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GENERALIZATIONS OF NAKAYAMA RING VII

(HEREDITARY RINGS)

Dedicated to Professor Takasi Nagahara on his 60th birthday

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We have studied left serial rings with $(*, 1)$ or $(*, 2)$ in [7] and [8] as a generalization of Nakayama ring (generalized uniserial ring).

In this note, we shall replace the assumption "left serial" to "hereditary", and give, in Sections 2~5, characterizations of an artinian hereditary ring with $(*, n)$ in terms of the structure of R; $n \leqslant 3$. In Section 6, we shall study another type of hereditary algebras over an algebraically closed field, i.e., right US-n hereditary algebras.

1. Hereditary rings

Throughout this paper we assume that a ring R is a left and right artinian ring with identity. We shall use the notations and terminologies given in [2]~[8]

First we recall the definition of $(*, n)$.

$(*, n)$ *Every maximal submodule of a direct sum of n hollow modules is also a direct sum of hollow modules* [2] *and* [4]

In this case we may restrict ourselves to a direct sum of hollow modules of a form eR/K, where e is a primitive idempotent and K is a submodule of eR [4].

Let R be an artinian hereditary ring. Then R is isomorphic to the ring of generalized tri-angular matrices over simple rings [1]. We are interested in a hereditary ring with $(*, n)$, and so we may assume that R is basic. Then

$$(1) \qquad R \approx \begin{pmatrix} \Delta_1 & M_{12} \cdots\cdots & M_{1n} \\ & \Delta_2 \; M_{23} \cdots & M_{2n} \\ & \ddots & \vdots \\ 0 & \ddots & \vdots \\ & & \Delta_n \end{pmatrix}$$

where the Δ_i are division rings and the M_{ij} are left Δ_i and right Δ_j modules. It is clear that $M_{ij} = e_i R e_j$ ($e_i = e_{ii}$ matrix units).

Lemma 1. *Let R be a hereditary ring as above. Then for any t, $\sum_{j\geq t}\oplus Re_j$ (resp. $\sum_{j<t}\oplus e_jR$) is an ideal and $R/\sum_{j\geq t}\oplus Re_j$ (resp. $R/\sum_{j<t}\oplus e_jR$) is also hereditary.*

Proof. This is clear from [1], Theorem 1.

Lemma 2. *Every non-zero element in* $\mathrm{Hom}_R(e_iR, e_jR)$ $(i\leqslant j)$ *is a monomorphism.*

Proof. Since e_iR is indecomposable and $f(e_iR)$ is projective for $f\in\mathrm{Hom}_R(e_iR, e_jR)$, this is clear.

Let R be a ring as (1). We may study hollow modules e_iR/A by the initial remark. Put $e=e_i$ and $H=\{h\,|\,M_{ih}\neq 0\}$, $J=\{j\,|\,M_{ij}=0\}$, and further put $E_i=\sum_{h\in H}e_h$, $R_i=E_iRE_i$ and $X_i=\sum_J\oplus e_jR\oplus\sum_{k<i}\oplus e_kR$. Since R is hereditary, $e_hRe_j=0$ for $h\in H$ and $j\in J$ (cf. [1]), and so X_i is a two sided ideal in R by Lemma 1 and $R_iX_i=0$. If $e_pRe_q\neq 0$ for $p\in H$, then $0\neq e_iRe_pe_pRe_q\subset e_iRe_q$ by [1], and so $q\in H$. Hence $e_pR=e_pRE_i$ and

$$R_i=E_iR \quad\text{and}\quad R_iX_i=0\,. \tag{2}$$

It is clear that $R=R_i\oplus X_i$ as R-modules and R_i is hereditary (cf. [1]). Hence every R_i-submodule in R_i is nothing but an R-submodule in R_i from (2). Further let $h_1<h_2<\cdots<h_p$ $(h_i\in H)$, then we note that $e_{h_1}Re_{h_q}\neq 0$ for all q. Therefore we obtain

Lemma 3. *Let R be a hereditary ring as in* (1) *and let R_i be as above. Then $(*, n)$ holds for any n hollow modules if and only if, for any i, the same holds on any R_i-modules. Further R_i satisfies $e_{h_1}Re_{h_q}\neq 0$ for all $h_q>h_1$.*

Next we shall observe a construction of hereditary (basic) rings. In order to make the observation clear, we shall first give an example.
Let

$$R=\begin{pmatrix} K_{11} & 0 & K_{13} & K_{14} & 0 & K_{16} & 0 & K_{18}\\ & K_{22} & 0 & K_{24} & 0 & K_{26} & 0 & K_{28}\\ & & K_{33} & K_{34} & 0 & 0 & 0 & 0\\ & & & K_{44} & 0 & 0 & 0 & 0\\ & & & & K_{55} & K_{56} & 0 & K_{58}\\ & 0 & & & & K_{66} & 0 & K_{68}\\ & & & & & & K_{77} & K_{78}\\ & & & & & & & K_{88}\end{pmatrix},$$

where $K_{ij}=K$ is a field.

We take non-zero entries in e_1R and put

$$R_1 = \begin{pmatrix} K_{11} & K_{13} & K_{14} & K_{16} & K_{18} \\ & K_{33} & K_{34} & 0 & 0 \\ & & K_{44} & 0 & 0 \\ & 0 & & K_{66} & K_{68} \\ & & & & K_{88} \end{pmatrix}$$

Since K_{22} does not appear in R_1 (since $M_{12}=0$), we take

$$R_2 = \begin{pmatrix} K_{22} & K_{24} & K_{26} & K_{28} \\ & K_{44} & 0 & 0 \\ 0 & & K_{66} & K_{68} \\ & & & K_{88} \end{pmatrix}$$

Since K_{55} does not appear in R_1 and R_2, put

$$R_5 = \begin{pmatrix} K_{55} & K_{56} & K_{58} \\ 0 & K_{66} & K_{68} \\ & & K_{88} \end{pmatrix}$$

Similarly to the above, we put

$$R_7 = \begin{pmatrix} K_{77} & K_{78} \\ 0 & K_{88} \end{pmatrix}$$

Then

$$A_{12} = \begin{pmatrix} K_{44} & 0 & 0 \\ 0 & K_{66} & K_{68} \\ 0 & 0 & K_{88} \end{pmatrix}$$

is the common components between R_1 and R_2. Similarly we can define

$$A_{15} = A_{25} = \begin{pmatrix} K_{66} & K_{68} \\ 0 & K_{88} \end{pmatrix}.$$

$$A_{17} = A_{27} = A_{57} = (K_{88}).$$

We note that the products in R of two components in R_i and R_j not contained in A_{ij} are zero. Now R_1 and R_2 are of right local type (see §5) and R_3 and R_4 are right serial. Further we know from the above note that R is the subring of $R_1 \oplus R_2 \oplus R_5 \oplus R_7$ given by identifying elements in the same K_{ij}, namely in A_{ij}. If we carefully observe the above constructions, we know that only some right ideals contained in $(1_i - e_1^{(i)})R_i$ are identified, where 1_i is the identity of R_i and

$e_1^{(i)}$ is the matrix unit in R_i.

We shall study the above fact in general. Let

$$(3) \qquad R = \begin{pmatrix} M_{11} M_{12} \cdots\cdots M_{1n} \\ \quad M_{22} \cdots\cdots M_{2n} \\ \quad \ddots \quad \vdots \\ 0 \quad \ddots \quad \vdots \\ \quad\quad M_{nn} \end{pmatrix}$$

where $M_{ii} = \Delta_i$ are division rings. We define R_i as before Lemma 3 and express R_i as

$$(4) \qquad R_i = \begin{pmatrix} M_{11}^{(i)} \cdots\cdots M_{1n_i}^{(i)} \\ 0 \ddots \quad \vdots \\ \quad \ddots \quad \vdots \\ \quad\quad M_{n_i n_i}^{(i)} \end{pmatrix}$$

where $M_{jk}^{(i)}$ is equal to some M_{lm} in (3) ($M_{11}^{(i)} = M_{ii}$ in (3)) and $M_{1k}^{(i)} \neq 0$ for all k.

