (\mu6)\ (1\overset{j}{2}\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots2\,|\,11)\ (j=2)\,, \\
(\mu7)\ (0\cdots\overset{j}{0}\cdots\overset{i}{0}1\overset{k}{2}\cdots2\,|\,11)\ (k=i+2)\,.
\end{array}
$$

Then such the triples $(\alpha,\beta';\mu)$ as in Case (2) of type AI are given in the following:

(1) $((\alpha1),(\beta2);(\mu1)),\ i-j\geq2$, (2) $((\alpha1),(\beta5);(\mu1)),\ j=1,\ i\geq3$,
(3) $((\alpha1),(\beta1);(\mu2))$, (4) $((\alpha1),(\beta4);(\mu2)),\ k=i+1$,
(5) $((\alpha1),(\beta2);(\mu3)),\ i-j=2$, (6) $((\alpha1),(\beta5);(\mu3)),\ j=1,\ i=3$,
(7) $((\alpha2),(\beta2);(\mu4)),\ j\geq2$, (8) $((\alpha2),(\beta3);(\mu4)),\ j\geq2,\ i=j+1$,
(9) $((\alpha2),(\beta1);(\mu5)),\ k-i\geq2$, (10) $((\alpha2),(\beta2);(\mu6)),\ j=2$,
(11) $((\alpha2),(\beta3);(\mu6)),\ j=2,\ i=j+1$,
(12) $((\alpha2),(\beta1);(\mu7)),\ k-i=2$, (13) $((\alpha3),(\beta2);(\mu4)),\ k=i+1,\ j\geq2$,
(14) $((\alpha3),(\beta3);(\mu4)),\ k=i+1,\ i=j+1.\ j\geq2$,
(15) $((\alpha3),(\beta2);(\mu6)),\ k=i+1,\ j=2$,

(16) $((\alpha 3), (\beta 3); (\mu 6)), j=2, i=3, k=4$,
(17) $((\alpha 4), (\beta 1); (\mu 2)), i-j=1$,
(18) $((\alpha 4), (\beta 4); (\mu 2)), i=j+1, k=i+1$,
(19) $((\alpha 5), (\beta 1); (\mu 5)), j=1, k-i\geq 2$,
(20) $((\alpha 5), (\beta 1); (\mu 7)), j=1, k-i=2$.

Lemma 2.4 is available for all cases and thus it follows that $\rho(u)\neq 0$.

We next suppose that u is decomposable. Put $u=a\,\omega_{\alpha_1}\otimes X_{\beta_1}+b\,\omega_{\alpha_2}\otimes X_{\beta_2}$. Then the weight λ is a root and Lemma 2.2 is available for all the cases. Hence it follows that $\rho(u)\neq 0$.

Case (3): $l(u)\geq 3$. By the same way as **Case (3)** for Case BII §4, we see that $\rho(u)\neq 0$.

Summing up the above arguments, we have the following result for the PSLA of Case (a); the homomorphism ρ is always injective. Similarly for the other cases ρ is always injective.

Theorem 6.11. *Let $\mathcal{V}$ be the G-orbit which corresponds to a PSLA in a family of type DII. Then the $\mathcal{V}$-geometry does not admit non-totally geodesic $\mathcal{V}$-submanifolds.*

Case DIII: The families $\mathcal{D}_j$ with pair (j, k)

Put $\sigma=\theta_l$ and $\tau=\theta_j$. Then, for each PSLA in $\mathcal{D}_j$, the corresponding symmetric space M and the totally geodesic $\mathcal{V}$-submanifold N are given as follows: (N is locally described.)

(a) $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$: $M=SO(2l)/U(l)$. In this case $N=\mathfrak{su}(l)/\mathfrak{s}(\mathfrak{u}(j)\oplus\mathfrak{u}(k))$;
(b) $\mathcal{V}=(\mathfrak{g}, \sigma, \sigma\tau)$: $M=SO(2l)/U(l)$. In this case $N=\mathfrak{so}(2j)/\mathfrak{u}(j)\oplus\mathfrak{so}(2k)/\mathfrak{u}(k)$;
(c) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma)$: $M=SO(2l)/SO(2j)\times SO(2k)$. In this case $N=\mathfrak{su}(l)/\mathfrak{s}(\mathfrak{u}(j)\oplus\mathfrak{u}(k))$.