We note first the following fact: Assume $M_{ab} \neq 0$ for some a and b. Put $I_a = \{x \mid M_{ax} \neq 0\}$ and $I_b = \{y \mid M_{by} \neq 0\}$. Since $M_{ab}R \approx e_b R^{(m)}$ (direct sum of m-copies of $e_b R$),

$$(5) \qquad I_a \subset I_b\,.$$

Starting with R_1 ($=R_{t_1}$), from the initial observation we can construct R_{t_h} so that $M_{11}^{(i)}$ does not appear on the diagonal of $R_{t_{h'}}$ for all $t_{h'} < i = t_h$ and so that each component M_{pq} in (3) appears at least once in some R_{t_s}. Take R_i and R_j ($t_h = i < j = t_{h'}$), and assume that $M_{kk'}^{(i)} = M_{ss'}^{(j)}$ ($= M_{pq}$ in (3)) are common components between R_i and R_j. Then $M_{kk}^{(i)} = M_{ss}^{(j)} = (M_{pp}$ in (3)) are also common ones between R_i and R_j by the definition of R_{t_h} and $R_{t_{h'}}$. We shall consider those components in (3). It is clear from (5) that

$$(6) \qquad e_k^{(i)} R_i = e_p R = e_s^{(j)} R_j\,.$$

Now let

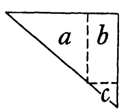
$$e_k^{(i)} R_i = (0 \cdots 0\, M_{kk}^{(i)}\, 0 \cdots M_{kk_2}^{(i)}\, 0 \cdots M_{kk_t}^{(i)}) = e_s^{(j)} R_j; \qquad M_{kk_l}^{(i)} \neq 0\,.$$

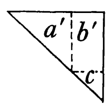
Then $e_{k_l}^{(i)} R_i = e_{s_l}^{(j)} R_j$ for all $l \leqslant t$ from (5). By A_{ij} we shall denote the right ideal whose components appear in R_i and R_j. Let I_i and I_j be as before (5) where $i = t_h$ and $j = t_{h'}$ and put $I_i \cap I_j = \{\pi_1 < \pi_2 < \cdots < \pi_s\}$. Then we know from the argument above that

(7)
i) $A_{ij} = \sum \oplus e_{\pi_k} R$,
ii) $A_{ij} e_p R = 0$ for $p \notin \{\pi_1, \cdots, \pi_s\}$,
and so
iii) the lattice of right R-modules of A_{ij} is equal to the lattice of right A_{ij}-modules of A_{ij}.

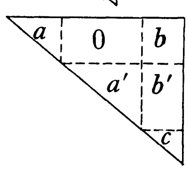
Finally we assume for some b $(1\leqslant b\leqslant n)$ that (M_{ab} in (3)) $=M^{(i)}_{uv}\neq 0$ and (M_{bc} in (3)) $=M^{(j)}_{xy}\neq 0$. Then $b\in I_i\cap I_j$ and so $M^{(k)}_{xy}\subset A_{ij}$ from (7)-i) and ii). Hence the product in R of an entry of R_i and one of R_j is zero if the latter (and hence two of them) is not contained in A_{ij}. Thus we can find a set $\{R_{i_t}\}$ of hereditary rings such that $e^{(i_t)}_1 R_{i_t} e^{(i_t)}_k\neq 0$ for all k and a set $\{A_{i_t,i_{t'}}\}$ of right ideals as (7), and R is the subring of $\sum\oplus R_{i_t}$ such that the entries in $A_{i_t,i_{t'}}$ of R_{i_t} are equal to the entries in $A_{i_t,i_{t'}}$ of $R_{i_{t'}}$. Conversely, let $\{R_i\}^m_{i=1}$ be a set of hereditary (basic) rings and $\{A_{ij}\}$ a set of right ideals in R_i and R_j which satisfy (7) where we replace R with R_i and R_j. Then we can easily show that the subring of $\sum\oplus R_i$ whose components in A_{ij} are identified for all i,j is a hereditary ring. We shall call such a ring the *patched ring* of $\{R_i\}$ with respect to (briefly w.r.t.) $\{A_{ij}\}$, (the name comes from the following examples).

We shall give some examples of the patched ring. In the following examples, tri-angules and squares mean tri-angular matrices and matrices over a field K, respectively and straight lines do vector spaces over K.

1 $R_1=$ 

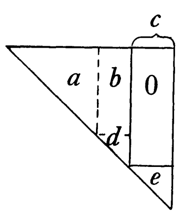
$R_2=$ 

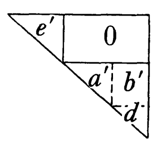
and $A_{11}=$ 

Then 

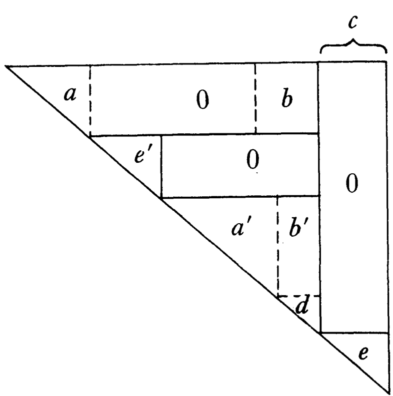
is the patched ring of R_1 and R_2 *w. r. t* A_{11},

2

$R_3=$ 

$R_4=$ 

and $B_{11}=$ 

Then 

is the patched ring of R_3 and R_4 w.r.t. B_{11}.

We note that R_1 and R_2 are left and right serial, but R is not left serial. R_3 and R_4 are of right local type, but R is not and $(*, 3)$ holds (see §§4 and 5). We shall show in §5 that every hereditary (basic) algebras over an algebraically closed field with $(*, 3)$ is obtained as the patched ring of R_1's and R_3's above.

Thus we obtain

Proposition 1. *Let R be a hereditary* (*basic*) *ring. Then R is the patched ring of hereditary rings $\{R_i\}$ such that $e_1^{(i)}R_ie_k^{(i)} \neq 0$ for all k, where $e_p^{(i)}$ is the matrix unit e_{pp} in R_i.*

REMARK 1. Let R be a hereditary ring which is one of R_i given in Proposition 1. Since $e_1Re_j \neq 0$, e_jR is monomorphic to e_1R. Hence, if the structure of e_1R is known as right R-modules, then we can see those of e_iR (cf. Theorem 2).

2. Hereditary rings with $(*,1)$

We shall first give some remarks on $(*, 1)$. If R satisfies $(*, 1)$, for $eJ^i \supset C$ $eJ/C = \sum_{i=1}^{n} \oplus A_i$, with A_i hollow. Since A_i is hollow, $A_iJ = \sum_j \oplus B_{ij}$ with B_{ij} hollow by $(*, 1)$. Hence $eJ^2/C = \sum_i \oplus A_iJ = \sum_i \sum_j \oplus B_{ij}$. By induction

(8) eJ^i/C is a direct sum of hollow modules.

In general, we assume that a module M is a direct sum of submodules M_i. For submodules N_i of M_i, we call $\sum_i \oplus N_i$ a *standard submodule* of M (with respect to the decomposition $\sum_i \oplus M_i$).

Proposition 2. *Let N be a finitely generated R-module. Then the following are equivalent*:

1) *N is a direct sum of hollow modules.*

2) *Let P be a projective cover of $N (P \xrightarrow{f} N)$. Then ker f is a standard submodule of P with respect to a suitable direct decomposition of indecomposable modules.*

3) *Let P' be projective and $f' : P' \to N$ an epimorphsim. Then ker f' is a standard submodule of P' as* 2).

Proof. Every hollow module is of a form eR/A. Hence 1)$\leftrightarrow$2) and 3)$\to$ 2) are clear.
2)$\to$3) Let

$$0 \to K' \to P' \to N \to 0$$

be exact with P' projective. Since P is a projective cover of N, there exist $g: P\to P'$ and $h: P'\to P$ such that $hg=1_p$. Let $P=\sum\oplus P_i$ and $\ker f=K=\sum\oplus K_i$ by 2), where the P_i are indecomposable and $K_i\subset P_i$. It is clear that $g(K)\oplus h^{-1}(0)=\sum\oplus g(K_i)\oplus h^{-1}(0)\subset\ker f'$ and $P'=g(P)\oplus h^{-1}(0)$. Hence $\ker f'=\sum\oplus g(K_i)\oplus h^{-1}(0)\subset\sum\oplus g(P_i)\oplus h^{-1}(0)=P'$.

We shall study, in this section, a hereditary ring with $(*, 1)$ as a right R-module. Hence we may assume that R is basic. We shall give a characterization of a hereditary ring with $(*, 1)$.

In the following, $\alpha, \beta, \cdots$ mean indices and $|i, \alpha, \beta, \cdots, \eta|$ means a natural number related with the index $(i, \alpha, \beta, \cdots, \eta)$. If R is a basic hereditary ring,

$$\begin{aligned} & J(e_iR) = e_iJ = N(i, \alpha)\oplus N(i, \beta)\oplus N(i, \gamma)\oplus\cdots, \\ & \qquad \text{where} \quad N(i, \alpha)\approx e_{|i,\alpha|}R,\ N(i, \beta)\approx e_{|i,\beta|}R,\ \cdots, \\ (9)\qquad & J(N(i, \alpha)) = N(i, \alpha, \alpha_1)\oplus N(i, \alpha, \alpha_1')\oplus\cdots, \\ & \qquad \text{where} \quad N(i, \alpha, \alpha_1)\approx e_{|i,\alpha,\alpha_1|}R,\ N(i, \alpha, \alpha_1')\approx e_{|i,\alpha,\alpha_1'|}R, \end{aligned}$$

and so on. It is clear that $i<|i, \alpha|<|i, \alpha, \alpha_1|<|i, \alpha, \alpha_1, \alpha_2|$ and so on, and

$$(10)\qquad e_iRe_j = M_{ij} = \sum_{|i,\cdots,\gamma|=j}\oplus N(i, \cdots, \gamma)e_j\,.$$

Theorem 1. *Let R be a hereditary (basic) ring and $N(i, \cdots, \gamma)$ be as in (9). Then the following conditions are equivalent:*

1) $(*, 1)$ *holds for any hollow right R-module.*

2) *The following conditions are satisfied.*

i) *Let $i<k=|i, \alpha|\leq j=|i, \beta|\,(\alpha\neq\beta)$, i.e., e_iJ contains two direct summands isomorphic to e_kR and e_jR, respectively. If $N(i, \alpha, \cdots, \gamma)$ and $N(i, \beta, \cdots, \gamma')$ with $|i, \alpha, \cdots, \gamma|=|i, \beta, \cdots, \gamma'|=h$ appear in (9), i.e., for some h, simultaneously $e_kRe_h\neq 0$ and $e_jRe_h\neq 0$, then e_jR is uniserial, and hence $[M_{jp}: \Delta_q]\leqslant 1$ for $q>j$. Further if we denote exactly $N(i, \alpha, \cdots, \gamma)$ as $N(i, \alpha, \alpha_2, \cdots, \alpha_t=\gamma)$, there exists a (unique) s such that $|i, \alpha, \alpha_2, \cdots, \alpha_s|=j$.*

ii) *If $M_{jq}=x\Delta_q\ (q>j)$, there exists an isomorphism σ of Δ_q onto Δ_j such that $x\delta=\sigma(\delta)x$ for all δ in Δ_q.*

3) *For any submodule A in e_iJ^k for any k, there exists a direct decomposition $e_iJ^k=\sum\oplus P_\alpha$ such that $A=\sum\oplus A_\alpha$; $A_\alpha\subset P_\alpha$ and P_α is indecomposable, i.e., A is a standard submodule of e_iJ^k with respect to the decomposition $\sum\oplus P_\alpha$.*

4) *For any submodule A in e_iJ, there exists a direct decomposition $e_iJ=\sum_\alpha\oplus N(i, \alpha)'$ such that $A=\sum\oplus A_\alpha$; $A_\alpha\subset N(i, \alpha)'$ and $N(i, \alpha)'\approx N(i, \alpha)$, i.e., A is a standard submodule of e_iJ with respect to the decomposition $\sum\oplus N(i, \alpha)'$,*

Proof. 1)$\to$2) Assume $(*, 1)$ and $i=1$ from Lemma 1. Put $i_1=|1, \alpha|$ and $i_2=|1, \beta|$. Assume $N(1, \alpha, \cdots, \gamma)$ and $N(1, \beta, \cdots, \gamma')$ appear in (9) for

$k=|1, \alpha, \cdots, \gamma|=|1, \beta, \cdots, \gamma'|$. Then $M_{1k}\neq 0$, $M_{i_1k}\neq 0$ and $M_{i_2k}\neq 0$. First we shall show $e_{i_2}R$ is monomorphic to $e_{i_1}R$ and $[M_{i_2k}: \Delta_k]=1$. If we can show that $e_{i_1}R$ contains a non-zero element y in $M_{i_1i_2}$, $e_{i_2}R\to yR\subset e_{i_1}R$ $(e_{i_2}\to y)$ is a monomorphism from Lemma 2. Hence we may assume $\Delta_{k+1}=\cdots=\Delta_n=0$ from Lemma 1. We shall identify $N(1, \alpha)$ with $e_{i_1}R$ (resp. $N(1, \beta)$ with $e_{i_2}R$). From the above assumption let $M_{i_2k}=\sum_{j=1}^{n}\oplus A_j$; the A_j are simple R-modules and $[A_j: \Delta_k]=1$. Since $e_{i_1}R\supset M_{i_1k}\supset N(1, \alpha, \cdots, \gamma)\neq 0$, there exists a natural homomorphism

$$f\colon\ M_{i_2k}/\sum_{j\geqslant 2}\oplus A_j\approx A_1\to M_{i_1k}\,.$$

From the assumption $(*, 1)$, f is extendible to an element h' in $\mathrm{Hom}_R(e_{i_2}R/\sum_{j\geqslant 2}\oplus A_j, e_{i_1}R)$ by [6], Theorem 4 (note that $\mathrm{Hom}_R(e_{i_1}R, e_{i_2}R/\sum_{j\geqslant 2}\oplus A_j)=0$ by Lemma 2 in case of $i_1=i_2$ and $j\geqslant 2$ and that we identify $e_{i_1}R$ and $e_{i_2}R$ with $N(1, \alpha)$ and $N(1, \beta)$, respectively). Consider a homomorphism

$$h\colon\ e_{i_2}R\to e_{i_2}R/\sum_{j\geqslant 2}\oplus A_j\xrightarrow{h'} e_{i_1}R\,.$$

Since $h\neq 0$ is a monomorphism by Lemma 2, $M_{i_2k}=A_1$. Therefore

(11) $\qquad e_{i_2}R$ is monomorphic to $e_{i_1}R$ and $[M_{i_2k}: \Delta_k]=1$, provided $M_{i_2k}\neq 0$.

We shall show similarly to (11) that $e_{i_2}R$ is uniserial. Put $e_{i_2}=e$ and $eJ^t\approx\sum_{j=1}^{v}\oplus e_{b(j)}R$ for some t, since R is hereditary. Let B be a simple submodule of $e_{b(1)}R$. Then we obtain a monomorphism of $(B\oplus\sum_{j\geqslant 2}\oplus e_{b(j)}R)/\sum_{j\geqslant 2}\oplus e_{b(j)}R\approx B$ to $e_{i_1}R$ (see (11)). From the argument before (11), $\sum_{j\geqslant 2}\oplus e_{b(j)}R=0$, and so $eJ^t\approx e_{b(1)}R$ and eJ^t/eJ^{t+1} is simple. Therefore eR is uniserial. Next assume $M_{i_2k}=x\Delta_k$ and we show ii). Hence we may assume $\Delta_{k+1}=\cdots=\Delta_n=0$ from Lemma 1. For any δ in Δ_k, define an endomorphism φ of M_{i_2k} by setting $\varphi(x\delta')=x\delta\delta'$. We may regard φ as an isomorphism of M_{i_2k} onto $N(1, \alpha, \cdots, \gamma)$ $(|1, \alpha, \cdots, \gamma|=k)$. Further, for an extension g (in $\mathrm{Hom}_R(eR, e_{i_1}R)\subset\mathrm{Hom}_R(eR, e_1R)$) of φ by [6], Theorem 4, $g(eRe)\subset e_1Re_{i_2}=M_{1i_2}=\sum\oplus N(1, \alpha, \cdots, \varepsilon)e_{i_2}$. Noting the structure (9) and $g(M_{i_2k})=\varphi(M_{i_2k})=N(1, \alpha, \cdots, \gamma)$, we obtain

(12) $\quad$ some $N(1, \alpha, \cdots, \varepsilon')$ contains $N(1, \alpha, \cdots, \gamma)$ and $N(1, \alpha, \cdots, \varepsilon')\approx eR$.

Therefore φ is extendible to an element in $\mathrm{Hom}_R(eR, eR)=\Delta_{i_2}$ (take the projection to $N(1, \alpha, \cdots, \varepsilon')$), which implies that there exists δ^* in Δ_{i_2} such that $\delta^*x=x\delta$. It is clear that the mapping: $\delta\to\sigma(\delta)=\delta^*$ is a monomorphism. We shall show that σ is an isomorphism. Let δ^{**} be an element in Δ_{i_2}. Since

$M_{i_2k}=x\Delta_k$ is a left Δ_{i_2}-module, $\delta^{**}x=x\delta''$ for some δ'' in Δ_k. Hence $\delta^{**}=\sigma(\delta'')$. The last part of i) is clear from (12) and its argument.