For the PSLA $(\mathfrak{g}, \sigma, \tau)$, the subsets $\Delta^+_{\mathfrak{k}+}, \Delta^+_{\mathfrak{k}-}, \Delta^+_{\mathfrak{p}+}, \Delta^+_{\mathfrak{p}-}$ of Δ^+ are given as follows:

$$(6.9)\qquad \begin{aligned}\Delta^+_{\mathfrak{k}+} &= \{\delta\in\Delta^+;\ \delta_l=0,\ \delta_j=0,2\}\\ &=\left\{\delta\in\Delta^+;\ \delta=\begin{matrix}(0\cdots01\cdots10\cdots\overset{j}{0}\cdots0\,|\,00)\\ (0\cdots\overset{j}{0}\cdots01\cdots10\cdots0\,|\,00)\\ (0\cdots\overset{j}{0}\cdots01\cdots1\,|\,10)\end{matrix}\right\},\\ \Delta^+_{\mathfrak{k}-} &= \{\delta\in\Delta^+;\ \delta_l=0,\ \delta_j=1\}\\ &=\left\{\delta\in\Delta^+;\ \delta=\begin{matrix}(0\cdots01\cdots\overset{j}{1}\cdots10\cdots0\,|\,00)\\ (0\cdots01\cdots\overset{j}{1}\cdots1\,|\,10)\end{matrix}\right\},\\ \Delta^+_{\mathfrak{p}+} &= \{\delta\in\Delta^+;\ \delta_l=1,\ \delta_j=0,2\}\end{aligned}$$

$$= \left\{ \delta\in\Delta^+;\ \delta = \begin{matrix} (0\cdots\overset{j}{0}\cdots01\cdots1\,|\,01) \\ (0\cdots\overset{j}{0}\cdots01\cdots1\,|\,11) \\ (0\cdots\overset{j}{0}\cdots01\cdots12\cdots2\,|\,11) \\ (0\cdots01\cdots12\cdots\overset{j}{2}\cdots2\,|\,11) \end{matrix} \right\},$$

$$\Delta^+_{\mathfrak{p}_-} = \{\delta\in\Delta^+;\ \delta_l = \delta_j = 1\}$$

$$= \left\{ \delta\in\Delta^+;\ \delta = \begin{matrix} (0\cdots01\cdots\overset{j}{1}\cdots1\,|\,01) \\ (0\cdots01\cdots\overset{j}{1}\cdots1\,|\,11) \\ (0\cdots01\cdots\overset{j}{1}\cdots12\cdots2\,|\,11) \end{matrix} \right\}.$$

Moreover the dominant weights in $\Delta_{\mathfrak{k}_-}$, $\Delta_{\mathfrak{p}_+}$, $\Delta_{\mathfrak{p}_-}$ are given by (6.10), (6.11), (6.12), respectively:

$$(1\cdots\overset{j}{1}\cdots1\,|\,10)\,,\quad -(0\cdots0\overset{j}{1}0\cdots0\,|\,00)\,. \tag{6.10}$$

$$\left\{ \begin{matrix} (0\cdots\overset{j}{0}12\cdots2\,|\,11)\,, & (12\cdots\overset{j}{2}\cdots2\,|\,11)\,, \\ -(0\cdots\overset{j}{0}\cdots0\,|\,01)\,, & -(0\cdots01\overset{j}{2}\cdots2\,|\,11)\,. \end{matrix} \right. \tag{6.11}$$

$$(1\cdots\overset{j}{1}2\cdots2\,|\,11)\,,\quad -(0\cdots0\overset{j}{1}\cdots1\,|\,01)\,. \tag{6.12}$$

We first see the injectivity of ρ for Case (a): $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$. Then ρ is a homomorphism of $(\mathfrak{p}^{\mathbf{C}}_-)^*\otimes\mathfrak{k}^{\mathbf{C}}_-$ to $\wedge^2(\mathfrak{p}^{\mathbf{C}}_-)^*\otimes\mathfrak{p}^{\mathbf{C}}_+$. The minus multiple of dominant weights in $\Delta_{\mathfrak{p}_-}$ are given by $(\alpha1)$, $(\alpha2)$ and the dominant weights in $\Delta_{\mathfrak{k}_-}$ are given by $(\beta1)$, $(\beta2)$:

$$(\alpha1)-(1\cdots\overset{j}{1}2\cdots2\,|\,11)\,,\qquad (\alpha2)\ \ (0\cdots0\overset{j}{1}\cdots1\,|\,01)\,,$$
$$(\beta1)\ \ (1\cdots\overset{j}{1}\cdots1\,|\,10)\,,\qquad (\beta2)-(0\cdots0\overset{j}{1}0\cdots0\,|\,00)\,.$$