2)$\to$1) Assume that i) and ii) are satisfied. We shall show that the condition ii) of [6], Theorem 4 is fulfiled, and so we may study a case $e=e_1$ by Lemma 1. Let

$$e_1J=N(1,\alpha)\oplus N(1,\beta)\oplus\cdots$$

and $C_1\supset D_1$ (resp. $C_2\supset D_2$) submodules in $N(1,\alpha)\approx e_{i_1}R$ (resp. $N(1,\beta)\approx e_{i_2}R$, $i_1\leq i_2$) such that C_1/D_1 is simple and f^{-1}: $C_1/D_1\approx C_2/D_2$. We shall show that f is extendible to an element in $\mathrm{Hom}_R(N(1,\beta)/D_2, N(1,\alpha)/D_1)$. First we note for any R-module E in e_kR,

$$E=E(\sum_{j\geqslant k}e_j)=\sum_{j\geqslant k}\oplus Ee_j \quad\text{and}\quad Ee_j\subset M_{kj}\,. \tag{13}$$

Since $C_1/D_1\approx C_2/D_2$, $N(1,\alpha,\cdots,\gamma)$ and $N(1,\beta,\cdots,\gamma')$ appear in e_1R for some $|1,\alpha,\cdots,\gamma|=|1,\beta,\cdots,\gamma'|=h$ from (13). Hence $N(1,\beta)\,(\approx e_{i_2}R)$ is uniserial by i) and $C_2=M_{i_2h}\oplus M_{i_2h_1}\oplus\cdots\oplus M_{i_2h_t}\supset D_2=M_{i_2h_1}\oplus\cdots\oplus M_{i_2h_t}$ from (13), where $h<h_1<\cdots<h_t$. We may identify $N(1,\alpha)$ with $e_{i_1}R$. Let $M_{i_2h}=x\Delta_h$ and take a representative $f(x)$ of $f(x+D_1)$ in M_{i_1h} from (13); $f(x)=\sum x_p$; $0\neq x_p\in N(1,\alpha,\cdots,\gamma_p)$ from (10) ($|1,\alpha,\cdots,\gamma_p|=h$). Since $x_p\neq 0$, $N(1,\alpha,\cdots,\gamma_p)\subset N(1,\alpha,\cdots,\delta_p)$ ($|i,\alpha,\cdots,\delta_p|=i_2$) from i), and $N(1,\alpha,\cdots,\delta_p)\neq N(1,\alpha,\cdots,\delta_{p'})$ if $p\neq p'$, since $e_{i_2}R$ is uniserial. Put $N=\sum\oplus N(1,\alpha,\cdots,\delta_p)\subset N(1,\alpha)$, $C_1'=C_1\cap N$ and $D_1'=D_1\cap N$, $f(x)$ being in C_1' and $f(x)\notin D_1$, $C_1=C_1'+D_1$, and so $C_1/D_1\approx C_1'/(C_1'\cap D_1)=C_1'/D_1'$. On the other hand, $x_p=x_pe_h$ for all p. Hence the mapping: $x_1\to x_p$ is extendible to an element g_p in $\mathrm{Hom}_R(N(1,\alpha,\cdots,\delta_1), N(1,,\alpha\cdots,\delta_p))\,(\approx\Delta_{i_2})$ from i) and ii). Then $N=N(1,\alpha,\cdots,\delta_1)\,(\sum_{q\geqslant 2}g_q)\oplus\sum_{q\geqslant 2}\oplus N(1,\alpha,\cdots,\delta_q)$ and $f(x)\in N(1,\alpha,\cdots,\delta_1)\,(\sum_{q\geqslant 2}g_q)\,(=N^*)$, where $T(u)$ means the graph of a module T with respect to a homomorphism u. Further $C_1/D_1\approx(C_1'\cap N^*)/(D_1'\cap N^*)$ as above. Now $C_1'\subset N^*\subset N^*\,(\approx e_{i_2}R)\subset N\subset N(1,\alpha)$ and $D_1'\cap N^*=J(C_1'\cap N^*)\approx D_2$. Hence we obtain the natural homomorphism

$$\begin{array}{l} N(1,\beta)/D_2\xrightarrow{u}N^*/(D_1'\cap N^*)\to N(1,\alpha)/(D_1'\cap N^*)\to \\ (x+D_2)\to f(x)+(D_1'\cap N^*)\to f(x)+(D_1'\cap N^*)\to \\ \quad N(1,\alpha)/D_1\,, \\ \quad (f(x)+D_1) \end{array}$$

where u is an extension of f given by i) and ii), which is an extension of f.

4)$\to$1) This is clear from the definition of $(*,1)$.

3)$\to$4) This is trivial.

1)$\to$3) This is clear from (8) and Proposition 2.

REMARK 2. We shall study the situation of 2)—ii) of Theorem 1. Let e_kR and e_jR be as in i). Assume

$$e_{j_1}R = (0\cdots\Delta_j\, 0\, M_{j_1j_2}\, 0\cdots M_{j_1j_3}\, 0\cdots M_{j_1j_t}\, 0)\,, \qquad (j=j_1 \text{ and } M_{pq}\neq 0)\,.$$

Then

$$(14)\qquad \begin{aligned} e_{j_2}R &= (0\cdots\cdots 0 \quad \Delta_{j_2} \quad 0\cdots\cdots M_{j_2j_3}\cdots\cdots M_{j_2j_t} \;\; 0) \\ &\approx (0\cdots\cdots 0 \;\; M_{j_1j_2} \;\; 0\cdots\cdots M_{j_1j_3}\cdots\cdots M_{j_1j_t} \;\; 0) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ e_{j_t}R &= (0\cdots\cdots 0\cdots\cdots\cdots 0 \cdots\cdots\cdots\cdots 0 \cdots \Delta_{j_t} \quad 0)\,, \end{aligned}$$

since $e_{j_1}R$ is uniserial. Further $M_{j_1j_s}=m'_{j_1j_s}\Delta_{j_s}$. In order to simplify the notations, we express j_i by i. Then $M_{ij}\neq 0$ for $i\leqslant j$. Every element in $\mathrm{End}_R(M_{1s}R/M_{1s+1}R)$ is extendible to an element in $\mathrm{End}_R(e_1R/M_{1s+1}R)$ by the proof after (12). Further, since $(0\cdots 0\ M_{ls}\cdots M_{lt})\approx(0\cdots M_{1s}\cdots M_{1t})$ for all l and s, every element in $\mathrm{End}_R(M_{ls}R/M_{ls+1}R)=\Delta_s$ is extendible to an element in $\mathrm{End}_R(e_lR/M_{ls+1}R)=\Delta_l$. Hence there exists an isomorphism $\varphi'_{ls}\colon \Delta_s\to\Delta_l$ (since $M_{ls}=m'_{ls}\Delta_s$, φ'_{ls} is an epimorphism) such that

$$(15)\qquad m'_{ls}x=\varphi'_{ls}(x)m'_{ls}\,, \qquad \text{where}\quad x\in\Delta_s \text{ and } M_{ls}=m'_{ls}\Delta_s$$

from the proof of Theorem 1. We fix generators $m_{i,i+1}$ of $M_{i,i+1}$ for all i and $\varphi_{i,i+1}\colon \Delta_{i+1}\to\Delta_i$ related with the fixed $m_{i,i+1}$ in (15). Then $m_{i,i+1}m_{i+1,i+2}\cdots m_{i+k,i+k+1}=m_{i,i+k+1}$ is a generator of $M_{i,i+k+1}$ and $\varphi_{i,i+k+1}=\varphi_{i,i+1}\cdots\varphi_{i+k,i+k+1}\colon \Delta_{i+k+1}\to\Delta_i$ is an isomorphism and satisfies (15) (cf [1], Lemma 13). Hence we may assume

$$(16)\qquad (e_{j_1}+\cdots+e_{j_t})R(e_{i_1}+\cdots+e_{j_t}) \approx \begin{pmatrix} \Delta_{j_1} & \Delta_{j_1} & \cdots\cdots & \Delta_{j_1} \\ & \Delta_{j_1} & \cdots\cdots & \Delta_{j_1} \\ & & \ddots & \vdots \\ 0 & & & \vdots \\ & & & \Delta_{j_1} \end{pmatrix}$$

Next assume that e_jR is uniserial only as in (14). Then by the similar argument as above, we obtain

$$(16')\qquad (e_{j_1}+\cdots+e_{j_t})R(e_{j_1}+\cdots+e_{j_t}) \approx \begin{pmatrix} \Delta_{j_1} & \Delta_{j_2} & \cdots\cdots & \Delta_{j_t} \\ & \Delta_{j_2} & \cdots\cdots & \Delta_{j_t} \\ & & \ddots & \vdots \\ 0 & & & \vdots \\ & & & \Delta_{j_t} \end{pmatrix}$$

and the $\varphi_{ij}\colon \Delta_i\to\Delta_j$ $(i<j)$ are monomorphisms (cf. [1], Lemma 13). By $\mathrm{T}_t(\Delta_{j_1})$ and $\mathrm{T}_t(\Delta_{j_1}, \Delta_{j_2}, \cdots, \Delta_{j_t})$ we denote the above rings (16) and (16'), respectively.

3. Hereditary rings with $(*, 2)$

We shall give a characterization of hereditary rings with $(*, 2)$.

Theorem 2. *Let R be a hereditary (basic) ring. Then $(*,2)$ holds for any two hollow right R-modules if and only if, for each e_i $(=e_{ii})$,*

$e_iJ=\sum_{k=1}^{n_i}\oplus A_k$, where the A_k are uniserial modules, which satisfy the following conditions:

i) If $A_k\not\approx A_{k'}$ for $k\neq k'$, any sub-factor modules of A_k are not isomorphic to ones of $A_{k'}$.

ii) If $A_k\approx A_{k'}$, $(\approx e_jR)$ $(k\neq k')$ and $M_{jp}=x\Delta_p$ $(j<p)$, there exists an isomorphism $\delta\colon \Delta_p\to\Delta_j$ as in 2)-ii) of Theorem 1.

Proof. Assume that $(*,2)$ holds. Then the A_i are uniserial by [8], Propoistion 7. As in the proof of Theorem 1, we consider a case $i=1$ from Lemma 1. Let

$$
\begin{aligned}
e_1J = {} & N_{11}\oplus N_{12}\oplus \cdots\cdots \oplus N_{1t_1}\\
& \oplus N_{21}\oplus N_{22}\oplus \cdots\cdots \oplus N_{2t_2}\\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
& \oplus N_{q1}\oplus N_{q2}\oplus \cdots\cdots \oplus N_{qt_q}\,,
\end{aligned}
\tag{17}
$$

where $N_{j1}\approx N_{js}\approx e_{i_j}R$ for all j, s and $N_{j1}\not\approx N_{j'1}$ if $j\neq j'$ and $i_1<i_2<\cdots<i_q$.

Assume that N_{21} contains a non-zero sub-factor module isomorphic to one of N_{11}. Then N_{21} is monomorphic (via g) to N_{11} by (13) and Theorem 1. It is clear that $N_{21}(g)\oplus N_{22}\oplus\cdots\oplus N_{2t_2}$ $(\approx N_{21}\oplus\cdots\oplus N_{2t_2})$ is a direct summand of e_1J. Hence from the assumptions (17) above and [8], Proposition 12, there exists j in e_1Je_1 $(=0)$ such that $(e+j)(N_{21}\oplus\cdots\oplus N_{2t_2})=N_{21}(g)\oplus N_{22}\oplus\cdots\oplus N_{2t_2}$. Hence g must be zero. ii) is clear from Theorem 1, since $(*,1)$ holds. Conversely, we assume i) and ii). Then $(*,1)$ holds by Theorem 1. We shall quote here the similar argument given in [8], Proposition 8. Let e be a primitive idempotent and let $eR/E_1\oplus eR/E_2$ be a direct sum of two hollow modules. We may consider only a maximal submodule M' $(\supset E_1\oplus E_2)$ in $F=eR\oplus eR$ (see [8], Proposition 8). There exists a unit x in eRe such that $F=eR(f)\oplus eR\supset M'=eR(f)\oplus eJ$, where $f(r)=xr$ for $r\in eR$. We shall define $g'\colon eR(f)\to eR$ by setting $g'(r+xr)=-xr$. Then $E_1\oplus E_2=E_1(f)(g')\oplus E_2$. Let $\varphi\colon F\to eR(f)\oplus eR/E_2$ be the natural epimorphism. Then $M=M'/(E_1\oplus E_2)=(eR(f)\oplus eJ/E_2)/(E_1(f)(g'))$. If we identify $eR(f)$ with eR, $M=(eR\oplus eJ/E_2)/\varphi(E_1(g))$, where $g=-f$. First we consider the structure of $\varphi(E_1(g))$. If eR/E_1 is simple, either $M'/(E_1\oplus E_2)\supset eR/E_1$ or $M'/(E_1\oplus E_2)\oplus eR/E_1=F/(E_1\oplus E_2)$. Hence $M'/(E_1\oplus E_2)$ is a direct sum of hollow modules, since $(*,1)$ holds. Therefore we may assume $E_1\subsetneqq eJ$. Let $eJ=\sum_{i=1}^{m}\oplus A_i$; the A_i are hollow. From i) of the theorem, we can express the index set $I=\{1,\cdots,m\}$ as the disjoint union $I=I_1\cup I_2\cup\cdots\cup I_p$ such that

$$A_i\approx A_j \text{ if } i,j\in I_t, \quad\text{and}\quad A_i\not\approx A_j \text{ if } i\in I_t,\ j\in I_{t'} \text{ and } t\neq t'\,.$$

We put $F_i=\sum_{I_i}\oplus A_k$ then $eJ=\sum_{i=1}^{p}\oplus F_i$, (cf. (17)). Since these F_i have the particular property above, $E_1=\sum_{i=1}^{p}\oplus C_i$; $C_i\subset F_i$, $E_2=\sum_{i=1}^{p}\oplus G_i$; $G_i\subset F_i$ and $\bar{g}(C_i)\subset F_i/G_i$, where $\bar{g}$ is induced from g. Hence

$$(18)\qquad M\approx(eR\oplus eJ/E_2)/\sum\oplus C_i(\bar{g})\,.$$

Next we consider $C_1(\bar{g})$. Assume that A_1 has the structure given in ii) of the theorem. Now A_1 has the structure of $e_{j_1}R$ in (16), and so every element in the endomorphism ring of sub-factor module T/L of A_1 is extendible to an element in $\mathrm{End}(A_1/L)$. Further $T_1/L_1\approx T_1'/L_1'$ for sub-factor modules T_1/L_1, T_1'/L_1' if and only if $T_1=T_1'$ (and $L_1=L_1'$). From this remark and the following fact: since $C_1(\bar{g})\subset eJ\oplus F_1/G_1$, for any submodule L in $eJ\oplus F_1$, $(eRJ\oplus F_1)/L\approx eR/X_1'\oplus F_1/G_1'$, where G_1' is a (standard) submodule of F_1 and X_1' is a submodule of eJ (cf. [8], Proposition 8), we can find an isomorphism:

$$(19)\qquad (eR\oplus eJ/E_2)/C_1(\bar{g})\approx eR/X_1'\oplus F_1/G_1'\oplus\sum_{k\neq 1}\oplus F_k/G_k$$
$$\text{and}\quad \sum\oplus C_i(\bar{g})/C_1(\bar{g})\subset eR/X_1'\oplus\sum_{k\neq 1}\oplus F_k/G_k\,,$$

(see the proof of Theorem 5 below and [8], Proposition 8).

Finally assume $F_1=A_1$, i.e., I_1 is a singlton. Then $C_1/X_1\approx\bar{g}(C_1)$, where $X_1=\bar{g}^{-1}(0)\cap C_1$. Since g is an isomorphism of A_1 to F_1 and A_1 is uniserial, $g(X_1)=G_1$. Hence we have the same situation as above (take g^{-1}). Accordingly we finally obtain from (19)

$$M\approx eR/\sum X_i'\oplus\sum\oplus F_i'/G_i'\colon\ F_i'\approx F_i\,,$$

which is a direct sum of hollow modules by Theorem 1.

Let R be a hereditary ring with $(*, 2)$. We shall assume $e_1R=(\Delta_1 M_{12}M_{13}\cdots M_{1n})$ and $M_{1j}\neq 0$ for all j from Lemma 3. $e_1J=(0\,M_{12}\cdots M_{1n})=\sum_{i=1}^{q}\oplus F_i$ as in the proof of Theorem 2. Following $\{F_i\}_{i=1}^{q}$ we divide the index set $\{2, 3, \cdots, n\}$ into q-parts $I=I_1\cup I_2\cup\cdots\cup I_q$ such that $F_ie_j\neq 0\leftrightarrow j\in I_i$. Then $I_i\cap I_j=\phi$ if $i\neq j$ by i) of Theorem 2. Put $|F_i/F_iJ|=p_i$. If $p_i=1$, F_i is uniserial, and so $F_i=m_{1i_1}\Delta_{i_1}\oplus m_{1i_2}\Delta_{i_2}\oplus\cdots\oplus m_{1i_t}\Delta_{i_t}$, where the i_s runs through over I_i and $\Delta_1\subset\Delta_{i_1}\subset\cdots\subset\Delta_{i_t}$ are division rings (see (16')). If $p_i\geqslant 2$, $F_i=(m_{1i_1}\Delta_{i_1})^{(P_i)}\oplus(m_{1i_2}\Delta_{i_1}{}^{(P_i)})\oplus\cdots\oplus(m_{1i_t}\Delta_{i_1})^{(P_i)}$, where $(m_{1i_t}\Delta_{i_1}{}^{(P_i)})$ means a direct sum of p copies of $m_{1i_1}\Delta_{i_1}$. Since $e_1Re_i\neq 0$ and R is hereditary, e_iR is monomorphic to e_1R by Lemma 2. On the other hand, the image of e_iR is a submodule of F_j for some j by i) of Theorem 2. Hence $e_iR\approx m_{1j_k}\Delta_{j_k}\oplus m_{1j_{k+1}}\Delta_{j_{k+1}}\oplus\cdots\oplus m_{1j_t}\Delta_{j_t}$ or $\approx m_{1j_k}\Delta_{j_k}\oplus m_{1j_{k+1}}\Delta_{j_k}\oplus\cdots\oplus m_{1j_t}\Delta_{j_k}$ $(1<i=j_k)$ from Lemma 2. Therefore R is determined by $\{F_i\}$, provided $e_1Re_i\neq 0$ for all i. Since R is hereditary and $I_i\cap I_j=\phi$ $(i\neq j)$, $M_{lm}=0$

if $l \in I_i$ and $m \in I_j$ $(i \neq j)$.