Case (1): $l(u)=1$. Represent u as follows: $u=a\,\omega_\alpha\otimes X_\beta$. Then the pair (α, β) is one of the pairs $((\alpha\ \mathrm{r}), (\beta\ \mathrm{s}))$, where $r, s=1, 2$. Applying Lemma 2.3 for each pair, we obtain that $\rho(u)\neq0$.

Case (2): $l(u)=2$. In this case there exists no decomposable u and thus we suppose that u is indecomposable. Consider the following elements in $\Delta_{\mathfrak{k}_+}$:

$$(\mu1)\ \ (10\cdots\overset{j}{0}\cdots0\,|\,00)\,,\qquad (\mu2)\ \ (0\cdots0\overset{j}{1}0\cdots0\,|\,00)\,,$$
$$(\mu3)\ \ (0\cdots01\overset{j}{0}\cdots0\,|\,00)\,,\qquad (\mu4)\ \ (0\cdots\overset{j}{0}\cdots0\,|\,10)\,.$$

Then such the triples $(\alpha, \beta'; \mu)$ as in Case (2) of type AI are given in the following:

(1) $((\alpha1), (\beta1); (\mu1))$, $j\geq2$, (2) $((\alpha1), (\beta2); (\mu1))$, $j=2$,
(3) $((\alpha1), (\beta2); (\mu2))$, $l-j\geq3$, (4) $((\alpha1), (\beta1); (\mu4))$, $j\leq l-2$,
(5) $((\alpha1), (\beta2); (\mu4))$, $j=l-2$, (6) $((\alpha2), (\beta1); (\mu3))$, $j=2$,

(7) $((\alpha 2), (\beta 2); (\mu 3))$, $j \geq 2$, (8) $((\alpha 2), (\beta 1); (\mu 4))$, $j \leq l-2$,
(9) $((\alpha 2), (\beta 2); (\mu 4))$, $l-j=2$.

Lemma 2.4 is available for all the cases and it thus follows that $\rho(u) \neq 0$.

Case (3): $l(u) \geq 3$. We see the weight spaces with dim≥ 3. Let λ be a weight in Λ and let α, β be weights such that $\lambda = -\alpha + \beta$, where $\alpha \in \Delta_{\mathfrak{p}_-}$ and $\beta \in \Delta_{\mathfrak{k}_-}$. Denote by a_k, b_k, λ_k the k-th components of α, β, λ, respectively. Sinc $a_j = \pm 1$ and $b_j = \pm 1$, it follows that $\lambda_j = 0, \pm 2$.

We first suppose that $\lambda_j = 0$. Then it follows by (6.9) that $\lambda_l = \pm 1$. We suppose that $\lambda_l = 1$. (For the case that $\lambda_l = -1$ we can similarly do the argument mentioned below.) It moreover follows by (6.9) that $\lambda_{l-1} = 0, \pm 1$.

Case (i): $\lambda_{l-1} = -1$. Then the pair $\binom{\alpha}{\beta}$ has the form $\left(\begin{array}{ccc|cc} \overset{j}{-1} & \cdots & -1 & 0 & -1 \\ -1 & \cdots & -1 & -1 & 0 \end{array}\right)$. If the weight space for λ has the dimension more than 3, it follows by Lemma 6.9 that

$$\lambda = (0 \cdots \overset{j}{0} \cdots 0 \mid -11)$$

and the pair $\binom{\alpha}{\beta}$ has the form

$$\left(\begin{array}{ccccccccc|cc} 0 & \cdots & 0 & -1 & \cdots & \overset{j}{-1} & \cdots & -1 & & 0 & -1 \\ 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & & -1 & 0 \end{array}\right). \tag{6.13}$$

Hence for a maximal vector u in this weight space, it follows by Lemma 2.4 that $\rho(u) \neq 0$.

Case (ii): $\lambda_{l-1} = 0$. Then the pair $\binom{\alpha}{\beta}$ has either of the following forms: $\left(\begin{array}{ccc|cc} \overset{j}{-1} & \cdots & -1 & 0 & -1 \\ -1 & & & 0 & 0 \end{array}\right)$, $\left(\begin{array}{ccc|cc} \overset{j}{-1} & & & -1 & -1 \\ -1 & \cdots & -1 & -1 & 0 \end{array}\right)$. If the weight space for λ has the dimension more than 3, it follows by Lemma 6.9 that

$$\lambda = (0 \cdots \overset{j}{0} \cdots 0 1 \cdots 1 \mid 01)$$

and the pair $\binom{\alpha}{\beta}$ has either of the following forms:

$$\left(\begin{array}{cccccccccccc|cc} 0 & \cdots & 0 & -1 & \cdots & \overset{j}{-1} & \cdots & -1 & -1 & \cdots & -1 & & 0 & -1 \\ 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & 0 & \cdots & 0 & & 0 & 0 \end{array}\right), \tag{6.14}$$
$$\left(\begin{array}{cccccccccccc|cc} 0 & \cdots & 0 & -1 & \cdots & \overset{j}{-1} & \cdots & -1 & -2 & \cdots & -2 & & -1 & -1 \\ 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & -1 & \cdots & -1 & & -1 & 0 \end{array}\right).$$

Hence for a maximal vector u in this weight space, it follows by Lemma 2.2 that $\rho(u) \neq 0$.

Case (iii): $\lambda_{l-1} = 1$. Then the pair $\binom{\alpha}{\beta}$ has the form $\left(\begin{array}{c|cc} \overset{j}{-1} & -1 & -1 \\ -1 & 0 & 0 \end{array}\right)$. If the weight space for λ has the dimension more than 3, it follows by Lemma

6.9 that

$$\lambda=(0\cdots\overset{j}{0}\cdots 01\cdots 12\cdots 2\,|\,11)$$

and the pair $\binom{\alpha}{\beta}$ has either of the following forms:

$$(6.15)\qquad \left(\begin{array}{ccccccccccccccccc|cc} 0 & \cdots & 0 & -1 & \cdots & \overset{j}{-1} & \cdots & -1 & -1 & \cdots & -1 & -2 & \cdots & -2 & -1 & -1 \\ 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right),$$

$$\left(\begin{array}{cccccccccccccc|cc} 0 & \cdots & 0 & -1 & \cdots & \overset{j}{-1} & \cdots & -1 & -2 & \cdots & -2 & -2 & \cdots & -2 & -1 & -1 \\ 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 & 0 & 0 \end{array}\right).$$

For a maximal vector u in this weight space, Lemma 2.2 is available if λ is a root and Lemma 2.4 is available if λ is not a root. Hence it follows that $\rho(u)\neq 0$.

We next suppose that $\lambda_j=2$. (For the case that $\lambda_j=-2$ we can similarly do the argument mentioned below.) Then it follows by (6.9) that $\lambda_l=1$ and $\lambda_{l-1}=0, 1, 2$.

Suppose that $\lambda_{l-1}=0$ (resp. $\lambda_{l-1}=2$). Then the pair $\binom{\alpha}{\beta}$ has the following form: $\left(\begin{array}{ccc|cc} \overset{j}{-1} & \cdots & -1 & 0 & -1 \\ 1 & & & 0 & 0 \end{array}\right)$ (resp. $\left(\begin{array}{ccc|cc} \overset{j}{-1} & & & -1 & -1 \\ 1 & \cdots & 1 & 1 & 0 \end{array}\right)$). By Lemma 6.9 the weight space with this λ has at most dimension 2.

Suppose that $\lambda_{l-1}=1$. Then the pair $\binom{\alpha}{\beta}$ has either of the following forms: $\left(\begin{array}{ccc|cc} \overset{j}{-1} & \cdots & -1 & 0 & -1 \\ 1 & \cdots & 1 & 1 & 0 \end{array}\right)$, $\left(\begin{array}{c|cc} \overset{j}{-1} & -1 & -1 \\ 1 & 0 & 0 \end{array}\right)$. If the weight space for λ has the dimension more than 3, it follows by Lemma 6.9 that

$$\lambda=(0\cdots 01\cdots 12\cdots\overset{j}{2}\cdots 2\,|\,11)$$

and the pair $\binom{\alpha}{\beta}$ has one of the following forms:

$$(6.16)\qquad \left(\begin{array}{cccccccccccccccc|cc} 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & \cdots & \overset{j}{-1} & \cdots & -1 & \cdots & -1 & 0 & -1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & 1 & 0 \end{array}\right),$$