Next let R_0 be a hereditary ring as in (1) and assume $R_0 \approx \sum \oplus S_i$ as rings. Then after renumbering $\{e_i = e_{ii}\}$, we may assume

$$R_0 = \begin{pmatrix} S_1 & & & \\ & S_2 & 0 & \\ & & \ddots & \\ & 0 & & \ddots \\ & & & & S_t \end{pmatrix}.$$

By E_i we denote the identity element in S_i. On the other hand, for any hereditary ring R as in (1)

$$R = e_1R \oplus R_0' \qquad \text{as } R\text{-modules},$$

where $R_0' = (1-e_1)R(1-e_1)$ and e_1R is a two-sided ideal of R by Lemma 1. If $R_0' \approx \sum \oplus S_i$ as above, $e_1J = \sum \oplus e_1RE_j$. Put $A_j = e_1RE_j$, and A_j is a right ideal in e_1R. We use those notations in the following theorem. Thus we obtain

Theorem 3. *Let R be a* (*basic*) *hereditary ring such that $e_1Re_j \neq 0$ for all j. Then the following conditions are equivalent*:

1) $(*, 2)$ *holds for any two hollow modules.*

2) *R/e_1R is a direct sum of right serial rings S_j*; 1) $S_j = \mathrm{T}_r(\Delta_{j_1}, \Delta_{j_2}, \cdots, \Delta_{j_r})$ *or* 2) $\mathrm{T}_r(\Delta_j)$ *and* $A_j = (\Delta_{j_1}, \Delta_{j_2}, \cdots, \Delta_{j_r})$ *in Case* 1), $A_j = (\Delta_j^{(p_j)}, \cdots, \Delta_j^{(p_j)})$ *is a left* Δ $(=e_1Re_1)$- *and right Δ_j-modules in Case* 2), *where $\Delta \subset \Delta_{j_1} \subset \cdots \subset \Delta_{j_r}$ are division rings.*

3) *R is isomorphic to*

$$\begin{pmatrix} \Delta & A_1 \cdots\cdots A_1 & & \\ & S_1 & & 0 \\ & & S_2 & \\ 0 & & & \ddots \\ & & & & S_r \end{pmatrix}. \tag{20}$$

where $S_k = \mathrm{T}_{r_k}(\Delta_{k1}, \Delta_{k2}, \cdots, \Delta_{kr_k})$ *or* $\mathrm{T}_{r_k}(\Delta_k)$.

Theorem 3′. *Let R be a* (*basic*) *hereditary ring. Then* $(*, 2)$ *holds if and only if R is a patched ring of hereditary rings given in* (20).

Lemma 4. *Let R be a hereditary and connected* (*basic*) *ring.* 1) *If R is a left serial ring, then $e_1Re_j \neq 0$ for all $j > 1$.* 2) *Conversely, if $e_1Re_j \neq 0$ for all j, and $[M_{ij}: \Delta_j] \leqslant 1$, $[M_{ij}: \Delta_i] \leqslant 1$ for all i and j, then R is left serial.*

Proof. 1) Let $e_1R = e_1\Delta \oplus M_{12} \oplus \cdots \oplus M_{1n}$. We divide the index set $\{2, 3, \cdots, n\}$ into two sets I, J such that $M_{1i} \neq 0$ provided $i \in I$ and $M_{1j} = 0$ provided $j \in J$. Take M_{1i} and consider M_{ji}. If $M_{ji} \neq 0$ for $j \in J$, $RM_{ji} \not\supset M_{1i}$, since $M_{1j}=0$. Hence $M_{ji}=0$ for all $i \in J$ by assumption. Hence $R=(e_{11}R \oplus \sum_{k \in I} \oplus e_k R) \oplus (\sum_{k' \in J} \oplus e_{k'} R)$ as rings from (2). Therefore $J=\phi$ by assumption.

2) Assume $0 \neq e_1 R e_j = \Delta_1 m_{1j} = m_{1j} \Delta_j$ for all j. Since R is hereditary, $e_1 J = \sum \oplus A_i$; the A_i are hollow and no sub-factor modules of A_i are isomorphic to any ones of A_j $(i \neq j)$ from (13) and the assumption $[M_{1j}: \Delta_j] \leqslant 1$. Similarly $J(A_i) = \sum \oplus A_{ij}$ and so on (cf. [7]). Hence any indecomposable (projective) module in eJ is equal to some $A_{i_1 i_2 \cdots i_t}$. Let $M_{it}=m_{it}\Delta_t=\Delta_i m_{it}$ and $M_{jt}=m_{jt}\Delta_t=\Delta_j m_{jt}$ $(i<j)$ for a fixed t. Then $m_{1i}e_iR$ and $e_{1j}e_jR$ have a common sub-factor module in e_1R. Hence e_jR is monomorphic to e_1R from the initial remark, and so $e_iRe_j \neq 0$, which implies $Rm_{it} \subset Rm_{jt}$. Therefore R is left serial.

Theorem 4. *Let R be a connected (basic) hereditary ring. Then R is a left serial ring with $(*,2)$ as right R-modules if and only if R is isomorphic to*

$$\begin{pmatrix} \Delta & \Delta \cdots\cdots \Delta & \Delta \cdots\cdots \Delta \cdots\cdots \Delta \\ & \mathrm{T}_{r_1}(\Delta_1) & & \\ & & \mathrm{T}_{r_2}(\Delta_2) & 0 \\ & 0 & & \ddots \\ & & & \mathrm{T}_{t_r}(\Delta_r) \end{pmatrix}$$

where $\Delta_i \subset \Delta$ are division rings.

Proof. Assume that R is a left serial ring with $(*,2)$ as right R-modules. Then R is isomorphic to the ring in (20) by Theorem 3 and Lemma 4. Since R is left serail, the A_i in (20) are isomorphic to Δ as left Δ-modules and $\Delta_{k1}=\Delta_{k2}=\cdots=\Delta_{kr_k}$ in (20). If we take a generator of A_i, we know $\Delta_i \subset \Delta$. The converse is clear from the structure of the diagram.

4. Hereditary rings with $(*,3)$

We have already obtained a characterization of artinian rings with $(*,3)$ and $|eJ/eJ^2| \leqslant 2$ in [5]. As is seen in [5], Theorem 1, the structure of such artinian rings is a little complicated. However if R is a hereditary ring with $|e_{ii}J/e_{ii}J^2| \leqslant 2$, we obtain the following theorem.

We quote here a particular property of a vector space (cf. [2] and [7]).

($\sharp$, m) *Let Δ_1 and Δ_2 be division rings and V a left Δ_1, right Δ_2-space. For any two right Δ_2-subspaces V_1, V_2 with $|V_1|=|V_2|=m$, there exists x in Δ_1 such that $xV_1=V_2$.*

Theorem 5. *Let R be a hereditary (basic) ring with $|eJ/eJ^2| \leqslant 2$ for each $e=e_i$. Then $(*,3)$ holds for any three hollow modules if and only if $eJ=A_1 \oplus A_2$ such that*

1) *The A_i are as in Theorem 2, and further if $A_1 \approx A_2$,* 2) $[\Delta : \Delta(A_1)]=2$ *and*

3) *eJ/eJ^2 satisfies* ($\sharp$, 1) *as a left Δ-module and right Δ'-module, where $A_1 \approx e_jR$, $\Delta = eRe$, $\Delta' = e_jRe_j$, and $\Delta(A_1) = \{x \mid \in \Delta, xA_1 \subset A_1\}$.*

Proof. Assume $eJ = A_1 \oplus B_1$ as in the theorem. If $A_1 \not\approx B_1$, $\Delta(C) = \Delta$ for every submodule C in eJ by i) of Theorem 2. Assume $A_1 \approx B_1$ ($\approx e_jR$). Then A_1 and B_1 have the structure of eR as in (16). For any C, there exists submodules $C_1 \supset D_1$ in A_1 and $C_2 \supset D_2$ in B_1 such that $f\colon C_1/D_1 \approx C_2/D_2$ and $C = \{x + D_1 + f(x) + D_2 \mid x \in C_1\}$. From (16), f is extendible to an element $g\colon A_1/D_1 \to B_1/D_2$. Since ($\sharp$, 1) is satisfied for $eJ/eJ^2 = u_1\Delta_j \oplus v_1\Delta_j$, there exist α in Δ and z in Δ_j such that $u_1 + g(u_1) = \alpha u_1 z + w$, $w \in eJ^2$. However, since u_1, v_1 are in $eJ - eJ^2$ and $u_1 e_j = u_1$, $v_1 = v_1 e_j$, $w = 0$. Hence $C = C_1(f) + D_1 \oplus D_2 = \alpha(C_1 \oplus D_2)$, (note that $D_1 \approx D_2$ and $\alpha(D_1 \oplus D_2) = D_1 \oplus D_2$ and that A_1 is uniserial). It is clear that $\Delta(A_1) \subset \Delta(C_1 \oplus D_1) = \Delta(\alpha^{-1}C) = \alpha^{-1}\Delta(C)\alpha$ and so $[\Delta\colon \Delta(C)] \leqslant 2$. Thus the conditions in [5], Theorem 1 are fulfiled, and hence ($*$, 3) holds by [5]. Theorem 2. Conversely, assume ($*$, 3) holds. Then 1) and 2) are clear from Theorem 2 and [5], Theorem 1. We shall show 3). We may assume from Lemma 1 and [2], Lemma 1 that $\Delta_{j+1} = \cdots = \Delta_n = 0$. Then 2) of [2], Theorem 1 is nothing but ($\sharp$, 1).