$$\left(\begin{array}{ccccccccccccccc|cc} 0 & \cdots & 0 & -1 & \cdots & -1 & -1 & \cdots & \overset{j}{-1} & \cdots & -1 & \cdots & -1 & 0 & -1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & 1 & 0 \end{array}\right),$$

$$\left(\begin{array}{cccccccccccccc|cc} 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & \cdots & \overset{j}{-1} & \cdots & -1 & -2 & \cdots & -2 & -1 & -1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 \end{array}\right),$$

$$\left(\begin{array}{cccccccccccccc|cc} 0 & \cdots & 0 & -1 & \cdots & -1 & -1 & \cdots & \overset{j}{-1} & \cdots & -1 & -2 & \cdots & -2 & -1 & -1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 \end{array}\right).$$

For a maximal vector u in this weight space, Lemma 2.2 is available if λ is a root and Lemma 2.4 is available if λ is not a root. Hence it follows that $\rho(u)\neq 0$.

We next see the injectivity of ρ for Case (c): $\mathcal{V}=(\mathfrak{g}, \tau, \sigma)$. Note that in this case ρ is a homomorphism of $(\mathfrak{p}_-^{\mathbf{C}})^*\otimes\mathfrak{p}_+^{\mathbf{C}}$ to $\wedge^2(\mathfrak{p}_-^{\mathbf{C}})^*\otimes\mathfrak{k}_-^{\mathbf{C}}$. The minus multiple of dominant weights in $\Delta_{\mathfrak{p}_-}$ are given by $(\alpha 1)$, $(\alpha 2)$ and the dominant weights in $\Delta_{\mathfrak{p}_+}$ are given by $(\beta 1)\sim(\beta 4)$:

$$\begin{array}{ll} (\alpha 1)-(1\cdots\overset{j}{1}2\cdots 2|11)\,, & (\alpha 2)\quad (0\cdots 0\overset{j}{1}\cdots 1|01)\,, \\ (\beta 1)\quad (0\cdots\overset{j}{0}12\cdots 2|11)\,, & (\beta 2)\quad (12\cdots\overset{j}{2}\cdots 2|11)\,, \\ (\beta 3)-(0\cdots\overset{j}{0}\cdots 0|01)\,, & (\beta 4)-(0\cdots 01\overset{j}{2}\cdots 2|11)\,. \end{array}$$

Case (1): $l(u)=1$. Represent u as follows: $u=a\,\omega_\alpha\otimes X_\beta$. Then the pair (α,β) is one of the pairs $((\alpha\,\mathrm{r}),(\beta\,\mathrm{s}))$, where $r=1, 2$ and $s=1, 2, 3, 4$. Applying Lemma 2.3 for each pair, we obtain that $\rho(u)\neq 0$.

Case (2): $l(u)=2$. We first suppose that u is indecomposable. Consider the following elements in $\Delta_{\mathfrak{k}_+}$:

$$\begin{array}{ll} (\mu 1)\quad (10\cdots\overset{j}{0}\cdots 0|00)\,, & (\mu 2)\quad (0\cdots\overset{j}{0}10\cdots 0|00)\,, \\ (\mu 3)\quad (0\cdots 01\overset{j}{0}\cdots 0|00)\,, & (\mu 4)\quad (0\cdots\overset{j}{0}\cdots 0|10)\,. \end{array}$$

Then such the triples $(\alpha,\beta';\mu)$ as in Case (2) of type AI are given in the following:

$$\begin{array}{ll} (1)\quad ((\alpha 1),(\beta 4);(\mu 1)),\ j=3\,, & (2)\quad ((\alpha 1),(\beta 3);(\mu 2)),\ j=l-3\,, \\ (3)\quad ((\alpha 2),(\beta 2);(\mu 3)),\ j=3\,, & (4)\quad ((\alpha 2),(\beta 1);(\mu 4)),\ j=l-3\,. \end{array}$$

Lemma 2.2 is available for all the cases and thus $\rho(u)\neq 0$.

We next suppose that u is decomposable. Put $u=a\,\omega_{\alpha_1}\otimes X_{\beta_1}+b\,\omega_{\alpha_2}\otimes X_{\beta_2}$. Then there exists one possible case when $l=4$, $j=2$, i.e., the pairs (α_i,β_i) are $((\alpha 1),(\beta 3))$ and $((\alpha 2),(\beta 2))$. In this case λ is a root and Lemma 2.2 is also available. Hence it follows that $\rho(u)\neq 0$.

Case (3): $l(u)\geq 3$. We see the weight spaces with $\dim\geq 3$. Let λ be a weight in Λ and let α,β be weights such that $\lambda=-\alpha+\beta$, where $\alpha\in\Delta_{\mathfrak{p}_-}$ and $\beta\in\Delta_{\mathfrak{p}_+}$. Since $a_j=\pm 1$ and $b_j=0, \pm 2$, it follows by (6.9) that $\lambda_j=\pm 1, \pm 3$.

We first suppose that $\lambda_j=1$. (For the case that $\lambda_j=-1$ we can similarly do the argument mentioned below.) Then it follows by (6.1) that $\lambda_l=0, 2$.

Case (i): $\lambda_l=0$. Then the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has either of the following forms: $\begin{pmatrix} & \overset{j}{-1} & \big| & \overset{l}{-1} \\ 0\cdots & 0 & \big| & -1 \end{pmatrix}$, $\begin{pmatrix} & \overset{j}{1} & \big| & 1 \\ & 2\cdots 2 & \big| & 1\,1 \end{pmatrix}$. If the weight space for λ has the dimension more than 3, it follows by Lemma 6.9 that

$$\lambda=(0\cdots 01\cdots\overset{j}{1}\cdots 10\cdots 0|00),\quad (0\cdots 01\cdots\overset{j}{1}\cdots 1|10)\,.$$

For the former λ the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has one of the following forms:

(6.17)
$$\begin{array}{ccccccccccccccccc} & & & & & j & & & & & & & & & & & \\ \multicolumn{17}{c}{\begin{pmatrix} 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & -2 & \cdots & -2 & -2 & \cdots & -2 & | & -1 & -1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & -1 & \cdots & -1 & -2 & \cdots & -2 & | & -1 & -1 \end{pmatrix}}\end{array},$$

$$\begin{array}{ccccccccccccccccc} & & & & & j & & & & & & & & & & & \\ \multicolumn{17}{c}{\begin{pmatrix} 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & -1 & \cdots & -1 & -2 & \cdots & -2 & | & -1 & -1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & -1 & \cdots & -1 & -2 & \cdots & -2 & | & -1 & -1 \end{pmatrix}}\end{array},$$

$$\begin{array}{ccccccccccccc} & & & & & j & & & & & & & \\ \multicolumn{13}{c}{\begin{pmatrix} 0 & \cdots & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & | & 0 & -1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & \cdots & -1 & | & 0 & -1 \end{pmatrix}}\end{array},$$

$$\begin{array}{cccccccccccccccc} & & & & & & & & j & & & & & & & \\ \multicolumn{16}{c}{\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 1 & 2 & \cdots & 2 & | & 1 & 1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 2 & \cdots & 2 & \cdots & 2 & 2 & \cdots & 2 & | & 1 & 1 \end{pmatrix}}\end{array},$$

$$\begin{array}{cccccccccccccccc} & & & & & & & & j & & & & & & & \\ \multicolumn{16}{c}{\begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & 1 & 2 & \cdots & 2 & | & 1 & 1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 2 & \cdots & 2 & \cdots & 2 & 2 & \cdots & 2 & | & 1 & 1 \end{pmatrix}}\end{array}.$$

Hence for a maximal vector u in this weight space, it follows by Lemma 2.2 that $\rho(u) \neq 0$. For the latter λ it similarly follows that $\rho(u) \neq 0$.

Case (ii): $\lambda_l = 2$. Then the pair $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ has the form $\begin{array}{cccc} & j & & l \\ \multicolumn{4}{c}{\begin{pmatrix} & -1 & | & -1 \\ 0 \cdots & 0 & | & 1 \end{pmatrix}}\end{array}$. If the weight space for λ has the dimension more than 3, it follows by Lemma 6.9 that

$$\lambda = (0 \cdots 0 1 \cdots \overset{j}{1} \cdots 1 \overbrace{2 \cdots 2}^{a} \overbrace{3 \cdots 3}^{b} | 12) \text{ or}$$
$$(0 \cdots 0 1 \cdots \overset{j}{1} \cdots 1 \overbrace{2 \cdots 2}^{a} \overbrace{3 \cdots 3}^{b} 4 \cdots 4 | 22)$$

where $a > 0$, $b > 0$, and the weight space for this λ has just dimension 3. For the latter λ the pair $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ has one of the following forms:

(6.18)
$$\begin{array}{cccccccccccccccccccc} & & & & & j & & & & & & & & & & & & & & \\ \multicolumn{20}{c}{\begin{pmatrix} 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -2 & \cdots & -2 & | & -1 & -1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 1 & 2 & \cdots & 2 & 2 & \cdots & 2 & | & 1 & 1 \end{pmatrix}}\end{array},$$

$$\begin{array}{cccccccccccccccccccc} & & & & & j & & & & & & & & & & & & & & \\ \multicolumn{20}{c}{\begin{pmatrix} 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & -1 & \cdots & -1 & -2 & \cdots & -2 & -2 & \cdots & -2 & | & -1 & -1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 2 & \cdots & 2 & | & 1 & 1 \end{pmatrix}}\end{array},$$

$$\begin{array}{cccccccccccccccccccc} & & & & & j & & & & & & & & & & & & & & \\ \multicolumn{20}{c}{\begin{pmatrix} 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots & -1 & -2 & \cdots & -2 & -2 & \cdots & -2 & -2 & \cdots & -2 & | & -1 & -1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 2 & \cdots & 2 & | & 1 & 1 \end{pmatrix}}\end{array}.$$

Hence for a maximal vector u in this weight space, it follows by Lemma 2.4 that $\rho(u) \neq 0$. For the former λ it similarly follows that $\rho(u) \neq 0$.

We next suppose that $\lambda_j = 3$. (For the case that $\lambda_j = -3$ we can similarly do the argument mentioned below.) Then, by (6.9), the pair $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ has the form

$\begin{array}{cccccc} j & & & & & \\ \multicolumn{6}{c}{\begin{pmatrix} -1 & & & | & & -1 \\ 2 & \cdots & 2 & | & 1 & 1 \end{pmatrix}}\end{array}$. If the weight space for λ has the dimension more than 3, it follows by Lemma 6.9 that

$$\lambda=(0\cdots0\ \overbrace{1\cdots1}^{a}\ \overbrace{2\cdots2}^{b}\ 3\cdots\overset{j}{3}\cdots3\,|\,12)\ \text{or}$$
$$(0\cdots0\ \overbrace{1\cdots1}^{a}\ \overbrace{2\cdots2}^{b}\ 3\cdots\overset{j}{3}\cdots34\cdots4\,|\,22)$$

where $a>0$, $b>0$, and the weight space for this λ has just dimension 3. For the latter λ the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has one of the following forms:

(6.19)

$$\begin{pmatrix}0\cdots0 & 0\cdots0 & 0\cdots0 & -1\cdots\overset{j}{-1}\cdots-1 & -2\cdots-2 & | & -1 & -1\\ 0\cdots0 & 1\cdots1 & 2\cdots2 & 2\ \cdots\ 2\ \cdots\ 2 & 2\ \cdots\ 2 & | & 1 & 1\end{pmatrix},$$

$$\begin{pmatrix}0\cdots0 & 0\cdots0 & -1\cdots-1 & -1\cdots\overset{j}{-1}\cdots-1 & -2\cdots-2 & | & -1 & -1\\ 0\cdots0 & 1\cdots1 & 1\ \cdots\ 1 & 2\ \cdots\ 2\ \cdots\ 2 & 2\ \cdots\ 2 & | & 1 & 1\end{pmatrix},$$

$$\begin{pmatrix}0\cdots0 & -1\cdots-1 & -1\cdots-1 & -1\cdots\overset{j}{-1}\cdots-1 & -2\cdots-2 & | & -1 & -1\\ 0\cdots0 & 0\ \cdots\ 0 & 1\ \cdots\ 1 & 2\ \cdots\ 2\ \cdots\ 2 & 2\ \cdots\ 2 & | & 1 & 1\end{pmatrix}.$$

For a maximal vector u in this weight space, it follows by Lemma 2.4 that $\rho(u)\neq0$. For the former λ it similarly follows that $\rho(u)\neq0$.

We last see the injectivity of ρ for Case (b): $\mathcal{V}=(\mathfrak{g}, \sigma, \sigma\tau)$. Note that in this case ρ is a homomorphism of $(\mathfrak{p}_+^C)^*\otimes\mathfrak{p}_-^C$ to $\wedge^2(\mathfrak{p}_+^C)^*\otimes\mathfrak{k}_-^C$. Hence we may regard roots α, β in this case as roots $-\beta$, $-\alpha$ in Case (c), respectively. We retain the notations in Case (c).