As in Lemma 3, if $e_1Re_j \neq 0$ for all j, R in Theorem 5 is isomorphic to

$$\begin{pmatrix} \Delta \ \Delta_1 \ \Delta_2 \cdots\cdots \Delta_r & \Delta_{r+1} \cdots\cdots\cdots\cdots \Delta_{r+s} \\ T_r\,(\Delta_1 \ \Delta_2 \cdots \Delta_r) & 0 \\ 0 \qquad 0 & T_s(\Delta_{r+1} \cdots \Delta_{r+s}) \end{pmatrix},$$

where $\Delta \subset \Delta_1 \subset \cdots \subset \Delta_r$ and $\Delta \subset \Delta_{r+1} \subset \cdots \subset \Delta_{r+s}$, or

$$\begin{pmatrix} \Delta & \Delta_1^{(2)} \ \Delta_1^{(2)} \cdots\cdots\cdots \Delta_1^{(2)} \\ 0 & T_r(\Delta_1) \end{pmatrix}.$$

where $\Delta_1^{(2)}$ is a left Δ and right Δ_1 space satisfying ($\sharp$, 1) and $[\Delta\colon \Delta(\Delta_1, \cdots, \Delta_1)] = 2$.

In the former ring, $eJ = A_1 \oplus A_2$ and $A_1 \not\approx A_2$. Hence ($*$, n) holds for all n by [5], Theorem 3. We do not know this fact for the latter ring.

5. Hereditary algebras

In this section we consider particular algebras over a field K such that

$$e_iRe_i/e_iJe_i = \bar{e}_iK \qquad \text{([2], Condition II}'') . \tag{21}$$

(e.g. an algebraically closed field.)

Under the assumption (21), every Δ_i in (1) is equal to K. In this case, if eR is uniserial, $[eRe'\colon K] \leqslant 1$ (cf. (14)). Hence

$$\mathrm{End}_R\,(A/A') \approx K \approx \mathrm{End}_R\,(eR/A') \tag{22}$$

for any submodules $A\supset A'$ in eR. Accordingly, from the proof of Theorem 2 (cf. [8], Theorem 2) we obtain

Theorem 6. *Let R be a hereditary K-algebra satisfying* (21). *Then the following conditions are equivalent:*

1) $(*, 2)$ *holds for any two hollow modules.*

2) *Every factor module of $eR\oplus eJ^{(m)}$ is a direct sum of hollow modules for each primitive idempotent e and any integer m.* (*It is sufficient in case $m=1$.*)

If every finitely generated R-module is a direct sum of hollow modules, R is called a ring of right local type [10]. It is clear from the definition that $(*, n)$ holds for a ring of *right local type*. By $T_n(\Delta)$ we denoted the ring of upper tri-angular matrices over a division ring Δ (see (14)).

Theorem 7. *Let R be a hereditary* (*basic*) *K-algebra satisfying* (21). *Then the following are equivalent:*

1) $(*, 3)$ *holds for any three hollow modules, and $e_1Re_j\neq 0$ for all j,* (*and hence $(*, n)$ holds for all n*).

2) *R is isomorphic to* $\begin{pmatrix} T_{m_1}(K) & K\ K\cdots K \\ 0 & T_{m_2}(K) \end{pmatrix}$.

3) *R is of right local type and connected.*

Proof. 1)→2). Since $|eJ/eJ^2|\leqslant 2$ from [4], Theorem 3, we obtain it from the remark after (21) and the last part in §4.

2)→3). It is clear that the ring in 2) is connected and of right local type from Lemma 4 and [10] (see [9]).

3)→1). $(*, 3)$ holds for any three hollow modules. Since R is left serial by [10], and connected, $M_{1j}\neq 0$ by Lemma 4.

Theorem 8. *Let R be a hereditary algebra as above. Then the following conditions are equivalent*:

1) $(*, 3)$ *holds for any three hollow right R-modules.*

2) $eJ=A_1\oplus A_2$, where the A_i are uniserial, and any non-trivial sub-factor modules of A_1 are not isomorphic to ones of A_2. In this case $(*, n)$ holds for all n.

3) *Let $\{N_i\}_{i=1}^k$ be any set of submodules in eR. Then every factor module of $\sum\oplus N_i^{(n_i)}$ is a direct sum of hollow modules.*

4) *Every factor modules of $eR^{(n)}\oplus eJ^{(m)}$ is a direct sum of hollow modules for any integers n and m.* (*It is sufficient in case $n=2$ and $m=1$*).

Proof. 1)↔2) This is clear from Theorem 5 and [2], Theorem 2′.

1)→3). Let $e=e_i$ and let R_i and X_i be as before Lemma 3. Then R_i is of a right local type by Theorem 7. Since $R_iX_i=0$ and $R/X_i=R_i$, every submodule in eR is an R_i-module. Hence every factor module of $\sum\oplus N_i^{(n_i)}$ is also an R_i-module. Therefore it is a direct sum of R_i-hollow (and hence R-hollow) modules.

3)→4). This is clear. (We can show directly 1)→4) in the similar manner to [8], Theorem 2, cf. the proof of Theorem 2.)

3)→1). Let $D=\sum_{i=1}^{3}\oplus eR/E_i$ and M a maximal submodule in D. Then $D'=eR^{(3)}$ contains the submodule M' such that $M'\supset\sum_{i=1}^{3}\oplus E_i$ and $M'/\sum\oplus E_i=M$. Since D' has the lifting property of simple modules modulo the radical, D' has a decomposition $\sum_{i=1}^{3}\oplus F_i$ such that $F_i\approx eR$ and $M'=F_1\oplus F_2\oplus J(F_3)$. Hence M is a factor module of $eR^{(2)}\oplus eJ$. Therefore M is a diect sum of hollow modules from 3).

Theorem 9. *Let R be as in Theorem* 8. *Then* $(*, 3)$ *holds for any three hollow modules if and only if R is the patched ring of serial rings* $\mathrm{T}_r(K)$ *and rings of right local type* $\begin{pmatrix}\mathrm{T}_{r'}(K) & K \; K\cdots K\\ 0 & \mathrm{T}_{r''},\ (K)\end{pmatrix}$.

Proof. This is clear from Proposition 1 and Theorem 7.

6. US-n algebras

We have studied special types of hereditary algebras in §5. We shall show, in this section, that they are related with US-n algebras defined in [4].

As another generalization of right serial ring (cf. $(*, n)$), we considered

$(**, n)$ *Every maximal submodule in a direct sum D of n hollow modules contains a non-zero direct summand of D* [4].

It is clear that if $D/J(D)$ is not homogeneous, D satisfies $(**, n)$. Hence we may restrict ourselves to hollow modules of a form eR/E, where e is a primitive idempotent and E is a submodule of eR. If $(**, n)$ holds for any direct sum of n hollow modules, we call R a right US-n ring [4]. We showed in [4] that R is right US-1 (resp. US-2) if and only if R is semisimple (resp. right uniserial). On the other hand,

Proposition 3 ([6], Proposition 8). *Let R be a right artinian ring. Then R is a right US-m ring for some m if and only if the number of isomorphism classes of hollow modules eR/A is finite and* $[\Delta:\Delta(A)]<\infty$.

If R is an algebra of finite dimension over a field K, $[\Delta:\Delta(A)]<\infty$. Hence from Proposition 3, we know that an algebra of finite representation type is a US-n algebra for some n. Further we note that if K is a finite field, R is a finite ring. Then, since there are only finite non-isomorphic hollow modules,

R is a US-n algebra. Hence we may assume that K is an infinite field.

From now on we assume that R is a K-algebra satisfying (21). Let e be a primitive idempotent in R. Let $\{A_1, A_2, \cdots, A_t\}$ be a set of submodules in eR such that $A_i \not\sim A_j$ for any pair i and j, where $A_i \sim A_j$ means that there exists a unit element x in eRe such that $xA_i \subset A_j$ or $xA_i \supset A_j$. Let $m(e)$ be the maximal number t among the above sets.

Proposition 4. *Let R be an algebra over K satisfying* (21). *Then R is a right US-m if and only if $m = \max_e \{m(e)\} + 1 < \infty$.*

Proof. This is clear from [3], Corollaries 1 and 2 of Theorem 2.

Theorem 10. *Let R be as above. We assume further $J^2 = 0$. Then R is a right US-m algebra if and only if eJ is square-free for each primitive idempotent e.*

Proof. Assume that R is right US-m. Since $J^2 = 0$, $eJ = \sum \oplus A_i$ the A_i are simple, i.e. $A_i \approx \bar{e}_i K$, (R is basic). If $A_i \approx A_j$, $(a_i + a_j k)K \approx A_i$ and $(a_i + a_j k)K \not\sim (a_i + a_j k')K$ for any $k \neq k'$ in K, where $A_i = a_i K$ ([6], Lemma 15). Then R is not right US-m for any m. Hence eJ is square-free. Conversely if eJ is square-free, every submodule in eJ is a sum of some A_i. Hence the number of hollow modules is finite, and so R is right US-m for some m from Proposition 4.