Case (1): $l(u)=1$. The pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ is one of the pairs $(-(\beta\, s), -(\alpha\, r))$, where $s=1, 2, 3, 4$ and $r=1, 2$. It follows by Lemma 2.3 that $\rho(u)\neq0$ for all cases.

Case (2): $l(u)=2$. We first suppose that u is indecomposable. Then the triples $(\alpha, \beta'; \mu)$ are given as follows:

(1) $(-(\beta4), -(\alpha1); (\mu1))$; (2) $(-(\beta3), -(\alpha1); (\mu2))$;
(3) $(-(\beta2), -(\alpha2); (\mu3))$; (4) $(-(\beta1), -(\alpha2); (\mu4))$.

Lemma 2.2 is available for all the cases and thus it follows that $\rho(u)\neq0$.

We next suppose that u is decomposable. Then there exists one possible case when $l=4, j=2$, i.e., Pairs (α_i, β_i) are $(-(\beta3), -(\alpha1))$ and $(-(\beta2), -(\alpha2))$. In this case λ is a root and Lemma 2.2 is also available. Hence it follows that $\rho(u)\neq0$.

Case (3): $l(u)\geq3$. Similarly to Case (3) in Case (c), we have the cases which correspond to (6.17), (6.18), (6.19). Lemma 2.2 is available for the former one and Lemma 2.4 is available for the other cases. Hence it follows that $\rho(u)\neq0$.

Summing up the above arguments, we have the following result for PSLA's in $\mathcal{D}_j$; the homomorphism ρ is always injective.

Theorem 6.12. *Let $\mathcal{V}$ be the G-orbit which corresponds to a PSLA in a family of type DIII. Then the $\mathcal{V}$-geometry does not admit non-totally geodesic $\mathcal{V}$-submanifolds.*

Case $\mathcal{D}_0$: The family $\mathcal{D}_{2\,;\,134}$.

In this case all PSLA's in $\mathcal{D}_0$ are equivalent to each other. Then the corresponding symmetric space M is $SO(8)/SO(4)\times SO(4)$ and the totally geodesic $\mathcal{V}$-submanifold N is locally four copies of $\mathfrak{su}(2)/\mathfrak{s}(\mathfrak{u}(1)\oplus\mathfrak{u}(1))$.

Put $\sigma=\theta_2$ and $\tau=\theta_{134}$ and consider the PSLA $(\mathfrak{g}, \sigma, \tau)$. Then it holds that

$$\Delta^+_{\mathfrak{k}_-} = \{(10\,|\,00), (00\,|\,10), (00\,|\,01), (12\,|\,11)\}\,,$$
$$\Delta^+_{\mathfrak{p}_-} = \{(11\,|\,00), (01\,|\,10), (01\,|\,01), (11\,|\,11)\}$$

and so a weight λ in Λ is one of the following:

$$\begin{aligned}&\pm(01\,|\,00),\ \pm(01\,|\,11),\ (11\,|\,01),\ (\pm(11\,|\,10),\ \pm(01\,|\,20)\,,\\ &\pm(1-1\,|\,-10),\ \pm(1-1\,|\,0-1),\ \pm(11\,|\,-10)\,,\\ &\pm(11\,|\,0-1),\ \pm(01\,|\,1-1),\ \pm(21\,|\,00),\ \pm(21\,|\,11)\,,\\ &\pm(11\,|\,21),\ \pm(01\,|\,02),\ \pm(11\,|\,12),\ \pm(23\,|\,11)\,,\\ &\pm(13\,|\,21),\ \pm(13\,|\,12),\ \pm(23\,|\,22)\,.\end{aligned}$$

Suppose that u is a maximal vector in this weight space. If λ is a root, Lemma 2.2 is available and thus $\rho(u)\neq 0$. If λ is not a root, the weight space has just dimension 1. It follows by Lemma 2.3 that $\rho(u)\neq 0$.

Theorem 6.13. *Let $\mathcal{V}$ be the G-orbit which corresponds to a PSLA in the family $\mathcal{D}_0$. Then the $\mathcal{V}$-geometry does not admit non-totally geodesic $\mathcal{V}$-submanifolds.*

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