Corollary. *Let R be as above. If R is right US-m, eJ^i/eJ^{i+1} is square-free for all i.*

Proof. It is clear that if R is right US-m, so is R/J^t for any t (cf. [4], Lemma 1). If $J^{n+1} = 0$, eJ^n is semisimple and hence we can employ the same argument given above. Therefore we obtain the corollary by induction on n and the initial remark.

It is clear that the converse is not true provided $J^2 \neq 0$.

Finally we study the ring of generalized tri-angular matrices over division rings Δ_j as (1). If R is a (basic) hereditary ring (more generally if $\operatorname{gl\,dim} R/J^2 < \infty$), R has the structure of (1) [1].

Theorem 11. *Let R be a (basic) algebra satisfying* (21). *Assume* $\operatorname{gl\,dim} R/J^2 < \infty$. *Then R is a US-m algebra for some m if and only if $[e_i R e_j : K] \leqslant 1$ for all i, j.*

Proof. Assume that R is a US-m algebra for some m. We may assume that $\Delta_{k+1} = \cdots = \Delta_k = 0$ in (1) by [4], Lemma 1. Let $M_{ik} = x_1 K \oplus x_2 K \oplus \cdots$. Then $[M_{ij} : K] \leqslant 1$ as the proof of Theorem 10. Conversely, if $[M_{ik} : K] \leqslant 1$, $e_i R$ contains only finitely many right ideals. Hence R is a US-m algebra for some m.

7. Examples

We shall give several examples of hereditary algebras with $(*, n)$.

Let $K \subset L$ be fields.

1. $\begin{pmatrix} K & L \\ 0 & K \end{pmatrix}$ is a hereditary ring with $(*, 2)$ and hence $(*, 1)$. (If $L \neq K$, $(*, 3)$ does not hold from Theorem 8.)

2.
$$\begin{pmatrix} K & \begin{pmatrix} K \\ K \end{pmatrix} & \begin{pmatrix} K \\ 0 \end{pmatrix} & \begin{pmatrix} K \\ K \end{pmatrix} \\ 0 & K & 0 & 0 \\ 0 & 0 & K & K \\ 0 & 0 & 0 & K \end{pmatrix}$$

is a hereditary ring with $(*, 1)$ but not $(*, 2)$. In this ring, eJ is a direct sum of uniserial modules (cf. [8], Theorem 3).

3. $\begin{pmatrix} K & L & L \\ 0 & L & 0 \\ 0 & 0 & L \end{pmatrix}$ is a hereditary ring satisfying $(*, n)$ for all n by Theorem 8

4.
$$\begin{pmatrix} K & K & \begin{pmatrix} K \\ 0 \\ K \end{pmatrix} & \begin{pmatrix} K \\ K \\ K \end{pmatrix} & \begin{pmatrix} K \\ K \\ K \end{pmatrix} \\ 0 & K & \begin{pmatrix} K \\ 0 \\ K \end{pmatrix} & \begin{pmatrix} K \\ K \\ 0 \end{pmatrix} & \begin{pmatrix} K \\ K \\ 0 \end{pmatrix} \\ 0 & 0 & K & K & K \\ 0 & 0 & 0 & K & K \\ 0 & 0 & 0 & 0 & K \end{pmatrix}$$

satisfies all conditions in Theorem 1 except the last one of i).

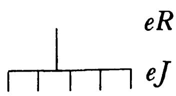
5. Let R be an algebra satisfying (21), and $\mathrm{gl\,dim}\, R/J^2 < \infty$. Then if R is right US-n, R is left US-m from Theorem 10 for some m. However $n \neq m$ in general. For example $R = \begin{pmatrix} K & 0 & K \\ & K & K \\ & & K \end{pmatrix}$. Then R is right US-2 and left US-3.

If R does not satisfy (21), then the above fact is not true. Let $L \supset K$ be fields with $[L : K] = 5$ (not small) and $R = \begin{pmatrix} K & L \\ 0 & L \end{pmatrix}$. Then R is right US-2 but not left US-n for any n.

6. Let K be a field. We can give the complete list of connected algebras given in Theorem 11, provided that R is hereditary and $|R/J|$ is enough small. For instance, let $|R/J| = 6$. We shall give some samples of them.

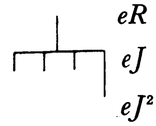
US-11 (and $(*, 2)$)

$$\begin{pmatrix} K & K & K & K & K & K \\ & K & 0 & 0 & 0 & 0 \\ & & K & 0 & 0 & 0 \\ 0 & & & K & 0 & 0 \\ & & & & K & 0 \\ & & & & & K \end{pmatrix}$$



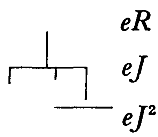
US-8 (and $(*, 2)$)

$$\begin{pmatrix} K & K & K & K & K & K \\ & K & 0 & 0 & 0 & 0 \\ & & K & 0 & 0 & 0 \\ 0 & & & K & 0 & 0 \\ & & & & K & K \\ & & & & & K \end{pmatrix}$$



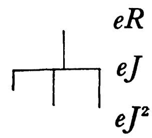
US-7 (and $(*, 1)$)

$$\begin{pmatrix} K & K & K & K & K & K \\ & K & 0 & 0 & 0 & 0 \\ & & K & 0 & 0 & 0 \\ 0 & & & K & K & K \\ & & & & K & 0 \\ & & & & & K \end{pmatrix}$$



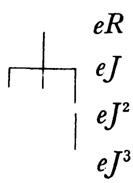
US-6 (and $(*, 2)$)

$$\begin{pmatrix} K & K & K & K & K & K \\ & K & 0 & 0 & 0 & 0 \\ & & K & K & 0 & 0 \\ 0 & & & K & 0 & 0 \\ & & & & K & K \\ & & & & & K \end{pmatrix}$$



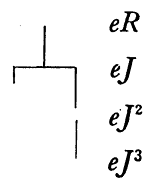
US-5 (and $(*, 2)$)

$$\begin{pmatrix} K & K & K & K & K & K \\ & K & 0 & 0 & 0 & 0 \\ & & K & 0 & 0 & 0 \\ 0 & & & K & K & K \\ & & & & K & K \\ & & & & & K \end{pmatrix}$$



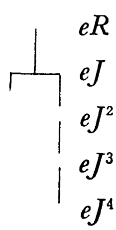
US-4 (and $(*, 3)$)

$$\begin{pmatrix} K & K & K & K & K & K \\ & K & K & 0 & 0 & 0 \\ & & K & 0 & 0 & 0 \\ 0 & & & K & K & K \\ & & & & K & K \\ & & & & & K \end{pmatrix}$$



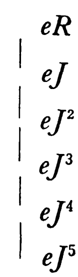
US-3 (and $(*, 3)$)

$$\begin{pmatrix} K & K & K & K & K & K \\ & K & 0 & 0 & 0 & 0 \\ & & K & K & K & K \\ 0 & & & K & K & K \\ & & & & K & K \\ & & & & & K \end{pmatrix}$$



US-2 (and $(*, 3)$)

$$\begin{pmatrix} K & K & K & K & K & K \\ & K & K & K & K & K \\ & & K & K & K & K \\ 0 & & & K & K & K \\ & & & & K & K \\ & & & & & K \end{pmatrix}$$



US-1 (and $(*, 3)$)

$$\begin{pmatrix} K & & & & & \\ & K & & & 0 & \\ & & K & & & \\ 0 & & & K & & \\ & & & & K & \\ & & & & & K \end{pmatrix},$$

where $e=e_1$.

We do not have US-9 and US-10 algebras under the assumption $|R/J|=6$.

References

[1] M. Harada: *Semiprimary hereditary rings and generalized triangular matrix rings,* Nagoya Math. J. **27** (1966), 463–489.

[2] ————: *On maximal submodules of a finite direct sum of hollow modules* III, Osaka J. Math. **21** (1984), 81–98.

[3] ———— and Y. Yukimoto: *On maximal submodules of a finite direct sum of hollow modules* IV, Osaka J. Math. **22** (1985), 321–326.

[4] M. Harada: *Generalizations of Nakayama ring* I, Osaka J. Math. **23** (1986), 181–200.

[5] ————: *Generalizations of Nakayama ring* II, Osaka J. Math. **23** (1986), 509–521.

[6] ————: *Generalizations of Nakayama ring* III, Osaka J. Math. **23** (1986), 523–539.

[7] ———— and Y. Baba: *Generalizations of Nakayama ring* IV, Osaka J. Math. **24** (1987), 139–145.

[8] M. Harada: *Generalizations of Nakayama ring* V, Osaka J. Math. **24** (1987), 373–389.

[9] T. Sumioka: *Tachikawa's theorem on algebras of left colocal type,* Osaka J. Math. **21** (1984), 629–648.

[10] H. Tachikawa: *On rings for which every indecomposable right module has a unique maximal submodule,* Math. Z. **71** (1959), 200–222.

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